ERROR CORRECTION IN MAIN MEMORY OF A HIGH-CAPACITY COMPUTER

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We examine the features and means of constructing uniform test matrices for modified Hamming codes having minimal ones that correct single and detect double independent errors and error bytes of length four. A number of uniform test matrices for (72,64)- and (137,128)-codes are presented which allow a maximally fast coder and decoder to be constructed in large-scale integrated circuits. The structure of devices for correcting errors in main memory of high-capacity computers are discussed.

1. Introduction

The use of codes that correct errors in computer memories substantially improves their reliability. In supercomputers, hundreds of millions of operations are executed per second on 64-bit words [1] whose main (operating) memory is on the order of a million words and a modified Hamming code of distance four [2-5] is used to improve reliability.

The operations of coding and correcting single errors must be completed during one computer operating cycle. This requirement is met by choosing an element base, a test matrix for the code, and logical and construction circuits for the coding and decoding devices. The coder and decoder are usually fabricated on LSI circuits. Therefore, the test matrix for the code selected must be sufficiently uniform and must at the same time meet the required device operating speed.

The features and means of constructing uniform test matrices for (72,64)- and (137,128)-codes that allow maximally fast LSI coders and decoders to be constructed will be discussed here, along with correcting single and detecting double independent errors and error bytes of length four. The structure of an LSI coder and decoder intended for correcting errors in main memory of high-capacity computers is examined.

2. Uniform Test Matrices for Modified Hamming Codes Having Minimal Ones

Matrices A_1 and A_2 are mutually uniform if they can be obtained individually from one another by replacing the rows. If the replacements are cyclic the matrices are cyclic mutually uniform.

A matrix P of size $r \times k$ is λ-uniform (cyclic λ-uniform) if a representation

$$P=\|A_1 A_2 \dots A_\lambda\|, \tag{1}$$

exists such that for any pair of numbers j_1 and j_2 ($1 \le j_1, j_2 \le \lambda$, and $k/\lambda = \nu$ is an integer) in an $r \times \nu$-matrix A_{j_1} and A_{j_2} are mutually uniform (cyclic mutually uniform).

The largest λ (hereafter denoted Λ) for which the representation, Eq. (1), with the properties indicated exists is the level of uniformity for the P matrix. Matrices with $\Lambda \ge 2$ are uniform at level Λ.

The number of different rows in the A_j, $j = 1, 2, \dots, \lambda$ matrices of Eq. (1) are designated f_λ. For uniform matrices, $f_\lambda \le r$. For cyclic uniform matrices, $f_\lambda = r$. The number f_λ may be much smaller than r.

The test matrix $H = \|PQ\|$ for an abbreviated $(2^{r-2} + r, 2^{r-2})$ Hamming code with a minimal distance of four, where P is a $r \times 2^{r-2}$ matrix corresponding to information symbols, Q is a square matrix of order r corresponding to the test symbols (for a systematic code $Q = I_r$, and I_r is a unit matrix) is uniform at level Λ if the matrix P is uniform at level Λ.

To determine the location of an error symbol the syndrome for the Hamming code word being decoded is compared with each column of the test matrix H.

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$H^{(1)} =$

1111	1111	0000	0000	0000	0000	0000	0000	0000	1111	0000	1111	0000	1111	0000	1111	0111	0000
0000	0000	1111	1111	0000	0000	0000	0000	0011	0011	0011	0011	0011	0011	0011	0011	1011	0000
0000	0000	0000	0000	1111	1111	0000	0000	0101	0101	0101	0101	0101	0101	0101	0101	1101	0000
0000	0000	0000	0000	0000	0000	1111	1111	0110	1001	0110	1001	0110	1001	0110	1001	1110	0000
0000	1111	0000	1111	0000	1111	0000	1111	1111	1111	0000	0000	0000	0000	0000	0000	0000	0111
0011	0011	0011	0011	0011	0011	0011	0011	0000	0000	1111	1111	0000	0000	0000	0000	0000	1011
0101	0101	0101	0101	0101	0101	0101	0101	0000	0000	0000	0000	1111	1111	0000	0000	0000	1101
0110	1001	0110	1001	0110	1001	0110	1001	0000	0000	0000	0000	0000	0000	1111	1111	0000	1110
1212	5003	3422	4314	4352	5366	6451	5001	1212	5002	3321	4304	4342	5356	6451	5001	6667	6677
4894	9741	0030	4517	6617	6903	2352	2385	6672	7529	2818	2395	8495	4781	4130	0163	5690	7812

a

$H^{(2)} =$

0111	1111	1000	0000	1000	0000	1000	0000	0000	1111	0000	1111	0000	1111	0000	1111	1000	0000
1000	0000	0111	1111	1000	0000	1000	0000	0011	0011	0011	0011	0011	0011	0011	0011	0100	0000
1000	0000	1000	0000	0111	1111	1000	0000	0101	0101	0101	0101	0101	0101	0101	0101	0010	0000
1000	0000	1000	0000	1000	0000	0111	1111	0110	1001	0110	1001	0110	1001	0110	1001	0001	0000
0000	1111	0000	1111	0000	1111	0000	1111	0111	1111	1000	0000	1000	0000	1000	0000	0000	1000
0011	0011	0011	0011	0011	0011	0011	0011	1000	0000	0111	1111	1000	0000	1000	0000	0000	0100
0101	0101	0101	0101	0101	0101	0101	0101	1000	0000	1000	0000	0111	1111	1000	0000	0000	0010
0110	1001	0110	1001	0110	1001	0110	1001	1000	0000	1000	0000	1000	0000	0111	1111	0000	0001

b

Fig. 1. Uniform minimal test matrices for a (72,64)-code.

The comparison circuits (decoders) for mutually uniform matrices are identical. The complexity of a decoder depends on the regularity of the columns; specifically, on the existence of common terms in the columns. The columns in cyclic mutually uniform matrices do not, generally speaking, have common and regular terms.

A numerical locator for a column $(h_\ell, \ldots, h_1)^T$ (T is the transpose symbol) is the number $N = \sum_{j=1}^{l} h_j 2^{j-1}$.

The regularity of the columns is defined by a rule associated with their numerical locators.

We will incorporate the following matrices into our examination. A matrix M_ν of size $\nu \times m$ consists of $m = 2^\nu$ different columns of length ν arranged in order of increasing numerical locator. A matrix $M_\nu'(M_\nu'')$ is obtained from M_ν by adding the rows of M_ν to the last row, which is an inversion of the sum (the sum) of rows in M_ν. The weight of each column in $M_\nu'(M_\nu'')$ is odd (even).

An $\mu \times \ell$-matrix L_N consisting if identical columns having numerical locator N is called a matrix locator.

We designate L'_{N_i} and L''_{N_j} to be the matrix locators whose columns are odd and even respectively.

Mutually uniform $\left\|\begin{matrix} M \\ L \end{matrix}\right\|$ matrices are constructed by respectively combining the M_ν, M'_ν, and M''_ν matrices having locators L_N, L'_{N_i}, and L''_{N_j}.

Thus, $\left\|\begin{matrix} M_\nu \\ L_{N_1} \end{matrix}\right\|$ and $\left\|\begin{matrix} M_\nu \\ L_{N_2} \end{matrix}\right\|$ matrices are mutually uniform if columns in the L_{N_1} and L_{N_2} matrices have identical weight.

The test matrix for a (72,64)-code is shown in Fig. 1a, and its structure can be given as

$$H^{(1)} = \|P^{(1)}Q^{(1)}\| = \left\|\begin{matrix} L_8' & L_4' & L_2' & L_1' & M_3'' & M_3'' & M_3'' & M_3'' \\ M_3'' & M_3'' & M_3'' & M_3'' & L_8' & L_4' & L_2' & L_1' \end{matrix} \; Q^{(1)}\right\|. \tag{2}$$

The level of uniformity for a $H^{(1)}$ matrix is eight.

By summing and replacing the rows of the $H^{(1)}$ matrix, a reduced-partitioned matrix $H^{(1)}_r$ is obtained (see Table 6 [5]). The $H^{(1)}_r$ matrix has the same level of uniformity as does $H^{(1)}$, but more ones in each row.

If we exchange locations for the first columns of all eight mutually uniform matrices in P with the corresponding columns of the $Q^{(1)}$ matrix we obtain the $H^{(2)}$ matrix of Fig. 1b. The $H^{(2)}$ matrix is a reduced-partitioned matrix convenient for decoding, but one column differs from those remaining in matrices obtained from L_i' matrices and complicates the transfer operation when decoding.

The number of logic levels in a logic circuit in which a test symbol or a syndrome symbol is computed depends on the number of ones in the corresponding row of the test matrix.

Computing the sum of N binary symbols in modulo two is done in a combinational circuit consisting of two-input or three-input adders as well as, possibly, combinations of these. The number of logic levels in each case is $[\log_2 N]$, $[\log_3 N]$ and $i + j$, respectively, where $[a]$ is the smallest integer greater than or equal to a, $N \le D$, and D is the smallest integer of the $2^{\ell}3^{m}$, $D = 2^{i}3^{j}$.

The time needed to code and compute a syndrome is determined by the number of ones in a row of the test matrix having the maximal number of ones.

The location of an error symbol is uniquely determined by $r - 1$ of r syndrome symbols.

A smaller number of ones in a test matrix produces, in principle, a smaller number of address in the coder and decoder and thereby greater reliability in these devices.

Therefore, it is desirable when meeting a required level of uniformity to choose a test matrix having the fewest ones and the least maximal number of ones in a row (by excluding a row of ones).

There is a row of ones in a test matrix for the standard Hamming code and only the number of ones in $r - 1$ rows can be minimized. Such a row does not exist in a test matrix for a modified Hamming code and all the columns have odd weight [2, 3].

For a given length n and number of test symbols r a modified Hamming code and its test matrix are minimal if the test matrix for the code contains a minimal number of ones μ_{min} and the number of ones in each row is not more than $[\mu_{min}/r]$.

There are 27 ones in every row of a minimal (72, 64)-code. The $H^{(1)}$, $H^{(1)}_{r}$, and $H^{(2)}$ matrixes for a (72, 64)-code are minimal and have a level of uniformity eight.

A minimal test matrix for a modified (137, 128)-Hamming code contains 481 ones and no more than 54 ones in each row. A minimal uniform test matrix for a (137,128)-code having a level of uniformity $\Lambda = 4$ is

$$H^{(3)} = \| A_1\ A_2\ A_3\ A_4\ I_9 \|, \qquad (3)$$

where

$$A_i = \left\| \begin{array}{c|cc} R_8^{2(i-1)} U_0 & R_8^{2(i-1)} U_1 & R_8^{i-1} F \\ J_0 & J_1 & \end{array} \right\|,$$

$$U_0 = \left\| \begin{array}{cccc} 1000 & 1000 & 1100 & 1101 \\ 1000 & 1001 & 0010 & 0111 \\ 0100 & 0011 & 0001 & 1011 \\ 0010 & 1100 & 0101 & 0101 \\ 0100 & 0100 & 1011 & 0101 \\ 0001 & 0010 & 0111 & 0010 \\ 0011 & 0010 & 1000 & 1110 \\ 0001 & 0101 & 0010 & 1010 \end{array} \right\|,$$

a U_1 matrix is obtained from a R_8U_0 matrix by eliminating the last column $F = (01000100)^T$, R_μ is a matrix for a cyclic row replacement of order μ, $J_0 = (J0)$, $J_1 = (J1)$, and $J = (111000000011100)$, $i = 1, 2, 3, 4$.

Increasing the level of uniformity increases the number of ones in a matrix. For a level of uniformity $\Lambda = 8$ the minimal number of ones in a uniform (cyclic uniform) test matrix for a systematic modified (137, 128)-Hamming code is 489 (505).

A uniform ($\Lambda = 8$) test matrix for a (137, 128)-code having these properties is

$$H^{(4)} = \left\| \begin{matrix} U_0 & R_8U_0 & R_8^2U_0 \ldots R_8^7U_0 & I_9 \\ J_0 & J_0 & J_0 \ldots \quad J_0 & \end{matrix} \right\|. \tag{4}$$

The maximal number of ones in rows of an $H^{(4)}$ matrix is 55.

A cyclic uniform test matrix for a (137, 128)-code is

$$H^{(5)} = \| A_1\, A_2 \ldots A_8\, I_9 \|, \tag{5}$$

where

$$A_1 = \left\| \begin{matrix} 1010 & 0001 & 0100 & 1011 \\ 1010 & 0010 & 0011 & 0101 \\ 1001 & 1000 & 0011 & 0101 \\ 0101 & 0001 & 0011 & 0110 \\ 0110 & 0010 & 1010 & 1100 \\ 0000 & 1101 & 0101 & 1001 \\ 0100 & 1000 & 1111 & 0010 \\ 0001 & 0100 & 0100 & 1011 \\ 0000 & 0110 & 1100 & 1110 \end{matrix} \right\|, \quad A_i = R_9^{i-1} A_1.$$

The $H^{(5)}$ matrix has a level of uniformity eight. There are 505 ones in this matrix and no more than 57 ones in each row.

Along with uniform matrices, we will examine almost uniform matrices. This substantially expands the class of test matrices that allow a coder and a decoder to be realized in LSI.

Matrices A and A' of size $\ell \times m$ ($m > \log_2 \ell$) are mutually almost uniform if they can be written

$$A = \left\| \begin{matrix} T_A \\ L \end{matrix} \right\| \quad \text{and} \quad A' = \left\| \begin{matrix} T_A' \\ L' \end{matrix} \right\|, \tag{6}$$

where T_A and T_A' are nonzero, mutually uniform matrices and L and L' are matrix locators.

The difference in column weights in the L and L' matrices is called the level of nonuniformity in the A and A' matrices. If the level of nonuniformity in the A and A' matrices is zero, these matrices are mutually uniform.

A matrix P of size $r \times k$ is almost λ-uniform if there is a representation of the type Eq. (1) in which the A_i and A_j matrices are mutually almost uniform.

The largest λ (designated Λ') for which such a representation exists is the level of uniformity for an almost uniform matrix. Almost uniform test matrices can be constructed for more than just Hamming codes. Thus, an almost uniform test matrix for a Bose–Chaudhuri–Hocquenghem (BCH) code having a minimal distance six was constructed for implementation in LSI [6]. When r is odd a test matrix for a H_r ($2^{r-2} + r$, 2^{r-2})-Hamming code with minimal distance four is frequently given as

$$H_r = \| P_r Q_r \| = \left\| \begin{array}{c:c:c:c} P_{r-1} & P_{r-1} & Q_{r-1} & \begin{matrix} q_1 \\ \vdots \\ q_{r-1} \end{matrix} \\ 1\ 1 \ldots 1 & 0\ 0 \ldots 0 & 0\ 0 \ldots 0 & 1 \end{array} \right\|, \tag{7}$$

where $H_{r-1} = \|P_{r-1}Q_{r-1}\|$ is a test matrix for a $(2^{r-3} + r - 1, 2^{r-3})$-Hamming code with d = 4, and $(q_1, \ldots, q_{r-1})^T$ is an arbitrary column of the Q_{r-1} matrix.

If P_{r-1} is a uniform matrix having a level of uniformity Λ, a matrix P_r will be almost uniform with a level of uniformity Λ' = 2Λ.

Thus, if for H_{r-1} a matrix $H^{(1)}$ is chosen a test matrix $H^{(6)}$ (see Fig. 1a) for a (137, 128)-code constructed according to Eq. (7) will have a level of uniformity Λ' = 16.

3. Uniform Test Matrices for a Modified Hamming Code That Corrects Single and Detects Double Errors and Error Bytes of Length Four

The main memory of a supercomputer is organized so that each bit in a code word is located in a separate LSI microcircuit. Microcircuits corresponding to one or more (two or four) word bits can be located on a substrate. In the latter case the problem arises of detecting error bytes while correcting single and detecting double errors [3]. The practically important problem of detecting error bytes of length four while correcting single and detecting double independent errors in (72, 64)- and (137, 128)-codes is examined in this section.

Test matrices for a (72, 64)-code that detect all error bytes of length four are not minimal and have a level of uniformity Λ ≤ 2 [3, 7]. Uniform test matrices having Λ = 4 and Λ = 8 that have minimal, or nearly minimal ones are constructed below.

A matrix

$$H^{(7)}=\|U^{(7)}\ T_4U^{(7)}\ T_4^2U^{(7)}\ T_4^3U^{(7)}\ Q^{(7)}\|, \tag{8}$$

where

$$U^{(7)}=\begin{Vmatrix} 1101 & 1110 & 1000 & 1100 \\ 0100 & 0100 & 1110 & 1000 \\ 1000 & 1000 & 0100 & 1000 \\ 0010 & 0001 & 0001 & 1010 \\ 0111 & 1011 & 0010 & 0011 \\ 0001 & 0001 & 1011 & 0010 \\ 0010 & 0010 & 0001 & 0010 \\ 1000 & 0100 & 0100 & 1010 \end{Vmatrix},\quad Q^{(7)}=\begin{Vmatrix} 0100 & 1100 \\ 1000 & 1100 \\ 1100 & 0100 \\ 1100 & 1000 \\ 0001 & 0011 \\ 0010 & 0011 \\ 0011 & 0001 \\ 0011 & 0010 \end{Vmatrix},$$

$$T_4=\begin{Vmatrix} R_4 & 0 \\ 0 & R_4 \end{Vmatrix},\quad R_4=\begin{Vmatrix} 0001 \\ 1000 \\ 0100 \\ 0010 \end{Vmatrix},$$

has a level of uniformity Λ = 4 and yields a minimal (72, 64)-code that corrects single and detects double independent errors and error bytes of length four.

To prove that a code is capable of detecting error bytes of length four we need only verify that the sum of three and four test matrix columns corresponding to any error byte are not equal to one matrix column. In $H^{(7)}$ the sums of any three columns are columns of the (2, 3) and (3, 2) type that do not occur in the matrix. (A column vector of length eight is a (ω_1, ω_2) column if the upper four bits of the column have weight ω_1 and the lower four have weight ω_2).

An $H^{(7)}$ matrix can be obtained from an $H^{(1)}$ matrix by replacing columns in the order shown in the two lower rows of Fig. 1a, where the notation $\frac{1}{7}$ in the example denotes the number 17.

A reduced-partitioned test matrix for a modified (72, 64)-Hamming code that detects error bytes of length four has no fewer than 29 ones in each row, because it cannot contain columns of the (3, 0) and (0, 3) type. A uniform Λ = 4 matrix $H^{(8)}$ having these properties is shown in Fig. 2a.

An $H^{(8)}$ matrix has the following structure

$$H^{(8)}=\|U^{(8)}T_4U^{(8)}T_4^2U^{(8)}T_4^3U^{(8)}I_8\|, \tag{9}$$

$$H^{(8)}=\left\|\begin{array}{cccc:cccc:cccc:cccc:cc}
1010 & 1010 & 1001 & 1111 & 0100 & 0100 & 0100 & 1110 & 0100 & 1001 & 0100 & 0011 & 1001 & 0100 & 1010 & 0001 & 1000 & 0000\\
1001 & 0100 & 1010 & 0001 & 1010 & 1010 & 1001 & 1111 & 0100 & 0100 & 0100 & 1110 & 0100 & 1001 & 0100 & 0011 & 0100 & 0000\\
0100 & 1001 & 0100 & 0011 & 1001 & 0100 & 1010 & 0001 & 1010 & 1010 & 1001 & 1111 & 0100 & 0100 & 0100 & 1110 & 0010 & 0000\\
0100 & 0100 & 0100 & 1110 & 0100 & 1001 & 0100 & 0011 & 1001 & 0100 & 1010 & 0001 & 1010 & 1010 & 1001 & 1111 & 0001 & 0000\\
1010 & 1010 & 0110 & 1111 & 0001 & 0001 & 0001 & 1011 & 0001 & 0110 & 0001 & 1100 & 0110 & 0001 & 1010 & 0100 & 0000 & 1000\\
0110 & 0001 & 1010 & 0100 & 1010 & 1010 & 0110 & 1111 & 0001 & 0001 & 0001 & 1011 & 0001 & 0110 & 0001 & 1100 & 0000 & 0100\\
0001 & 0110 & 0001 & 1100 & 0110 & 0001 & 1010 & 0100 & 1010 & 1010 & 0110 & 1111 & 0001 & 0001 & 0001 & 1011 & 0000 & 0010\\
0001 & 0001 & 0001 & 1011 & 0001 & 0110 & 0001 & 1100 & 0110 & 0001 & 1010 & 0100 & 1010 & 1010 & 0110 & 1111 & 0000 & 0001
\end{array}\right\|$$

a

$$H^{(9)}=\left\|\begin{array}{cc:cc:cc:cc:cc:cc:cc:cc:cc}
0001 & 1110 & 1000 & 0111 & 1100 & 0011 & 0011 & 1100 & 0111 & 1000 & 0000 & 1111 & 0000 & 1111 & 1110 & 0001 & 1000 & 0000\\
0011 & 1100 & 0001 & 1110 & 1000 & 0111 & 1100 & 0011 & 1110 & 0001 & 0111 & 1000 & 0000 & 1111 & 0000 & 1111 & 0100 & 0000\\
1100 & 0011 & 0011 & 1100 & 0001 & 1110 & 1000 & 0111 & 0000 & 1111 & 1110 & 0001 & 0111 & 1000 & 0000 & 1111 & 0010 & 0000\\
1000 & 0111 & 1100 & 0011 & 0011 & 1100 & 0001 & 1110 & 0000 & 1111 & 0000 & 1111 & 1110 & 0001 & 0111 & 1000 & 0001 & 0000\\
0111 & 1000 & 0000 & 1111 & 0000 & 1111 & 1110 & 0001 & 0001 & 1110 & 1000 & 0111 & 1100 & 0011 & 0011 & 1100 & 0000 & 1000\\
1110 & 0001 & 0111 & 1000 & 0000 & 1111 & 0000 & 1111 & 0011 & 1100 & 0001 & 1110 & 1000 & 0111 & 1100 & 0011 & 0000 & 0100\\
0000 & 1111 & 1110 & 0001 & 0111 & 1000 & 0000 & 1111 & 1100 & 0011 & 0011 & 1100 & 0001 & 1110 & 1000 & 0111 & 0000 & 0010\\
0000 & 1111 & 0000 & 1111 & 1110 & 0001 & 0111 & 1000 & 1000 & 0111 & 1100 & 0011 & 0011 & 1100 & 0001 & 1110 & 0000 & 0001
\end{array}\right\|$$

b

Fig. 2. Uniform test matrices for a (72, 64)-code with d = 4 that detect error bytes of length four.

where

$$U^{(8)}=\|B_{01}\ B_{02}\ B_{10}\ D_{1303}\|,$$

$$B_{ij}=\left\|\begin{array}{c:c:c:c} m_i+m_j & m_i+m_j+p & m_i & m_j \\ m_i & m_j & m_i+m_j & m_i+m_j+p \end{array}\right\|,$$

$$m_0=\left\|\begin{array}{c}1\\0\\0\\0\end{array}\right\|,\quad m_1=\left\|\begin{array}{c}0\\1\\0\\0\end{array}\right\|,\quad m_2=\left\|\begin{array}{c}0\\0\\1\\0\end{array}\right\|,\quad m_3=\left\|\begin{array}{c}0\\0\\0\\1\end{array}\right\|,\quad p=\left\|\begin{array}{c}1\\1\\1\\1\end{array}\right\|,$$

$$D_{sfkr}=\left\|\begin{array}{c:c:c:c} m_k+m_r & m_k+m_r & m_s+p & m_f+p \\ m_s+p & m_f+p & m_k+m_r & m_k+m_r \end{array}\right\|,$$

the "+" sign in operations on the p and m_u, u = 0, 1, 2, 3 column vectors means bit-by-bit addition in modulo two.

In a $U^{(8)}$ matrix we can replace any submatrix B_{ij} or D_{sjkr} with $T_4{}^uB_{ij}$ or $T_4{}^uD_{sfkr}$ respectively. This leads to replacing the columns and redistributing the ones in rows of the $H^{(8)}$ matrix. Thus, having replaced D_{1303} with $T_4{}^3D_{1303}$, we obtain a matrix $U^{(8)}$ having seven ones in each row. Other columns of weight five may be included in $H^{(8)}$, having replaced D_{1303} with the matrices D_{1323}, D_{1202}, D_{1201}, D_{1212}, and D_{1203}.

Increasing the level of uniformity of a test matrix increases the number of ones it contains. The $H^{(9)}$ test matrix of Fig. 2b allows errors of length four to be detected and and contains 33 ones in each row. The $H^{(9)}$ matrix is uniform at level $\Lambda = 8$ and is almost uniform at level $\Lambda' = 16$.

The structure of the $H^{(9)}$ matrix is

$$H^{(9)}=\left\|\begin{array}{cccc:cccc:c} F & R_4F & R_4^2F & R_4^3F & G & R_4G & R_4^2G & R_4^3G & \multirow{2}{*}{I_8} \\ G & R_4G & R_4^2G & R_4^3G & F & R_4F & R_4^2F & R_4^3F & \end{array}\right\|, \tag{10}$$

where $F=\|\varphi\bar{\varphi}\|$ and $G=\|g\bar{g}\|$; the bars indicate that all elements of the matrix are inverted; $\varphi=\|m_2+m_3,\ m_2,\ m_1,\ m_0+m_1\|$ and $g=\|m_1,\ m_0+m_1,\ m_0+m_1,\ m_0\|$.

The construction, Eq. (7), is used to build a test matrix for a (137, 128)-code with d = 4 that detects all error bytes of length four. Having substituted this matrix into $H^{(7)}$ we obtain an almost uniform matrix $H^{(10)}$ having a level of uniformity $\Lambda' = 8$.

4. Structure of the Coder and Decoder

To obtain the maximal speed promised by operations paralleling and the possibility of computing and recording data simultaneously the main memory of a high-capacity computer is

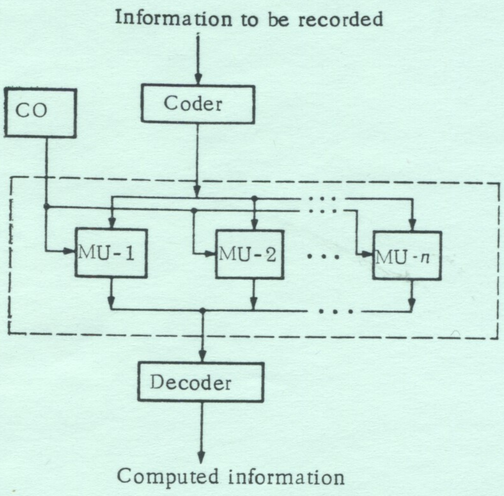

Fig. 3. Block diagram of the main memory in a high-capacity computer (MU-1, MU-2, ..., MU-n are independent memory units, and CU is the control unit).

composed of n ($n \geq 8$) memory units. The block diagram of the main memory in a high-capacity computer is shown in Fig. 3.

The coding and decoding devices (coder and decoder) are separate devices of the pipeline type. The next word arrives at the device input and the previous word arrives at the output on every clock pulse.

The time required for coding and detecting single errors must be less than one computer operating cycle and is measured in nanoseconds [1, 4]. These times are governed not only by the number of logic levels (the coder and decoder are combinational circuits), but depend on the lengths of the conductors connecting the functional elements. The number of junctions between microcircuits also affects this time.

Therefore, constructing the devices on LSI allows greater operating speed to be obtained than is possible with medium- and small-scale integration. For reasons of technology and construction, it is desirable to use one type of LSI to construct the coder and decoder. At the same time, a coder and decoder, each constructed on its own type of LSI, can be effective in specific situations. When this is done circuits that perform other functions, specifically, multiplexing, are included in an LSI coder.

Hamming code coder and decoder circuits other than LSI contain medium- and small-scale integrated circuits [5, 8]. The approach examined in this section for achieving maximal speed consisting of constructing the coder and decoder only on LSI without using medium- and small-scale integration and minimizing the number of junctions between the microcircuits.

If the test matrix for the code being used is uniform or almost uniform the coder and decoder may be constructed of the same type of units, the connections between and within which are regular and nearly minimal.

Single errors in the information symbols are corrected and double errors and errors of even weight are detected during decoding (as in standard algorithms [5, 8]), along with all odd weight errors that are potentially detectable by the code [3]. When this happens the syndrome is compared with each column of the test matrix and if the syndrome does not agree with any of them, the decoder aborts the decoding operation.

This approach is considered in an example of a coder and decoder for a minimal (72, 64)-code with a uniform test matrix ($\Lambda = 8$). The block diagram of a decoder consisting of eight identical LSI is shown in Fig. 4.

A word (a_1', ..., a_{64}', c_1', ..., c_8') arriving at the decoder input is distributed around the LSI as follows: the a_1', ..., a_8', c_2', c_3', c_4' symbols arrive at the input register of the first LSI, the a_9', ..., a_{16}', c_1', c_3', c_4' symbols arrive at the input register of the second LSI, etc.

TABLE 1. Characteristics of Test Matrices for (72, 64)- and (137, 128)-Hamming Codes

Code	Matrix	Level of uniformity	Number of ones in the matrix	Maximal number of ones per row	f_λ	Number ~~of layers~~ codewords with wt. four	δ	Matrix properties	Distribution of ones ~~ones~~ in the matrix rows	Comments
(72,64)	$H^{(1)}$	8	216	27	5	8408	0,5639	U, M. ML	(8, 0, 0, 0, 4, 4, 4, 4; 3)	Fig. 1, a
	$H_r^{(1)}$	8	328	41	5	8408	0,5639	U, UL, RP	(0, 8, 8, 8, 4, 4, 4, 4; 1)	Table 6 [5]
	$H^{(2)}$	8	216	27	6	8408	0,5639	U, RP, M	(7, 1, 1, 1, 4, 4, 4, 4; 1)	Fig. 1, b
	H_d	8	264	33	5	8912	0,5977	U, ML, RP	(8, 0, 0, 8, 4, 4, 4, 4; 1)	Fig.. 3(*d*)[3]
	H_c	8	216	27	8	8392	0,5628	CU, M, RP	(8, 3, 3, 2, 1, 3, 3, 3; 1)	Fig.. 5[2]
	H_k	2	248	31	8	8200	0,5500	U, RP, OB	(10, 20; 1), (11, 19; 1)	Fig. 2[7]
	$H^{(7)}$	4	216	27	8	8408	0,5639	U, M, OB	(9, 5, 4, 6; 3)	(~~9~~ 8)
	$H^{(8)}$	4	232	29	8	8256	0,5537	U, RP, OB	(10, 6, 6, 6; 1)	Fig. 2, a
	$H^{(9)}$	8	264	33	7	8912	0,5977	U, RP, OB	(4, 4, 4, 4, 4, 4, 4, 4; 1)	Fig. 2, b
(137,128)	$H^{(3)}$	4	481	54	9	56354	0,5377	U, M, RP	(13, 13, 13, 13; 1), (13, 13, 13, 14; 1)	(3)
	$H^{(4)}$	8	489	55	9	56252	0,5367	U, RP	(6, 6, 6, 6, 6, 6, 6, 6; 1), (7, 6, 7, 6, 7, 7, 7, 7; 1)	(4)
	$H^{(5)}$	8	505	57	9	55792	0,5323	CU, RP	(7, 7, 6, 7, 7, 7, 7, 7; 1) (7, 7, 7, 7, 7, 7, 7, 7; 1)	(5)
	$H^{(6)}$	16	476	65	5	57339	0,5471	AU, E/O		(7), Fig. 1, a
	$H^{(10)}$	8	476	65	9	57339	0,5471	AU, OB, E/O		(7), (8)

The functions

$$\varphi_1^{(l)}=a'_{\varkappa+2}+a'_{\varkappa+3}+a'_{\varkappa+5}+a'_{\varkappa+8},\qquad \varphi_2^{(l)}=a'_{\varkappa+2}+a'_{\varkappa+4}+a'_{\varkappa+6}+a'_{\varkappa+8},$$
$$\varphi_3^{(l)}=a'_{\varkappa+3}+a'_{\varkappa+4}+a'_{\varkappa+7}+a'_{\varkappa+8},\qquad \varphi_4^{(l)}=a'_{\varkappa+5}+a_{\varkappa+6}+a'_{\varkappa+7}+a'_{\varkappa+8},\qquad (12)$$
$$\varkappa=8(l-1)$$

are defined in the ℓ-th (ℓ = 1, 2, ..., 8) LSI in unit 3 of the syndrome preparation computer.

Values of the $\varphi_1^{(l)},\ldots,\varphi_4^{(l)}$ functions arrive at the output of the ℓ-th LSI and are then distributed to the inputs of all the LSI. The functions $\varphi_5^{(l)}=\sum_{p=1}^{8}a'_{\varkappa+p}$ computed in unit 2 of the syndrome preparation computer are used only in the ℓ-th LSI. At the ν-th (ν = 1, ..., 4) LSI of the ν-syndrome symbol computer (unit 4) input arrive the $\varphi_\nu^{(5)}$, $\varphi_\nu^{(6)}$, $\varphi_\nu^{(7)}$ and $\varphi_\nu^{(8)}$. At the μ-th (μ = 5, ..., 8) LSI of the μ-syndrome symbol computer input arrive the $\varphi_{\mu-4}^{(1)}, \varphi_{\mu-4}^{(2)}, \varphi_{\mu-4}^{(3)}$ and $\varphi_{\mu-4}^{(4)}$.

A function

$$S_\nu=\sum_{p=5}^{8}\varphi_\nu^{(p)}+\varphi_5^{(\nu)}+u_\nu,\qquad S_\mu=\sum_{p=1}^{4}\varphi_{\mu-4}^{(p)}+\varphi_5^{(\mu)}+u_\mu,\qquad (13)$$

where $u_\nu=\sum_{p=1}^{4}c_p'+c_\nu'$ and $u_\mu=\sum_{p=5}^{8}c_p'+c_\mu'$, $\nu=1,\ldots,4;\ \mu=5,\ldots,8$ is determined at the syndrome symbol computer in each LSI.

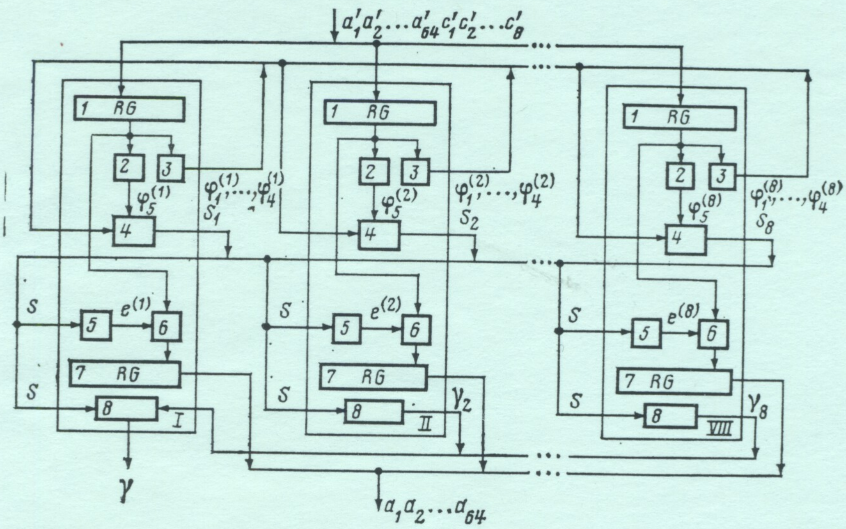

Fig. 4. Block diagram of a decoder for a (72, 64)-code fabricated on LSI circuits [1) input register; 2, 3) syndrome preparation computers; 4) syndrome symbol computer; 5) error position symbol shaper; 6) corrector; 7) output register; 8) error analyzer].

Values of the syndrome symbols arrive at the error position symbol generator and errors analyzer in each LSI. In the errors position symbol generator (decoder) in the i-th LSI for the syndrome $S = (S_1, S_2, \ldots, S_8)$ a double vector $\ell^{(i)} = (\ell_{1+8(i-1)}, \ldots, \ell_{8+8(i-1)})$, is generated such that all, or all but one symbol ℓ_t are zero. In the latter case the t-th column of a test matrix $H^{(1)}$ must be equal to the transposed syndrome S^T.

In the corrector of the i-th LSI, in the error symbols position generator from which a vector $\ell^{(i)}$ having a nonzero ℓ_t is obtained, $a_t = a_t' + \ell_t$ errors are corrected and corrected information symbols arrive at the output register.

In the errors analyzer of the i-th LSI the syndrome is compared with the test matrix columns corresponding to the information symbols $a'_{1+8(i-1)}, \ldots, a'_{8+8(i-1)}$ and test symbol c_i. If S^T is not equal to one of these columns a γ_i signal that single errors are absent from the i-th LSI is generated. The signals that single errors are absent from all LSI arrive at the errors analyzer of one of the LSI (in Fig. 4 this is the first LSI) and if these signals are equal to eight and $S \neq 0$, a decoding abort signal γ is generated.

A test symbol c_i is computed from (compare $H^{(1)}$ and $H_r^{(1)}$)

$$c_\nu = \sum_{p=5}^{8} \varphi_\nu^{(p)} + u_\nu^\bullet, \quad c_\mu = \sum_{p=1}^{4} \varphi_{\mu-4}^{(p)} + u_\mu^\bullet, \tag{14}$$

where

$$u_\nu^\bullet = \sum_{p=1}^{4} \varphi_5^{(p)} + \varphi_5^{(\nu)}, \quad u_\mu^\bullet = \sum_{p=5}^{8} \varphi_5^{(p)} + \varphi_5^{(\mu)}; \ \nu=1,\ldots,4; \ \mu=5,\ldots,8,$$

in the syndrome symbol computer during coding.

Replacing the columns of a test matrix is equivalent to reswitching LSI inputs. Therefore, the coding and decoding devices for a (72, 64)-code with test matrices $H^{(1)}$, $H_r^{(1)}$, $H^{(2)}$ and $H^{(7)}$ can be constructed on a single type of LSI.

Comparative characteristics of (72, 64)- and (137, 128)-codes examined here and optimal in terms of uniformity are shown in Table 1. The folowing designations are used in the column "matrix properties": U — uniform, CU — cyclic uniform, AU — almost uniform, M — minimal, ML — numbered with a matrix locator, RP — reduced-partitioned, OB — detects error bytes at the same time it corrects single and detects double independent errors, and E/O — has columns of equal, as well as odd, weight (if this designation is absent all columns in a matrix have odd weight).

The (72, 64)- and (137, 128)-Hamming codes shown in Table 1 correct single and detect double independent errors and, in a number of cases, error bytes of length four. The decoding algorithm examined here even allows a substantial number of triple errors to be detected at the same time. The fraction of incorrectly decoded triple errors is $\delta = 4A_4/C_n^3$, where A_4 is the number of code words of weight four [2]. For a minimal (72, 64)-code, $A_4 \geq 8392$ [9]. For arbitrary (137, 128)- and (72, 64)-Hamming codes with d = 4, $A_4 \geq 55182$ and $A_4 \geq 8157$ respectively [10]. Numerical values of A_4 shown in Table 1 were calculated on a BESM-6 computer. The distribution of ones in the $H^{(6)}$ and $H^{(10)}$ matrix rows were obtained by applying Eq. (7) to $H^{(1)}$ and $H^{(7)}$.

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