J. Geom. 107 (2016), 89–117

0047-2468/16/010089-29
published online June 16, 2015
DOI 10.1007/s00022-015-0277-z

Journal of Geometry

Upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane based on computer search

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Abstract. In the projective plane $\mathrm{PG}(2,q)$, upper bounds on the smallest size $t_2(2,q)$ of a complete arc are considered. For a wide region of values of q, the results of computer search obtained and collected in the previous works of the authors and in the present paper are investigated. For $q \leq 301813$, the search is complete in the sense that all prime powers are considered. This proves new upper bounds on $t_2(2,q)$ valid in this region, in particular

$$\begin{array}{lcr} t_2(2,q) < 0.998\sqrt{3q\ln q} & \text{for} & 7 \leq q \leq 160001; \\ t_2(2,q) < 1.05\sqrt{3q\ln q} & \text{for} & 7 \leq q \leq 301813; \\ t_2(2,q) < \sqrt{q}\ln^{0.7295} q & \text{for} & 109 \leq q \leq 160001; \\ t_2(2,q) < \sqrt{q}\ln^{0.7404} q & \text{for} & 160001 < q \leq 301813. \end{array}$$

The new upper bounds are obtained by finding new small complete arcs in $\mathrm{PG}(2,q)$ with the help of a computer search using randomized greedy algorithms and algorithms with fixed (lexicographical) order of points (FOP). Also, a number of sporadic q's with $q \leq 430007$ is considered. Our investigations and results allow to conjecture that the 2-nd and 3-rd bounds above hold for all $q \geq 109$. Finally, random complete arcs in $\mathrm{PG}(2,q)$, $q \leq 46337$, q prime, are considered. The random complete arcs

The research of D. Bartoli, G. Faina, S. Marcugini, and F. Pambianco was supported in part by Ministry for Education, University and Research of Italy (MIUR) (Project "Geometrie di Galois e strutture di incidenza") and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA-INDAM). The work of D. Bartoli was also supported by the European Community under a Marie-Curie Intra-European Fellowship (FACE Project: number 626511). The research of A.A. Davydov and A.A. Kreshchuk was made in IITP RAS and supported by a Grant from the Russian Science Foundation (Project No. 14-50-00150).

and complete arcs obtained by the algorithm FOP have the same region of sizes; this says on the common nature of these arcs.

Mathematics Subject Classification. Primary 51E21, 51E22; Secondary 94B05.

Keywords. Projective planes, complete arcs, small complete arcs.

1. Introduction. The main results

Let $\mathrm{PG}(2,q)$ be the projective plane over the Galois field $\mathbb{F}_q$ of q elements. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(2,q)$. For an introduction to projective geometries over finite fields see [45, 68, 69].

The relationship among the theory of n-arcs, coding theory, and mathematical statistics is presented in [46, 47], see also [53]. In particular, a complete arc in a plane $\mathrm{PG}(2,q)$, the points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [74, Sec.7], [76] and they are related to optimal coverings arrays [43], superregular matrices [49], and quantum codes [19].

A point set $S \subset \mathrm{PG}(2,q)$ is 1-*saturating* if any point of $\mathrm{PG}(2,q) \setminus S$ is collinear with two points in S [17, 27, 28, 38, 77]. The points of a 1-saturating set in $\mathrm{PG}(2,q)$ form a parity check matrix of a *linear covering code* with codimension 3, Hamming distance 3 or 4, and covering radius 2. An open problem is to find small 1-saturating sets (respectively, short covering codes). A complete arc in $\mathrm{PG}(2,q)$ is, in particular, a 1-saturating set; often the smallest known complete arc is the smallest known 1-saturating set [17, 27, 28, 38, 62]. Let $\ell_1(2,q)$ be the smallest size of a 1-saturating set in $\mathrm{PG}(2,q)$. In [21, 52], for q large enough, by probabilistic methods the following upper bound is proved (for $3\sqrt{2}$ see [21, p.24]):

$$\ell_1(2,q) < 3\sqrt{2}\sqrt{q \ln q} < 5\sqrt{q \ln q}. \tag{1.1}$$

One of the main problems in the study of projective planes, which is also of interest in coding theory, is the finding of the spectrum of possible sizes of complete arcs. In particular, the value of $t_2(2,q)$, the smallest size of a complete arc in $\mathrm{PG}(2,q)$, is interesting. Finding estimates of the minimum size $t_2(2,q)$ is a hard open problem, see e.g. [48, Sect.4.10].

This paper is devoted to *upper bounds* for $t_2(2,q)$.

Surveys of results on the sizes of plane complete arcs, methods of their construction, and the comprehension of the relating properties can be found in [6, 10, 12, 15, 23, 24, 32, 38, 44–48, 50, 54, 63–65, 68–71, 73–75].

Problems connected with small complete arcs in $\mathrm{PG}(2,q)$ are considered in [1–18,20,23–28,30–42,44–48,50,51,54–58,61–63,65,66,68–75,78–81], see also references therein.

The exact values of $t_2(2,q)$ are known only for $q \leq 32$; see [31,41,45,56–58] and the paper [17] where the equalities $t_2(2,31) = t_2(2,32) = 14$ are established.

The following lower bounds hold (see [2,20,66,68] and references therein):

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \text{for any } q, \\ \sqrt{3q}+\frac{1}{2} & \text{for } q=p^h,\ p \text{ prime},\ h=1,2,3. \end{cases} \tag{1.2}$$

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Desarguesian) projective plane $\mathcal{P}_q$ of order q. In [50], for q *large enough*, the following result is proved by *probabilistic methods* (we take into account the remark [50, p. 320] that all logarithms in [50] have the natural base):

$$t(\mathcal{P}_q) \leq \sqrt{q}\ln^C q,\ C \leq 300, \tag{1.3}$$

where C is a constant independent of q (so-called universal or absolute constant). The authors of [50] conjecture that the constant C can be reduced to $C = 10$. A survey and analysis of random constructions for geometrical objects can be found in [21,34,52].

In $\mathrm{PG}(2,q)$, complete arcs are obtained by *algebraic constructions* (see [47, p. 209]) with sizes approximately $\frac{1}{3}q$ [1,10,51,70–72,81], $\frac{1}{4}q$ [10,51,73], $2q^{0.9}$ for $q > 7^{10}$ [70], and $2.46q^{0.75}\ln q$ for big prime q [42]. It is noted in [34, Sec. 8] that the smallest size of a complete arc in $\mathrm{PG}(2,q)$ obtained via algebraic constructions is

$$cq^{3/4} \tag{1.4}$$

where c is a universal constant [73, Sec.3], [74, Th.6.8].

Thus, there is a substantial gap between the known upper bounds and the lower bounds on $t_2(2,q)$, see (1.2)–(1.4). The gap is essentially reduced if one consider the lower bound (1.2) for complete arcs and the upper bound (1.1) for 1-saturating sets. However, though complete arcs are 1-saturating sets they represent a narrower class of objects. Therefore, for complete arcs, one may not use the bound (1.1) directly. Nevertheless, the common nature of complete arcs and 1-saturating sets allows to hope for upper bounds on $t_2(2,q)$ similar to (1.1). The hope is supported by numerous experimental data, see below.

In [50, p.313], it is noted (with reference to the work [20]) that in a preliminary report of 1989, Fisher obtained by computer search complete arcs in many planes of small orders and conjectured that the average size of a complete arc in $\mathrm{PG}(2,q)$ is about $\sqrt{3q\log q}$.

In [6], see also [5], an attempt to obtain a theoretical upper bound on $t_2(2,q)$ with the main term of the form $c\sqrt{q\ln q}$, where c is a small universal constant, is done. The reasonings of [6] are based on explanation of the working mechanism of a step-by-step greedy algorithm for constructing complete arcs in $\mathrm{PG}(2,q)$ and on quantitative estimations of the algorithm. For more than half of the steps of the iterative process, these estimations are proved rigorously. The natural (and well-founded) conjecture that they hold for the rest of steps is done, see [6, Conject.2]. As a result the following conjectural upper bounds are formulated:

Conjecture 1.1 [6]. *In* PG$(2, q)$*, under conjecture given in* [6, Conject. 2]*, the following estimates hold:*

$$t_2(2, q) < \sqrt{q}\sqrt{3\ln q + \ln\ln q + \ln 3} + \sqrt{\frac{q}{3\ln q}} + 3, \tag{1.5}$$

$$t_2(2, q) < 1.87\sqrt{q\ln q} < 1.08\sqrt{3q\ln q}. \tag{1.6}$$

Moreover, in [6] it is conjectured that the upper bounds (1.5), (1.6) hold for all q without any extra conditions.

Denote by $\bar{t}_2(2, q)$ the smallest *known* size of a complete arc in PG$(2, q)$. Clearly,

$$t_2(2, q) \leq \bar{t}_2(2, q). \tag{1.7}$$

From [10,12], see also references therein, it follows that $\bar{t}_2(2, q) < 4\sqrt{q}$ for $q \leq 841$ and $q = 857$. Complete $(4\sqrt{q} - 4)$-arcs are obtained for odd $q = p^2$ with $p \leq 41$ and $p = 7^2$ [35,40] and for $q = 2^6, 2^8, 2^{10}$ [26,30]. So,

$$t_2(2, q) < 4\sqrt{q} \quad \text{for } q \leq 841,\ q = 857, 31^2, 2^{10}, 37^2, 41^2, 7^4.$$

For $q \leq 151$, a number of improvements of $\bar{t}_2(2, q)$, in comparison with [10,12], are given in [62] and cited in [4].

For $q \leq 13627$ and sporadic $q \leq 45893$, the values of $\bar{t}_2(2, q)$ (up to January 2013) are collected in [10,12] where the following results are obtained:

$$t_2(2, q) < 4.5\sqrt{q} \text{ for } q \leq 2647,\ q = 2659, 2663, 2683, 2693, 2753, 2801;$$
$$t_2(2, q) < 5\sqrt{q} \text{ for } q \leq 9497,\ q = 9539, 9587, 9613, 9623, 9649, 9689, 9923, 9973.$$

Let Q_1, Q_2, Q_3, Q_4, and Q be the following sets of values of q:

$$Q_1 = \{2 \leq q \leq 49727,\ q \text{ prime power}\} \cup \{2 \leq q \leq 150001,\ q \text{ prime}\}$$
$$\cup \{40 \text{ sporadic prime } q\text{'s in the interval } [150503 \ldots 410009]\}; \tag{1.8}$$
$$Q_2 = \{49727 < q \leq 150001,\ q = p^h,\ h \geq 2,\ p \text{ prime}\} \cup \{430007\}; \tag{1.9}$$
$$Q_3 = \{150001 < q \leq 160001,\ q \text{ prime power}\}; \tag{1.10}$$
$$Q_4 = \{34 \text{ sporadic } q'\text{s in the interval } [160801 \ldots 430007]\}; \tag{1.11}$$
$$Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \{2 \leq q \leq 160001, q \text{ prime power}\}$$
$$\cup \{34 \text{ sporadic } q'\text{s in the interval } [160801 \ldots 430007]\}. \tag{1.12}$$

For $q \in Q_1$, the values of $\bar{t}_2(2, q)$ (up to August 2014) are collected in the work [4]. The values of q in the interval $[150503 \ldots 410009]$, corresponding to the 2-nd row of the formula (1.8), are given in [4, Tab.6]. For $q \in Q_2$, the values of $\bar{t}_2(2, q)$ are obtained in [6] and collected in Table 1 of this work. For $q \in Q_3$ the values of $\bar{t}_2(2, q)$ are obtained in the present paper via computer

Table 1 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2, q)$ of complete arcs in planes $\mathrm{PG}(2, q)$, $49729 \le q \le 160801$, q non-prime

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
49729	1246	50653	1262	51529	1272	52441	1284
54289	1311	57121	1349	58081	1363	59049	1370
63001	1417	65536	1453	66049	1462	68921	1497
69169	1497	72361	1534	73441	1552	76729	1589
78125	1606	78961	1614	79507	1623	80089	1624
83521	1665	85849	1691	94249	1785	96721	1810
97969	1820	100489	1847	103823	1878	109561	1942
113569	1976	117649	2011	120409	2036	121801	2055
124609	2077	128881	2116	130321	2130	131072	2141
134689	2163	139129	2208	143641	2253	146689	2275
148877	2297	151321	2312	157609	2371	160801	2388

Table 2 The smallest known sizes $\bar{t}_2 = \bar{t}_2(2, q)$ of complete arcs in planes $\mathrm{PG}(2, q)$, $q = 124477, 128683$, and $160801 \le q \le 430007$, q sporadic

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
124477	**2087**	**128683**	**2117**	**160801**	**2388**	**161009**	**2401**
162007	**2398**	163003	2423	**164011**	**2424**	**165001**	**2433**
166013	2448	167009	2455	168013	2463	169003	2472
170503	2481	**178169**	**2530**	180001	2555	185021	2591
190027	2635	200003	2711	210011	2780	250007	3065
260003	3129	**262144**	**3066**	270001	3188	280001	3262
290011	3325	300007	3384	**330017**	**3568**	350003	3685
360007	3740	370003	3767	380041	3819	390001	3883
400009	3937	410009	4000	420001	4041	430007	4087

search[1] and collected in [7]. The values of q in the interval $[160801 \ldots 430007]$ corresponding to Q_4 and the 2-nd row of the formula (1.12) are given in Table 2 of this work, where results that are new or smaller in comparison with [4] are written in bold font. They are obtained in [4] and in this work. For $q = 2^{18}$ the result of [25] is used, see (1.13) below. Note that $160801 = 401^2$.

From results collected in [4], we have

$$t_2(2, q) < 5.5\sqrt{q} \text{ for } q \le 38557, \ \ q \ne 36481, 37537, 37963, 38039, 38197.$$

For even $q = 2^h$, small complete arcs in planes are a base for inductive infinite families of small complete caps in the projective spaces $\mathrm{PG}(N, q)$, see [26]. For $h \le 15$, the smallest known sizes of complete arcs in $\mathrm{PG}(2, 2^h)$ are collected in

[1]The computer search for $q \in Q_3$ [see (1.10)] and $q \in L_2 \cup L_3 \cup L_4$ [see (1.15)–(1.17) below] is done using computational resources of Multipurpose computing complex of National Research Centre "Kurchatov Institute" (http://computing.kiae.ru/).

[4], see also [10,12,24,26,30] and references therein. For $q = 2^{16}, 2^{17}$ complete arcs are obtained in [6]. Finally, complete $(6\sqrt{q} - 6)$-arcs in $\mathrm{PG}(2, 4^{2h+1})$, are constructed in [25]; for $h \leq 4$ it is proved that they are complete. This gives a complete 3066-arc in $\mathrm{PG}(2, 2^{18})$. Sizes of the arcs for $h \geq 11$ are as follows:

$$\begin{aligned} &\bar{t}_2(2, 2^{11}) = 199,\ \bar{t}_2(2, 2^{12}) = 300,\ \bar{t}_2(2, 2^{13}) = 449,\ \bar{t}_2(2, 2^{14}) = 665,\\ &\bar{t}_2(2, 2^{15}) = 987,\ \bar{t}_2(2, 2^{16}) = 1453,\ \bar{t}_2(2, 2^{17}) = 2141,\ \bar{t}_2(2, 2^{18}) = 3066. \end{aligned} \tag{1.13}$$

Note that, for $q \in Q$, in the works [4,6,10,12] and in the present paper, most of the values $\bar{t}_2(2, q)$ have been obtained by computer search using randomized greedy algorithms described in [10,12,15,23,24,29]. The step-by-step greedy algorithm adds to the arc in every step a point providing the maximum possible (for the given step) number of new covered points.

Another way for obtaining of small plane complete arcs is applying a step-by-step algorithm with fixed order of points (FOP), see [8,9,13,15,16,18]. The algorithm FOP fixes a particular order on points of $\mathrm{PG}(2, q)$. In every step the algorithm FOP adds to an incomplete running arc the next point in this order not lying on bisecants of this arc. For both prime and non-prime q, a lexicographical order of points can be used, see Sect. 3. We call *lexiarcs* the arcs obtained by the algorithm FOP with lexicographical order of points. Note that the sizes of complete arcs obtained by the algorithm FOP vary insignificantly with respect to the order of points, see [15,16,18].

Let $t_2^L(2, q)$ be the size of a complete lexiarc in $\mathrm{PG}(2, q)$.

Let $L_1 - L_4$ and L be the following sets of values of q:

$$\begin{aligned} L_1 &= \{q \leq 67993,\ q \text{ prime}\}\\ &\quad \cup \{43 \text{ sporadic prime } q\text{'s in the interval } [69997 \ldots 190027]\}; \end{aligned} \tag{1.14}$$

$$L_2 = \{q \leq 67993,\ q = p^h,\ h \geq 2,\ p \text{ prime}\}; \tag{1.15}$$

$$L_3 = \{67993 < q \leq 301813,\ q \text{ prime power}\}; \tag{1.16}$$

$$L_4 = \{19 \text{ sporadic } q\text{'s in the interval } [310249 \ldots 430007]\}; \tag{1.17}$$

$$\begin{aligned} L &= L_1 \cup L_2 \cup L_3 \cup L_4 = \{q \leq 301813,\ q \text{ prime power}\}\\ &\quad \cup \{19 \text{ sporadic } q\text{'s in the interval } [310249 \ldots 430007]\}. \end{aligned} \tag{1.18}$$

For $q \in L_1$, the values of $t_2^L(2, q)$ are collected in the work [13]. The values of q in the interval $[69997 \ldots 190027]$ corresponding to the 2-nd row of the formula (1.14) are given in [13, Tab.2]. A few values of $t_2^L(2, q)$ for prime $q \in L_3$ are given in [9]. The rest of values of $t_2^L(2, q)$ for $q \in L_3$ and all the values for $q \in L_2 \cup L_4$ are obtained in the present paper, see Footnote 1 above. The values of $t_2^L(2, q)$ for $q \in L_2$ and for *non-prime* $q \in L_3$ are given in Table 3 of this work. (Thus, Table 3 contains sizes $t_2^L(2, q)$ for all non-prime q's in the region $[4 \ldots 300763]$). All values of $t_2^L(2, q)$ for $q \in L_3$ are given in [8]. The values of $t_2^L(2, q)$ corresponding to L_4 are presented in Table 4 of this work.

TABLE 3 The sizes $t_2^L = t_2^L(2, q)$ of complete lexiarcs in planes PG$(2, q)$, $4 \leq q \leq 300763$, q non-prime

q	t_2^L	q	t_2^L	q	t_2^L	q	t_2^L
4	6	8	6	9	8	16	12
25	14	27	14	32	17	49	24
64	26	81	31	121	40	125	40
128	43	169	49	243	59	256	64
289	68	343	77	361	80	512	99
529	97	625	108	729	120	841	129
961	140	1024	142	1331	173	1369	172
1681	195	1849	207	2048	219	2187	227
2197	228	2209	228	2401	242	2809	261
3125	283	3481	296	3721	310	4096	328
4489	344	4913	362	5041	369	5329	383
6241	416	6561	429	6859	442	6889	445
7921	483	8192	491	9409	525	10201	551
10609	562	11449	590	11881	600	12167	606
12769	626	14641	680	15625	691	16129	710
16384	718	16807	722	17161	733	18769	776
19321	787	19683	795	22201	847	22801	859
24389	887	24649	896	26569	935	27889	959
28561	975	29791	996	29929	997	32041	1040
32761	1055	32768	1048	36481	1114	37249	1127
38809	1158	39601	1174	44521	1245	49729	1316
50653	1333	51529	1341	52441	1360	54289	1385
57121	1424	58081	1447	59049	1448	63001	1504
65536	1531	66049	1542	68921	1579	69169	1575
72361	1627	73441	1637	76729	1680	78125	1684
78961	1704	79507	1712	80089	1720	83521	1755
85849	1776	94249	1866	96721	1891	97969	1910
100489	1947	103823	1980	109561	2036	113569	2087
117649	2119	120409	2132	121801	2149	124609	2184
128881	2214	130321	2231	131072	2240	134689	2272
139129	2327	143641	2358	146689	2388	148877	2408
151321	2419	157609	2479	160801	2504	161051	2506
167281	2553	175561	2629	177147	2638	177241	2649
185761	2713	187489	2721	192721	2774	196249	2785
201601	2831	205379	2858	208849	2891	212521	2919
214369	2934	218089	2956	226981	3020	229441	3029
237169	3097	241081	3119	249001	3174	253009	3206
259081	3236	262144	3258	271441	3328	273529	3358
279841	3391	292681	3461	299209	3505	300763	3512

Table 4 The sizes $t_2^L = t_2^L(2, q)$ of complete lexiarcs in planes $\mathrm{PG}(2, q)$, $310249 \le q \le 430007$, sporadic q

q	t_2^L	q	t_2^L	q	t_2^L	q	t_2^L
310249	3570	312509	3599	316969	3626	320009	3638
323761	3652	326041	3673	330017	3690	332929	3710
340007	3752	344569	3775	350003	3809	360007	3883
370003	3928	380041	3978	390001	4045	400009	4089
410009	4159	420001	4206	430007	4262		

It should be emphasized that in the present work we use the computer search results for *all prime power* q in the regions $q \le 160001$ with greedy algorithms and $q \le 301813$ with the algorithm FOP. In this sense, we say that the *computer search* in the noted regions is *complete*.

In the works of the authors [3,12,15,16,18], non-standard types of upper bounds on $t_2(2, q)$ are proposed. They are based on *decreasing functions* $c(q)$ and $\varphi(q; D)$ defined by the following relations:

$$t_2(2, q) = \sqrt{q}\ln^{c(q)} q, \quad c(q) = \frac{\ln(t_2(2, q)/\sqrt{q})}{\ln\ln q}, \tag{1.19}$$

$$t_2(2, q) = D\sqrt{q}\ln^{\varphi(q;D)} q, \tag{1.20}$$

where D is a universal constant independent of q.

In this paper, we propose one more form of the upper bound on $t_2(2, q)$ connected with an upper estimate of a function $h(q)$ given by the equality

$$t_2(2, q) = h(q)\sqrt{3q\ln q}. \tag{1.21}$$

The following theorem summarizes the main results of this paper using the mentioned types of upper bounds. The theorem is based on the complete computer search the results of which are collected in [4,7,8,10,12–14] and in Tables 1, 2, 3, and 4 of the present paper.

Theorem 1.2. *Let $t_2(2, q)$ be the smallest size of a complete arc in the projective plane $\mathrm{PG}(2, q)$. Let Q_4 and L_4 be the sets of values of q given by relations (1.11) and (1.17) and Tables 2 and 4. In $\mathrm{PG}(2, q)$ the following hold.*

A. *There are the following upper bounds with a fixed degree of logarithm of q:*
(i)

$$t_2(2, q) < 0.998\sqrt{3q\ln q} \quad \text{for} \quad 7 \le q \le 160001, \tag{1.22}$$
$$q = 401^2, 161009, 162007, 164011, 165001, 178169, 2^{18}, 380041;$$

$$t_2(2, q) < 1.007\sqrt{3q\ln q} \quad \text{for} \quad q \in Q_4; \tag{1.23}$$

$$t_2(2, q) < 1.05\sqrt{3q\ln q} \quad \text{for} \quad 7 \le q \le 301813, \ q \in L_4. \tag{1.24}$$

(ii)

$$t_2(2, q) < \sqrt{q}\ln^{0.7295} q \quad \text{for} \quad 109 \le q \le 160001, \ q \in Q_4; \tag{1.25}$$

$$t_2(2,q) < \sqrt{q}\ln^{0.7404} q \quad \textit{for} \ \ 160001 < q \leq 301813, \ \ q \in L_4. \tag{1.26}$$

B. *There are the following upper bounds with a decreasing degree of logarithm of q:*
 (i) *Let $t_2(2,q) = \sqrt{q}\ln^{c(q)} q$. We have*

$$c(q) < \frac{0.27}{\ln q} + 0.7 \quad \textit{for} \ \ 19 \leq q \leq 160001, \ \ q \in Q_4; \tag{1.27}$$

$$c(q) < \frac{0.6}{\ln \ln q} + \frac{1}{2} \quad \textit{for} \ \ 160001 < q \leq 301813, \ \ q \in L_4. \tag{1.28}$$

 (ii) *Let $t_2(2,q) = 0.6\sqrt{q}\ln^{\varphi(q;0.6)} q$. We have*

$$\varphi(q;0.6) < \frac{1.5}{\ln q} + 0.802 \quad \textit{for} \ \ 19 \leq q \leq 160001, \ \ q \in Q_4. \tag{1.29}$$

For $q \leq 160001$, $q \in Q_4$, complete arcs in PG$(2,q)$ *satisfying the upper bounds* (1.22), (1.23), (1.25), (1.27), *and* (1.29) *can be constructed with the help of the step-by-step greedy algorithm which adds to the arc in every step a point providing the maximum possible (for the given step) number of new covered points.*

For $160001 < q \leq 301813$, $q \in L_4$, *complete arcs in* PG$(2,q)$ *satisfying the upper bounds* (1.24), (1.26), *and* (1.28) *can be constructed as lexiarcs with the help of the algorithm FOP with lexicographical order of points. The only exception is $q = 178169$ for which a complete arc satisfying* (1.24) *can be obtained by the greedy algorithm.*

A number of calculations executed for sporadic $q \leq 430007$ strengthen the confidence for validity of the bounds of Theorem 1.2. Also, it is important that the bounds (1.22)–(1.29) are close to the conjectural (but well-founded) bounds of [6], see (1.5), (1.6) in Conjecture 1.1. On the whole, our investigations and results (see also figures below) allow to conjecture the following.

Conjecture 1.3. *The upper bounds* (1.24)–(1.29) *hold for all $q \geq 109$.*

Some results of this work were briefly presented in [3,9].

The paper is organized as follows. In Sect. 2, an approach to create a randomized greedy algorithm is described. Results and bounds connected with smallest known complete arcs in PG$(2,q)$ are given. In Sect. 3, the algorithm with fixed order of points (FOP) is presented. For both prime and non-prime q, a lexicographical order of points is described. Results and bounds based on lexiarcs are given and compared with the ones from Sect. 2. In Sect. 4, results for random complete arcs in PG$(2,q)$ are presented. In Sect. 5, the results of the present paper are discussed.

2. Greedy algorithms. The smallest known complete arcs

2.1. Randomized greedy algorithms

The most of the smallest known complete arcs *for* $q \in Q$ *used in the present paper* are obtained by computer search based on randomized greedy algorithms [4,10,12,15,23,24,29], with the exception of (relatively few) theoretically constructed complete arcs, see, for example, [25,26,30,35,39,40,62] and references therein.

In every step a randomized greedy algorithm maximizes an objective function f but some steps are executed in a random manner. The number of these steps, their ordinal numbers, and some other parameters of the algorithm have been taken intuitively. Also, if the same maximum of f can be obtained in distinct ways, one way is chosen randomly.

We begin to construct a complete arc by using a starting point set S_0. In the i-th step one point is added to the set S_{i-1} and we obtain a point set S_i. As the value of the objective function f we consider *the number of covered* points in $\mathrm{PG}(2,q)$, that is, points that lie on bisecants of S_i.

On every "random" i-th step we take $d_{q,i}$ *randomly chosen points* of $\mathrm{PG}(2,q)$ *not covered by* S_{i-1} and compute the objective function f adding each of these $d_{q,i}$ points to S_{i-1}. The point providing the maximum of f is included into S_i. On every "non-random" j-th step we consider *all points not covered by* S_{j-1} and add to S_{j-1} the point providing the maximum of f.

As S_0 we can use a subset of points of an arc obtained in previous stages of the search.

A generator of random numbers is used for a random choice. To get arcs with distinct sizes, the starting conditions of the generator are changed for the same set S_0. In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the arc from which the fixed points have been taken.

In order to obtain arcs with new sizes, sufficiently many attempts should be made with randomized greedy algorithms. "Predicted" sizes could be useful for understanding if a good result has been obtained. For $\mathrm{PG}(2,q)$, the predicted sizes can be obtained in distinct ways including upper bounds considered in Introduction and complete arc sizes for smaller q calculated before. See, for example, approaches used in [12]. If the result is not close to the predicted size, the attempts are continued.

We obtain small complete arcs in $\mathrm{PG}(2,q)$ in two stages.

At the 1-st stage, we take the frame as S_0 and create a starting complete arc K_0 using in the beginning of the process δ_q "random" steps with distinct $d_{q,i}$. All the next steps are "non-random".

At the 2-nd stage we execute n_q attempts to get a complete arc. For every attempt, the starting conditions of the random generator are other than in the previous ones. But the set S_0 is the same. Two or three of the first five steps of every attempt are "random" , the rest of them are "non-random".

The values of $d_{q,i}$, δ_q, and n_q are given intuitively depending on q and (for $d_{q,i}$) on $|S_{i-1}|$ and on the stage of the process. Of course, CPU performance affects the algorithm parameters choice.

Thus, arc sizes obtained by the randomized greedy algorithms depend on many factors. Nevertheless, we repeat that, for $q \in Q$, the most of the smallest known complete arcs are obtained with the help of these algorithms.

2.2. Upper bounds on $t_2(2, q)$, $q \in Q$, based on the smallest known complete arcs

For bounds, we use data on the smallest known complete arc sizes $\overline{t}_2(2, q)$, $q \in Q$, obtained and collected in [3,4,6,7,9–12] and in this work.

In Table 1, for $49729 \le q \le 160801$, q non-prime, the smallest known sizes $\overline{t}_2(2, q)$ (for short $\overline{t}_2$) of complete arcs in PG$(2, q)$ are collected. All $q \in Q_2$, see (1.9), are included to this table.

In Table 2, for 34 sporadic q's in the interval [160801...430007] and for $q = 124477, 128683$, the smallest known sizes $\overline{t}_2 = \overline{t}_2(2, q)$ of complete arcs in PG$(2, q)$ are written. All $q \in Q_4$, see (1.11), are included to this table. In bold font we write results that are new or smaller in comparison with [4], see (1.8). The new results for $\overline{t}_2(2, 124477)$ and $\overline{t}_2(2, 128683)$ satisfy the bound (1.22).

Using sizes $\overline{t}_2(2, q)$ for $q \in Q$, see [4,7] and Tables 1 and 2, we obtain the bottom curve on Fig. 1 where vertical lines mark regions of complete search for all q's prime power, in particular, $q \le 160001$ for $\overline{t}_2(2, q)$. About lexiarcs sizes on Fig. 1 see Sect. 3.

Bounds of Theorem 1.2 and Theorem 2.1 below are not shown on Fig. 1 as, on the scale of this figure, the curves of the bounds and of $\overline{t}_2(2, q)$ almost coincide with each other. At the same time, non-standard types of upper bounds defined by (1.19)–(1.21) allow us to show bounds more expressive.

Figure 2 shows values of $\overline{h}(q) = \overline{t}_2(2, q)/\sqrt{3q \ln q}$ for the smallest known complete arcs in PG$(2, q)$, $q \in Q$ (the bottom solid curve). The bounds (1.22) and (1.23) are shown by dashed lines $y = 0.998$ and $y = 1.007$. Vertical lines mark regions of complete search for all q's prime power, in particular, $q \le 160001$ for $\overline{t}_2(2, q)$. About lexiarcs sizes on Fig. 2 see Sect. 3.

A possible explanation of the fact that the bound (1.23) is worse than the bound (1.22) could be the following: for $q \in Q_4$, the parameters $d_{q,i}$, δ_q, and n_q of the randomized greedy algorithm are chosen inefficiently as these q's are relatively large and we tried to reduce the time of the calculations.

Figure 3 shows values $\overline{t}_2(2, q)/\sqrt{q \ln q}$ for the smallest known complete arcs in PG$(2, q)$, $q \in Q$ (the bottom solid curve) and the following upper bounds: $\ln^{0.2295} q$ (the top dashed-dotted curve), $C(q) = (\ln q)^{\overline{c}_{up}(q)-0.5}$ with $\overline{c}_{up}(q) =$

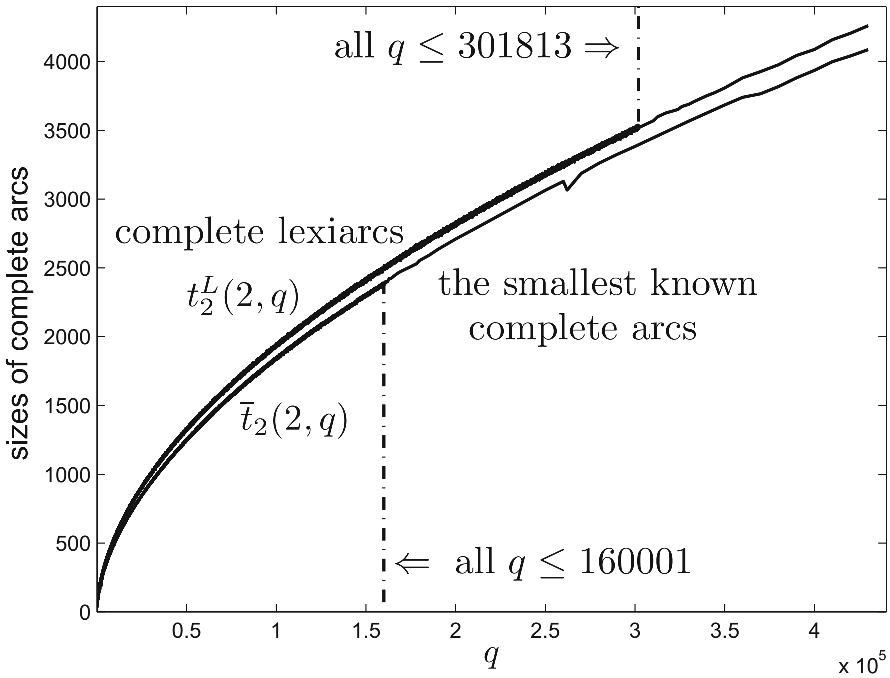

FIGURE 1 Sizes $t_2^L(2,q)$ of complete lexiarcs in PG$(2,q)$, $q \in L$ (*top curve*), the smallest known sizes $\overline{t}_2(2,q)$ of complete arcs in PG$(2,q)$, $q \in Q$ (*bottom curve*). *Vertical lines* mark regions of complete search for all q's prime power

$\frac{0.27}{\ln q} + 0.7$ (the 2-nd dashed-dotted curve), $\Phi(q) = 0.6(\ln q)^{\overline{\varphi}(q;0.6)-0.5}$ with $\overline{\varphi}(q;0.6) = \frac{1.5}{\ln q} + 0.802$ (the 3-rd dashed-dotted curve), $0.998\sqrt{3} = 1.729$ (dashed line A), $1.007\sqrt{3} = 1.744$ (dashed line B). Correlations of these bounds are given by Theorem 2.1.

Theorem 2.1. *Let $t_2(2,q)$ be the smallest size of a complete arc in the projective plane* PG$(2,q)$. *Let Q be the set of values of q given by* (1.12). *In* PG$(2,q)$ *the following upper bounds hold:*

$$t_2(2,q) < \min\left\{\sqrt{q}\ln^{0.7295} q,\ \sqrt{q}\ln^{\overline{c}_{up}(q)} q,\ 0.6\sqrt{q}\ln^{\overline{\varphi}(q;0.6)} q\right\},\ q \in Q, \quad (2.1)$$

where

$$\overline{c}_{up}(q) = \frac{0.27}{\ln q} + 0.7, \quad \overline{\varphi}(q;0.6) = \frac{1.5}{\ln q} + 0.802,$$

$$\min\left\{\sqrt{q}\ln^{0.7295} q,\ \sqrt{q}\ln^{\overline{c}_{up}(q)} q,\ 0.6\sqrt{q}\ln^{\overline{\varphi}(q;0.6)} q\right\}$$
$$= \begin{cases} \sqrt{q}\ln^{0.7295} q & \text{if} \quad 109 \le q \le 9437 \\ \sqrt{q}\ln^{\overline{c}_{up}(q)} q & \text{if} \quad 9437 < q \le 88873. \\ 0.6\sqrt{q}\ln^{\overline{\varphi}(q;0.6)} q & \text{if} \quad 88873 < q \end{cases}$$

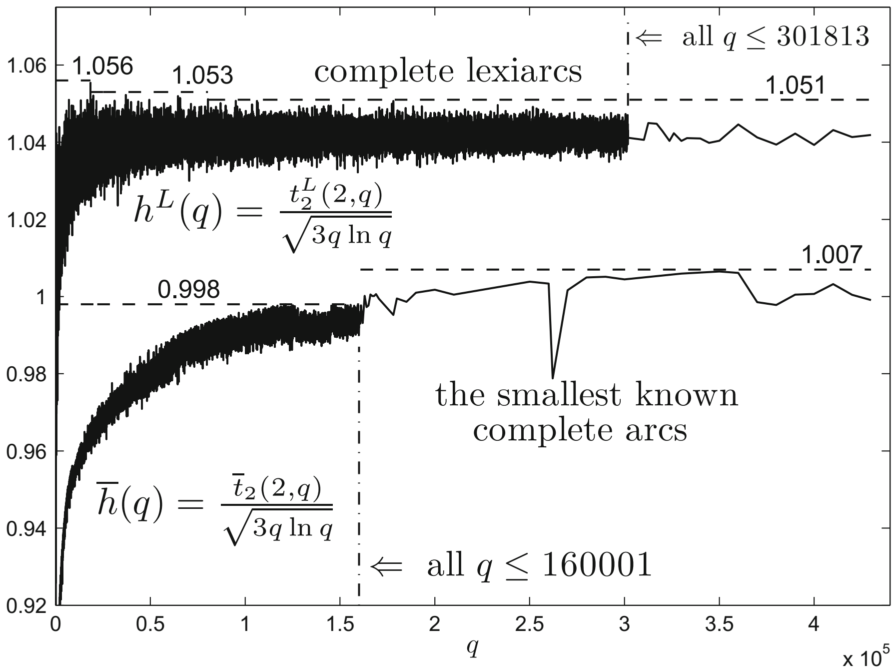

FIGURE 2 Values of $h^L(q) = t_2^L(2,q)/\sqrt{3q\ln q}$ for complete lexiarcs in PG$(2,q)$, $q \in L$ (*top solid curve*) and $\overline{h}(q) = \overline{t}_2(2,q)/\sqrt{3q\ln q}$ for the smallest known complete arcs in PG$(2,q)$, $q \in Q$ (*bottom solid curve*) and the corresponding upper bounds (*dashed lines*). *Vertical lines* mark regions of complete search for all q's prime power

Using Fig. 3 and Theorem 2.1, we obtain Fig. 4 where values and bounds connected with the decreasing function $c(q)$ are presented. Let

$$\overline{t}_2(2,q) = \sqrt{q}\ln^{\overline{c}(q)} q$$

where $\overline{c}(q)$ is a decreasing function of q. Values of $\overline{c}(q)$ for the smallest known complete arcs in PG$(2,q)$, $q \in Q$, are shown by the bottom solid curve. The upper bound (1.25) is given by the dashed line $y = 0.7295$. Finally, the bound (1.27) is shown by the dashed-dotted curve $\overline{c}_{up}(q) = \frac{0.27}{\ln q} + 0.7$. About lexiarcs sizes on Fig. 4 see Sect. 3.

Remark 2.2. In Figs. 1, 2, 3, 4, a "tooth" connected with $\overline{t}_2(2,2^{18}) = 3066$ is provided by the construction of [25]. It reminds to the reader that, in principle, theoretical constructions are able to give a better result than greedy algorithms. On the other side, as a rule, the order of arc size from a theoretical construction and a greedy algorithm is similar. We have $2^{18} = 262144$; the nearest known greedy algorithm's result is $\overline{t}_2(2,260003) = 3129$, see Table 2. The difference between 3129 and 3066 is ≈2 %.

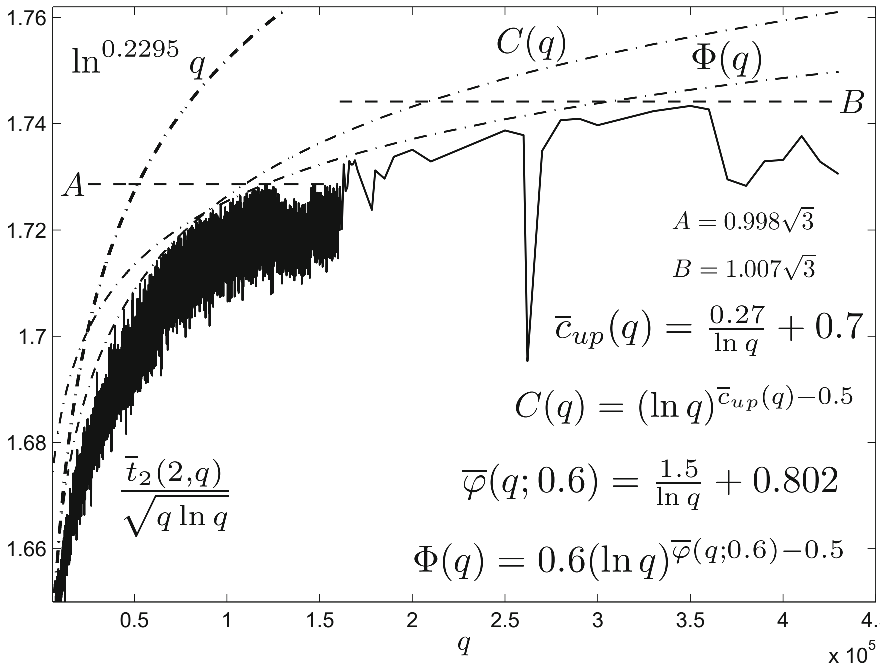

FIGURE 3 Values $\overline{t}_2(2,q)/\sqrt{q\ln q}$ for the smallest known complete arcs in $\mathrm{PG}(2,q)$, $q \in Q$ (*bottom solid curve*) and upper bounds: $\ln^{0.2295} q$ (*top dashed-dotted curve*), $C(q) = (\ln q)^{\overline{c}_{up}(q)-0.5}$ with $\overline{c}_{up}(q) = \frac{0.27}{\ln q} + 0.7$ (the *second dashed-dotted curve*), $\Phi(q) = 0.6(\ln q)^{\overline{\varphi}(q;0.6)-0.5}$ with $\overline{\varphi}(q;0.6) = \frac{1.5}{\ln q} + 0.802$ (the *third dashed-dotted curve*), $0.998\sqrt{3} = 1.729$ (*dashed line* A), $1.007\sqrt{3} = 1.744$ (*dashed line* B)

3. Algorithm with fixed order of points (FOP). Lexiarcs

3.1. Algorithm FOP

We use results and approaches of the works [8,9,13,15,16,18]. Consider the projective plane $\mathrm{PG}(2,q)$ and fix a particular order on its points. The algorithm builds a complete arc *iteratively*.

Let $K^{(i-1)}$ be the arc obtained on the $(i-1)$-th step. On the next step, the first point in the fixed order not lying on the bisecants of $K^{(i-1)}$ is added to $K^{(i-1)}$. Suppose that the points of $\mathrm{PG}(2,q)$ are ordered as $A_1, A_2, \ldots, A_{q^2+q+1}$. Consider the empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in the j-th step, as extension of the root. We put

$$K^{(0)} = \emptyset,\ K^{(1)} = \{A_1\},\ K^{(2)} = \{A_1, A_2\},\ m(1) = 2, \tag{3.1}$$

$$K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},\ m(j) = \min\{i \in [m(j-1)+1, q^2+q+1] \mid$$

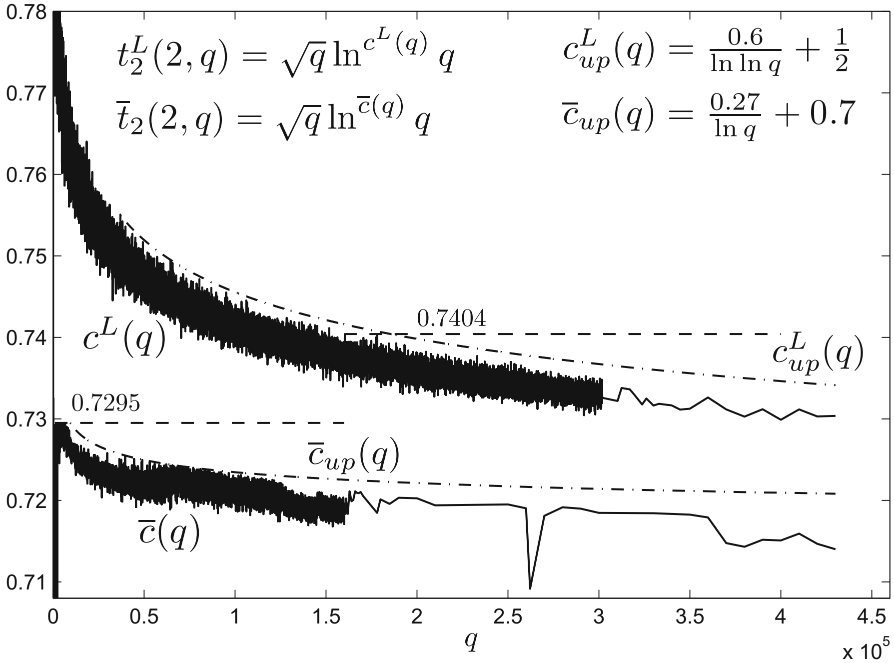

FIGURE 4 Values and bounds connected with decreasing function $c(q)$. Upper bounds "0.7404" and "0.7295" (*dashed lines*), upper bound $c^L_{up}(q) = \frac{0.6}{\ln\ln q} + \frac{1}{2}$ (*top dashed-dotted curve*), values of $c^L(q)$ for complete lexiarcs in $\mathrm{PG}(2,q)$, $q \in L$ (*top solid curve*), upper bound $\overline{c}_{up}(q) = \frac{0.27}{\ln q} + 0.7$ (*bottom dashed-dotted curve*), values of $\overline{c}(q)$ for the smallest known complete arcs in $\mathrm{PG}(2,q)$, $q \in Q$ (*bottom solid curve*)

$$\nexists\, P, Q \in K^{(j)} : A_i, P, Q \text{ are collinear}\},$$

i.e $m(j)$ is the minimum subscript i such that the corresponding point A_i is not saturated by $K^{(j)}$. The process ends when a complete arc is obtained.

Remark 3.1. In Coding Theory, so called *greedy codes* (or *lexicographical codes,* or *lexicodes*) are considered, see e.g. [22,59,60,67,82] and references therein. These codes are constructed in two ways. The first kind of greedy codes is considered in the most works on this topic. In order to obtain a q-ary code of length n with minimum distance d one writes all q-ary n-vectors in a list using a certain order. The first vector of the list should be included to the code. Then step-by-step, one takes the next vector from the list which has distance d or more from all vectors already chosen.

In another kind, see [22,59,60], one creates a *parity check matrix* of a q-ary code with codimension r and distance d. All q-ary r-vectors are written as columns in a list in some order. The first column of the list should be included into the matrix. Then step-by-step, one takes the next column from the list which cannot be represented as a linear combination of $d-2$ or smaller columns already chosen. The process is finished when no new column may be included to the matrix.

If a point of $\mathrm{PG}(2,q)$ is treated as a 3-vector column then formally FOP algorithm is an algorithm of the second kind creating a parity check matrix with $r=3, d=4$. But this viewpoint is only formal. In Coding Theory, for given r, d, the aim is to get a *long* code while our goal is to obtain a *short* complete arc. Moreover, our estimates and computer search show that for $r=3, d=4$ FOP algorithm gives "bad" codes, essentially shorter than "good" codes corresponding to ovals and hyperovals. Finally, note that we do not use a word "greedy" in a name of FOP algorithm, as in Projective Geometry the terms "greedy algorithm" and "randomized greedy algorithm" are traditionally connected with other approaches, see [10,12,23,29] and Sect. 2.

3.2. Lexicographical order of points

In the beginning, we consider q prime. Let the elements of the field $\mathbb{F}_q = \{0,1,\ldots,q-1\}$ be treated as integers modulo q. Let the points A_i of $PG(2,q)$ be represented in homogenous coordinates so that

$$A_i = (x_0^{(i)}, x_1^{(i)}, x_2^{(i)}),\ x_j^{(i)} \in \mathbb{F}_q, \tag{3.2}$$

where the leftmost non-zero element is 1. For A_i, we put

$$i = x_0^{(i)} q^2 + x_1^{(i)} q + x_2^{(i)}. \tag{3.3}$$

So, the homogenous coordinates of a point A_i are treated as its number i written in the q-ary scale of notation. Recall that the points of $\mathrm{PG}(2,q)$ are ordered as $A_1, A_2, \ldots, A_{q^2+q+1}$.

It is important that for such lexicographical order for prime q, the size $t_2^L(2,q)$ of a lexiarc depends on q only. No other factors affect its size and structure.

Now let $q=p^m$, p prime, $m \geq 2$. Let $F_q(x)$ be a primitive polynomial of $\mathbb{F}_{p^m}$ and let α be a root of $F_q(x)$. Elements of $\mathbb{F}_{p^m}$ are represented by integers as follows

$$\mathbb{F}_{p^m} = \mathbb{F}_q = \{0, 1=\alpha^0, 2=\alpha^1, \ldots, q-1=\alpha^{q-2}\}.$$

Then we again use (3.2) and (3.3). However for non-prime q the size $t_2^L(2,q)$ of a lexiarc depends on q and on the form of the polynomial $F_q(x)$. In this work we use primitive polynomials that are created by the program system MAGMA by default. In any case, the choice of the polynomial changes the lexiarc size inessentially.

We have noted in Introduction that, in general, the sizes of complete arcs obtained by Algorithm FOP vary insignificantly with respect to the order of points. In particular, in [15,16,18] the so-called Singer order of points (based on the Singer group of collineations) is considered and it is shown that in the region $5000 < q \leq 40009$, q prime, the difference between the sizes of complete

TABLE 5 The first 24 points of complete lexiarcs in PG$(2,q)$, q prime

i	a_0	a_1	a_2	$q_0(i)$	i	a_0	a_1	a_2	$q_0(i)$	i	a_0	a_1	a_2	$q_0(i)$
1	0	0	1	2	2	0	1	0	2	3	1	0	0	2
4	1	1	1	2	5	1	2	3	5	6	1	3	2	7
7	1	4	5	11	8	1	5	4	11	9	1	6	8	17
10	1	7	11	23	11	1	8	6	31	12	1	9	13	37
13	1	10	12	37	14	1	11	7	41	15	1	12	22	137
16	1	13	16	79	17	1	14	17	101	18	1	15	21	71
19	1	16	9	229	20	1	17	14	151	21	1	18	10	199
22	1	19	27	239	23	1	20	18	197	24	1	21	15	251

arcs obtained with the lexicographical order and the Singer order is less then 2 % and none of the two orders gives the smallest size for all q.

3.3. Starting points of lexiarcs in PG$(2, q)$, q prime

Proposition 3.2. *Let q be a prime. Then the i-th point of a lexiarc in* PG$(2,q)$ *is the same for all $q \geq q_0(i)$ where $q_0(i)$ is large enough.*

Proof. Suppose that, in (3.1), at a certain step i we have $K^{(i)} \setminus K^{(i-1)} = \{P\}$, with $P = A_s$. A point $Q = A_r \notin K^{(i)}$ will be the next chosen point in the extension process if and only if all the points A_j with $j \in [s+1, r-1]$ are covered by $K^{(i)}$. That is, for any $j \in [s+1, r-1]$ at least one of the determinants of the coordinates of the points P_1, P_2, A_j, with $P_1, P_2 \in K^{(i)}$, is equal to zero modulo q. This can happen only for two reasons:

1. $\det(P_1, P_2, A_j) = 0$: we say that A_j is "absolutely" covered by $K^{(i)}$;
2. $\det(P_1, P_2, A_j) = m \neq 0$, with $m \equiv 0 \bmod q$: we say that A_j is "simply" covered by $K^{(i)}$.

It is clear that, for q large enough, q does not divide any of the possible $m = \det(P_1, P_2, A_j)$ and then, at least for i "small", the points covered are just the absolutely covered points. Therefore, when q is large enough the lexiarcs share a certain number of points. $\square$

The values of $q_0(i)$ can be found with the help of calculations based on the proof of Proposition 3.2. Also, we can directly consider lexiarcs constructed by the algorithm FOP for a convenient region of q.

Example 3.3. Values of $q_0(i)$, $i \leq 24$, together with the homogenous coordinates (a_0, a_1, a_2) of the common points, are given in Table 5. So, for all prime $q \geq 251$ the first 24 points of a lexiarc are as in Table 5. Since $t_2^L(2, 251) = 63$ [13], we know $24/63 \approx 38\,\%$ of lexiarc points for $q = 251$. For growing q this percentage decreases (relatively slowly). For instance, for $q \approx 270000$ it is $\approx$20 %.

3.4. Upper bounds on $t_2(2, q)$, $q \in L$, based on complete lexiarcs

For bounds, we use data on lexiarc sizes $t_2^L(2, q)$, $q \in L$, obtained and collected in [8,9,13,15,16,18] and in this work.

In Table 3, for $4 \leq q \leq 300763$, q non-prime, the sizes $t_2^L(2, q)$ (for short t_2^L) of complete lexiarcs in $\mathrm{PG}(2, q)$ are collected. All $q \in L_2$ and all non-prime $q \in L_3$, see (1.15) and (1.16), are included to this table.

In Table 4, for 19 sporadic q's in the interval [310249...430007], the sizes $t_2^L = t_2^L(2, q)$ of complete lexiarcs in $\mathrm{PG}(2, q)$ are written. All $q \in L_4$, see (1.17), are included to this table.

Using sizes $t_2^L(2, q)$ for $q \in L$, see [8,13] and Tables 3 and 4, we obtain the top curve on Fig. 1. Bounds of Theorem 1.2 are not shown here as, on the scale of this figure, curves of the bounds and of $t_2^L(2, q)$ almost coincide with each other.

Figure 2 represents values of $h^L(q) = t_2^L(2, q)/\sqrt{3q \ln q}$ for complete lexiarcs in $\mathrm{PG}(2, q)$, $q \in L$ (the top solid line) and the corresponding upper bounds (dashed lines $y = 1.056$, $y = 1.053$, $y = 1.051$).

Let

$$t_2^L(2, q) = \sqrt{q} \ln^{c^L(q)} q, \tag{3.4}$$

where $c^L(q)$ is a decreasing function of q, cf. (1.19). By the 2-nd formula of (1.19), we have

$$c^L(q) = \frac{1}{2} + \frac{\ln d + 0.5 \ln 3}{\ln \ln q} \quad \text{for} \quad t_2^L(2, q) = d\sqrt{3q \ln q},$$

whence (cf. (3.7) and Figs. 2, 5)

$$c^L(q) < \frac{1}{2} + \frac{0.6}{\ln \ln q} \quad \text{for} \quad t_2^L(2, q) \leq 1.052\sqrt{3q \ln q}.$$

We denote

$$c_{up}^L(q) = \frac{1}{2} + \frac{0.6}{\ln \ln q}.$$

Data for $q \in L$ give the following theorem.

Theorem 3.4. *Let $t_2(2, q)$ be the smallest size of a complete arc in the projective plane $\mathrm{PG}(2, q)$. Let $t_2^L(2, q)$ be the size of the lexiarc in $\mathrm{PG}(2, q)$. Let L_4 be the set of values of q given by (1.17) and Table 4. Finally, let the functions $c(q)$ and $c^L(q)$ be as in (1.19) and (3.4). In $\mathrm{PG}(2, q)$ the following upper bound is provided by complete lexiarcs:*

$$c(q) \leq c^L(q) < \frac{1}{2} + \frac{0.6}{\ln \ln q} \quad \textit{for} \quad 37087 \leq q \leq 301813, \ q \in L_4. \tag{3.5}$$

Figure 4 shows values of $c^L(q)$ for complete lexiarcs in $\mathrm{PG}(2, q)$, $q \in L$ (the top solid curve). The bound (1.26) is given by the dashed line $y = 0.7404$. The bound (1.28) is represented by the top dashed-dotted curve $c_{up}^L(q) = 0.6/\ln \ln q + 0.5$.

We define a function $h^L(q)$.

$$t_2^L(2,q) = h^L(q)\sqrt{3q\ln q}. \tag{3.6}$$

Data for $q \in L$ give two theorems on upper and lower bounds on $h^L(q)$.

Theorem 3.5. *Let $t_2(2,q)$ be the smallest size of a complete arc in the projective plane $\mathrm{PG}(2,q)$. Let $t_2^L(2,q)$ be the size of the lexiarc in $\mathrm{PG}(2,q)$. Let L_4 be the set of values of q given by (1.17) and Table 4. Finally, let the functions $h(q)$ and $h^L(q)$ be as in (1.21) and (3.6). In $\mathrm{PG}(2,q)$ the following upper bounds are provided by complete lexiarcs:*

$$h(q) \le h^L(q) < \begin{cases} 1.056 & \text{for} \quad 16 \le q \le 18443 \\ 1.053 & \text{for} \quad 18443 < q \le 80407 \\ 1.051 & \text{for} \quad 80407 < q \le 178169; \\ 1.050 & \text{for} \quad 178169 < q \le 257249 \\ 1.048 & \text{for} \quad 257249 < q \le 301813, \ q \in L_4 \end{cases} \tag{3.7}$$

$$h(q) \le h^L(q) < 1.05 \text{ for } 80407 < q \le 301813, \ q \ne 178169. \tag{3.8}$$

Theorem 3.6. *Let L be the set of values of q given by (1.18). Let the function $h^L(q)$ be as in (3.6). In $\mathrm{PG}(2,q)$ the following lower bounds on $h^L(q)$ hold.*

$$h^L(q) > \begin{cases} 1.023 & \text{for} \quad 11971 \le q \le 34583 \\ 1.028 & \text{for} \quad 34583 < q \le 70451 \\ 1.032 & \text{for} \quad 70451 < q \le 159349 \\ 1.033 & \text{for} \quad 159349 < q \le 192133 \\ 1.034 & \text{for} \quad 192133 < q \le 291829 \\ 1.035 & \text{for} \quad 291829 < q \le 301813, \ q \in L_4 \end{cases} . \tag{3.9}$$

Figure 5 illustrates Theorems 3.5 and 3.6 and presents the value $h^L(q) = t_2^L(2,q)/\sqrt{3q\ln q}$, $q \in L$ (solid curve), and its upper and lower bounds (dashed-dotted lines) in more detail than Fig. 2. Also, the middle line $y = h^L_{mid} = 1.042$ is shown. The percentage for a bound B is calculated as

$$\frac{B - h^L_{mid}}{h^L_{mid}} 100\,\%, \quad h^L_{mid} = 1.042. \tag{3.10}$$

By Theorems 3.5 and 3.6 and Fig. 5, we have the following observation.

Observation 3.7. *For $q \in L$, $11971 \le q$, the values of $h^L(q) = t_2^L(2,q)/\sqrt{3q\ln q}$ oscillate around the horizontal line $y = 1.042$ with a small amplitude. For growing q, the oscillation amplitude decreases. Upper bounds on the amplitude decrease from 1.4% to 0.6% while lower bounds change from −1.8% to −0.7%, where the percentage corresponds to (3.10).*

Remark 3.8. The oscillation (with decreasing amplitude) of $h^L(q)$ around a horizontal line is an interesting enigma that should be investigated and explained.

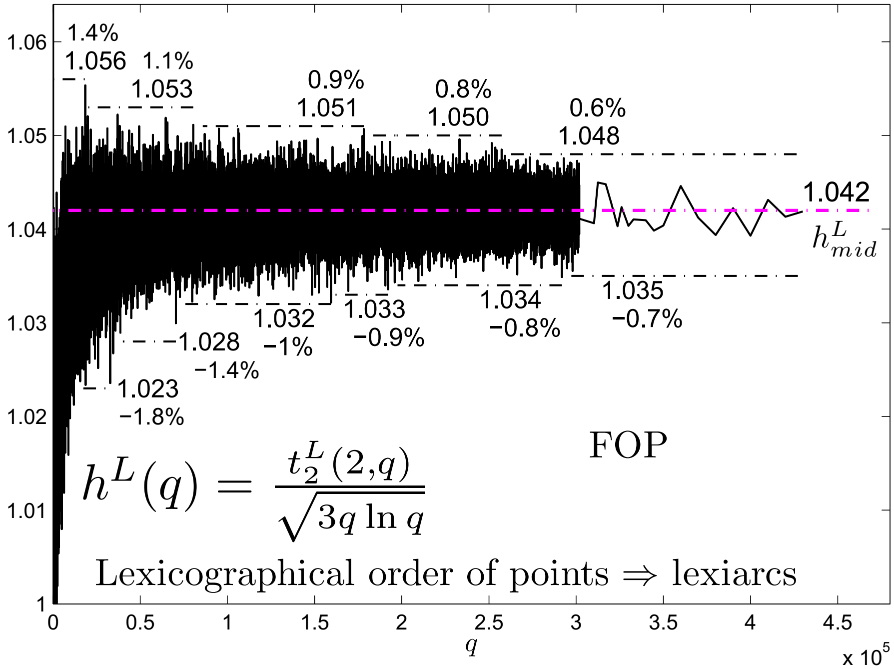

FIGURE 5 Values of $h^L(q) = t_2^L(2,q)/\sqrt{3q\ln q}$ for complete lexiarcs in $\mathrm{PG}(2,q)$, $q \in L$ (*solid curve*) and the corresponding upper and lower bounds (*dashed-dotted lines*)

Corollary 3.9. *In* $\mathrm{PG}(2,q)$ *the following upper bound holds:*

$$t_2(2,q) < 1.05\sqrt{3q\ln q} \quad \textit{for} \quad 7 \le q \le 301813, \;\; q \in L_4. \tag{3.11}$$

Proof. By Table 2, $\bar{t}_2(2,178169) = 2530$, whence $h(178169) < \overline{h}(178169) < 0.996$. Now the assertion follows from (3.8). □

It is useful to compare complete lexiarc sizes with the smallest known sizes. Differences $\Delta(q)$ in percentage between sizes of complete lexiarcs and the smallest known sizes of complete arcs in $\mathrm{PG}(2,q)$ are shown in Fig. 6 in the form

$$\Delta(q) = \frac{t_2^L(2,q) - \bar{t}_2(2,q)}{t_2^L(2,q)} 100\,\%.$$

One can see that the differences are relatively small, in particular, for $q \ge 21559$ we have $\Delta(q) < 8\,\%$. Moreover, for $q \gtrapprox 90000$ the values of $\Delta(q)$ are placed in the region $3.7 \ldots 5.7\,\%$. A "tooth" on the graphics $\Delta(q)$ corresponds to $\bar{t}_2(2,2^{18}) = 3066$ [25]. But even in this special case, we have $\Delta(q) < 6\,\%$.

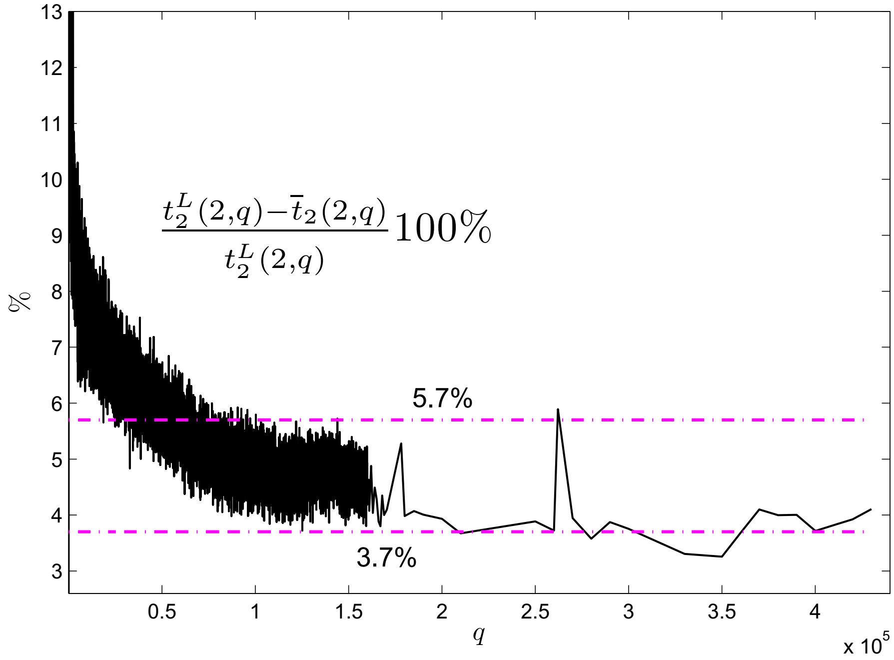

FIGURE 6 Differences in percentage between sizes $t_2^L(2,q)$ of complete lexiarcs and the smallest known sizes $\bar{t}_2(2,q)$ of complete arcs in PG$(2,q)$

4. Random complete arcs

In this section we consider random complete arcs in PG$(2,q)$. The random arcs are constructed iteratively. The next point of an incomplete running arc is taken randomly among points that are not covered by the arc bisecants. Let $t_2^R(2,q)$ be the size of a complete random arc in PG$(2,q)$. The values of $t_2^R(2,q)$ for $q \leq 46337$, q prime, are collected in the work [14].

We define a function $h^R(q)$.

$$t_2^R(2,q) = h^R(q)\sqrt{3q\ln q}. \tag{4.1}$$

Values of $h^R(q) = t_2^R(2,q)/\sqrt{3q\ln q}$ and upper bound $y = 1.054$ for $q \leq 46337$, q prime, are shown in Fig. 7.

It is useful to compare complete lexiarc sizes with the sizes of complete random arcs. Differences $\Delta^{LR}(q)$ in percentage between sizes of complete lexiarcs and complete random arcs in PG$(2,q)$ for $q \leq 46337$, q prime, are shown in Fig. 8 in the form

$$\Delta^{LR}(q) = \frac{t_2^L(2,q) - t_2^R(2,q)}{t_2^L(2,q)} 100\,\%.$$

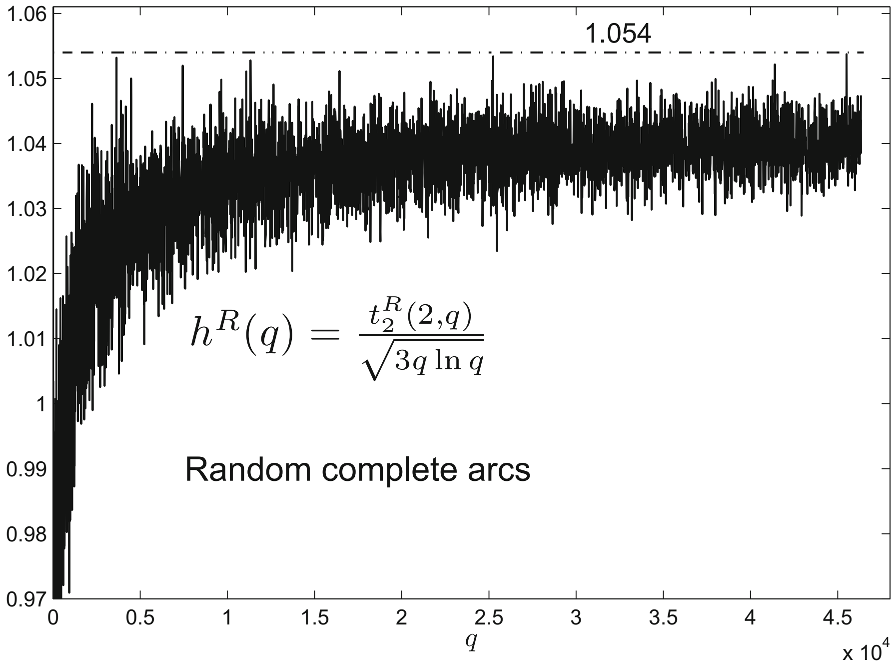

FIGURE 7 Values of $h^R(q) = t_2^R(2,q)/\sqrt{3q\ln q}$ for complete random arcs in $\mathrm{PG}(2,q)$, $q \leq 46337$, q prime (*solid curve*) and the upper bound $y = 1.054$ (*dashed-dotted line*)

One can see that the differences are relatively small, they are in the region $\approx \pm 2\,\%$. Moreover, the differences oscillate around the horizontal line $y = 0$. It means that *complete lexiarcs and random arcs have the same nature*. This is expected, as the lexicographical order of points is a random order in the geometrical sense.

5. Conclusion

The present paper is devoted to upper bounds on the smallest size $t_2(2,q)$ of a complete arc in the projective plane $\mathrm{PG}(2,q)$ based on computer search. We use the results of computer search obtained and collected in the previous works of the authors [3–16, 18, 23, 24] and in the present paper. Also, we develop the approaches proposed in the noted papers to construct new bounds. The search for small complete arcs in $\mathrm{PG}(2,q)$ is executed in a wide region of all prime power $q \leq 301813$ and a number of $q \leq 430007$. To find small complete arcs, randomized greedy algorithms [4, 10, 12, 15, 23, 24, 29] and algorithms with fixed (lexicographical) order of points (FOP) [8, 9, 13, 15, 16, 18] are used.

For $q \leq 301813$, the computer search is complete, i.e. it has been performed for *all* q prime power. This *proves* that the upper bounds on $t_2(2,q)$ are valid, at

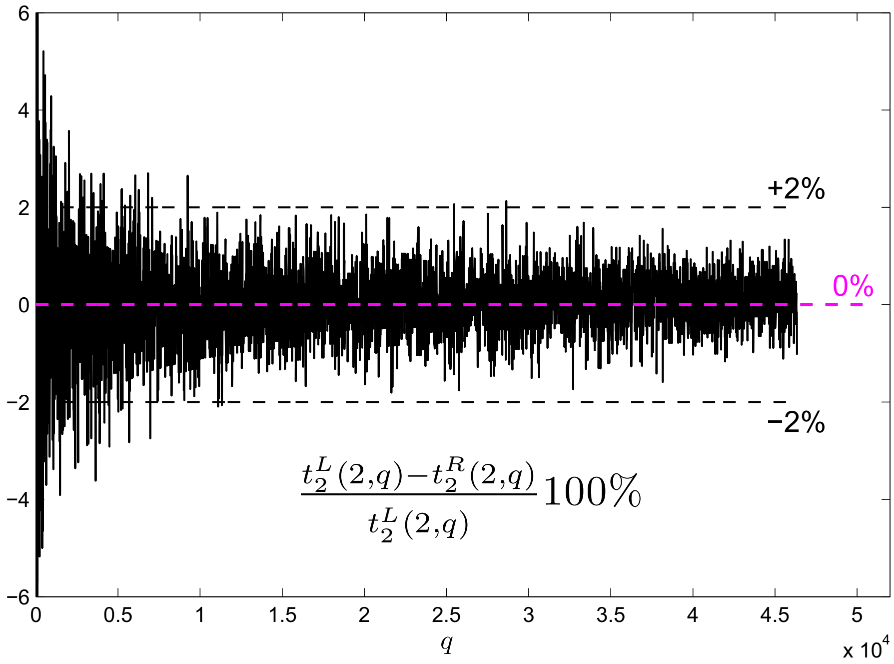

FIGURE 8 Differences in percentage between sizes $t_2^L(2,q)$ of complete lexiarcs and sizes $t_2^R(2,q)$ of random complete arcs in $\mathrm{PG}(2,q)$

least, in this region, see (1.22)–(1.29) of Theorem 1.2. A number of calculations executed for sporadic $q \leq 430007$ strengthen the confidence for validity of these bounds for big q's, see also Figs. 2, 3, 4, 5. Moreover, the bounds of Theorem 1.2 are close to the conjectural bounds of [6] cited in Conjecture 1.1. By all the arguments, we conjecture that the bounds (1.24)–(1.29) hold for all $q \geq 109$, see Conjecture 1.3.

Complete arcs obtained by greedy algorithms have smaller sizes than lexiarcs, however the greedy algorithms take essentially greater computer time than the algorithm FOP. This is why the complete computer search for all q's prime power with the help of greedy algorithms is done for $q \leq 160001$ whereas the complete search by algorithm FOP is executed for $q \leq 301813$.

On the other hand, the differences between sizes $t_2^L(2,q)$ of complete lexiarcs and the smallest known sizes $\bar{t}_2(2,q)$ of complete arcs in $\mathrm{PG}(2,q)$ are relatively small; they are $\lesssim 6\,\%$, see Fig. 6. Therefore, the *algorithm FOP* with lexicographical order of the points seems to be *better for the computer search for large q*.

Moreover, investigations of lexiarcs for large q could help understanding of the enigma connected with oscillation (with decreasing amplitude) of $h^L(q)$ around a horizontal line, see Fig. 5, Observation 3.7, and Remark 3.8.

It would also be useful to understand the structure of lexiarcs, in particular, the connection with the initial part of an arc that is the same for all lexiarcs with greater q, see Sect. 3.3.

Finally, further investigations of random arcs and their "similarity" to lexiarcs, see Figs. 5, 7, and 8, would be interesting.

References

[1] Abatangelo, V.: *A class of complete* $[(q+8)/3]$*-arcs of* $\mathrm{PG}(2,q)$*, with* $q=2^h$ *and* $h(\geq 6)$ *even*. Ars Combin. **16**, 103–111 (1983)

[2] Ball, S.: *On small complete arcs in a finite plane*. Discrete Math. **174**, 29–34 (1997)

[3] Bartoli, D., Davydov, A., Faina, G., Kreshchuk, A., Marcugini, S., Pambianco, F.: *Two types of upper bounds on the smallest size of a complete arc in the plane* $PG(2,q)$. In: Proceedings of 7th international workshop on optimal codes and related topics, OC2013, Albena, Bulgaria, pp. 19–25, http://www.moi.math.bas.bg/oc2013/a4 (2013)

[4] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F.: *Tables of sizes of small complete arcs in the plane* $PG(2,q), q\leq 410009$. arXiv:1312.2155v2 [math.CO] (2014)

[5] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F.: *Conjectural upper bounds on the smallest size of a complete arc in* $PG(2,q)$ *based on an analysis of step-by-step greedy algorithms*. In: Proceedings of the 14th international workshop on algebraic and combinatorial coding theory, ACCT2014, Svetlogosk, Russia, pp. 24–31, http://www.moi.math.bas.bg/acct2014/a4 (2014)

[6] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F.: *Upper bounds on the smallest size of a complete arc in* $PG(2,q)$ *under a certain probabilistic conjecture*. Problems Inform. Transmission **50**, 320–339 (2014)

[7] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F.: *Tables of sizes of small complete arcs in the plane* $PG(2,q)$ *for all* $q\leq 160001$ *and sporadic* q *in the interval* $[160801\ldots 430007]$. arXiv, in preparation

[8] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F.: *Tables of sizes of complete lexiarcs in the plane* $PG(2,q)$ *for all* $q\leq 301813$ *obtained by an algorithm with fixed order of points (FOP)*. arXiv, in preparation

[9] Bartoli, D., Davydov, A.A., Faina, G., Kreshchuk, A.A., Marcugini, S., Pambianco, F., Tkachenko, I.: *Upper bounds on the smallest sizes of a complete arc in PG(2,q) based on computer search*. In: Proceedings of the 14th international workshop on algebraic and combinatorial coding theory, ACCT2014, Svetlogosk, Russia, pp. 32–40, http://www.moi.math.bas.bg/acct2014/a5 (2014)

[10] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *On sizes of complete arcs in* $\mathrm{PG}(2,q)$. Discrete Math. **312**, 680–698 (2012)

[11] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *New upper bounds on the smallest size of a complete arc in the plane* $\mathrm{PG}(2,q)$. In: Proceedings of the XIII international workshop on algebraic and combin. coding theory, ACCT2012, Pomorie, Bulgaria, pp. 60–66, http://www.moi.math.bas.bg/moiuser/~ACCT2012/b10 (2012)

[12] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *New upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane*. J. Geom. **104**, 11–43 (2013)

[13] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *Tables of sizes of small complete arcs in the plane* $PG(2,q)$*,* $q \leq 190027$*, obtained by an algorithm with fixed order of points (FOP)*. arXiv:1404.0469 [math.CO] (2014)

[14] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *Tables of sizes of random complete arcs in the plane* $PG(2,q)$. arXiv:1405.5862 [math.CO] (2014)

[15] Bartoli, D., Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *New types of estimates for the smallest size of complete arcs in a finite Desarguesian projective plane*. J. Geom. **106**, 1–17 (2015)

[16] Bartoli, D., Davydov, A.A., Marcugini, S., Pambianco, F.: *New type of estimations for the smallest size of complete arcs in* $\mathrm{PG}(2,q)$. In: Proceedings of the XIII international workshop on algebraic and combin. coding theory, ACCT2012, Pomorie, Bulgaria, pp. 67–72, http://www.moi.math.bas.bg/moiuser/~ACCT2012/b11 (2012)

[17] Bartoli, D., Faina, G., Marcugini, S., Pambianco, F.: *On the minimum size of complete arcs and minimal saturating sets in projective planes*. J. Geom. **104**, 409–419 (2013)

[18] Bartoli, D., Faina, G., Marcugini, S., Pambianco, F., Davydov, A.A.: *A new algorithm and a new type of estimate for the smallest size of complete arcs in* $PG(2,q)$. Electron. Notes Discrete Math. **40**, 27–31 (2013)

[19] Bartoli, D., Marcugini, S., Pambianco, F.: *New quantum caps in* $\mathrm{PG}(4,4)$. J. Combin. Des. **20**, 448–466 (2012)

[20] Blokhuis, A.: *Blocking sets in Desarguesian planes*. In: Miklós, D., Sós, V.T., Szónyi, T. (eds.) Erdős is eighty, Bolyai society mathematical studies. **2**, pp. 133–155 (1996)

[21] Boros, E., Szőnyi, T., Tichler, K.: *On defining sets for projective planes.* Discrete Math. **303**, 17–31 (2005)

[22] Cohen, G., Litsyn, S., Zemor, G.:*On greedy algorithms in coding theory*. IEEE Trans. Inform. Theor. **42**, 2053–2057 (1996)

[23] Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *Computer search in projective planes for the sizes of complete arcs*. J. Geom. **82**, 50–62 (2005)

[24] Davydov, A.A., Faina, G., Marcugini, S., Pambianco, F.: *On sizes of complete caps in projective spaces* $\mathrm{PG}(n,q)$ *and arcs in planes* $\mathrm{PG}(2,q)$. J. Geom. **94**, 31–58 (2009)

[25] Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: *On sharply transitive sets in* $\mathrm{PG}(2,q)$. Innov. Incid. Geom. **6–7**, 139–151 (2009)

[26] Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: *New inductive constructions of complete caps in* $\mathrm{PG}(N,q)$*, q even*. J. Combin. Des. **18**, 176–201 (2010)

[27] Davydov, A.A., Giulietti, M., Marcugini, S., Pambianco, F.: *Linear nonbinary covering codes and saturating sets in projective spaces.* Adv. Math. Commun. **5**, 119–147 (2011)

[28] Davydov, A.A., Marcugini, S., Pambianco, F.: *On saturating sets in projective spaces.* J. Combin. Theory Ser. A. **103**, 1–15 (2003)

[29] Davydov, A.A., Marcugini, S., Pambianco, F.: *Complete caps in projective spaces* $\mathrm{PG}(n, q)$. J. Geom. **80**, 23–30 (2004)

[30] Faina, G., Giulietti, M.: *On small dense arcs in Galois planes of square order.* Discrete Math. **267**, 113–125 (2003)

[31] Faina, G., Marcugini, S., Milani, A., Pambianco, F.: *The spectrum of the values* k *for which there exists a complete* k*-arc in* $PG(2, q)$ *for* $q \leq 23$. Ars Combin. **47**, 3–11 (1997)

[32] Faina, G., Pambianco, F.: *On the spectrum of the values* k *for which a complete* k*-cap in* $\mathrm{PG}(n, q)$ *exists.* J. Geom. **62**, 84–98 (1998)

[33] Faina, G., Pambianco, F.: *On some 10-arcs for deriving the minimum order for complete arcs in small projective planes.* Discrete Math. **208–209**, 261–271 (1999)

[34] Gács, A., Szőnyi, T.: *Random constructions and density results.* Des. Codes Cryptogr. **47**, 267–287 (2008)

[35] Giulietti, M.: *Small complete caps in* $\mathrm{PG}(2, q)$ *for* q *an odd square.* J. Geom. **69**, 110–116 (2000)

[36] Giulietti, M.: *Small complete caps in Galois affine spaces* J. Algebr. Combin. **25**, 149–168 (2007)

[37] Giulietti, M.: *Small complete caps in* $\mathrm{PG}(N, q)$, q *even.* J. Combin. Des. **15**, 420–436 (2007)

[38] Giulietti, M.: *The geometry of covering codes: small complete caps and saturating sets in Galois spaces.* In: Blackburn, S.R., Holloway, R., Wildon, M. (eds.) Surveys in Combinatorics 2013. London Math. Soc. Lect. Note Series Cambridge Univ. Press **409**, 51–90 (2013)

[39] Giulietti, M., Korchmáros, G., Marcugini, S., Pambianco, F.: *Transitive* $\mathbf{A_6}$*-invariant* k*-arcs in* $\mathrm{PG}(2, q)$. Des. Codes Cryptogr. **68**, 73–79 (2013)

[40] Giulietti, M., Ughi, E.: *A small complete arc in* $\mathrm{PG}(2, q), q = p^2, p \equiv 3 \pmod 4$. Discrete Math. **208–209**, 311–318 (1999)

[41] Gordon, C.E.: *Orbits of arcs in* $PG(N, K)$ *under projectivities.* Geom. Dedic.**42**, 187–203 (1992)

[42] Hadnagy, E.: *Small complete arcs in* $\mathrm{PG}(2, p)$. Finite Fields Appl. **5**, 1-12 (1999)

[43] Hartman, A., Raskin, L.: *Problems and algorithms for covering arrays.* Discrete Math. **284**, 149–156 (2004)

[44] Hirschfeld, J.W.P.: *Maximum sets in finite projective spaces.* In: Lloyd, E.K. (ed.) Surveys in Combinatorics, London Math. Soc. Lecture Note Ser. **82**, pp. 55–76. Cambridge University Press, Cambridge (1983)

[45] Hirschfeld, J.W.P.: *Projective geometries over finite fields*, 2nd edn. Clarendon Press, Oxford (1998)

[46] Hirschfeld, J.W.P., Storme, L.: *The packing problem in statistics, coding theory and finite projective spaces.* J. Statist. Planning Infer. **72**, 355–380 (1998)

[47] Hirschfeld, J.W.P., Storme, L.: *The packing problem in statistics, coding theory and finite geometry: update 2001.* In: Blokhuis, A., Hirschfeld, J.W.P. et al. (eds.) Finite Geometries, Developments of Mathematics, vol. 3, Proceedings of

the Fourth Isle of Thorns Conference, Chelwood Gate, 2000, pp. 201–246. Kluwer Academic Publisher, Boston (2001)

[48] Hirschfeld, J.W.P., Thas, J.A.: *Open problems in finite projective spaces.* Finite Fields Appl. **32**, 44–81 (2015)

[49] Keri, G.: *Types of superregular matrices and the number of n-arcs and complete n-arcs in* PG(r, q). J. Combin. Des. **14**, 363–390 (2006)

[50] Kim, J.H., Vu, V.: *Small complete arcs in projective planes.* Combinatorica **23**, 311–363 (2003)

[51] Korchmáros, G.: *New examples of complete k-arcs in* PG$(2, q)$. Europ. J. Combin. **4**, 329–334 (1983)

[52] Kovács, S.J.: *Small saturated sets in finite projective planes.* Rendiconti di Matematica. Ser. VII (Roma). **12**, 157–164 (1992)

[53] Landjev, I.N.: *Linear codes over finite fields and finite projective geometries.* Discrete Math. **213**, 211–244 (2000)

[54] Lombardo-Radice, L.: *Sul problema dei k-archi completi di $S_{2,q}$.* Boll. Un. Mat. Ital. **11**, 178–181 (1956)

[55] Lisoněk, P., Marcugini, S., Pambianco, F.: *Constructions of small complete arcs with prescribed symmetry.* Contrib. Discrete Math. **3**, 14–19 (2008)

[56] Marcugini, S., Milani, A., Pambianco, F.: *Minimal complete arcs in* PG$(2, q), q \leq 29$. J. Combin. Math. Combin. Comput. **47**, 19–29 (2003)

[57] Marcugini, S., Milani, A., Pambianco, F.: *Complete arcs in PG$(2, 25)$: the spectrum of the sizes and the classification of the smallest complete arcs.* Discrete Math. **307**, 739–747 (2007)

[58] Marcugini, S., Milani, A., Pambianco, F.: *Minimal complete arcs in* PG$(2, q)$, $q \leq 32$. In: Proceedings XII international workshop on algebraic and combin. coding theory, ACCT2010, Novosibirsk, Russia, pp. 217–222 (2010)

[59] Monroe, L.: *Binary greedy codes.* Congr. Numer. **104**, 49–63 (1994)

[60] Monroe, L., Pless, V.: *Greedy generation of non-binary codes.* In: Proceedings of the IEEE international Symposium on Inform. Theory, ISIT (1995)

[61] Östergård, P.R.J.: *Computer search for small complete caps.* J. Geom. **69**, 172–179 (2000)

[62] Pace, N.: *On small complete arcs and transitive $A5$-invariant arcs in the projective plane $PG(2, q)$.* J. Combin. Des. **22**, 425–434 (2014)

[63] Pellegrino, G.: *Un'osservazione sul problema dei k-archi completi in $S_{2,q}$, con $q \equiv 1 (mod 4)$.* Atti Accad. Naz. Lincei Rend. **63**, 33–44 (1977)

[64] Pellegrino, G.: *Sugli archi completi dei piani* PG$(2, q)$*, con q dispari, contenenti $(q + 3)/2$ punti di una conica.* Rend. Mat. **12**, 649–674 (1992)

[65] Pellegrino, G.: *Archi completi, contenenti $(q+1)/2$ punti di una conica, nei piani di Galois di ordine dispari.* Rend. Circ. Mat. Palermo (2) **62**, 273–308 (1993)

[66] Polverino, O.: *Small minimal blocking sets and complete k-arcs in* PG$(2, p^3)$. Discrete Math. **208–209**, 469–476 (1999).

[67] Pless, V.S.: *Coding constructions.* In: Pless, V.S., Human, W.C., Brualdi, R.A. (eds.) Handbook of Coding Theory, pp. 141-176. Sect. 7. Elsevier, Amsterdam, The Netherlands (1998)

[68] Segre, B.: *Le geometrie di Galois.* Ann. Mat. Pura Appl. **48**, 1–97 (1959)

[69] Segre, B.: *Introduction to Galois geometries.* Atti Accad. Naz. Lincei Mem. **8**, 133–236 (1967)

[70] Szőnyi, T.: *Small complete arcs in Galois planes.* Geom. Dedicata **18**, 161–172 (1985)

[71] Szőnyi, T.: *Note on the order of magnitude of k for complete k-arcs in* $\mathrm{PG}(2,q)$. Discrete Math. **66**, 279–282 (1987)

[72] Szőnyi, T.: *Arcs in cubic curves and 3-independent subsets of abelian groups.* In: Colloquia Mathematica Societatis Janos Bolyai **52**. Combinatorics, Eger, pp. 499–508 (1987)

[73] Szőnyi, T.: *Complete arcs in Galois planes: survey.* Quaderni del Seminario di Geometrie Combinatorie, Università degli studi di Roma, La Sapienza, **94** (1989)

[74] Szőnyi, T.: *Arcs, caps, codes and 3-independent subsets.* In: Faina, G., et al. (eds.) Giornate di Geometrie Combinatorie, Università degli studi di Perugia, pp. 57–80. Perugia (1993)

[75] Szőnyi, T.: *Some applications of algebraic curves in finite geometry and combinatorics.* In: Bailey, R.A. (ed.) Surveys in Combinatorics, pp. 198–236. Cambridge University Press, Cambridge (1997)

[76] Thas, J.A.: *M.D.S. codes and arcs in projective spaces: a survey.* Le Matematiche (Catania) **47**, 315–328 (1992)

[77] Ughi, E.: *Saturated configurations of points in projective Galois spaces.* European J. Combin. **8**, 325–334 (1987)

[78] Ughi, E.: *The values* $\sqrt{2q}$ *and* $log_2 q$*: their relationship with k-arcs.* Ars Combin. **57**, 201–207 (2000)

[79] Ughi, E.: *Small almost complete arcs.* Discrete Math. **255**, 367–379 (2002)

[80] Voloch, J.F.: *On the completeness of certain plane arcs.* Eur. J. Combin. **8**, 453–456 (1987)

[81] Voloch, J.F.: *On the completeness of certain plane arcs II.* Eur. J. Combin. **11**, 491–496 (1990)

[82] van Zanten, A.J., Suparta, I.N.: *On the construction of linear q-ary lexicodes.* Des. Codes Crypt. **37**, 15–29 (2005)

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Received: March 1, 2015.