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On constructions and parameters of symmetric configurations v_k

Alexander A. Davydov · **Giorgio Faina** ·
Massimo Giulietti · **Stefano Marcugini** ·
Fernanda Pambianco

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Abstract The spectrum of possible parameters of symmetric configurations is investigated. We propose some new constructions, mainly based on Finite Geometry and on extension methods. New parameters are provided, both for general symmetric configurations and for cyclic symmetric configurations. For several values of k, new upper bounds on the minimum integer E such that for each $v \geq E$ there exists a (cyclic) symmetric configuration v_k are obtained.

Keywords Configurations in Combinatorics · Symmetric configurations · Cyclic configurations · Golomb rulers · Projective geometry · LDPC codes

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A. A. Davydov
Institute for Information Transmission Problems (Kharkevich institute), Russian Academy of Sciences
Bol'shoi Karetnyi per. 19, GSP-4, Moscow, 127994, Russian Federation
E-mail: adav@iitp.ru

G. Faina · M. Giulietti · S. Marcugini · F. Pambianco (⊠)
Dipartimento di Matematica e Informatica, Università degli Studi di Perugia,
Via Vanvitelli 1, Perugia, 06123, Italy

G. Faina
E-mail: faina@dmi.unipg.it

M. Giulietti
E-mail: giuliet@dmi.unipg.it

S. Marcugini
E-mail: gino@dmi.unipg.it

F. Pambianco
E-mail: fernanda@dmi.unipg.it Tel.: +39-075-5855006 Fax: +39-075-5855024

1 Introduction

A configuration (v_r, b_k) is an incidence structure of v points and b lines such that each line contains k points, each point is contained in r lines, and two distinct points are joined by at most one line. If $v = b$ the configuration is *symmetric*, and it is usually denoted as a configuration v_k. For a thorough introduction to configurations, see [32–34] and the references therein.

A symmetric configuration v_k is *cyclic* if there exists a cyclic permutation group acting regularly both on the set of its points and on the set of its blocks. Equivalently, v_k is cyclic if one of its incidence matrices is circulant.

Any incidence matrix $\mathbf{M}(v,k)$ of a symmetric configuration v_k is a $v \times v$ binary matrix with k elements equal to 1 in each row and column; moreover, the 2×2 matrix consisting of all 1s is not a submatrix of $\mathbf{M}(v,k)$. Two incidence matrices of the same configuration may differ by a permutation on the rows and the columns.

A matrix $\mathbf{M}(v,k)$ can be considered as a biadjacency matrix of the *Levi graph* of the configuration v_k, which is a k-regular bipartite graph without multiple edges [33, Sec. 7.2]. The graph has girth at least six, i.e. it does not contain 4-cycles. Interestingly, such graphs are useful for the construction of bipartite-graph codes that can be treated as *low-density parity-check* (LDPC) *codes*. If $\mathbf{M}(v,k)$ is *circulant*, then the corresponding LDPC code is *quasi-cyclic*; it can be encoded with the help of shift-registers with relatively small complexity; see [5,6,17,19,25,37,38,42] and the references therein. Matrices $\mathbf{M}(v,k)$ consisting of square circulant submatrices have a number of useful properties; for instance, they are more suitable for LDPC codes implementation. We say that a binary matrix $\mathbf{A}$ is *block double-circulant* (BDC for short) if $\mathbf{A}$ consists of square circulant blocks whose weights give rise to a circulant matrix (see Definition 1). A configuration v_k with a BDC incidence matrix $\mathbf{M}(v,k)$ is called a *BDC symmetric configuration*.

Cyclic configurations are considered, for instance, in [3,4,17–19,23,28,37,38, 44,48]. A standard method for constructing cyclic configurations is based on *Golomb rulers* [20,23,28,51–53]. In fact, a circulant $v \times v$ binary matrix with first row $(a_0, a_1, \ldots, a_{v-1})$ is an incidence matrix of a cyclic symmetric configuration v_k if and only if the set of indices j such that $a_j = 1$ is a (v,k) modular Golomb ruler. Let $L_{\overline{\mathrm{G}}}(k)$ be the length of the *shortest known* Golomb ruler $\overline{\mathrm{G}}_k$; then for all v such that $v \geq 2L_{\overline{\mathrm{G}}}(k)+1$, there exists a cyclic symmetric configuration v_k; see [28, Sec. 4],[45]. We call the value $G(k) = 2L_{\overline{\mathrm{G}}}(k)+1$ the *Golomb bound*.

It is well known that $v \geq k^2 - k + 1$ for configurations v_k, and that equality holds if and only if there exists a projective plane of the order $k-1$ [28,33]. We call $P(k) = k^2 - k + 1$ the *projective plane bound*.

For a positive integer k, let $v_\delta(k)$ denote the smallest v for which a cyclic symmetric configuration v_k exists. Also, let $E(k)$ be the least integer such that for any $v \geq E(k)$, there exists a symmetric configuration v_k; similarly, let $E_c(k)$ denote the least integer such that for any $v \geq E_c(k)$, there exists a cyclic v_k. Then

$$k^2 - k + 1 = P(k) \leq E(k) \leq E_c(k) \leq G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1. \quad (1.1)$$

$$k^2 - k + 1 = P(k) \leq v_\delta(k) \leq E_c(k) \leq G(k) = 2L_{\overline{\mathrm{G}}}(k) + 1. \quad (1.2)$$

The aim of this work is twofold:

- to describe new constructions, paying special attention to both cyclic and BDC symmetric configurations;
- to investigate the *spectrum* of possible parameters of symmetric configurations v_k in the interval

$$k^2 - k + 1 = P(k) \leq v < G(k) = 2L_{\overline{G}}(k) + 1. \tag{1.3}$$

Configurations, both symmetric and non-symmetric, admitting an incidence matrix consisting of square circulant blocks have been considered in [3,4,17–19,37,38,49]. In [3,4,18,19] methods for obtaining new BDC indicence matrices from a given one are described. We remark that the notion of a BDC matrix is not explicitly used in the papers [3,4]; however it is easy to see that this is equivalent to that of a $\mathbb{Z}_\mu$-scheme, as described in [3] (see Remarks 1 and 2 in Section 3). What is essentially new here with respect to the methods described in [3,4] is the systematic use of subgroups of the automorphism group of a cyclic configuration, which may produce several distinct BDC indicence matrices from the same configuration. This explains how classical configurations, such as projective planes or punctured affine planes, provide a number of new examples (see Theorem 3, together with the examples in Section 4). Other achievements here are some improvements on the known upper bounds on $E(k)$ and $E_c(k)$, and several new parameters for cyclic and non-cyclic configurations v_k (see Sections 6 and 7).

The *Generalized Martinetti Construction* (Construction GM) proposed in [22] plays a key role for the investigation of the spectrum of possible parameters of symmetric configurations as it provides, for a fixed k, intervals of values of v for which a v_k exists. Construction GM has been considered also in [3,5,6][1]. To be successfully applied, Construction GM needs a convenient starting incidence matrix. To this end, BDC matrices turn out to be particularly useful; see Corollary 1. In this work new starting matrices are considered, see Example 6(ii), as well as those originally proposed in [3,6].

For applications, including Coding Theory, it is sometimes useful to have different matrices $\mathbf{M}(v,k)$ for the same v and k. This is why we attentively consider various constructions, even when they provide configurations with the same parameters.

The paper is organized as follows. In Section 2, we briefly summarize some constructions and parameters of configurations v_k. Preliminaries on BDC matrices are given in Section 3. Section 4 is the core of the paper and contains our new constructions of block double-circulant symmetric configurations $\mathbf{M}(v,k)$. In Section 5, methods for constructing matrices admitting extensions are described. In Sections 6 and 7, our results on the spectra of parameters of cyclic and non-cyclic configurations are reported.

We remark that some results from the present work were published without proofs in [17,18].

[1] The authors of the papers [5,6] (represented here by Davydov) regret that the paper [22] is not cited in [5,6]; the reason is that, unfortunately, the authors did not know the paper [22] during the preparation of [5,6].

2 Some known configurations v_k with $P(k) \le v < G(k)$

In this section we briefly summarize a list of pairs (v,k) for which a (cyclic) symmetric configuration v_k is known to exist, see Equations (2.1)–(2.16), and that will be used later in the paper. Infinite families of configurations v_k given in this section are described in [1–8,10,13,16–19,22–24,26,28–34,44,48,50,51,55]; see also the references therein.

Throughout the paper, q is a prime power and p is a prime. Let $\mathbb{F}_q$ be the Galois field of q elements. Let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. For a positive integer u, let $\mathbf{0}_u$ be the zero $u \times u$ matrix. Also, a permutation matrix of order u will be denoted by $\mathbf{P}_u$.

We first recall a basic result on symmetric configurations.

Theorem 1 [3, Sec. 2],[31, Sec. 5.2],[34, Sec. 2.5], [48] *If a (cyclic) configuration v_k exists, then for each δ with $0 \le \delta < k$ there exists a (cyclic) configuration $v_{k-\delta}$.*

The value δ appearing in Equations (2.1)–(2.16) is connected with Theorem 1. When a reference is given, it usually refers to the case $\delta = 0$.

The projective plane $PG(2,q)$ of order q provides configurations

$$\text{cyclic } v_k : v = q^2 + q + 1, \quad k = q + 1 - \delta, \quad q + 1 > \delta \ge 0. \tag{2.1}$$

We recall that the punctured affine plane of order q is the affine plane $AG(2,q)$ where the origin and all the lines through the origin are dismissed. Punctured affine planes are also called elliptic (Desarguesian) semiplanes of type L. They are cyclic configurations giving rise to

$$\text{cyclic } v_k : v = q^2 - 1, \quad k = q - \delta, \quad q > \delta \ge 0. \tag{2.2}$$

Configurations with parameters (2.3) below arise from Ruzsa's construction [50], [20, Sec. 5.4], [21], [51, Th. 19.19].

$$\text{cyclic } v_k : v = p^2 - p, \quad k = p - 1 - \delta, \quad p - 1 > \delta \ge 0. \tag{2.3}$$

The non-cyclic families (2.4) and (2.5) below are described in [1, Constructions (i),(ii), p. 126] and [26, Constructions 3.2,3.3, Rem. 3.5]; see also the references therein and [5], [6, Sec. 3], [19, Sec. 7.3], [24,28,48].

$$v_k : v = q^2 - qs, \quad k = q - s - \delta, \quad q > s \ge 0, \quad q - s > \delta \ge 0; \tag{2.4}$$

$$v_k : v = q^2 - (q-1)s - 1, \quad k = q - s - \delta, \quad q > s \ge 0, \quad q - s > \delta \ge 0. \tag{2.5}$$

For q a *square*, in [1, Conjec. 4.4, Rem. 4.5, Ex. 4.6], [3, Th. 6.4], and [26, Construction 3.7, Th. 3.8], families of non-cyclic configuration v_k with parameters (2.6) are provided; see also [19, Ex. 8]. Taking $c = q - \sqrt{q}$, we obtain (2.7); see also [17, Ex. 2(ii)], [24].

$$v_k : v = c(q + \sqrt{q} + 1), k = \sqrt{q} + c - \delta, c = 2, 3, \dots, q - \sqrt{q}, \delta \ge 0; \tag{2.6}$$

$$v_k : v = q^2 - \sqrt{q}, \quad k = q - \delta, \quad q > \delta \ge 0. \tag{2.7}$$

In [24, Th. 1.1], a non-cyclic family with parameters

$$v_k : v = 2p^2, \quad k = p + s - \delta, \quad 0 < s \le q + 1, \quad q^2 + q + 1 \le p, \quad p + s > \delta \ge 0 \tag{2.8}$$

is given. In [19, Sec. 6] a construction of non-cyclic family with parameters

$$v_k \;:\; v = c(q-1),\;\; k = c-\delta,\;\; c = 2,3,\ldots,b,\;\; b = q \text{ if } \delta \geq 1, \quad (2.9)$$
$$b = \left\lceil \frac{q}{2} \right\rceil \text{ if } \delta = 0,\; c > \delta \geq 0,$$

is described.

In [17, Sec. 2], [19, Sec. 3], a geometrical construction which uses point orbits under the action of a collineation group is described. Families of non-cyclic configurations v_k with the following parameters are given in [19, Exs 2, 3].

$$v_k : v = \frac{q(q-1)}{2},\;\; k = \frac{q+1}{2} - \delta,\;\; \frac{q+1}{2} > \delta \geq 0,\; q \text{ odd.} \quad (2.10)$$
$$v_k : v = \frac{q(q+1)}{2},\;\; k = \frac{q-1}{2} - \delta,\;\; \frac{q-1}{2} > \delta \geq 0,\; q \text{ odd.} \quad (2.11)$$
$$v_k : v = q^2 + q - q\sqrt{q},\; k = q - \sqrt{q},\; q - \sqrt{q} > \delta \geq 0,\; q \text{ square.} \quad (2.12)$$

In [7, 8, 10], non-cyclic families with parameters (2.13)–(2.16) are described in connection with graph theory; see also [4] for another construction of (2.14).

$$v_k : v = q^2 - rq - 1,\;\; k = q - r - \delta,\;\; q - r > \delta \geq 0,\; q - 3 \geq r \geq 0. \quad (2.13)$$
$$v_k : v = q^2 - q - 2,\;\; k = q - 1 - \delta,\;\; q - 1 > \delta \geq 0. \quad (2.14)$$
$$v_k : v = tq - 1,\;\; k = t - \delta,\;\; t > \delta \geq 0,\;\; q > t \geq 3. \quad (2.15)$$
$$v_k : v = tq - 2,\;\; k = t - \delta,\;\; t > \delta \geq 0,\;\; q > t \geq 3. \quad (2.16)$$

Use of Construction GM to obtain parameters for symmetric configurations will be considered in Section 5.

Some known results on existence and non-existence of sporadic symmetric configurations will be mentioned in Sections 6 and 7.

3 Preliminaries on block double-circulant incidence matrices

We first present the notion of a block double-circulant matrix, originally introduced in [18]; see also [17, 19]. Recall that the weight of a circulant binary matrix is the number of 1s in each its rows.

Definition 1 Let $v = td$. A binary $v \times v$ matrix $\mathbf{A}$ is said to be a *block double-circulant matrix* (*BDC matrix* for short) if

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_{0,0} & \mathbf{C}_{0,1} & \ldots & \mathbf{C}_{0,t-1} \\ \mathbf{C}_{1,0} & \mathbf{C}_{1,1} & \ldots & \mathbf{C}_{1,t-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}_{t-1,0} & \mathbf{C}_{t-1,1} & \ldots & \mathbf{C}_{t-1,t-1} \end{bmatrix}, \quad (3.1)$$

where $\mathbf{C}_{i,j}$ is a *circulant* $d \times d$ binary matrix for all i, j, and submatrices $\mathbf{C}_{i,j}$ and $\mathbf{C}_{l,m}$ with $j - i \equiv m - l \pmod t$ have the same weight. The matrix

$$\mathbf{W}(\mathbf{A}) = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & \dots & w_{t-2} & w_{t-1} \\ w_{t-1} & w_0 & w_1 & w_2 & \dots & w_{t-3} & w_{t-2} \\ w_{t-2} & w_{t-1} & w_0 & w_1 & \dots & w_{t-4} & w_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1 & w_2 & w_3 & w_4 & \dots & w_{t-1} & w_0 \end{bmatrix} \tag{3.2}$$

is a *circulant* $t \times t$ matrix whose entry in position i, j is the *weight* of $\mathbf{C}_{i,j}$. $\mathbf{W}(\mathbf{A})$ is called the *weight matrix* of $\mathbf{A}$. The vector $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, w_1, \dots, w_{t-1})$ is called the *weight vector* of $\mathbf{A}$.

Remark 1 If in Definition 1 the matrices $C_{i,j}$ were assumed to be right-circulant and not left-circulant, then they would have been the sum of some right-circulant permutation matrices. A right-circulant $d \times d$ permutation matrix is always associated to a permutation of the set $\{1, 2, \dots, d\}$ in the subgroup generated by the cycle $(1\,2\,3\,\dots\,d)$. Then the notion of a BDC matrix is substantially equivalent to that of a $\mathbb{Z}_d$-scheme, as defined in [3].

Let $\mathbf{A}$ be as in Definition 1. In addition, assume that $\mathbf{A}$ is the incidence matrix of a symmetric configuration v_k with $k = \sum_{i=0}^{t} w_i$. We present three simple techniques for obtaining other BDC incidence matrices of symmetric configurations from $\mathbf{A}$.

(i) For each $h \in \{0, 1, \dots, t-1\}$, in each row of every submatrix $\mathbf{C}_{i,j}$ with $j - i \equiv h \pmod t$ replace $\delta_h \geq 0$ values of 1 with zeros, in such a way that the obtained submatrix is still circulant. As a result a BDC incidence matrix of a configuration $v'_{k'}$ with

$$v' = v,\ k' = k - \sum_{h=0}^{t-1} \delta_h,\ 0 \leq \delta_h \leq w_h,\ w'_h = w_h - \delta_h, \overline{\mathbf{W}}(\mathbf{A}') = (w'_0, \dots, w'_{t-1}) \tag{3.3}$$

is obtained.

(ii) Fix some non-negative integer $j \leq t-1$. Let m be such that $w_m \leq w_h$ for all $h \neq j$. Cyclically shift all block rows of $\mathbf{A}$ to the left by j block positions. A matrix $\mathbf{A}^*$ with $\overline{\mathbf{W}}(\mathbf{A}^*) = (w^*_0 = w_j, \dots, w^*_u = w_{u+j \pmod t}, \dots, w^*_{t-1} = w_{j-1})$ is obtained. By applying (i), construct a matrix $\mathbf{A}^{**}$ with $w^{**}_0 = w^*_0 = w_j$, $w^{**}_h = w_m$, $h \geq 1$. Now remove from $\mathbf{A}^{**}$ $t - c$ block rows and columns from the bottom and the right. In this way an incidence $cd \times cd$ BDC matrix $\mathbf{A}'$ of a configuration $v'_{k'}$ is obtained with

$$v' = cd, k' = w_j + (c-1)w_m, c = 1, 2, \dots, t, \overline{\mathbf{W}}(\mathbf{A}') = (w_j, w_m, \dots, w_m). \tag{3.4}$$

(iii) Let t be even. Let $\mathbf{A}^*$ be as in (ii). Let $w_{\mathrm{O}}, w_{\mathrm{E}}$ be weights such that $w_{\mathrm{O}} \leq w^*_h$ for odd h and $w_{\mathrm{E}} \leq w^*_h$ for even h. By applying (i), construct a matrix $\mathbf{A}^{**}$ with $w^{**}_0 = w^*_0 = w_j$, $w^{**}_h = w_{\mathrm{O}}$ for odd h, $w^{**}_h = w_{\mathrm{E}}$ for even $h \geq 2$. Let $f = 1, 2, \dots, t/2$. From $\mathbf{A}^{**}$ remove $t - 2f$ block rows and columns from the bottom and the right. An incidence $2fd \times 2fd$ BDC matrix $\mathbf{A}'$ of a configuration $v'_{k'}$ is obtained with

$$v' = 2fd, k' = w_j + w_{\mathrm{O}} + (f-1)(w_{\mathrm{E}} + w_{\mathrm{O}}), \overline{\mathbf{W}}(\mathbf{A}') = (w_j, w_{\mathrm{O}}, \underbrace{w_{\mathrm{E}}, w_{\mathrm{O}}, \dots, w_{\mathrm{E}}, w_{\mathrm{O}}}_{f-1 \text{ pairs}}). \tag{3.5}$$

Remark 2 Construction (i) in this section essentially follows from Theorem 1; in [3] it is referred to as 1-factor deletion. Construction (ii) is essentialy a different formulation of Proposition 4.3 in [3], which is stated in terms of $\mathscr{S}_\mu$-schemes. Apart from terminology, the only difference is that here the case $m > 1$ is considered. Other methods for obtaining families of symmetric configurations from **A** can be found in [19, Sec. 4].

4 Constructions of block double-circulant symmetric configurations

The aim of this section is the construction of block double-circulant incidence matrices of a cyclic symmetric configuration. Our main tool will be a general method based on the action of the automorphism group of the configuration, originally presented in [17,19]; it can be viewed as the geometrical counterpart of a procedure proposed in [18,37,38] for circulant matrices, and called *decomposition of a circulant matrix*. Here, after summarizing some of the results from [17,19], we apply the procedure to cyclic configurations $(q^2+q+1)_{q+1}$ arising from projective planes $PG(2,q)$ (Section 4.1), as well as to cyclic configurations $(q^2-1)_q$ associated to affine planes $AG(2,q)$ (Section 4.2). We then briefly discuss BDC incidence matrices from Ruzsa's sequences (Section 4.3).

For a cyclic symmetric configuration v_k, viewed as an incidence structure $\mathscr{I} = (\mathscr{P},\mathscr{L})$, let σ be a permutation of $\mathscr{P}$ mapping lines to lines, such that the group S generated by σ acts regularly on both $\mathscr{P}$ and $\mathscr{L}$. Let $\mathscr{P} = \{P_0,\dots,P_{v-1}\}$ and $\mathscr{L} = \{\ell_0,\dots,\ell_{v-1}\}$. Arrange indices so that $\sigma : P_i \mapsto P_{i+1 \pmod v}$ and $\ell_i = \sigma^i(\ell_0)$. Clearly, $P_i = \sigma^i(P_0)$ holds.

For any divisor d of v, the group S has a unique cyclic subgroup $\widehat{S}_d$ of order d, namely the group generated by σ^t where $t = v/d$. Let $O_0,O_1,\dots,O_{t-1}$ (resp. $L_0,L_1,\dots,L_{t-1}$) be the orbits of $\mathscr{P}$ (resp. $\mathscr{L}$) under the action of $\widehat{S}_d$. Clearly, $|O_i| = |L_i| = d$ for any i. We arrange indices so that $P_0 \in O_0$, $O_w = \sigma^w(O_0)$, $\ell_0 \in L_0$, $L_w = \sigma^w(L_0)$. For each $i = 0,1,\dots,t-1$,

$$O_i = \{P_i,\sigma^t(P_i),\sigma^{2t}(P_i),\dots,\sigma^{(d-1)t}(P_i)\},\ L_i = \{\ell_i,\sigma^t(\ell_i),\sigma^{2t}(\ell_i),\dots,\sigma^{(d-1)t}(\ell_i)\}.$$

Equivalently, O_i (resp. L_i) consists of d points P_u (resp. d lines L_u) with u equal to i modulo t. Let

$$w_u = |\ell_0 \cap O_u|,\ u = 0,1,\dots,t-1. \qquad (4.1)$$

Clearly, $w_0 + w_1 + \ldots + w_{t-1} = k$.

Theorem 2 [19] *Let $\mathscr{I} = (\mathscr{P},\mathscr{L})$ be a cyclic symmetric configuration v_k with $v = td$. Let d, t, $\hat{S}_d$, O_i, L_i be as above.*

(i) *For any i and j, every line of the orbit L_i meets the orbit O_j in the same number of points $w_{j-i \pmod t}$ where w_u is defined by* (4.1).

(ii) *The incidence matrix of $\mathscr{I}$ is a block double-circulant matrix* **A** *of type* (3.1) *where $\mathbf{C}_{i,j}$ is a* circulant *$d \times d$ matrix of weight $w_{j-i \pmod t}$, with w_u as in* (4.1).

In order to use Theorem 2 effectively, one should find the intersection numbers of the orbits of the cyclic subgroup $\widehat{S}_d$.

4.1 BDC incidence matrices from projective planes

For cyclic projective planes, useful results on intersection numbers of orbits of the cyclic subgroup $\widehat{S}_d$ numbers are given in [15, 19] and in some of the references therein.

Example 1 We consider the projective plane $PG(2,q)$ as a cyclic symmetric configuration $(q^2+q+1)_{q+1}$ [55], [19, Sec. 5], [20, Sec. 5.5], [51, Th. 19.15]. In this case the group S is a Singer group of $PG(2,q)$.

(i) Let $t=3$, $t|(q^2+q+1)$, $p\equiv 2 \pmod 3$, and let $\{i_0,i_1,i_2\}=\{0,1,2\}$. In [19, Prop. 4] the following is proven: $w_{i_0}=(q+2\sqrt{q}+1)/3$, $w_{i_1}=w_{i_2}=(q-\sqrt{q}+1)/3$, if $q=p^{4m+2}$; $w_{i_0}=(q-2\sqrt{q}+1)/3$, $w_{i_1}=w_{i_2}=(q+\sqrt{q}+1)/3$, if $q=p^{4m}$. We can use (ii) of Section 3. By (3.4) with $c=2$, we obtain families of configurations v_k with parameters

$$v_k : v=2\frac{q^2+q+1}{3},\ k=\frac{2q+\sqrt{q}+2}{3},\ q=p^{4m+2},\ p\equiv 2 \pmod 3;$$
$$v_k : v=2\frac{q^2+q+1}{3},\ k=\frac{2q-\sqrt{q}+2}{3},\ q=p^{4m},\ p\equiv 2 \pmod 3.$$

(ii) Let $q=p^{2m}$ be a square. Let t be a prime divisor of q^2+q+1. Then t divides either $q+\sqrt{q}+1$ or $q-\sqrt{q}+1$. Assume that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$. By [19, Prop. 6], in this case $w_0=(q+1\pm(1-t)\sqrt{q})/t$, $w_1=w_2=\ldots=w_{t-1}=(q+1\pm\sqrt{q})/t$. We can use (ii) of Section 3. By (3.4), we obtain a family of configurations v_k with parameters

$$v_k\ :\ v=c\frac{q^2+q+1}{t},\ k=\frac{q+1\pm(1-t)\sqrt{q}}{t}+(c-1)\frac{q+1\pm\sqrt{q}}{t}, \qquad (4.2)$$
$$c=1,2,\ldots,t,\ q=p^{2m},\ t \text{ prime}.$$

The hypothesis that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$ holds e.g. in the following cases: $q=3^4$, $t=7$; $q=2^8$, $t=13$; $q=5^4$, $t=7$; $q=2^{12}$, $t=19$; $q=3^8$, $t=7$; $q=2^{16}$, $t=13$; $q=17^4$, $t=7$; $p\equiv 2 \pmod t$, $t=3$.

(iii) Let q be a square. Let $v\geq 1$, $v|(q-\sqrt{q}+1)$, and $t=\frac{1}{v}(q-\sqrt{q}+1)$. Then $d=v(q+\sqrt{q}+1)$ and, by [19, Prop. 7], we have $w_0=\sqrt{q}+v$, $w_1=w_2=\ldots=w_{t-1}=v$. Now using (ii) of Section 3, for $v=1$ we obtain a family of configurations with parameters (2.6). The orbits $O_0,O_1,\ldots,O_{t-1}$ are Baer subplanes. It should be noted that such confugurations admit an extension, see Section 5 and Example 6(iii) below.

(iv) In rows marked by "*PG*" of Table 1, parameters of configurations v'_n with BDC incidence matrices are given. We use (ii) of Section 3. The starting weights w_i^* are obtained by computer forming orbits of subgroups $\widehat{S}_d$ of a Singer group of $PG(2,q)$. The values k',v' are calculated by (3.4). Then the smallest value $k^\#$ for which $v'<G(k^\#)$ is found. As a result, each row of Table 1 provides configurations v'_n with $v'<G(n)$, $n=k',k'-1,\ldots,k^\#$, see (i) of Section 3 and (3.3).

Table 1 Parametrs of configurations v'_n with BDC incidence matrices, $v' < G(n)$, $n = k', k'-1, \ldots, k^{\#}$, from the cyclic projective plane $PG(2,q)$ and the cyclic punctured affine plane of order q

plane	q	t	d	w_i^*	c	f	k'	v'	$G(k')$	$k^{\#}$	$G(k^{\#})$
PG	25	3	217	12,7,7	2		19	434	493	19	493
PG	25	7	93	8,3,3,3,3,3,3	4		17	372	399	17	399
PG	25	7	93	8,3,3,3,3,3,3	5		20	465	567	19	493
PG	25	7	93	8,3,3,3,3,3,3	6		23	558	745	20	567
PG	32	7	151	0,5,5,6,5,6,6	6		25	906	961	25	961
PG	37	3	469	16,9,13	2		25	938	961	25	961
PG	43	3	631	19,13,12	2		31	1262	1495	30	1361
PG	49	3	817	21,16,13	2		34	1634	1877	33	1719
PG	97	3	3169	39,28,31	2		67	6338	7639	62	6431
PG	107	7	1651	24,15,15,13,15,13,13	6		89	9906	13557	75	9965
PG	107	7	1651	24,15,15,13,15,13,13	5		76	8255	10179	69	8291
PG	109	3	3997	43,36,31	2		74	7994	9507	69	8291
PG	109	7	1713	8,15,15,19,15,19,19	6		83	10278	12041	77	10409
AG	16	3	85	8,4,4	2		12	170	171	12	171
AG	31	3	320	14,9,8	2		22	640	713	21	667
AG	37	3	456	16,9,12	2		25	912	961	25	961
AG	49	4	600	16,12,9,12	3		34	1800	1877	34	1877
AG	49	6	400	4,9,12,8,8,8	5		36	2000	2011	36	2011
AG	53	4	702	17,12,10,14	3		37	2106	2199	37	2199
AG	79	3	2080	32,25,22	2		54	4160	4747	50	4189
AG	79	6	1040	8,14,13,14,18,12	5		56	5200	5451	56	5451
AG	79	6	1040	18,12,8,14,13,14		2	50	4160	4189	50	4189
AG	107	8	1431	17,15,10,13,10,12,17,13	7		77	10017	10409	76	10179
AG	107	8	1431	17,15,10,13,10,12,17,13		3	73	8586	9027	71	8661

Remark 3 In [49, Prop. 3, Th. 9], parity check matrices of LDPC codes based on the Hermitian curve in $PG(2,q^2)$ and consisting of square cyclic submatrices are constructed by geometrical tools that can be considered as special cases of the more general approach of Theorem 2. The mentioned parity check matrices are incidence matrices of non-symmetric configurations. It is possible that by dismissing some units in the matrix, BDC configurations could be obtained. This problem is not considered here, nor in [49]. Note that the matrix of [49, Prop. 3] uses points belonging to the Hermitian curve, whereas the point set of the symmetric configuration in [19, Ex. 3], whose parameters are as in (2.12), coincides with the complement of the same curve.

4.2 BDC incidence matrices from punctured affine planes

The cyclic punctured affine planes $(q^2-1)_q$, see [13],[20, Sec. 5.6],[51, Th. 19.17], as well as [19, Ex. 5, Sec. 6], is a cyclic symmetric configuration where S is an affine Singer group of $AG(2,q)$.

Example 2 Consider the (q^2-1,q) modular Golomb ruler obtained from the cyclic punctured affine plane of order q [13]. Let $q^2-1 = td$. In [43] it is proven that if t is a divisor of $q+1$ then exactly $t-1$ values of w_h are equal to $\frac{q+1}{t}$, and there is precisely one $w_{h_0} = \frac{q+1}{t} - 1$. In [43] only proper divisors t of $q+1$ are considered. Actually, the same arguments work for $t = q+1$; in this case $\overline{\mathbf{W}}(\mathbf{A}) = (0,1,1,\dots,1)$ can be assumed. For comparison, see Example 3 below.

In [19,43] useful results for t a divisor of $q+1$ are obtained. Theorem 3 below extends our knowledge on this topic and gives new results for t dividing $q-1$.

Theorem 3 *Let q be an odd square. Consider the cyclic symmetric configuration $(q^2-1)_q$ associated to the punctured affine plane of order q. Let t be a divisor of $q-1$, and let $d = (q^2-1)/t$. Let $\mathbf{A}$ be an incidence BDC matrix of this configuration as in Theorem* 2(ii). *Let $w_0, w_1, \dots, w_{t-1}$ be the weights of the circulant $d \times d$ blocks of* $\mathbf{A}$.

(i) *Let $t = \sqrt{q}+1$. Then $w_0 = 1$, $w_j = \sqrt{q}-1$ for j odd, $w_j = \sqrt{q}+1$ for j even, $j = 1,2,\dots,\sqrt{q}$.*
(ii) *Let $t = \frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q} \equiv 1 \pmod 4$. Then $w_0 = \sqrt{q}$, $w_1, w_2, \dots, w_{t-1} = 2\sqrt{q}$.*
(iii) *Let $t = \frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q} \equiv 3 \pmod 4$. Then $w_0 = \sqrt{q}+2$, $w_j = 2\sqrt{q}-2$ for j odd, $w_j = 2\sqrt{q}+2$ for j even, $j = 1,2,\dots,\frac{1}{2}(\sqrt{q}+1)-1$.*
(iv) *Let $t = \frac{1}{4}(\sqrt{q}+1)$, $\sqrt{q} \equiv 3 \pmod 4$.*
- *If $\frac{1}{4}(\sqrt{q}+1)$ is odd, then $w_0 = 3\sqrt{q}$, $w_1, w_2, \dots, w_{t-1} = 4\sqrt{q}$.*
- *If $\frac{1}{4}(\sqrt{q}+1)$ is even, then $w_0 = 3\sqrt{q}+4$, $w_j = 4\sqrt{q}-4$ for j odd, $w_j = 4\sqrt{q}+4$ for j even, $j = 1,2,\dots,\frac{1}{4}(\sqrt{q}+1)-1$.*

Proof. Let ξ be a primitive element of $\mathbb{F}_{q^2}$. Let $\omega = \xi^{\frac{q+1}{2}}$, and $\theta = \omega^{q+1}$. Identify a point $(x,y) \in AG(2,q)$ with the element $z = x + \omega y \in \mathbb{F}_{q^2}$. As $\omega^{q-1} = -1$ it is straightforward to check that $z^{q+1} = x^2 + \theta y^2$.

We need to consider the orbits of $\mathbb{F}_{q^2}^*$ under the action of the cyclic group generated by σ^t where $\sigma(\xi^i) = \xi^{i+1}$. The orbit O_j of the element ξ^j is $\{\xi^j, \xi^{j+t}, \xi^{j+2t}, \dots, \xi^{j+(\frac{q^2-1}{t}-1)t}\}$. Let μ be a primitive element in $\mathbb{F}_q$.

(i) Let $t = \sqrt{q}+1$. For each $z \in O_j$ we have $z^{q+1} = \xi^{j(q+1)}(\xi^{(q+1)(\sqrt{q}+1)})^h$ for some h. Also, ξ^{q+1} is a primitive element of $\mathbb{F}_q$ and $(\xi^{(q+1)(\sqrt{q}+1)})^h \in \mathbb{F}_{\sqrt{q}}^*$. This means that for each $j = 0,\dots,\sqrt{q}$, the orbit O_j consists precisely of the elements z such that $z^{q+1} \in \mu^j \mathbb{F}_{\sqrt{q}}^*$. Therefore, the orbit O_j in $AG(2,q)$ consists of the union of the $\sqrt{q}-1$ conics with equation $x^2 + \theta y^2 = \mu^j \alpha$ where $\alpha \in \mathbb{F}_{\sqrt{q}}^*$.

A number of lemmas are necessary to compute the sizes w_j of the intersections $|O_j \cap \ell|$, where ℓ is any line of $AG(2,q)$ not passing through the origin. Choose the line $\ell : x = 1$.

Lemma 1 $w_0 = 1$.

Proof We prove that $|\ell \cap O_0| = 1$. Note that O_0 consists of the conics $C_\alpha : x^2 + \theta y^2 = \alpha$. The line $x = 1$ meets the conic C_α in one point if $\alpha = 1$. If $\alpha \neq 1$ then the intersection is empty since θ is not a square in $\mathbb{F}_q$ (and α is a square since it is an element of $\mathbb{F}_{\sqrt{q}}$). □

Lemma 2 *Let N_j be the number of non-squares in the set $\mu^j \mathbb{F}_{\sqrt{q}}^* - 1$. Then $w_j = 2N_j$.*

Proof The conic $C_\alpha : x^2+\theta y^2 = \mu^j\alpha$ meets the line $x=1$ in 0 or 2 points. The latter case occurs precisely when $\mu^j\alpha-1$ is not a square. □

In order to compute the integers N_j, the following lemmas will be useful.

Lemma 3 *The collection of sets* $H_\beta = \{\frac{\mu}{1-\mu\beta}\mathbb{F}^*_{\sqrt{q}}-1 \mid \beta\in\mathbb{F}_{\sqrt{q}}\}$ *coincides with* $\{\mu^j\mathbb{F}^*_{\sqrt{q}}-1 \mid j=1,\ldots,\sqrt{q}\}$.

Proof We only need to show that the sets $\frac{\mu}{1-\mu\beta}\mathbb{F}^*_{\sqrt{q}}$ are pairwise distinct. Assume on the contrary that $\frac{\mu}{1-\mu\beta}=\alpha\frac{\mu}{1-\mu\gamma}$ for $\beta,\gamma\in\mathbb{F}_{\sqrt{q}}$, $\alpha\in\mathbb{F}^*_{\sqrt{q}}$. Then $\alpha(1-\mu\beta)=1-\mu\gamma$ that is $\mu(-\alpha\beta+\gamma)=1-\alpha$. If $\alpha\beta-\gamma=0$ then $\alpha=1$ and hence $\beta=\gamma$. If $\alpha\beta-\gamma\neq 0$ then $\mu\in\mathbb{F}_{\sqrt{q}}$, which is a contradiction. □

Lemma 4 *Fix* $\beta\in\mathbb{F}_{\sqrt{q}}$ *and* $j\in\{1,\ldots,\sqrt{q}\}$*. Assume that the set* $\frac{\mu}{1-\mu\beta}\mathbb{F}^*_{\sqrt{q}}-1$ *coincides with* $\mu^j\mathbb{F}^*_{\sqrt{q}}-1$*. Then* $1-\mu\beta$ *is a square if and only if* j *is odd.*

Proof Note that $\frac{\mu}{1-\mu\beta}\mathbb{F}^*_{\sqrt{q}}-1$ coincides with $\mu^j\mathbb{F}^*_{\sqrt{q}}-1$ if and only if $\mu^{j-1}(1-\mu\beta)\in\mathbb{F}_{\sqrt{q}}$. Then $\mu^{j-1}(1-\mu\beta)$ is a square. Whence the assertion follows. □

Let M_β be the number non-squares in the set $\frac{\mu}{1-\mu\beta}\mathbb{F}^*_{\sqrt{q}}-1$. By the previous lemma, the set of integers M_β coincides with the set of integers N_j. Let $A=M_0=N_1$. Next we show that every M_β is related to A.

Lemma 5 *If* $1-\mu\beta$ *is a square in* $\mathbb{F}_q$ *then* $M_\beta=A$*. If* $1-\mu\beta$ *is not a square in* $\mathbb{F}_q$ *then* $M_\beta=\sqrt{q}-A$*.*

Proof A is the number of non-squares in the set $H_0=\{\mu\alpha-1 \mid \alpha\in\mathbb{F}^*_{\sqrt{q}}\}$. For each $\beta\in\mathbb{F}_{\sqrt{q}}$, this set coincides with $\{\mu(\alpha+\beta)-1 \mid \alpha\in\mathbb{F}^*_{\sqrt{q}},\alpha\neq-\beta\}\cup\{\mu\beta-1\}$. But since $\mu(\alpha+\beta)-1=\mu\alpha+\mu\beta-1=(1-\mu\beta)(\frac{\mu}{1-\mu\beta}\alpha-1)$ we have that

$$H_0=(1-\mu\beta)\{\frac{\mu}{1-\mu\beta}\alpha-1 \mid \alpha\in\mathbb{F}^*_{\sqrt{q}},\alpha\neq-\beta\}\cup\{\mu\beta-1\},$$

that is $H_0=(1-\mu\beta)H_\beta\setminus\{-1\}\cup\{\mu\beta-1\}$. Two cases have to be distinguished.

a) $1-\mu\beta$ is a square. Then either $\frac{1}{\mu\beta-1}$ and $\mu\beta-1$ are both squares or are both non-squares. It follows that the number of non-squares in H_0 equals the number of non-squares in H_β. Therefore, $M_\beta=A$.

b) $1-\mu\beta$ is not a square. As -1 is a square, $\mu\beta-1$ is not a square. The number of non-squares in H_0 equals the number of squares in H_β plus 1. Then $M_\beta=\sqrt{q}-A$. □

As a corollary to Lemmas 4 and 5, the following result is obtained.

Lemma 6 *If* j *is odd then* $N_j=A$*. If* j *is even then* $N_j=\sqrt{q}-A$*.*

By the above lemma *only two possibilities occur for w_j, namely* $2A$ *and* $2(\sqrt{q}-A)$. Next we calculate A. Let u_1 be the number of $\beta's$ such that $1-\mu\beta$ is a square in $GF(q)$. Then, from $w_0+w_1+\ldots+w_{\sqrt{q}}=q$, we obtain
$q=1+2u_1A+2(\sqrt{q}-u_1)(\sqrt{q}-A)=1+4u_1A+2q-2\sqrt{q}(u_1+A)$.
Hence,

$$0=q+1+4u_1A-2\sqrt{q}(u_1+A)=(\sqrt{q}-2u_1)(\sqrt{q}-2A)+1$$

Since both $\sqrt{q}-2u_1$ and $\sqrt{q}-2A$ are integers, the only possibility is that they are both equal to ±1. This implies that $A=\frac{\sqrt{q}\pm1}{2}$, $u_1=\frac{\sqrt{q}\mp1}{2}$ is the only solution. So, we have proven that the integers $w_0,\ldots,w_{\sqrt{q}}$ are such that: 1 occurs precisely once; the integer $\sqrt{q}+1$ occurs $\frac{\sqrt{q}-1}{2}$ times; the integer $\sqrt{q}-1$ occurs $\frac{\sqrt{q}+1}{2}$ times. Finally, since $w_j=w_{j'}$ if $j=j' \pmod 2$ and the number of odd integers in $[1,\sqrt{q}]$ is greater than that of even integers, the assertion of Theorem 3(i) follows.

(ii) Assume that $t=\frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q}\equiv 1 \pmod 4$. An orbit here is the union of two orbits O_j of case (i). More precisely, an orbit consists of the union of the $2(\sqrt{q}-1)$ conics with equation

$$x^2+\theta y^2=\mu^j\alpha, \qquad \alpha\in\mathbb{F}^*_{\sqrt{q}}\cup\mu^{\frac{\sqrt{q}+1}{2}}\mathbb{F}^*_{\sqrt{q}}.$$

Equivalently, an orbit here is the union $O_j\cup O_{j+\frac{\sqrt{q}+1}{2}}$ for some $j=0,\ldots,\frac{\sqrt{q}-1}{2}$. Assume that $j>0$. Since j and $j+\frac{\sqrt{q}+1}{2}$ are different modulo 2, we have that the number of points of ℓ_i in this orbit is $(\sqrt{q}-1)+(\sqrt{q}+1)=2\sqrt{q}$. If $j=0$, since $\frac{\sqrt{q}+1}{2}$ is odd we have that ℓ_1 meets $O_0\cup O_{\frac{\sqrt{q}+1}{2}}$ in $\sqrt{q}$ points.

(iii) Assume that $t=\frac{1}{2}(\sqrt{q}+1)$, $\sqrt{q}\equiv 3 \pmod 4$. Again, an orbit here is the union $O_j\cup O_{j+\frac{\sqrt{q}+1}{2}}$ for some $j=0,\ldots,\frac{\sqrt{q}-1}{2}$. Assume that $j>0$. Since j and $j+\frac{\sqrt{q}+1}{2}$ are equal modulo 2, we have that the number of points of ℓ_i in this orbit is $2\sqrt{q}-2$ if j is odd, $2\sqrt{q}+2$ if j is even. If $j=0$, since $\frac{\sqrt{q}+1}{2}$ is even we have that ℓ_1 meets $O_0\cup O_{\frac{\sqrt{q}+1}{2}}$ in $\sqrt{q}+2$ points.

(iv) Assume that $t=\frac{1}{4}(\sqrt{q}+1)$, $\sqrt{q}\equiv 3 \pmod 4$. An orbit here is the union $O_j\cup O_{j+\frac{\sqrt{q}+1}{4}}\cup O_{j+\frac{\sqrt{q}+1}{2}}\cup O_{j+3\frac{\sqrt{q}+1}{4}}$, for some $j=0,\ldots,\frac{\sqrt{q}-3}{4}$. Then it is easy to deduce the assertion of Theorem 3(iv). □

Example 3 We consider the the cyclic symmetric configuration $(q^2-q)_q$ associated to the punctured affine plane of order q.

(i) Let t be a divisor of $q+1$. In [19, Prop. 10] it is proven that $w_0=(q+1)/t-1$, $w_1=w_2=\ldots=w_{t-1}=(q+1)/t$, cf. Example 2 which uses results of [43] obtained by a different approach. Putting $t=q+1$ we obtain $d=q-1$ and $w_0=0$, $w_1=w_2=\ldots=w_q=1$. Now using (ii) in Section 3 one can obtain a family of configurations with parameters (2.9). It can be noted that when $\nu=1$ the configuration admits an extension, see Section 5 and Example 6(i) below.

(ii) Let q be an odd square. Let $t=\sqrt{q}+1$, $d=(\sqrt{q}-1)(q+1)$. By Theorem 3(i) and (iii) of Section 3 one can obtain a family of configurations with parameters

$$v_k: v=2f(\sqrt{q}-1)(q+1),\ k=(2f-1)\sqrt{q},\ f=1,2,\ldots,\frac{\sqrt{q}+1}{2},\ q \text{ odd square.} \tag{4.3}$$

By using Theorem 3(ii),(iii),(iv), together with (iii) of Section 3, we obtain the same parameters as in (4.3). But the structure of an incidence matrix $\mathbf{M}(v,k)$ is different from that arising from Theorem 3(i).

(iii) In rows marked by "*AG*" of Table 1, parameters of configurations v'_n with BDC incidence matrices are given. We use the both (ii) and (iii) of Section 3. The starting weights w_i^* are obtained by computer through the constructions of the orbits of subgroups $\widehat{S}_d$ of the affine Singer group. For k' and $k^{\#}$ see Example 1(iv).

More detailed tables similar to Table 1 can be found in [16].

4.3 BDC incidence matrices from Ruzsa's sequences

Example 4 For a prime p, let g be a primitive element of $\mathbb{F}_p$. The following Ruzsa's sequence [50], [20, Sec. 5.4], [51, Th. 19.19] forms a $(p^2-p,p-1)$ modular Golomb ruler:

$$e_h=ph+(p-1)g^h \pmod{p^2-p},\ h=1,2,\ldots,p-1,\ v=p^2-p. \tag{4.4}$$

(i) By the result in [54, Tab. 5], for a proper divisor t of $p-1$ we have $w_u=\frac{p-1}{t}$ for each $u=0,\ldots,t-1$. Then the block double-circulant indicence matrix of the associated configuration has a weight vector $\overline{\mathbf{W}}(\mathbf{A})=(\frac{p-1}{t},\ldots,\frac{p-1}{t})$.

(ii) There are two different possibilities to get a binary weight vector from (4.4).

a) Fix $t=p-1$, $d=p$. Then for each $u=0,1,\ldots,t-1$ there is precisely one element e_h such that $e_h\equiv u \pmod t$. We have $e_{p-1}\equiv 0 \pmod t$ and $e_h\equiv h \pmod t$, $h=1,2,\ldots,p-2$. Therefore, $\overline{\mathbf{W}}(\mathbf{A})=(1,1,\ldots,1)$.

b) Fix $t=p$, $d=p-1$. In this case $e_u\not\equiv 0 \pmod t$ for all u. Also, for each $u=1,2,\ldots,t-1$ there is precisely one element e_h such that $e_h\equiv u \pmod t$. We have $e_h\equiv u \pmod s$ if and only if $-g^h\equiv u \pmod p$. So, $\overline{\mathbf{W}}(\mathbf{A})=(0,1,1,\ldots,1)$.

Case (ii) in Example 4 is of particular interest. In fact, BDC matrices such that each weight w_h is in $\{0,1\}$ admit an extension; see Section 5 and Example 6(ii) below.

5 Constructing configurations v_k admitting an extension

A classical construction for configurations v_3 due to Martinetti goes back to 1887 [46]; see also [3,6,12,14,22,29], [34, Sec. 2.4, Fig. 2.4.1]. A *Generalized Martinetti Construction* (Construction GM) for configurations v_k, $k\geq 3$, was proposed in [22]. In [3,5,6] Construction GM is used to obtain a wide spectrum of parameters v,k. To be successfully applied, Construction GM, as described in Theorem 4 below, needs a convenient starting incidence matrix. To this end, BDC matrices turn out to be particularly useful; see Corollary 1. In this section new starting matrices are considered, see Example 6(ii), as well as those originally proposed in [3,5,6].

Construction GM can be presented from different points of view, see [3,22]. Here we focus on an approach based on incidence matrices, which will be used in Section 7 for obtaining new values of v,k.

Definition 2 Let $\mathbf{M}(v,k)$ be an incidence matrix of a symmetric configuration v_k. In $\mathbf{M}(v,k)$, we consider a set $\mathscr{A}$ consisting of $k-1$ rows corresponding to pairwise disjoint lines of v_k, together with $k-1$ columns corresponding to pairwise non-collinear points of v_k. If the $(k-1)\times(k-1)$ submatrix $\mathbf{C}(\mathscr{A})$ formed by the intersection of the rows and columns of $\mathscr{A}$ is a permutation matrix $\mathbf{P}_{k-1}$ then $\mathscr{A}$ is called an *extending aggregate* (or *E-aggregate*). The matrix $\mathbf{M}(v,k)$ *admits an extension* if it contains at least one E-aggregate. The matrix $\mathbf{M}(v,k)$ *admits θ extensions* if it contains θ E-aggregates that do not intersect each other.

Theorem 4 (Procedure E) *Let $\mathbf{M}(v,k)$ be an incidence matrix of a symmetric configuration v_k. Assume that $\mathbf{M}(v,k)$ admits θ extensions, for some $\theta \geq 1$.*

(i) *An incidence matrix $\mathbf{M}(v+\theta,k)$ of a symmetric configuration $(v+\theta)_k$ can be obtained from $\mathbf{M}(v,k)$ by adding θ rows and θ columns.*

(ii) *If $\theta \geq k-1$, then any $k-1$ new rows and $k-1$ new columns added to $\mathbf{M}(v,k)$ according to (i) form an E-aggregate.*

In Theorem 4(i), the form of the added rows/columns can be found in [22, Tab. 1] or in [6, Enlargement 2a, p. 310].

Establishing whether a configuration admits an extension or not is not an easy task in the general case. This leads us to consider distinguished classes of incidence matrices.

Definition 3 Let $v = td$, $t \geq k$, $d \geq k-1$, and let v_k be a symmetric configuration. Let $\mathbf{M}(v,k)$ be an incidence matrix of v_k, viewed as a $t\times t$ block matrix, every block being of type $d\times d$. We say that $\mathbf{M}(v,k)$ has *Structure E* if every $d\times d$ block is either a permutation matrix $\mathbf{P}_d$ or the zero $d\times d$ matrix $\mathbf{0}_d$.

Lemma 7 *Let $v = td,\ t \geq k,\ d \geq k-1$, and let v_k be a symmetric configuration. Assume that $\mathbf{M}(v,k)$ is an incidence matrix of v_k having Structure E.*

(i) *The matrix $\mathbf{M}(v,k)$ admits $\theta(t,d,k) := t\cdot\lfloor d/(k-1)\rfloor \geq t \geq k$ extensions.*

(ii) *Let $\mathbf{M}(v+\theta(t,d,k),k)$ be the matrix obtained from (i) by applying $\theta(t,d,k)$ extensions. Then $\mathbf{M}(v+\theta(t,d,k),k)$ above admits $\theta_2(t,d,k) := \lfloor\theta(t,d,k)/(k-1)\rfloor \geq$ 1 extensions.*

Proof (i) We need to provide $\theta(t,d,k)$ pairwise disjoint E-aggregates of $\mathbf{M}(v,k)$. For each block of type $\mathbf{P}_d$, one can easily define a set $\mathscr{E}$ of $\lfloor d/(k-1)\rfloor$ disjoint E-aggregates consisting of rows and columns with non-trivial intersection with $\mathbf{P}_d$. Let $\mathbf{B}$ be the binary $t\times t$ matrix where each entry corresponds to a $d\times d$ block in $\mathbf{M}(v,k)$: an entry is 1 if the corresponding block is of type $\mathbf{P}_d$. Each row and each column of $\mathbf{B}$ has weight k. Therefore, it is possible to obtain a permutation matrix $\mathbf{P}_t$ by dismissing some units in $\mathbf{B}$. The sets $\mathscr{E}$ defined from blocks $\mathbf{P}_d$ corresponding to units of $\mathbf{P}_t$ are clearly disjoint, and their union gives $t \cdot \lfloor d/(k-1) \rfloor$ not intersecting E-aggregates.

(ii) Theorem 4(ii) can be applied $\theta_2(t,d,k)$ times. □

Corollary 1 *Let $v = td$, $t \geq k$, $d \geq k-1$, and let v_k be a symmetric configuration. Assume that an incidence matrix $\mathbf{A}$ of v_k is a BDC matrix as in* (3.1) *with weight vector $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, \ldots, w_{t-1})$.*

(i) *If all the weights w_u belong to the set $\{0,1\}$, then $\mathbf{A}$ admits $t+1$ extensions.*

(ii) *If $\overline{\mathbf{W}}(\mathbf{A}) = (0,1,1,\ldots,1)$ or $\overline{\mathbf{W}}(\mathbf{A}) = (1,1,\ldots,1)$ then one can obtain a family of symmetric configurations v_k with parameters* (5.1) *or* (5.2)*, respectively*

$$v_k : v = cd + \theta,\ k = c-1-\delta,\ c = 2,3,\ldots,t,\ \theta = 0,1,\ldots,c+1,\ \delta \geq 0. \quad (5.1)$$

$$v_k : v = cd + \theta,\ k = c-\delta,\ c = 2,3,\ldots,t,\ \theta = 0,1,\ldots,c+1,\ \delta \geq 0. \quad (5.2)$$

Proof The matrix $\mathbf{A}$ has clearly Structure E. Then (i) follows from Lemma 7, together with Theorem 4(i). For (ii), we use (ii) of Section 3. □

In order to obtain a configuration having Structure E from a given one, sometimes the procedures described in Section 3 are useful; see Example 6(iii) below.

Example 5 The following infinite family of symmetric configurations v_k is obtained in [3, Th. 6.2], [6, Th. 1(i)] with use of Construction GM:

$$v_k : v = q^2 - qs + \theta,\ k = q-s-\Delta,\ q > s \geq 0, q-s > \Delta \geq 0, \theta = 0,1,\ldots,q-s+1. \quad (5.3)$$

Example 6 (i) We consider the punctured affine plane of order q as a cyclic configuration $(q^2-1)_q$; see Examples 2 and 3(i). Let $t = q+1$, $d = q-1$, $w_0 = 0$, $w_1 = w_2 = \ldots = w_q = 1$. By Corollary 1(ii) we obtain a family of symmetric configurations v_k with

$$v_k : v = c(q-1) + \theta,\ k = c-1-\delta,\ c = 2,3,\ldots,q+1,\ \theta = 0,1,\ldots,c+1,\ \delta \geq 0. \quad (5.4)$$

We remark that configurations with parameters (5.4) from the punctured affine plane were first obtained in [3, Th. 6.3] by using Construction GM; see also [18, Eqn. (8)].

(ii) We consider Ruzsa's configuration $(p^2-p)_{p-1}$; see Example 4(ii). Put $t = p-1$, $d = p$. Then $w_0 = \ldots = w_{p-2} = 1$. By Corollary 1(iii) we obtain a family with

$$v_k : v = cp + \theta,\ k = c-\delta,\ c = 2,3,\ldots,p-1,\ \theta = 0,1,\ldots,c+1,\ \delta \geq 0,\ p \text{ prime}. \quad (5.5)$$

If $t = p$, $d = p-1$ then $w_0 = 0$, $w_1 = \ldots = w_{p-1} = 1$. We obtain a family with

$$v_k : v = c(p-1) + \theta,\ k = c-1-\delta,\ c = 2,3,\ldots,p,\ \theta = 0,1,\ldots,c+1,\ \delta \geq 0,\ p \text{ prime}. \quad (5.6)$$

(iii) Let q be a square. We consider $PG(2,q)$ as a cyclic configuration $(q^2+q+1)_{q+1}$, see Example 1(iii). Let $v = 1$, $t = q - \sqrt{q} + 1$, $d = q + \sqrt{q} + 1$, $w_0 = \sqrt{q}+1$, $w_1 = w_2 = \ldots = w_{t-1} = 1$. By (i) of Section 3 we can put $w'_0 = 1$ and obtain the weight vector $(1,1,\ldots,1)$. Now, by Corollary 1(iii), we obtain a family with

$$v_k : v = c(q+\sqrt{q}+1) + \theta,\ k = c-\delta,\ c = 2,3,\ldots,q-\sqrt{q}+1,$$
$$\theta = 0,1,\ldots,c+1,\ \delta \geq 0,\ q \text{ square}. \quad (5.7)$$

6 The spectrum of parameters of cyclic symmetric configurations

Current data on the existence of cyclic configurations v_k, $k \leq 37$, are given in Table 2. The value $v_\delta(k)$ is defined in Introduction. In the second column, the exact values of $v_\delta(k)$ are marked by the dot "**.**". For $k \leq 16$, the exact values of $v_\delta(k)$ are taken from [27, Tab. IV], [35, Tab. 2], [53, Tab. 1a], [56]. Also, if $k-1$ is a power prime then $v_\delta(k) = P(k) = k^2 - k + 1$. The remaining entries in the second column are lower bounds for $v_\delta(k)$. It is well known that $P(k) \leq v_\delta(k)$, and that a *cyclic* symmetric configuration $(k^2 - k + 1)_k$ exists if and only if a *cyclic* projective plane of order $k-1$ exists. By [11], no cyclic projective planes exist with non-prime power orders $\leq 2 \cdot 10^9$. Therefore for $k = 25, 27, 36$ we write the lower bound for $v_\delta(k)$ as $k^2 - k + 2$. Also, we apply Theorem 5 that gives $k^2 - k + 3 \leq v_\delta(k)$ for $k = 19, 21, 22, 23, 26, 29, 31, 34, 35, 37$.

Theorem 5 [30, Th. 2.4] *There is no symmetric configuration* $(k^2 - k + 2)_k$ *if* $5 \leq k \leq 10$ *or if neither* k *or* $k-2$ *is a square.*

In the third column of Table 2, in a row "k", the known (resp. new) values of v for which a cyclic symmetric configuration v_k exists are written in normal (resp. bold) font. Also, an entry $\overline{v}$ (with v in italic font) means that no configuration v_k exists while $\overline{v^c}$ (again with v in italic font) indicates that no *cyclic* configuration v_k exists.

For $k \leq 15$, we use data from [23, 28, 30, 33, 39, 44, 53], including those reported in (2.1)–(2.3). An entry of the form "$t+$" in a row "n" of [53, Tab. 1] indicates the existence of *all* cyclic symmetric configurations v_n with $v \geq t$. The entries v_a, v_b, and v_c mean, respectively, that (2.1),(2.2), and (2.3) are applied. We use the following *non-existence* results: $\overline{32_6}$ [28, Th. 4.8]; $\overline{33_6}$ [39]; $\overline{34^c_6}$, $\overline{59^c_8}$-$\overline{62^c_8}$ [44]; $\overline{75^c_9}$-$\overline{79^c_9}$, $\overline{81^c_9}$- $\overline{84^c_9}$ [23]; $\overline{93^c_{10}} - \overline{106^c_{10}}$, $\overline{121^c_{11}} - \overline{132^c_{11}}$, $\overline{134^c_{11}}$, $\overline{135^c_{12}} - \overline{155^c_{12}}$, $\overline{157^c_{12}}$, $\overline{160^c_{12}}$, $\overline{169^c_{13}} - \overline{182^c_{13}}$, $\overline{184^c_{13}} - \overline{192^c_{13}}$, $\overline{185^c_{14}} - \overline{224^c_{14}}$, $\overline{256^c_{15}} - \overline{260^c_{15}}$, $\overline{263^c_{15}}$ [53, Tab. 1]; see also Theorem 5.

The values of k for which the spectrum of parameters of cyclic symmetric configurations v_k is completely known are indicated by a dot "**.**"; the corresponding values of $E_c(k)$ are sharp and they are noted by the dot "**.**" too.

In order to widen the ranges of parameter pairs $\{v, k\}$ for which a cyclic symmetric configuration v_k exists, we consider a number of procedures that allow to define a new modular Golomb ruler from a known one. We recall a result from [51], which describes a method to construct different rulers with the same parameters.

Theorem 6 [51] *If* $(a_1, a_2, \ldots, a_k)$ *is a* (v, k) *modular Golomb ruler and* m *and* b *are integers with* $\gcd(m, v) = 1$ *then* $(ma_1 + b \pmod v), ma_2 + b \pmod v), \ldots, ma_k + b \pmod v))$ *is also a* (v, k) *modular Golomb ruler.*

It should be noted that a (v, k) modular Golomb ruler can also be a $(v + \Delta, k)$ modular Golomb ruler for some integer Δ [28]. However, this possibility does not depend on parameters v and k only. This is why Theorem 6 can be useful for our purposes.

For $k \leq 83$, we performed a computer search starting from the (v, k) modular Golomb rulers corresponding to (2.1)–(2.3). For projective and affine planes, we got concrete rulers from [52]. For Ruzsa's construction, we used (4.4). For every starting

Table 2 Values of v for which a cyclic symmetric configuration v_k exists, $v_\delta(k) \le v \le G(k) - 1$, $k \le 37$

k	$v_\delta(k)$	$v_\delta(k) \le v \le G(k) - 1$	$E_c(k)$	$G(k)$
5.	21.	$21_a, \overline{22}$	23.	23
6.	31.	$31_a, \overline{32}, \overline{33}, \overline{34^c}$	35.	35
7.	48.	$48_b, 49, 50$	48.	51
8.	57.	$57_a, \overline{58}, \overline{59^c} - \overline{62^c}, 63_b, 64 - 68$	63.	69
9.	73.	$73_a, \overline{74}, \overline{75^c} - \overline{79^c}, 80_b, \overline{81^c} - \overline{84^c}, 85 - 88$	85.	89
10.	91.	$91_a, \overline{92}, \overline{93^c} - \overline{106^c}, 107 - 109, 110_c$	107.	111
11.	120.	$120_b, \overline{121^c} - \overline{132^c}, 133_a, \overline{134^c}, 135 - 144$	135.	145
12.	133.	$133_a, \overline{134}, \overline{135^c} - \overline{155^c}, 156_c, \overline{157^c}, 158, 159, \overline{160^c}, 168_b, 161 - 170$	161.	171
13.	168.	$168_b, \overline{169^c} - \overline{182^c}, 183_a, \overline{184^c} - \overline{192^c}, 193 - 212$	193.	213
14.	183.	$183_a, \overline{184}, \overline{185^c} - \overline{224^c}, 225 - 254$	225.	255
15	255.	$255_b, \overline{256^c} - \overline{260^c}, \overline{263^c}, 267 - 271, 272_c, 273_a, 274 - 287, 288_b, 289 - 302$	267	303
16	255.	255_b, 272_c, 273_a, 288_b, 307_a, **313**, **317**, **318**, **320-354**	**320**	355
17	273.	273_a, $\overline{274}$, 288_b, 307_a, 342_c, **343**, **349**, **353**, 360_b, 381_a, **356-398**	**356**	399
18	307.	307_a, 342_c, 360_b, 381_a, **389**, **391**, **395 − 398**, **401**, **403-432**	**403**	433
19	≥ 345	360_b, 381_a, **445**, **450**, **453**, **455-458**, **460-492**	**460**	493
20	381.	381_a, $\overline{382}$, **482**, **497**, **498**, **501-503**, 506_c, **505-509**, 528_b, 553_a, **511-566**	**511**	567
21	≥ 423	506_c, 528_b, 553_a, **586**, **589**, **591**, **592**, **594**, **595**, **597**, **598**, 624_b, **600-666**	**600**	667
22	≥ 465	506_c, 528_b, 553_a, 624_b, **633**, **637**, **640-642**, 651_a, **644-712**	**644**	713
23	≥ 509	528_b, 553_a, 624_b, 651_a, **683**, **686-688**, **692**, **695-700**, 728_b, **702-744**	**702**	745
24	553.	553_a, $\overline{554}$, 624_b, 651_a, 728_b, **738**, **739**, **742**, **747-749**, **752**, **753**, **755**, 757_a, 812_c, 840_b, **757-850**	**757**	851
25	≥ 602	624_b, 651_a, 728_b, 757_a, 812_c, **830**, 840_b, 871_a, 930_c, 960_b, **837-960**	**837**	961
26	651.	651_a, $\overline{652}$, 728_b, 757_a, 812_c, 840_b, 871_a, **885**, **888**, **895**, **900**, **903**, **905-907**, **910-913**, **915-917**, **919-925**, **927**, 930_c, 960_b, **929-984**	**929**	985
27	≥ 704	728_b, 757_a, 812_c, 840_b, 871_a, 930_c, 960_b, **970**, **971**, **972**, **975**, **977**, **978**, **985**, **987**, **988**, **991**, 993_a, **993-997**, **1000**, **1001**, **1003-1015**, 1023_b, 1057_a, **1017-1106**	**1017**	1107
28	757.	757_a, $\overline{758}$, 812_c, 840_b, 871_a, 930_c, 960_b, 993_a, **1006**, 1023_b, **1045**, **1051**, **1053**, 1057_a, **1063-1067**, **1070-1072**, **1074**, **1075**, **1077**, **1079-1170**	**1079**	1171
29	≥ 815	840_b, 871_a, 930_c, 960_b, 993_a, 1023_b, 1057_a, **1091**, **1127**, **1135**, **1137**, **1141**, **1143**, **1145**, **1146**, **1151-1246**	**1151**	1247
30	871.	871_a, $\overline{872}$, 930_c, 960_b, 993_a, 1023_b, 1057_a, **1196**, **1198-1201**, **1206**, **1207**, **1216**, **1217**, **1219-1224**, 1332_c, **1226-1360**	**1226**	1361
31	≥ 933	960_b, 993_a, 1023_b, 1057_a, **1298**, **1309**, **1314**, **1315**, **1320**, **1321**, **1324**, **1325**, 1332_c, **1330-1335**, **1339-1346**, 1368_b, **1348-1494**	**1348**	1495
32	993.	993_a, $\overline{994}$, 1023_b, 1057_a, 1332_c, **1366**, 1368_b, **1383**, **1388**, **1391-1395**, **1397**, **1398**, **1400**, **1401**, **1403**, 1407_a, **1406-1409**, **1411-1414**, **1416**, **1420**, **1421**, **1424-1434**, **1436-1568**	**1436**	1569
33	1057.	1057_a, $\overline{1058}$, 1332_c, 1368_b, 1407_a, **1492**, **1506**, **1507**, **1515**, **1518**, **1520**, **1521**, **1528**, **1529**, **1533**, **1535**, **1537**, **1540**, **1542**, **1543**, **1545**, **1547-1553**, **1555-1559**, 1640_c, 1680_b, **1561-1718**	**1561**	1719
34	≥ 1125	1332_c, 1368_b, 1407_a, 1640_c, **1664**, **1665**, **1670**, **1676**, 1680_b, **1686**, **1693**, **1698**, **1699**, **1702**, **1705**, **1708-1712**, **1714**, **1717**, **1721**, 1723_a, **1723-1726**, **1728**, **1730-1742**, **1744-1752**, 1806_c, 1848_b, **1754-1876**	**1754**	1877
35	≥ 1193	1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, **1777**, **1781**, **1783**, **1788**, **1792**, **1793**, **1795**, **1798**, **1800-1803**, 1806_c, **1805-1807**, **1810**, **1812-1815**, 1848_b, 1893_a, **1817-1974**	**1817**	1975
36	≥ 1262	1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, **1853**, **1855**, **1860**, **1867-1870**, **1872-1876**, **1878**, **1882-1884**, 1893_a, **1886-2010**	**1886**	2011
37	≥ 1335	1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, **1892**, 1893_a, **1910**, **1922**, **1930**, **1934**, **1938**, **1943**, **1944**, **1947-1953**, **1957**, **1959**, **1960**, **1962**, **1963**, **1965-1967**, **1969**, 2162_c, **1972-2198**	**1972**	2199

Table 3 Upper bounds on the cyclic existence bound $E_c(k)$, $38 \leq k \leq 83$

k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$	k	$E_c(k)$	$G(k)$
38	**2172**	2293	46	**3280**	3407	54	**4513**	4747	62	**6150**	6431	70	**8125**	8435	78	**10395**	10599
39	**2330**	2505	47	**3429**	3609	55	**5195**	5197	63	**6611**	6783	71	**8288**	8661	79	**10800**	10817
40	**2438**	2565	48	**3642**	3775	56	**5341**	5451	64	**6796**	7055	72	**8694**	8947	80	**10977**	11127
41	**2544**	2611	49	**3839**	3917	57	**5501**	5547	65	**6853**	7187	73	**8813**	9027	81	**11396**	11435
42	**2628**	2795	50	**3871**	4189	58	**5551**	5703	66	**7279**	7515	74	**8965**	9507	82	**11443**	11629
43	**2860**	3015	51	**4233**	4381	59	**5612**	5823	67	**7359**	7639	75	**9883**	9965	83	**11593**	12041
44	**2916**	3193	52	**4359**	4541	60	**5687**	6039	68	**7463**	7913	76	**10023**	10179			
45	**3165**	3375	53	**4463**	4695	61	**5994**	6269	69	**8111**	8291	77	**10229**	10409			

(v,k) modular ruler we first considered all possible m with $\gcd(m,v)=1$, and applied Theorem 6 for all $b<v$ to get new rulers with the same parameters v and k. Then, we checked whether new rulers are also $(v+\Delta,k)$ modular rulers for some Δ.

In Table 2, for $16 \leq k \leq 37$, the values of v obtained in this work are given in bold font; the entry **v-w** notes an interval of sizes from **v** to **w** without gaps. If an already known value lies within an interval **v-w** obtained in this work, then it is written immediately before the interval. Also, data on the non-existence by Theorem 5 are written in italic font, in the form $\overline{v}$.

In Table 3, for $38 \leq k \leq 83$, the upper bounds on the cyclic existence bound $E_c(k)$ obtained in this work are listed. For each k in the interval $E_c(k), E_c(k)+1, \ldots, G(k)-1$, at least one cyclic symmetric configuration v_k has been obtained. These configurations are new. We obtained also many other new cyclic symmetric configurations v_k for $38 \leq k \leq 83$; see [16].

The upper bounds on $E_c(k)$ in Table 2, for $16 \leq k \leq 37$, and all upper bounds in Table 3 are new. They are written in bold font.

Some new cyclic symmetric configurations important for Table 5 of Section 7 are given in Table 4 where we give the first rows of their incidence matrices; these rows may be the same for distinct v.

More detailed tables for cyclic symmetric configurations can be found in [16].

7 The spectrum of parameters of symmetric (not necessarily cyclic) configurations

The known results regarding parameters of symmetric configurations can be found in [1–10, 12–14, 16–19, 22–24, 26–35, 39, 41, 44–51, 53, 56]; see also the references therein. The known families of configurations are described in Section 2 and in (5.3). New families are given in Sections 4 and 5. In Table 5, for $k \leq 37$, $P(k) \leq v < G(k)$, values of v for which a symmetric configuration v_k from one of the families of Sections 2–5 exists are given. An entry of type $v_{\text{subscript}}$ indicates that relations (2.*i*),(3.*j*),(4.*k*) or Tables 1, 2 are used: more precisely v_a indicates that v is obtained from (2.1), and similarly $v_b \to (2.2)$, $v_c \to (2.3)$, $v_f \to (2.5)$, $v_g \to (2.6)$, $v_h \to (2.7)$, $v_j \to (2.8)$, $v_k \to (2.9)$, $v_\lambda \to (2.13)-(2.16)$, $v_m \to (5.3)$, $v_P \to (4.3)$, $v_r \to (5.4)$, $v_S \to (5.5)$, $v_T \to (5.6)$, $v_W \to$ Table 1, $v_y \to$ Table 2 with $k \leq 15$, $v_Z \to$ Table 2 with $k > 15$. Here capital letters in subscripts remark new results and constructions while

Table 4 New cyclic symmetric configurations v_k

k	v	the 1-st row of incidence matrix
17	382	0,25,69,81,88,89,112,123,126,128,141,174,196,202,206,223,232
17	383 385 386 388	0,5,15,34,35,42,73,75,86,89,98,134,151,155,177,183,201
17	384	0,68,70,84,90,107,111,120,139,151,185,186,193,196,211,244,249
17	387	0,5,7,17,52,56,67,80,81,100,122,138,159,165,168,191,199
18	389	0,10,27,28,35,50,74,94,103,105,108,146,159,165,191,195,207,228
18	391	0, 1, 4,15,35,42,75,85,94,111,133,139,141,157,162,194,206,219
18	395	0,71,73,81,85,88 117 118 141 167 172 183 192 210 231 244 250 272
18	396	0,79,89,106,107,114,129,153,173,182,184,187,225,238,244,270,274,286
18	397	0,36,50,56,59,74,78,85,122,139,147,149,179,180,192,226,231,247
18	398	0,18,30,32,71,84,90,93,119,127,152,169,176,192,196,197,207,243
18	401	0,71,94,104,136,160,164,176,195,217,238,243,256,263,290,292,293,301
18	404	0, 2,10,14,17,46,47,70,96,101,112,121,139,160,173,179,201,236
18	405 407	0,2,10,22,53,56,82,83,89,98,130,148,153,167,188,192,205,216
18	406	0,49,59,62,97,99,117,141,173,180,184,192,201,206,207,237,253,278
18	408	0,14,15,27,50,53,60,81,97,115,137,139,145,156,188,208,213,217
18	409	0,91,193,195,203,215,246,249,275,276,282,291,323,341,346,360,381,385
18	410	0,36,68,103,110,111,121,124,130,161,176,200,202,225,230,247,259,263
18	411	0,93,195,197,205,217,248,251,277,278,284,293,325,343,348,362,383,387

lowercase letters note the known ones. An entry $v_{\text{subscript}_1\cdot\text{subscript}_2\cdot\dots}$ with more than one subscript means that the same value can be obtained from different constructions. An entry of type $v_{\text{subscript}_1\cdot\text{subscript}_2\cdot\dots} - v'_{\text{subscript}_1\cdot\text{subscript}_2\cdot\dots}$ indicates that a whole interval of values from v to v' can be obtained from the constructions corresponding to the subscripts. We use the following known results on the existence of sporadic symmetric configurations: 45_7 [9]; 98_{10} [24, Tab. 1]; 135_{12} [47, Ex. 3.10]; 34_6 [41], see also [4]. The non-existence of configuration 112_{11} is proven in [40]. The non-existence of the plane $PG(2,10)$ implies the non-existence of configuration 111_{11}. The values of k for which the spectrum of parameters of symmetric configurations v_k is completely known are indicated by a dot "**.**"; the corresponding values of $E(k)$ are exact and they are indicated by a dot as well.

To save space, in Table 5 for given v,k, we do not write all constructions providing the configuration v_k, but often we describe *a few variants* for illustration. Also, to conserve space, for $k > 29$ we do not give any subscripts.

The *filling of the interval* $P(k)$–$G(k)$ is expressed as a percentage in the last column. It is interesting to note that such a percentage is quite high, and that most gaps occur for v close to $k^2 - k + 1$.

We note that a number of parameters obtained in this work, see Sections 4–6, are *new*. For $k \leq 37$, these new parameters are as follows: $382_{17} - 388_{17}$, 389_{18}, $391_{18}, 395_{18} - 398_{18}, 401_{18}$, $404_{18} - 411_{18}$. All these configurations are cyclic; they are given in Table 4 of Section 6. New parameters of configurations v_k and new bounds on $E(k)$ are written in Table 5 in bold font. Note that parameters of some new families are too big to be included in Table 5, see e.g. Examples 1 and 3, Theorem 3, configuration of (5.7). Recall also (see Introduction) that from the standpoint of applications, including Coding Theory, it is useful to have different matrices $\mathbf{M}(v,k)$ for the same v and k.

More detailed tables for parameters of symmetric configurations v_k can be found in [16].

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Table 5 The current known parameters of symmetric configurations v_k (cyclic and non-cyclic)

k	$P(k)$	$P(k) \leq v < G(k)$	$E(k) \leq$	$G(k)$	filling
3.	7	7	7.	7	100%
4.	13	13	13.	13	100%
5.	21	$21_a, \overline{22}$	23.	23	100%
6.	31	$31_a, \overline{32}, \overline{33}, 34$	34.	35	100%
7	43	$\overline{43}, \overline{44}, 45, 48_{b\cdot f\cdot r}, 49_{m\cdot r}, 50_{m\cdot r}$	48	51	75%
8	57	$57_a, \overline{58}, 63_{b\cdot f\cdot r}, 64_{m\cdot r} - 68_{m\cdot r}$	63	69	67%
9	73	$73_a, \overline{74}, 78_{g\cdot h}, 80_{b\cdot f\cdot r}, 81_{m\cdot r} - 88_{m\cdot r}$	80	89	75%
10	91	$91_a, \overline{92}, 98_j, 107_y - 109_y, 110_{c\cdot f\cdot k\cdot m\cdot r\cdot S\cdot T}$	107	111	35%
11	111	$\overline{111}, \overline{112}, 120_{b\cdot f\cdot r}, 121_{m\cdot r} - 133_{m\cdot r}, 135_y - 142_y, 143_{m\cdot y\cdot S}, 144_{m\cdot y\cdot r\cdot S\cdot T}$	135	145	76%
12	133	$133_a, \overline{134}, 135, 154_\lambda, 155_\lambda, 156_{m\cdot r\cdot S\cdot T} - 169_{m\cdot r\cdot S\cdot T}, 170_{m\cdot r\cdot T\cdot W}$	154	171	52%
13	157	$\overline{158}, 168_{b\cdot f\cdot r}, 169_{m\cdot r} - 183_{m\cdot r}, 189_g, 193_y - 209_y, 210_{m\cdot y\cdot r} - 212_{m\cdot y\cdot r}$	193	213	68%
14	183	$183_a, \overline{184}, 210_g, 222_\lambda, 223_\lambda, 224_m, 225_{m\cdot y\cdot r} - 254_{m\cdot y\cdot r}$	222	255	50%
15	211	$\overline{211}, \overline{212}, 231_g, 238_\lambda, 239_\lambda, 240_{m\cdot r} - 266_{m\cdot r}, 267_{m\cdot y\cdot r} - 302_{m\cdot y\cdot r}$	238	303	73%
16	241	$252_{g\cdot h}, 255_{b\cdot f\cdot r}, 256_{m\cdot r} - 321_{m\cdot r}, 322_{\lambda\cdot r}, 323_{m\cdot r\cdot S} - 354_{m\cdot r\cdot S}$	255	355	89%
17	273	$273_a, \overline{274}, 288_{b\cdot f\cdot r}, 289_{m\cdot r} - 307_{m\cdot r}, 321_\lambda, 322_\lambda, 323_{m\cdot S}, 372_W,$ $324_{m\cdot r} - 381_{m\cdot r}, \mathbf{382}_Z - \mathbf{388}_Z, 389_{\lambda\cdot Z}, 390_{\lambda\cdot Z}, 391_{m\cdot S\cdot Z} - 398_{m\cdot S\cdot Z}$	**321**	399	79%
18	307	$307_a, 340_\lambda, 341_\lambda, 342_{m\cdot r} - 381_{m\cdot r}, \mathbf{389}_Z, \mathbf{391}_Z, \mathbf{395}_Z - \mathbf{398}_Z, \mathbf{401}_Z,$ $403_{g\cdot Z}, \mathbf{404}_Z - \mathbf{411}_Z, 412_{\lambda\cdot Z}, 413_{\lambda\cdot Z}, 414_{m\cdot S} - 432_{m\cdot S}$	**403**	433	63%
19	343	$\overline{344}, 360_{b\cdot f\cdot r}, 361_{m\cdot r} - 381_{m\cdot r}, 434_{g\cdot W}, 435_\lambda, 436_\lambda, 437_{m\cdot S} - 457_{m\cdot S},$ $458_{\lambda\cdot r\cdot T}, 459_{\lambda\cdot r\cdot T}, 465_W, 460_{m\cdot r\cdot T} - 492_{m\cdot r\cdot T}$	434	493	54%
20	381	$381_a, \overline{382}, 458_\lambda, 459_\lambda, 465_W, 460_{m\cdot S} - 481_{m\cdot S}, 482_{\lambda\cdot r\cdot T}, 558_W,$ $483_{m\cdot r\cdot T} - 566_{m\cdot r\cdot T}$	458	567	59%
21	421	$\overline{422}, 481_\lambda, 482_\lambda, 483_{m\cdot S}, 558_W, 640_W, 484_{m\cdot r} - 666_{m\cdot r}$	481	667	76%
22	463	$\overline{463}, \overline{464}, 504_\lambda, 505_\lambda, 506_{m\cdot r} - 573_{m\cdot r}, 574_{\lambda\cdot r}, 558_W, 575_{m\cdot r} - 712_{m\cdot r}, 640_W$	504	713	84%
23	507	$\overline{507}, \overline{508}, 528_{b\cdot f\cdot r}, 529_{m\cdot r} - 553_{m\cdot r}, 558_{g\cdot W}, 573_\lambda, 574_\lambda, 575_m, 576_{m\cdot r} - 744_{m\cdot r}$	573	745	84%
24	553	$553_a, \overline{554}, 589_g, 598_\lambda, 599_\lambda, 600_{m\cdot r} - 673_{m\cdot r}, 674_{\lambda\cdot r}, 675_{m\cdot r} - 850_{m\cdot r}$	598	851	85%
25	601	$620_{g\cdot h}, 624_{b\cdot f\cdot P\cdot r}, 625_{m\cdot r} - 651_{m\cdot r}, 673_\lambda, 674_\lambda, 675_m, 676_{m\cdot r} - 960_{m\cdot r},$ $906_W, 912_W, 938_W$	673	961	88%
26	651	$651_a, \overline{652}, 700_\lambda, 701_\lambda, 702_{m\cdot r} - 781_{m\cdot r}, 782_{\lambda\cdot r}, 783_{m\cdot r} - 984_{m\cdot r}$	700	985	85%
27	703	$728_{b\cdot f\cdot r}, 729_{m\cdot r} - 757_{m\cdot r}, 781_\lambda, 782_\lambda, 783_{m\cdot S}, 784_{m\cdot r} - 1065_{m\cdot r},$ $1066_{r\cdot T\cdot Z} - 1072_{r\cdot T\cdot Z}, 1073_{m\cdot S\cdot Z} - 1103_{m\cdot S\cdot Z}, 1104_{r\cdot T\cdot Z}, 1105_{\lambda\cdot r\cdot T\cdot Z},$ $1106_{\lambda\cdot r\cdot T\cdot Z}$	781	1107	88%
28	757	$757_a, \overline{758}, 810_\lambda, 811_\lambda, 812_{m\cdot r} - 1065_{m\cdot r}, 1066_{r\cdot T} - 1072_{r\cdot T},$ $1073_{m\cdot S} - 1103_{m\cdot S}, 1104_{r\cdot T\cdot Z} - 1109_{r\cdot T\cdot Z}, 1110_{m\cdot r\cdot S\cdot Z} - 1141_{m\cdot r\cdot S\cdot Z},$ $1142_{r\cdot T\cdot Z} - 1145_{r\cdot T\cdot Z}, 1146_{\lambda\cdot r\cdot T\cdot Z}, 1147_{m\cdot r\cdot S\cdot Z} - 1170_{m\cdot r\cdot S\cdot Z}$	810	1171	87%
29	813	$\overline{814}, 840_{b\cdot f\cdot r}, 841_{m\cdot r} - 871_{m\cdot r}, 897_\lambda, 898_\lambda, 899_{m\cdot S}, 900_{m\cdot r} - 1057_{m\cdot r},$ $1071_\lambda, 1072_\lambda, 1073_{m\cdot S} - 1103_{m\cdot S}, 1104_{r\cdot T} - 1109_{r\cdot T}, 1110_{m\cdot S} - 1141_{m\cdot S},$ $1142_{r\cdot T} - 1146_{r\cdot T}, 1147_{m\cdot S} - 1179_{m\cdot S}, 1180_{r\cdot T\cdot Z} - 1183_{r\cdot T\cdot Z},$ $1184_{m\cdot S} - 1219_{m\cdot S}, 1220_{r\cdot T\cdot Z}, 1221_{m\cdot S} - 1246_{m\cdot S}$	1071	1247	85%
30	871	$871, \overline{872}, 928 - 1057, 1108 - 1360$	1108	1361	78%
31	931	$\overline{931}, \overline{932}, 960 - 1057, 1145 - 1494$	1145	1495	79%
32	993	$993, \overline{994}, 1023 - 1057, 1182 - 1568$	1182	1569	73%
33	1057	$1057, \overline{1058}, 1219 - 1718$	1219	1719	75%
34	1123	$\overline{1123}, \overline{1124}, 1256 - 1876$	1256	1877	82%
35	1191	$\overline{1192}, 1293 - 1407, 1433 - 1974$	1433	1975	83%
36	1261	$1330 - 1407, 1474 - 2010$	1474	2011	82%
37	1333	$\overline{1334}, 1368 - 1407, 1515 - 2198$	1515	2199	83%

Key to Table 5: $a \to (2.1)$, $b \to (2.2)$, $c \to (2.3)$, $f \to (2.5)$, $g \to (2.6)$, $h \to (2.7)$, $j \to (2.8)$, $k \to (2.9)$, $\lambda \to (2.13)-(2.16)$, $m \to (5.3)$, $P \to (4.3)$, $r \to (5.4)$, $S \to (5.5)$, $T \to (5.6)$, $W \to$ Table 1, $y \to$ Table 2 with $k \leq 15$, $Z \to$ Table 2 with $k > 15$

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