STRONG TRAJECTORY AND GLOBAL $W^{1,p}$-ATTRACTORS FOR THE DAMPED-DRIVEN EULER SYSTEM IN $\mathbb{R}^2$

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Abstract. We consider the damped and driven two-dimensional Euler equations in the plane with weak solutions having finite energy and enstrophy. We show that these (possibly non-unique) solutions satisfy the energy and enstrophy equality. It is shown that this system has a strong global and a strong trajectory attractor in the Sobolev space H^1. A similar result on the strong attraction holds in the spaces $H^1 \cap \{u : \ \|\operatorname{curl} u\|_{L^p} < \infty\}$ for $p \ge 2$.

1. Introduction

In this work we study the following dissipative Euler equations

$$\begin{cases} \partial_t u + (u, \nabla)u + \nabla p + ru = g, \\ u\big|_{t=0} = u_0, \quad \operatorname{div} u = 0 \end{cases} \tag{1.1}$$

in the whole plane $x \in \mathbb{R}^2$. This system with linear damping term ru, where $r > 0$ is the Rayleigh or Ekman friction coefficient, is relevant in large scale geophysical models [24]. See also [7], where this hydrodynamic system is derived under certain assumptions from the Boltzmann equations.

This system has attracted considerable attention over the last years. Our interest in this work is in the time dependent problem, but for the moment we observe that the regularity and uniqueness of the stationary solutions of (1.1) in a bounded domain in $\mathbb{R}^2$ were studied in Sobolev spaces [4], [28]. In the framework of the Lagrangian approach the stability of stationary solutions in Hölder spaces was studied in [32].

2000 *Mathematics Subject Classification.* 35B40, 35B41, 35Q35.

Key words and phrases. Damped Euler equations, global and trajectoty attractors.

The research of V. Chepyzhov and A. Ilyin was carried out in the Institute for Information Transmission Problems, Russian Academy of Science, at the expense of the Russian Science Foundation (project 14-50-00150).

Closely related to (1.1) is the Navier–Stokes perturbation of it

$$\begin{cases} \partial_t u + (u,\nabla)u + \nabla p + ru = \nu\Delta u + g, \\ u\big|_{t=0} = u_0, \quad \operatorname{div} u = 0, \end{cases} \tag{1.2}$$

which is studied in the vanishing viscosity limit $\nu \to 0$.

We have to distinguish two fundamentally different phase spaces in which systems (1.1) and (1.2) are studied in case of unbounded domains $\Omega \subseteq \mathbb{R}^2$. Namely, the usual Sobolev spaces (so that, for instance in the case of L^2, we are dealing with finite energy solutions) and the uniformly local spaces in which the Sobolev norms on balls of arbitrary fixed radius are uniformly bounded with respect to all translations.

In the case of periodic boundary conditions the estimate of the fractal dimension of the global attractor for (1.2) was obtained in [18], where it was shown that the corresponding dynamical system possesses a global attractor $\mathcal{A}$ (in L^2) with fractal dimension satisfying the estimate

$$\dim_f \mathcal{A} \le \frac{3}{8}\frac{\|\nabla g\|^2}{\nu r^3}. \tag{1.3}$$

It was also shown in [18] that if $r > 0$ is fixed, then the rate of growth of the dimension estimate, as $\nu \to 0$, is sharp, and the upper bounds were supplemented with a lower bound of the order $1/\nu$. See also [20] for the estimates of the same order for number of the degrees of freedom expressed in terms of various finite dimensional projections.

Since estimate (1.3) does not explicitly contain the length scale, it is reasonable to expect that this estimate survives in $\mathbb{R}^2$ in the phase space of finite energy solutions. It has recently been shown in [19] that this is indeed the case. Furthermore, due to the scale invariance of $\mathbb{R}^2$, the estimate was imbedded in the family of estimates depending on the norm of the right-hand side g in the whole family of homogeneous Sobolev spaces:

$$\dim_f \mathcal{A} \le \frac{1-s^2}{64\sqrt{3}}\left(\frac{1+|s|}{1-|s|}\right)^{|s|}\frac{1}{r^{2+s}\,\nu^{2-s}}\|g\|^2_{\dot H^s}, \qquad s\in[-1,1].$$

In particular, for $s = 1$ we obtain

$$\dim_f \mathcal{A} \le \frac{1}{16\sqrt{3}}\frac{\|\nabla g\|^2}{\nu\, r^3},$$

which up to a constant agrees with (1.3).

In the case of uniformly local spaces (where the energy and higher Sobolev norms in the whole $\mathbb{R}^2$ are infinite) one of the main issues is the proof of the dissipative estimate. For the uniformly local spaces in the viscous case $\nu > 0$ the global attractors for (1.2) in the strong topology were constructed in [39], see also [37, 38] for similar results in channel-like domains. In the inviscid case the strong attractor for (1.1) in the uniformly local H^1 space was recently constructed in [12].

For the damped-driven Euler equations (1.1) on the torus in the phase space of finite energy (and enstrophy) solutions (where the solutions are not necessarily unique) the weak H^1 attractors were constructed in [17] using the approach of [1] (see also [5]). The corresponding weak H^1 trajectory attractors were constructed in [8] and in [13] for the non-autonomous case of (1.1).

A method for proving the strong attraction and compactness for the attractors of equations in unbounded or non-smooth domains was proposed and developed in [3, 16, 23, 27]. The method essentially uses the energy equality for the corresponding equation for the proof of the asymptotic compactness of the solution semigroup. In our case this method was applied in [11] for the proof of the existence of the strong H^1 trajectory attractor for (1.1) on the 2D torus. We point that the uniqueness of the solutions lying on the attractor for system (1.1) with $\operatorname{curl} g \in L^\infty$ was used in (1.1) for the proof of the enstrophy equality.

In this work we obtain the energy (more precisely, enstrophy) equality in the Sobolev spaces $W^{1,p}$, $2 \le p < \infty$ without the assumption on g that guarantees the uniqueness on the attractor. Instead we use the fact that in the 2D case the vorticity satisfies the transport equation, and the corresponding enstrophy equality directly follows from the results of [15].

In Section 2 we give the definition of the weak solution and prove that a solution exists and satisfies the energy equality. However, the regularity of it is insufficient for the straightforward proof of the enstrophy equality.

In Section 3 we show that the vorticity $\omega = \operatorname{curl} u$ satisfies in the weak sense the transport equation

$$\partial_t \omega + (u, \nabla)\omega + r\omega = \operatorname{curl} g(x),$$

where $u \in L^\infty(0, T; H^1)$ is a weak solution of (1.1). The results of [15] then immediately imply the required enstrophy equality.

In Section 4 we construct the weak trajectory attractor for system (1.1) and describe its structure.

In Section 5 we recall the definition of the global attractor for a system with possibly non-unique solutions. Based on the enstrophy equality obtained in Section 3 we prove the existence of the global attractor that is compact in H^1, and then show that the weak trajectory attractor constructed in Section 4 is, in fact, the strong trajectory attractor.

In Section 6 we prove further regularity of the attractor under the assumption that $\operatorname{curl} g \in L^p$, $p \ge 2$. Namely, we prove the compactness and attraction in the norm of the space $H^1 \cap \mathcal{E}_p$, where

$$\mathcal{E}_p := \{u,\ \operatorname{div} u = 0,\ \|u\|_{\mathcal{E}_p} := \|u\|_{L^2} + \|\operatorname{curl} u\|_{L^p} < \infty\}.$$

Finally, in Section 7 we give a variant of the Yudovich's proof [34, 35] of the uniqueness if $\operatorname{curl} g \in L^\infty$ (see also [11]).

2. Weak solutions for damped Euler system in $\mathbb{R}^2$

We study the following 2D damped-driven Euler system:

$$\begin{cases} \partial_t u + (u, \nabla)u + \nabla p + ru = g(x), \\ \operatorname{div} u := \partial_1 u^1 + \partial_2 u^2 = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad t \geq 0. \end{cases} \tag{2.1}$$

Here, $u = u(t, x) = (u^1(t, x), u^2(t, x))$ is the unknown velocity vector field, $p = p(t, x)$ is the unknown pressure, $g(x) = (g^1(x), g^2(x))$ is the given external force, $\nabla = (\partial_1, \partial_2)$, and $\partial_i := \partial/\partial x_i,\ i = 1, 2$. Next,

$$(u, \nabla)v = u^1\partial_1 v + u^2\partial_2 v$$

is the bilinear term, and $r > 0$ is the given parameter that describes the Ekman damping term ru. Without loss of generality we shall assume that $\operatorname{div} g = 0$.

We study weak solutions of the system (2.1) with *finite* energy. We set

$$\mathcal{H} := \{v \in (L_2(\Omega))^2,\ \operatorname{div} v = 0\}.$$

The norm in the space $\mathcal{H}$ is denoted by $\|\cdot\|$. We shall use standard notation for the Sobolev spaces $W^{\ell,p} = W^{\ell,p}(\mathbb{R}^2)$. As usual, $H^s = W^{s,2}$. Along with the space $\mathcal{H} = \mathcal{H}^0$ we shall use the standard scale of spaces $\mathcal{H}^s \subset (H^s)^2$. In particular, $\mathcal{H}^1 = \{u \in (H^1)^2,\ \operatorname{div} u = 0\}$. We clearly have that $\mathcal{H}^{-s} = (\mathcal{H}^s)^*$ is the dual space to $\mathcal{H}^s$ for $s > 0$. The norm in $\mathcal{H}^s$ is denoted by $\|\cdot\|_s$.

In two (and three) dimensions we have for the vector Laplacian

$$\Delta u = (\Delta u^1, \Delta u^2) = \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u, \tag{2.2}$$

so that integrating by parts we obtain for $u \in \mathcal{H}^1$

$$\|u\|_1^2 = \|u\|^2 + \|\nabla u\|^2 = \|u\|^2 + \|\operatorname{div} u\|^2 + \|\operatorname{curl} u\|^2 = \|u\|^2 + \|\operatorname{curl} u\|^2. \tag{2.3}$$

We shall also use the spaces $\mathcal{W}^{1,p}$, $1 < p < \infty$, with norm

$$\|u\|_{\mathcal{W}^{1,p}} = \|u\|_{L^p} + \|\operatorname{curl} u\|_{L^p}. \tag{2.4}$$

Smooth divergence free vector functions $v \in (C_0^\infty(\mathbb{R}^2))^2$, $\operatorname{div} v = 0$ are dense in $\mathcal{W}^{1,p}$.

The functional formulation of the system (2.1) is similar to that of the Navier–Stokes system [30]. The trilinear form

$$b(u, v, w) := \int_{\mathbb{R}^2} ((u, \nabla)v, w)dx \tag{2.5}$$

is continuous on $\mathcal{H}^1\times\mathcal{H}^1\times\mathcal{H}^1$:

$$b(u,v,w)\le C\|u\|_1\|v\|_1\|w\|_1, \tag{2.6}$$

and for a fixed $u,v\in\mathcal{H}^1$ defines a bounded linear functional $B(u,v)$ on $\mathcal{H}^1$:

$$B(u,v):\mathcal{H}^1\to\mathcal{H}^{-1},\quad b(u,v,w)=\langle B(u,v),w\rangle$$

with

$$\|B(u,v)\|_{-1}\le C\|u\|_1\|v\|_1. \tag{2.7}$$

In fact, we have more, namely, for any $\varepsilon>0$

$$\|B(u,v)\|_{-\varepsilon}\le C_\varepsilon\|u\|_1\|v\|_1, \tag{2.8}$$

since by the Hölder inequality $u\nabla v\in L^{2-\varepsilon}$, and by the Sobolev imbedding and duality $L^{2-\varepsilon}\supset H^{-\varepsilon/(2-\varepsilon)}$.

We now define a weak solution of the system (2.1) in the space $\mathcal{H}^1$. Let $g\in\mathcal{H}^{-1}$.

Definition 2.1. A function $u(t,x)\in L^\infty(0,T;\mathcal{H}^1)$ is a weak solution of (2.1) if for every $w\in\mathcal{H}^1$ the following equality holds in the sense of distributions $\mathcal{D}'((0,T))$:

$$\frac{d}{dt}(u,w)+b(u,u,w)+r(u,w)=(g,w), \tag{2.9}$$

or, equivalently, u satisfies the equation

$$\partial_t u+B(u,u)+ru=g \tag{2.10}$$

in the space $\mathcal{D}'((0,T);\mathcal{H}^{-1})$ of distributions with values in $\mathcal{H}^{-1}$. Both equalities (2.9), and (2.10) hold if and only if for every smooth test function $\varphi(t,x)\in C_0^\infty((0,T)\times\mathbb{R}^2)$, $\operatorname{div}\varphi=0$, the following integral identity holds

$$-\int_0^T(u,\partial_t\varphi)dt-\int_0^T\sum_{i=1}^2(u^iu,\partial_i\varphi)dt+r\int_0^T(u,\varphi)dt=\int_0^T(g,\varphi)dt. \tag{2.11}$$

Let $u(t,x)\in L^\infty(0,T;\mathcal{H}^1)$ be an arbitrary weak solution of (2.1). Equation (2.10) and inequality (2.7) imply that

$$\|\partial_t u\|_{-1}\le\|B(u,u)\|_{-1}+r\|u\|_{-1}+\|g\|_{-1}\le C\|u\|_1^2+\|g\|_{-1}. \tag{2.12}$$

Therefore, $\partial_t u\in L^\infty(0,T;\mathcal{H}^{-1})$ and $u\in C([0,T];\mathcal{H}^{-1})$. Applying [30, Lemma III.1.4] we conclude that $u\in C_w([0,T];\mathcal{H}^1)$. Hence, the initial condition is meaningful:

$$u|_{t=0}=u_0\in\mathcal{H}^1. \tag{2.13}$$

Theorem 2.2. *Let $g \in \mathcal{H}^1$ and $u_0 \in \mathcal{H}^1$. Then the problem (2.1), (2.13) has a weak solution $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$, which satisfies the following estimate:*

$$\|u(t)\|_1^2 = \|u(t)\|^2 + \|\nabla u(t)\|^2 \le \|u(0)\|_1^2 e^{-rt} + r^{-2}\|g\|_1^2. \tag{2.14}$$

Proof. The proof is standard and uses the vanishing viscosity method (alternatively, one can use the Galerkin method with respect to expanding family of bounded domains) (see [5, 8, 11], see also [22, 21, 29, 30, 14]). Therefore we restrict ourselves to discussing a priori estimates for weak solutions of the problem (2.1) and (2.13).

The L^2-norm estimate contained in (2.14) is obtained by taking the scalar product of (2.1) with u and using the orthogonality relation

$$((u,\nabla v),v) = -\frac{1}{2}\int_{\mathbb{R}^2} |v|^2 \operatorname{div} u \, dx = 0, \tag{2.15}$$

which gives

$$\|u(t)\|^2 \le \|u(0)\|^2 e^{-rt} + r^{-2}\|g\|^2. \tag{2.16}$$

Next, we estimate the H^1-norm and use the orthogonality relation $((u,\nabla u),\Delta u) = 0$ specific to the two-dimensional case [29]. In fact, using the invariant expressions for the Laplacian (2.2) and for the advection term

$$(u,\nabla)u = \operatorname{curl} u \times u + \frac{1}{2}\nabla|u|^2, \tag{2.17}$$

and setting

$$\omega := \operatorname{curl} u = \partial_1 u^2 - \partial_2 u^1,\ u^\perp := (-u^2, u^1),\ \nabla^\perp := (-\partial_2, \partial_1), \tag{2.18}$$

we find that

$$\begin{aligned} ((u,\nabla u),\Delta u) = (\operatorname{curl} u \times u, \operatorname{curl}\operatorname{curl} u) = (\omega u^\perp, \nabla^\perp \omega) = \\ = (\omega u, \nabla\omega) = \frac{1}{2}\int_{\mathbb{R}^2} u\nabla\omega^2 dx = 0. \end{aligned} \tag{2.19}$$

We take the scalar product of (2.1) with Δu and integrating by parts and using (2.19) we obtain

$$\|\nabla u(t)\|^2 \le \|\nabla u(0)\|^2 e^{-rt} + r^{-2}\|\nabla g\|^2. \tag{2.20}$$

Summing (2.16) and (2.20) we finally have (2.14).

Finally, we observe that since the orthogonality relations (2.15) and (2.19) are 'bilinear', the formal argument above can be justified by Galerkin approximation solutions. Alternatively, we can consider more regular solutions of the corresponding Navier–Stokes perturbation of this system. □

Corollary 2.3. *Let $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ be an arbitrary weak solution of* (2.1)*. Then the function $\|u(t)\|^2$ is absolutely continuous and the following energy equality holds*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + r\|u(t)\|^2 = (g, u(t)). \tag{2.21}$$

Proof. The established regularity $u(t,x) \in L^\infty(0,T;\mathcal{H}^1) \subset L^2(0,T;\mathcal{H}^1)$ of the weak solution and (2.12) giving $\partial_t u(t,x) \in L^\infty(0,T;\mathcal{H}^{-1}) \subset L^2(0,T;\mathcal{H}^{-1})$ show that we can take the scalar product of (2.1) with u to obtain

$$(\partial_t u, u) + r(u,u) = (g,u). \tag{2.22}$$

It now follows from [30, Lemma III.1.2] that

$$(\partial_t u, u) = \frac{1}{2}\frac{d}{dt}\|u\|^2 \text{ for a.e. } t \in [0,T]$$

and the function $\|u(t)\|^2$ is absolutely continuous. Therefore we obtain from (2.22) the *energy* equality (2.21). □

Remark 2.4. If we assume additional regularity of the solution u, namely, $u(t,x) \in L^2(0,T;\mathcal{H}^2)$, then arguing as in Corollary 2.3 one can show that the following *enstrophy* equality holds

$$\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 + r\|\nabla u(t)\|^2 = (\nabla g, \nabla u(t)). \tag{2.23}$$

Summing (2.21) and (2.23) and taking into account that $\|\nabla u\|^2 = \|\operatorname{curl} u\|^2$, we obtain the equality for the full H^1-norm

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2 + \|\operatorname{curl} u\|^2\right) + r\left(\|u(t)\|^2 + \|\operatorname{curl} u\|^2\right) = \\ = (u(t), g) + (\operatorname{curl} u(t), \operatorname{curl} g) \equiv (u(t), g)_1. \end{aligned} \tag{2.24}$$

In the next section we establish the enstrophy equality (2.23) and, hence, equality (2.24) for an *arbitrary* weak solution $u \in L^\infty(0,T;\mathcal{H}^1)$ of the problem (2.1), (2.13) (not necessarily constructed by, say, the Galerkin method). This will be the key tool in the construction of the strong trajectory attractor for the damped-driven 2D Euler system.

Remark 2.5. The uniqueness of a weak solution of the problem (2.1) and (2.13) for $g \in \mathcal{H}^1$ and $u_0 \in \mathcal{H}^1$ in the space $L^\infty(0,T;\mathcal{H}^1)$ is not proved. The situation is the same as in the case of the classical conservative 2D Euler system with $r = 0$. Therefore our goal is to construct strong global and trajectory attractors for the system (2.1) with possibly non-unique solutions.

3. Vorticity equation and enstrophy equality

Let $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ be an arbitrary weak solution of system (2.1). We consider the following (scalar) function called the *vorticity*

$$\omega(t,x) := \operatorname{curl} u(t,x) = \partial_1 u^2(t,x) - \partial_2 u^1(t,x).$$

We clearly have $\omega \in L^\infty(0,T;H)$. We now derive the equation for $\omega(t,x)$. Using (2.17), where $\operatorname{curl} u \times u = \omega u^\perp$, see (2.18), we rewrite the system (2.1) in the equivalent form

$$\begin{cases} \partial_t u^1 - \omega u^2 + \partial_1 p + \frac{1}{2}\partial_1 |u|^2 + r u^1 = g^1, \\ \partial_t u^2 + \omega u^1 + \partial_2 p + \frac{1}{2}\partial_2 |u|^2 + r u^2 = g^2, \\ \operatorname{div} u = 0. \end{cases} \tag{3.1}$$

Taking the (distributional) derivative ∂_2 of the first equation of the system (3.1) and ∂_1 of the second equation, we obtain

$$\begin{cases} \partial_t \partial_2 u^1 - \partial_2(\omega u^2) + \partial_2 \partial_1 p + \frac{1}{2}\partial_2 \partial_1 |u|^2 + r\partial_2 u^1 = \partial_2 g^1, \\ \partial_t \partial_1 u^2 + \partial_1(\omega u^1) + \partial_1 \partial_2 p + \frac{1}{2}\partial_1 \partial_2 |u|^2 + r\partial_1 u^2 = \partial_1 g^2. \end{cases}$$

We now subtract the first equation from the second and obtain

$$\partial_t \left(\partial_1 u^2 - \partial_2 u^1\right) + \partial_1(\omega u^1) + \partial_2(\omega u^2) + r\left(\partial_1 u^2 - \partial_2 u^1\right) = \partial_1 g^2 - \partial_2 g^1.$$

Here we have used the commutation property $\partial_2\partial_1 = \partial_1\partial_2$ for the distributional derivatives. The resulting equation for the vorticity ω reads

$$\partial_t \omega + \sum_{j=1}^{2} \partial_j(\omega u^j) + r\omega = \operatorname{curl} g(x). \tag{3.2}$$

This equation holds in the sense of distributions, that is, for every smooth test function $\varphi(t,x) \in C_0^\infty((0,T)\times\mathbb{R}^2)$ the following integral identity holds:

$$\begin{aligned} -\int_0^T \omega \partial_t \varphi dt - \int_0^T \int_{\mathbb{R}^2} \omega \sum_{j=1}^{2} u^j \partial_j \varphi dx dt + \\ + r\int_0^T \int_{\mathbb{R}^2} \omega\varphi dx dt = \int_0^T \int_{\mathbb{R}^2} \varphi \operatorname{curl} g dx dt. \end{aligned} \tag{3.3}$$

It is well known that if $f_1, f_2 \in H^1$, then $\partial_i(f_1 f_2) = (\partial_i f_1) f_2 + f_1(\partial_i f_2) \in L^1_{loc}$ is the distributional derivative ∂_i of the product $f_1 f_2$. Recall that $u \in L^\infty(0,T;\mathcal{H}^1)$ and $\operatorname{div} u = 0$. Therefore we have

$$\sum_{j=1}^{2} \partial_j \left(u^j \varphi\right) = \varphi \operatorname{div} u + \sum_{j=1}^{2} u^j \partial_j \varphi = \sum_{j=1}^{2} u^j \partial_j \varphi.$$

Consequently, we can rewrite identity (3.3) as follows

$$
\begin{aligned}
-\int_0^T \omega\partial_t\varphi dt - \int_0^T\int_{\mathbb{R}^2}\omega\sum_{j=1}^2\partial_j\left(u^j\varphi\right)dxdt + \\
+r\int_0^T\int_{\mathbb{R}^2}\omega\varphi dxdt = \int_0^T\int_{\mathbb{R}^2}\varphi\,\mathrm{curl}\,g\varphi dxdt.
\end{aligned}
\tag{3.4}
$$

We have proved the following result.

Proposition 3.1. *Let $g \in \mathcal{H}^1$. If $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ is a weak solution of* (2.1)*, then the corresponding vorticity function $\omega = \mathrm{curl}\,u$ is the solution of the equation*

$$\partial_t\omega + (u,\nabla)\omega + r\omega = \mathrm{curl}\,g(x). \tag{3.5}$$

in the sense of the integral identity (3.4)*.*

We note that in equation (3.2) we have $u \in L^\infty(0,T;\mathcal{H}^1)$ so that $\omega \in L^\infty(0,T;L^2)$. Using the embedding $H^1(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ for any $q \geq 2$, we observe that the product $\omega \cdot u^j$ belongs to $L^\infty(0,T;L^{2-\varepsilon})$ for any $\varepsilon > 0$. Therefore, $\sum_{j=1}^2 \partial_j(\omega u^j) \in L^\infty(0,T;W^{-1,2-\varepsilon})$ and the equation (3.2) holds the space of distributions $\mathcal{D}'((0,T);W^{-1,2-\varepsilon})$. We also note that since $\mathrm{div}\,u = 0$, the weak formulations of (3.2) and (3.5), namely, (3.3) and (3.4), respectively, are equivalent.

We now formulate the main theorem of this section.

Theorem 3.2. *Let $\omega(t,x) \in L^\infty(0,T;L^2)$ be a solution of the linear transport equation* (3.5) *in the sense of* (3.3) *(or* (3.4)*), where $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ is an arbitrary function (not necessarily a weak solution of* (2.1)*). Then the function $\|\omega(t)\|^2$ is absolutely continuous and satisfies the differential equation*

$$\frac{1}{2}\frac{d}{dt}\|\omega(t)\|^2 + r\|\omega(t)\|^2 = (\omega(t),\mathrm{curl}\,g) \quad \text{a.e. on } (0,T). \tag{3.6}$$

Proof. This follows from the proof of [15, Theorem II.2 and equation (26)], the only difference being that in [15] the unforced case ($g = 0$) is considered. □

Corollary 3.3. *Let $g \in \mathcal{H}^1$. If $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ is a weak solution of* (2.1)*, then the function $\|u(t)\|_1^2 = \|u(t)\|^2 + \|\nabla u(t)\|^2$ is absolutely continuous and*

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) + r\left(\|u(t)\|^2 + \|\nabla u(t)\|^2\right) = \\
= (u(t),g) + (\nabla u(t),\nabla g)
\end{aligned}
\tag{3.7}
$$

almost everywhere on $t \in (0,T)$. In addition, $u \in C([0,T];\mathcal{H}^1)$ and $\partial_t u \in C([0,M];\mathcal{H}^{-\varepsilon})$ for every $\varepsilon > 0$.

Proof. We recall that $\|\nabla u\| = \|\operatorname{curl} u\|$ for all $u \in \mathcal{H}^1$ (see (2.3)). Therefore, the equalities (2.21) and (3.6) prove the full H^1-norm equality (2.24) (equivalently, (3.7)) for any weak solution of the damped Euler system (2.1).

Next, we set $F(u) = -B(u,u) - ru + g$. Then $\partial_t u = F(u)$, see (2.10). It follows from (2.8) that the operator F is continuous from $\mathcal{H}^1$ to $\mathcal{H}^{-\varepsilon}$. Therefore $\partial_t u$ is bounded in $L^\infty(0,T;\mathcal{H}^{-\varepsilon})$, and, hence, $u \in C([0,T];\mathcal{H}^{-\varepsilon})$. Since u is bounded in $L^\infty(0,T;\mathcal{H}^1)$, it follows that u is weakly continuous in $\mathcal{H}^1$, see, for instance, [30, Lemma III.1.4]. Now the continuity with respect to time of the norms $\|u(t)\|$ and $\|\nabla u(t)\|$ proved in Corollary 2.3 and Theorem 3.2 and the weak continuity of u in $\mathcal{H}^1$ imply the strong continuity: $u \in C([0,T];\mathcal{H}^1)$. This also gives that $\partial_t u \in C([0,T];\mathcal{H}^{-\varepsilon})$. □

4. Weak trajectory attractor for damped Euler system

We consider the following two linear spaces

$$\mathcal{F}_+^\infty = L^\infty(\mathbb{R}_+;\mathcal{H}^1) \quad \text{and} \quad \mathcal{F}_+^{\mathrm{loc}} = L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H}^1).$$

Recall that $z(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{H}^1)$ if and only if $z(t,x) \in L^\infty(0,M;\mathcal{H}^1)$ for all $M > 0$. We clearly have $\mathcal{F}_+^\infty \subset \mathcal{F}_+^{\mathrm{loc}}$.

The space $\mathcal{F}_+^{\mathrm{loc}}$ is equipped with the following local weak topology $\Theta_+^{\mathrm{loc,w}}$. By definition, a sequence $z_n \in \mathcal{F}_+^{\mathrm{loc}}$ converges in $\Theta_+^{\mathrm{loc,w}}$ to an element z if $z_n \rightharpoonup z$ $*$-weakly in $L^\infty(0,M;\mathcal{H}^1)$ for every $M > 0$. Sometimes, when it causes no ambiguity, we denote by $\Theta_+^{\mathrm{loc,w}}$ the space $\mathcal{F}_+^{\mathrm{loc}}$ with topology $\Theta_+^{\mathrm{loc,w}}$.

We now define the *trajectory space* $\mathcal{K}_+$ for the equation (2.1).

Definition 4.1. The trajectory space $\mathcal{K}_+$ consists of all functions from $\mathcal{F}_+^\infty = L^\infty(\mathbb{R}_+;\mathcal{H}^1)$ that are weak solutions of system (2.1) on every interval $(0,M)$. The elements from $\mathcal{K}_+$ are called the *trajectories*.

It follows from Theorem 2.2 that for any $u_0 \in \mathcal{H}^1$ there is at least one trajectory $u \in \mathcal{K}_+$ such that $u(0) = u_0$. Thus, $\mathcal{K}_+$ is non-empty.

Proposition 4.2. *The trajectory space $\mathcal{K}_+$ is sequentially closed in the topology $\Theta_+^{\mathrm{loc,w}}$ and, in addition, $\mathcal{K}_+ \subset C_b(\mathbb{R}_+;\mathcal{H}^1)$.*

Proof. Let $\{u_n(t,x)\}$ be a sequence from $\mathcal{K}_+$ and let $u_n(t,x) \to u(t,x)$ in $\Theta_+^{\mathrm{loc,w}}$ for some $u \in \mathcal{F}_+^\infty$, i.e.,

$$u_n(t,x) \rightharpoonup u(t,x) \quad *\text{-weakly in } L^\infty(0,M;\mathcal{H}^1),\ \forall M > 0. \tag{4.1}$$

We claim that $u \in \mathcal{K}_+$. The functions $u_n(t,x)$ satisfy (2.1) in the sense of distributions, that is,

$$-\int_0^M (u_n, \partial_t\varphi)dt - \int_0^M \sum_{i=1}^2 (u_n^i u_n, \partial_i\varphi)dt + r\int_0^M (u_n,\varphi)dt = \int_0^M (g,\varphi)dt \tag{4.2}$$

for every $\varphi(t,x) \in C_0^\infty((0,M)\times\mathbb{R}^2)$, $\operatorname{div}\varphi = 0$. Let us show that $u(t,x)$ is also a distributional solution of (2.1). We fix an arbitrary $M > 0$ and $\varphi \in C_0^\infty((0,M)\times\mathbb{R}^2)$, $\operatorname{div}\varphi = 0$. Let the ball $B_0^R := \{x \in \mathbb{R}^2, |x| \le R\}$ contain the support of φ.

It follows from (4.1) that $\{u_n(t,x)\}$ is bounded in $L^\infty(0,M;\mathcal{H}^1)$. Then due to (2.12) $\{\partial_t u_n(t,x)\}$ is bounded in $L^\infty(0,M;\mathcal{H}^{-1})$. Passing to a subsequence, we may assume that

$$\partial_t u_n(t,x) \rightharpoonup \partial_t u(t,x) \quad *\text{-weakly in } L^\infty(0,M;\mathcal{H}^{-1}).$$

Applying Aubin compactness theorem (see, for instance, [6], [9]) and restricting the functions to the balls B_0^R, we conclude that

$$u_n \to u \quad \text{strongly in } L^2(0,M;\mathcal{H}(B_0^R)). \tag{4.3}$$

Recall that $L^2(0,M;\mathcal{H}(B_0^R)) \subset L^2((0,M)\times B_0^R)^2$ and therefore

$$u_n(t,x) \to u(t,x) \quad \text{for a.e. } (t,x) \in (0,M)\times B_0^R. \tag{4.4}$$

Consider the trilinear term in the left-hand side of (4.2). It follows from (4.4) that for $i = 1,2$

$$u_n^i(t,x)u_n(t,x) \to u^i(t,x)u(t,x) \quad \text{for a.e. } (t,x) \in (0,M)\times B_0^R. \tag{4.5}$$

Recall that the sequence $\{u_n\}$ is bounded in $L^\infty(0,M;\mathcal{H}^1)$. Then, due to the embedding $\mathcal{H}^1 \subset L^4(B_0^R)^2$, we see that

$$\{u_n^i u_n\} \quad \text{is bounded in } L^\infty(0,M;L^2(B_0^R)^2) \tag{4.6}$$

and in $L^2((0,M)\times B_0^R)^2$ as well. Applying [22, Lemma 1.1.3] on weak convergence, we conclude from (4.5) and (4.6) that

$$u_n^i u_n \rightharpoonup u^i u \quad \text{weakly in } L^2((0,M)\times B_0^R)^2.$$

Therefore, since $\operatorname{supp}\varphi \subset B_0^R$,

$$\int_0^M \sum_{i=1}^2 (u_n^i u_n, \partial_i\varphi)dt \to \int_0^M \sum_{i=1}^2 (u^i u, \partial_i\varphi)dt \quad \text{as } n\to\infty.$$

Finally, we see that (4.3) allows to pass to the limit as $n \to \infty$ in the remaining terms of (4.2) and we obtain the equality

$$-\int_0^M (u, \partial_t\varphi)dt - \int_0^M \sum_{i=1}^2 (u^i u, \partial_i\varphi)dt + r\int_0^M (u,\varphi)dt = \int_0^M (g,\varphi)dt.$$

Since the function φ and the number $M > 0$ are arbitrary, the function u is a weak solution of (2.1), that is, $u \in \mathcal{K}_+$. Hence, $\mathcal{K}_+$ is sequentially closed in the topology $\Theta_+^{\mathrm{loc,w}}$.

The embedding $\mathcal{K}_+ \subset C_b(\mathbb{R}_+; \mathcal{H}^1)$ follows from Corollary 3.3. $\square$

Remark 4.3. Any ball $\mathcal{B}(0, R) = \Big\{ z \in \mathcal{F}_+^\infty,\ \|z\|_{\mathcal{F}_+^\infty} \le R \Big\}$ is a compact subset in the topology $\Theta_+^{\mathrm{loc,w}}$ and the corresponding space is metrizable and complete (see [9, 26]). Therefore the intersection of the trajectory space $\mathcal{K}_+$ with $\mathcal{B}(0, R)$ is compact in the topology $\Theta_+^{\mathrm{loc,w}}$.

On the other hand, the entire space $\mathcal{F}_+^{\mathrm{loc}}$ and its subspace $\mathcal{F}_+^\infty$ are not metrizable in the topology $\Theta_+^{\mathrm{loc,w}}$.

We now consider the translation semigroup $\{T(h)\} := \{T(h), h \ge 0\}$ acting on the spaces $\mathcal{F}_+^{\mathrm{loc}}$ and $\mathcal{F}_+^\infty$ by the formula $T(h)z(t) = z(t+h)$. The translations $T(h)$ clearly map the trajectory space $\mathcal{K}_+$ into itself:

$$T(h)\mathcal{K}_+ \subseteq \mathcal{K}_+,\ \forall h \ge 0.$$

Indeed, if $u(t), t \ge 0$, is a weak solution of (2.1) on every interval $(0, M)$ then the function $u(t+h), t \ge 0$ is also a weak solution of (2.1) since the equation is autonomous.

Proposition 4.4. *The ball $\mathcal{B}(0, R_0)$ in $\mathcal{F}_+^\infty$ with radius $R_0 = \sqrt{2} r^{-1} \|g\|_1$ is an absorbing set of the semigroup $\{T(h)\}$ acting on $\mathcal{K}_+$, that is, for any subset $B \subset \mathcal{K}_+$ bounded in $\mathcal{F}_+^\infty$ there is a number $h_1 = h_1(B)$ such that $T(h)B \subset \mathcal{B}(0, R_0)$ for all $h \ge h_1$.*

Proof. It follows from Corollary 3.3 that any weak solution satisfies the enstrophy equality (3.7). This shows that any weak solution satisfies the dissipative estimate (2.14). (In Theorem 2.2 we proved that a weak solution constructed by the Galerkin approximations satisfies (2.14).) It now follows from inequality (2.14) that if B is a bounded set of trajectories from $\mathcal{K}_+$ and $\|u\|_{\mathcal{F}_+^\infty} \le R$ for all $u \in B$, then

$$\|T(h)u\|_{\mathcal{F}_+^\infty}^2 = \operatorname*{ess\,sup}_{t \ge 0} \|u(h+t)\|_1^2 \le R^2 e^{-rh} + r^{-2}\|g\|_1^2 \le 2r^{-2}\|g\|_1^2 = R_0^2,$$

for all $h \ge h_1$, where $e^{-rh_1} \le r^{-2}\|g\|_1^2/R^2$. That is, $T(h)u \subset \mathcal{B}(0, R_0)$ for $h \ge h_1$. Hence, the ball $\mathcal{B}(0, R_0)$ is absorbing. $\square$

We now recall the definition of the weak trajectory attractor (see [9, 10, 31]).

Definition 4.5. A set $\mathfrak{A} \subset \mathcal{K}_+$ is called a weak trajectory attractor of (2.1) if

(1) $\mathfrak{A}$ is compact in $\Theta_+^{\mathrm{loc,w}}$ and bounded in $\mathcal{F}_+^\infty$;
(2) $\mathfrak{A}$ is strictly invariant, that is, $T(h)\mathfrak{A} = \mathfrak{A}$ for all $h \ge 0$;

(3) $\mathfrak{A}$ attracts any bounded trajectory set $B \subset \mathcal{K}_+$, that is, for any neighborhood $\mathcal{O}(\mathfrak{A})$ of $\mathfrak{A}$ in the topology $\Theta_+^{\mathrm{loc,w}}$, there is $h_1 = h_1(B, \mathcal{O}) \geq 0$ such that $T(h)B \subset \mathcal{O}(\mathfrak{A})$ for all $h \geq h_1$.

Similarly to the space $\mathcal{F}_+^{\mathrm{loc}}$ and the space $\mathcal{F}_+^{\infty}$ with topology $\Theta_+^{\mathrm{loc,w}}$, we consider the space $\mathcal{F}^{\infty} = L^{\infty}(\mathbb{R}; \mathcal{H}^1)$ and the space $\mathcal{F}^{\mathrm{loc}} = L^{\infty}_{\mathrm{loc}}(\mathbb{R}; \mathcal{H}^1)$ with topology $\Theta^{\mathrm{loc,w}}$ defined on the entire time-axis.

Definition 4.6. The kernel $\mathcal{K}$ of the system (2.1) is the union of all functions $u(t, x) \in L^{\infty}(\mathbb{R}; \mathcal{H}^1)$ that are weak solutions of (2.1) on any interval $(-M, M)$. The elements of $\mathcal{K}$ are often called complete bounded trajectories of the system (2.1).

Proposition 4.7. *The kernel $\mathcal{K}$ of system (2.1) belongs to $C_b(\mathbb{R}; \mathcal{H}^1)$.*

The next theorem proves the existence and describes the structure of the weak trajectory attractor for damped 2D Euler system.

Theorem 4.8. *Let the external force $g \in \mathcal{H}^1$. Then the equation* (2.1) *has the weak trajectory attractor and*

$$\mathfrak{A} = \Pi_+ \mathcal{K}, \tag{4.7}$$

where $\mathcal{K}$ is the kernel of (2.1) *and Π_+ denotes the restriction operator onto the semiaxis $\mathbb{R}_+$. Besides, $\mathfrak{A}$ is bounded in $C_b(\mathbb{R}_+; \mathcal{H}^1)$ and the following inequality holds:*

$$\|u\|_{C_b(\mathbb{R}_+; \mathcal{H}^1)} \leq r^{-1} \|g\|_1, \ \ \forall u \in \mathfrak{A}. \tag{4.8}$$

Proof. To apply the general theorem on the existence and the structure of trajectory attractors (see, e.g., [9]), we have to verify that our trajectory space $\mathcal{K}_+$ contains, for the translation semigroup $\{T(h)\}$, an attracting (or absorbing) set which is bounded in $\mathcal{F}_+^{\infty}$ and compact in the topology $\Theta_+^{\mathrm{loc,w}}$. We have proved in Propositions 4.2 and 4.4 that such absorbing set is $\mathcal{B}(0, R_0) \cap \mathcal{K}_+$. See also Remark 4.3. Therefore, there exists a trajectory attractor $\mathfrak{A}$ for system (2.1) with structure (4.7). We also have that $\mathfrak{A} \subset C_b(\mathbb{R}_+; \mathcal{H}^1)$, while inequality (4.8) follows from (2.14). □

Using Corollary 3.3 we obtain the following estimate for the time derivatives of the solutions lying on the attractor.

Corollary 4.9. *For any $u \in \mathfrak{A}$, the following inequality holds*

$$\|\partial_t u\|_{C_b(\mathbb{R}_+; \mathcal{H}^{-\varepsilon})} \leq C_\varepsilon, \tag{4.9}$$

where $\varepsilon > 0$ and C_ε depends on r, $\|g\|_1$ and is independent of u.

Corollary 4.10. *The trajectory attractor* $\mathfrak{A}$ *is compact in the space* $C_{\rm loc}(\mathbb{R}_+;\mathcal{H}^{1-\delta}_{\rm loc})$ *for any* $\delta>0$. *Moreover, for any bounded set* $B\subset\mathcal{K}_+$

$$\operatorname{dist}_{C([0,M];\mathcal{H}^{1-\delta}_{\rm loc})}(T(h)B,\mathfrak{A})\to 0\ (h\to+\infty),\ \forall M>0. \tag{4.10}$$

(Here and below, $\operatorname{dist}_{\mathcal{M}}(X,Y)$ *denotes the Hausdorff semidistance from a set* X *to a set* Y *in a metric space* $\mathcal{M}$*).*

Proof. In fact, if $B\subset\mathcal{K}_+$ is a bounded set of trajectories $u(t,x)$, then any sequence $\{u_n\}\subset B$ is bounded in $C([0,M];\mathcal{H}^1)$ and, due to Corollary 4.9, the sequence $\{\partial_t u_n\}$ is bounded in the space $C([0,M];\mathcal{H}^{-1})$. Applying Aubin–Lions–Simon compactness theorem (see, for instance, [6], [9]) and restricting functions to the balls $B_0^R:=\{x\in\mathbb{R}^2,|x|\le R\}$, we conclude that $\{u_n\}$ is precompact in $C([0,M];\mathcal{H}^{1-\delta}(B_0^R))$ for any $\delta>0$ and $R<\infty$. Therefore, we have the compact embedding of every set $B\subset\mathcal{K}_+$ of bounded trajectories into

$$C_{\rm loc}(\mathbb{R}_+;\mathcal{H}^{1-\delta}_{\rm loc}),\ \text{for every } \delta>0.$$

This property implies (4.10). In particular, taking $B=\mathfrak{A}$ we obtain that $\mathfrak{A}$ is compact in $C_{\rm loc}(\mathbb{R}_+;\mathcal{H}^{1-\delta}_{\rm loc})$. □

In the next section we prove that Corollary 4.10 holds for the limiting $\delta=0$, furthermore, we remove the locality condition with respect to the spatial variable $x\in\mathbb{R}^2$.

5. Strong global and trajectory attractors

Our main result in this section relies heavily on the construction of the strong *global* attractor for the system (2.1), whose definition we now recall. Since, generally speaking, the solutions of the system are non-unique, we have to accordingly modify the classical definition [2, 9, 14, 29].

Definition 5.1. ([9, 31]) A set $\mathcal{A}\subset\mathcal{H}^1$ is called the global attractor of system (2.1) if

(1) $\mathcal{A}$ is compact in $\mathcal{H}^1$: $\mathcal{A}\Subset\mathcal{H}^1$;
(2) $\mathcal{A}$ is attracting: for every set $B\subset\mathcal{K}^+$ that is bounded in $C_b(\mathbb{R}^+;\mathcal{H}^1)$

$$\operatorname{dist}_{\mathcal{H}^1}(B(t),\mathcal{A})\to 0 \text{ as } t\to\infty;$$

(3) $\mathcal{A}$ is the minimal set (with respect to inclusion) among all compact attracting sets.

Here $B(t)$ is the (well-defined) section at time $t\ge 0$ of a set of bounded trajectories B:

$$B(t)=\{u(t),\ u\in B\}\subset\mathcal{H}^1.$$

We now formulate the main result of this work (in the $W^{1,2}$ case) in terms of the global and trajectory attractors. We first consider the global attractor.

In Theorem 4.8 we have shown that the weak trajectory attractor $\mathfrak{A}$ exists and is bounded in $C_b(\mathbb{R}_+;\mathcal{H}^1)$.

We set

$$\mathcal{A} := \mathfrak{A}(0) = \{u(0),\ u \in \mathfrak{A}\} \subset \mathcal{H}^1.$$

Theorem 5.2. *The set $\mathcal{A}$ is the global attractor.*

Proof. We fix a bounded trajectory set $B \subset \mathcal{K}_+$. Let $h_n \to +\infty$, and for each n let $u_n \in T(h_n)B$ be an arbitrary h_n-shifted trajectory. By Theorem 4.8, since $\mathfrak{A}$ is the weak trajectory attractor, passing to a subsequence of $\{h_n\}$ which we label the same, we may assume that

(1) there is $u \in \mathfrak{A}$ such that $u_n \rightharpoonup u$ in the (weak) topology $\Theta_+^{\mathrm{loc,w}}$;
(2) $u_n(0) \rightharpoonup u(0)$ weakly in $\mathcal{H}^1$;
(3) there are weak solutions $U_n(t)$ of equation (2.1) defined for $[-h_n, +\infty)$ such that $U_n(t) = u_n(t)$ for all $t \ge 0$ and

$$\|U_n(t)\|_{\mathcal{H}^1} \le C,\ \forall t \ge -h_n, \tag{5.1}$$

where $C = C(B)$ is independent of n;
(4) there is a bounded complete trajectory $U \in \mathcal{K}$ such that $U_n \rightharpoonup U$ in the (weak) topology $\Theta^{\mathrm{loc,w}}$ and $U(t) = u(t)$ for all $t \ge 0$.

To prove the theorem we have to show that

$$u_n(0) \to u(0) \text{ strongly in } \mathcal{H}^1 \text{ as } n \to \infty. \tag{5.2}$$

In fact, since $u_n(0) \rightharpoonup u(0)$ weakly in $\mathcal{H}^1$, it is sufficient to prove that

$$\|u_n(0)\|_1 \to \|u(0)\|_1 \text{ as } n \to \infty. \tag{5.3}$$

The functions $U_n(t), t \ge -h_n$ satisfy the H^1-norm equality (3.7)), that is,

$$\frac{d}{dt}\|U_n(t)\|_1^2 + 2r\|U_n(t)\|_1^2 = 2(U_n(t), g)_1.$$

We multiply this equation by e^{2rt} and integrate in t from $-h_n$ to 0:

$$\|U_n(0)\|_1^2 = \|U_n(h_n)\|_1^2 e^{-2rh_n} + 2\int_{-h_n}^0 (U_n(t), g)_1 e^{2rt}dt. \tag{5.4}$$

We want to pass to the limit $n \to \infty$ in this equality. Recall that $U_n(t)$ are uniformly bounded (see (5.1)) and $U_n(\cdot) \rightharpoonup U(\cdot)$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R};\mathcal{H}^1)$. Then we have that

$$\int_{-h_n}^0 (U_n(t), g)_1 e^{2rt}dt \to \int_{-\infty}^0 (U(t), g)_1 e^{2rt}dt \text{ as } n \to \infty. \tag{5.5}$$

Using (5.5) in (5.4), where $\|U_n(h_n)\|_1^2$ are uniformly bounded, and the condition $u_n(0) = U_n(0)$ we obtain that

$$\lim_{n\to\infty} \|u_n(0)\|_1^2 = 2\int_{-\infty}^{0} (U(t), g)_1 e^{2rt} dt. \tag{5.6}$$

But the H^1-norm equality (3.7) also holds for the complete trajectory $U(t), t \in \mathbb{R}$:

$$\frac{d}{dt}\|U(t)\|_1^2 + 2r\|U(t)\|_1^2 = 2(U(t), g)_1.$$

We multiply this equation by e^{2rt} and integrate from $-\infty$ to 0:

$$\|u(0)\|_1^2 = 2\int_{-\infty}^{0} (U(t), g)_1 e^{2rt} dt \quad \text{(recall that } u(0) = U(0)). \tag{5.7}$$

Hence, equalities (5.6) and (5.7) imply convergence (5.3), which gives the strong convergence $u_n(0) \to u(0)$ in $\mathcal{H}^1$ as $n \to \infty$ and proves (5.2) since

$$\lim_{n\to\infty} \|u_n(0) - u(0)\|_1^2 = \lim_{n\to\infty} \big(\|u_n(0)\|_1^2 + \|u(0)\|_1^2 - 2(u_n(0), u(0))_1\big) = 0.$$

Next, the compactness of $\mathcal{A}$ follows from the above argument if we take $B = \mathfrak{A}$. Finally, the minimality property follows from the fact that every compact attracting set must contain $\mathcal{K}(0) = \mathfrak{A}(0) = \mathcal{A}$ (see (4.7)). The proof is complete. □

We now prove that the weak trajectory attractor constructed in Theorem 4.8 is in fact a strong trajectory attractor.

We consider the trajectory space $\mathcal{K}_+ \subset C_b(\mathbb{R}_+; \mathcal{H}^1)$ and the weak trajectory attractor $\mathfrak{A} \subset \mathcal{K}_+$ of the system (2.1). In Theorem 4.8, we have established that $\mathfrak{A}$ exists and is bounded in $C_b(\mathbb{R}_+; \mathcal{H}^1)$. On the trajectory space $\mathcal{K}_+$ we consider now the local strong topology $\Theta_+^{\text{loc,s}}$. By definition, a sequence $\{z_n\} \subset \mathcal{K}_+$ converges to z in $\Theta_+^{\text{loc,s}}$ if, for every $M > 0$, $z_n(t, x) \to z(t, x)$ strongly in $C([0, M]; \mathcal{H}^1)$, that is

$$\max_{t\in[0,M]} \|z_n(t, \cdot) - z(t, \cdot)\|_{\mathcal{H}^1} \to 0 \ (n \to \infty).$$

It is clear that the topology $\Theta_+^{\text{loc,s}}$ is metrizable and this metric space is complete.

Theorem 5.3. *Let $g \in \mathcal{H}^1$. Then, the trajectory attractor $\mathfrak{A}$ is compact in $\Theta_+^{\text{loc,s}}$ and, for any trajectory set $B \subset \mathcal{K}_+$, bounded in $C_b(\mathbb{R}_+; \mathcal{H}^1)$, the h-shifted set $T(h)B$ tends to $\mathfrak{A}$ as $h \to \infty$ in $\Theta_+^{\text{loc,s}}$, that is, for every $M > 0$*

$$\operatorname{dist}_{C([0,M];\mathcal{H}^1)} (T(h)B, \mathfrak{A}) \to 0 \text{ as } h \to +\infty. \tag{5.8}$$

Proof. To prove the theorem, we have to verify that

$$\|u_n(\cdot) \to u(\cdot)\|_{C([0,M];\mathcal{H}^1)} \to 0 \tag{5.9}$$

for every fixed $M > 0$, where the $u_n(\cdot) \in T(h_n)B$ and $u(\cdot) \in \mathfrak{A}$ were defined in the beginning of the proof of Theorem 5.2. We note that $B_M = \bigcup_{s\in[0,M]} T(s)B$ is a bounded trajectory set. Then, by Theorem 5.2

$$\operatorname{dist}_{\mathcal{H}^1}(B_M(h), \mathcal{A}) \to 0 \text{ as } h \to +\infty.$$

Now, since, $B_M(h)$ contains all the time shifts of length $s \le M$ of the trajectories u_n, it follows that

$$\sup_{t\in[0,M]} \operatorname{dist}_{\mathcal{H}^1}(u_n(t), \mathcal{A}) \to 0 \text{ as } n \to \infty. \tag{5.10}$$

By Theorem 5.2, the set $\mathcal{A}$ is compact in $\mathcal{H}^1$. Hence, for any $\varepsilon > 0$, there is a finite ε-net $\{w_j\}_{j=1}^N$ for $\mathcal{A}$ in $\mathcal{H}^1$. By the density of compactly supported functions in $H^1(\mathbb{R}^2)$ we can assume that the supports of the w_j's are contained in B_R^0, where $R = R(N)$ is sufficiently large. Let P_N denote the orthogonal projection in $\mathcal{H}^1$ onto the finite-dimensional linear span of $\{w_j\}_{j=1}^N$ and let $Q_N = 1 - P_N$. Then

$$\|Q_N u(t)\|_1 \le \varepsilon, \ \forall t \in [0, M], \tag{5.11}$$

since $u \in \mathfrak{A}$. Due to (5.10),

$$\limsup_{n\to\infty} \|Q_N u_n(t)\|_1 \le \varepsilon, \ \forall t \in [0, M]. \tag{5.12}$$

Recall that the sequence $u_n \rightharpoonup u$ $*$-weakly in $L^\infty(0, M; \mathcal{H}^1)$ and $\partial_t u_n \rightharpoonup \partial_t u$ $*$-weakly in $L^\infty(0, M; \mathcal{H}^{-1})$ for any $M > 0$. Then, by Aubin–Lions–Simon theorem $u_n \to u$ strongly in $C([0, M]; \mathcal{H}(B_r^0))$ for every $r < \infty$. We set $r = R(N)$ and obtain

$$\sup_{t\in[0,M]} \|P_N u_n(t) - P_N u(t)\| \to 0 \text{ as } n \to \infty,$$

and, hence,

$$\sup_{t\in[0,M]} \|P_N u_n(t) - P_N u(t)\|_1 \to 0 \text{ as } n \to \infty, \tag{5.13}$$

because P_N is a finite dimensional projection and the norms of $\mathcal{H}$ and $\mathcal{H}^1$ on $\operatorname{span}\{w_j\}$ are equivalent. Finally, for an arbitrary $\varepsilon > 0$ we chose the corresponding projections P_N and Q_N for which we have (5.11) and (5.12). Then, thanks to (5.12) and (5.13), we find $n_1 \in \mathbb{N}$ such that

$$\sup_{t\in[0,M]} \|Q_N u_n(t)\|_1 \le \varepsilon \quad \text{and} \quad \sup_{t\in[0,M]} \|P_N u_n(t) - P_N u(t)\|_1 \le \varepsilon,$$

for all $n \ge n_1$, which together with (5.11) and (5.12) implies that

$$\begin{aligned}
&\sup_{t\in[0,M]} \|u_n(t) - u(t)\|_1 \le \\
&\sup_{t\in[0,M]} (\|P_N u_n(t) - P_N u(t)\|_1 + \|Q_N u_n(t) - Q_N u(t)\|_1) \le \\
&\sup_{t\in[0,M]} \|P_N u_n(t) - P_N u(t)\|_1 + \sup_{t\in[0,M]} \|Q_N u_n(t)\|_1 + \\
&\qquad + \sup_{t\in[0,M]} \|Q_N u(t)\|_1 \le 3\varepsilon,
\end{aligned}$$

for all $n \ge n_1$. Therefore, we obtain (5.9) and Theorem 5.3 is proved. □

Using the strong convergence to $\mathfrak{A}$ and Corollary 3.3, we obtain the following result.

Corollary 5.4. *For an arbitrary bounded trajectory set $B \subset \mathcal{K}_+$ the corresponding time derivative set*

$$\partial_t B = \{\partial_t u,\ u \in B\}$$

converges to $\partial_t \mathfrak{A}$ in the strong topology $C_{loc}(\mathbb{R}_+; \mathcal{H}^{-\varepsilon})$ for every $\varepsilon > 0$.

6. Additional regularity and $W^{1,p}$-attractors

In this section, we show that the trajectory attractor $\mathfrak{A}$ for the dissipative Euler system (2.1) constructed above is more regular when the external force g has some additional regularity.

We define the Banach space $\mathcal{E}_p$ for $2 \le p \le \infty$:

$$\mathcal{E}_p := \{u,\ \operatorname{div} u = 0,\ \|u\|_{\mathcal{E}_p} := \|u\|_{L^2} + \|\operatorname{curl} u\|_{L^p} < \infty\}. \tag{6.1}$$

We observe that for $\operatorname{div} u = 0$ and $1 < p < \infty$ we have the equivalence of the norms

$$\|\nabla u\|_{L^p} \sim \|\operatorname{curl} u\|_{L^p}.$$

In fact, since $|\operatorname{curl} u(x)| \le |\nabla u(x)|$ point-wise, it remains to show that $\|\nabla u\|_{L^p} \le c(p)\|\operatorname{curl} u\|_{L^p}$. The proof of this inequality (along with the information on the constant $c(p)$) is contained in Lemma 7.2.

Therefore, in view of the Gagliardo–Nirenberg inequality

$$\|u\|_{L^p(\mathbb{R}^2)} \le c_1(p)\|u\|^{\alpha}_{L^2(\mathbb{R}^2)}\|\nabla u\|^{1-\alpha}_{L^p(\mathbb{R}^2)}, \quad \alpha = 1/(p-1), \quad 2 \le p,$$

the $\mathcal{E}_p$-norm dominates the $W^{1,p}$-norm for $2 \le p < \infty$:

$$\|u\|_{L^p(\mathbb{R}^2)} + \|\nabla u\|_{L^p(\mathbb{R}^2)} \le c_2(p)\left(\|u\|_{L^2(\mathbb{R}^2)} + \|\operatorname{curl} u\|_{L^p(\mathbb{R}^2)}\right). \tag{6.2}$$

We need the following theorem, which similarly to Theorem 3.2 essentially uses the results in [15].

Theorem 6.1. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_p$ and let $\omega_0 \in L^2 \cap L^p$ for $p \geq 2$. Then the weak solution ω of the transport equation* (3.5) *with initial condition $\omega(0) = \omega_0$ and a given vector field $u(t,x) \in L^\infty(0,T;\mathcal{H}^1)$ satisfies the equation*

$$\frac{1}{p}\frac{d}{dt}\|\omega(t)\|_{L^p}^p + r\|\omega(t)\|_{L^p}^p = \int_{\mathbb{R}^2} \omega(t,x)\cdot|\omega(t,x)|^{p-2}\operatorname{curl} g(x)dx, \quad (6.3)$$

and in addition to $\omega \in C_b(\mathbb{R}_+;L^2)$ we have $\omega \in C_b(\mathbb{R}_+;L^p)$.

Proof. As before the assertion of this theorem follows from the proof of [15, Theorem II.2 and equation (26)]. The hypotheses of the cited theorem are satisfied since our vector field u belongs to $L^\infty(0,T;\mathcal{H}^1)$ and therefore $u \in L^\infty(0,T;W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2)) \subset L^1(0,T;W^{1,q}_{\mathrm{loc}}(\mathbb{R}^2))$ for all $q = p' = p/(p-1) \leq 2$, as required in [15]. □

Theorem 6.2. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_p$ for some $2 \leq p < \infty$. Then the trajectory attractor $\mathfrak{A}$ of system* (2.1) *belongs to $C_b(\mathbb{R}_+;\mathcal{E}_p)$, and for every $u \in \mathfrak{A}$ the following estimate holds*

$$\|u\|_{C_b(\mathbb{R}_+;\mathcal{E}_p)} \leq \frac{1}{r}\|g\|_{\mathcal{E}_p}. \quad (6.4)$$

Proof. Let $u \in \mathfrak{A}$. According to (4.7), there is a bounded complete trajectory $U \in \mathcal{K}$ such that $\Pi_+ U = u$. The corresponding vorticity function $\omega(t,x) = \operatorname{curl} U(t,x)$ satisfies the equation

$$\partial_t\omega + (U,\nabla)\omega + r\omega = \operatorname{curl} g(x), \ \ t \in \mathbb{R}.$$

We consider the following Cauchy problem for $\tau \geq 0$:

$$\partial_t W^\tau + (U,\nabla)W^\tau + rW^\tau = \operatorname{curl} g(x), \quad W^\tau|_{t=-\tau} = 0.$$

Since $\operatorname{curl} g(x) \in L^2 \cap L^p$, it follows from Theorem 6.1 that this linear transport problem has a unique weak solution $W^\tau \in C_b([-\tau,\infty);L^2) \cap C_b([-\tau,\infty);L^p)$ satisfying for $t \geq \tau$ the following equation:

$$\frac{1}{p}\frac{d}{dt}\|W^\tau(t)\|_{L^p}^p + r\|W^\tau(t)\|_{L^p}^p = \int_{\mathbb{R}^2} W^\tau(t,x)\cdot|W^\tau(t,x)|^{p-2}\operatorname{curl} g(x)dx.$$

Using Hölder's inequality we obtain

$$\frac{1}{p}\frac{d}{dt}\|W^\tau\|_{L^p}^p + r\|W^\tau\|_{L^p}^p \leq \|\operatorname{curl} g\|_{L^p}\|W^\tau\|_{L^p}^{p-1}.$$

Using Young's inequality with parameter $\delta > 0$:

$$ab \leq \delta^{-\frac{p}{q}}a^p/p + \delta b^q/q$$

with $a = \|\operatorname{curl} g(x)\|_{L^p}$, $b = \|W^\tau\|_{L^p}^{p-1}$, and $\delta = r$, we obtain

$$\frac{d}{dt}\|W^\tau\|_{L^p}^p + pr\|W^\tau\|_{L^p}^p \leq r^{-(p-1)}\|\operatorname{curl} g(x)\|_{L^p}^p + (p-1)r\|W^\tau\|_{L^p}^p,$$

that is

$$\frac{d}{dt}\|W^\tau\|_{L^p}^p + r\|W^\tau\|_{L^p}^p \le r^{-(p-1)}\|\operatorname{curl} g(x)\|_{L^p}^p.$$

Multiplying both sides by e^{rt} and integrating over $[-\tau, t]$ we obtain the inequality for $t \ge -\tau$

$$\|W^\tau(t)\|_{L^p}^p \le \|W^\tau(-\tau)\|_{L^p}^p e^{-r(t+\tau)} + (1 - e^{-r(t+\tau)})r^{-p}\|\operatorname{curl} g\|_{L^p}^p.$$

Since $W^\tau(-\tau) = 0$, this gives

$$\|W^\tau(t)\|_{L^p} \le r^{-1}\|\operatorname{curl} g\|_{L^p}, \quad t \ge -\tau. \tag{6.5}$$

Consider now the function $V^\tau(x,t) = \omega(t,x) - W^\tau(t,x)$, which is the solution of the following linear homogeneous transport problem

$$\partial_t V^\tau + (U, \nabla)V^\tau + rV^\tau = 0, \quad V^\tau|_{t=-\tau} = U(-\tau), \quad t \ge -\tau.$$

¿From [15] (see also Theorem 3.2), it follows that this problem has a unique solution satisfying the equation

$$\frac{1}{2}\frac{d}{dt}\|V^\tau(t)\|^2 + r\|V^\tau(t)\|^2 = 0,\ t \ge -\tau.$$

Since $U \in C_b(\mathbb{R}; \mathcal{H}^1)$, we have

$$\|V^\tau(t)\|^2 \le \|V^\tau(-\tau)\|^2 e^{-2r(t+\tau)} = \|U^\tau(-\tau)\|^2 e^{-2r(t+\tau)} \to 0, \tag{6.6}$$

as $\tau \to \infty$. We see from (6.5) and (6.6) that for a fixed t we have

$$\omega - V^\tau \to \omega' \quad \text{weakly in } L^p,$$
$$\omega - V^\tau \to \omega \quad \text{strongly in } L^2,$$

as $\tau \to \infty$, where $\omega' \in L^p$ and $\|\omega'\|_{L^p} \le r^{-1}\|\operatorname{curl} g\|_{L^p}$.

Using here a simple fact that if $a_n \to a$ weakly in L^2 and $a_n \to a'$ weakly in L^p, then $a = a' \in L^2 \cap L^p$, we obtain that $\omega(t)$ also belongs to L^p and for all $t \ge 0$ satisfies

$$\|\omega(t)\|_{L^p} \le r^{-1}\|\operatorname{curl} g\|_{L^p}. \tag{6.7}$$

To derive (6.4) from (6.7), we recall that every $u \in \mathfrak{A}$ satisfies as before the inequality

$$\|u(t)\|_{L^2} \le r^{-1}\|g\|_{L^2}, \quad t \ge 0,$$

(see (2.16)) and that the norm in $\mathcal{E}_p$ is given by (6.1). □

The next result extends Theorem 5.3 to the $W^{1,p}$-case (more precisely, to the $\mathcal{E}_p$-case).

Theorem 6.3. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_p$, $2 \le p < \infty$. Then the trajectory attractor $\mathfrak{A}$ is compact in $C_{\text{loc}}(\mathbb{R}_+; \mathcal{H}^1 \cap \mathcal{E}_p)$ and if a trajectory set $B \subset \mathcal{K}_+$ is bounded in $C_b(\mathbb{R}_+; \mathcal{H}^1 \cap \mathcal{E}_p)$, then for every $M > 0$*

$$\operatorname{dist}_{C([0,M];\mathcal{H}^1 \cap \mathcal{E}_p)}(T(h)B, \mathfrak{A}) \to 0 \quad \text{as } h \to +\infty. \tag{6.8}$$

The proof the theorem is similar to the proof of Theorem 5.3 and is based on the proof of the corresponding result for the global attractor.

Theorem 6.4. *The set*

$$\mathcal{A} := \mathfrak{A}(0) = \{u(0),\ u \in \mathfrak{A}\} \subset \mathcal{H}^1 \cap \mathcal{E}_p$$

is the global attractor in the norm of $\mathcal{H}^1 \cap \mathcal{E}_p$.

Proof. We use the notation introduced in the proof of Theorem 5.2, in which we have shown that $u_n(0) \to u(0)$ strongly in $\mathcal{H}^1$ as $n \to \infty$. Now we have, in addition, that

$$\operatorname{curl} u_n(0) \to \operatorname{curl} u(0) \tag{6.9}$$

weakly in L^p, and we have to show that the convergence (6.9) is, in fact, strong.

For the functions $U_n(t,x), t \ge -h_n$ from the proof of Theorem 5.2 we have the equality (6.3), more precisely, the functions $\omega_n = \operatorname{curl} U_n$ satisfy for $t \ge -h_n$

$$\frac{1}{p}\frac{d}{dt}\|\omega_n(t)\|^p_{L^p} + r\|\omega_n(t)\|^p_{L^p} = \int_{\mathbb{R}^2} \omega_n(t,x)\cdot|\omega_n(t,x)|^{p-2}\operatorname{curl} g(x)dx, \tag{6.10}$$

Multiplying by e^{2rpt} and integrating from $-h_n$ to 0 we obtain:

$$\|\omega_n(0)\|^p_{L^p} = \|\omega_n(h_n)\|^p_{L^p} e^{-2rph_n} + 2p\int_{-h_n}^0\int_{\mathbb{R}^2}\omega_n|\omega_n|^{p-2}\operatorname{curl} g\, e^{2rt}dxdt.$$

Since $\omega_n \to \omega$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, L^p)$, it follows that $\omega_n|\omega_n|^{p-2} \to \omega|\omega|^{p-2}$ $*$-weakly in $L^\infty_{\mathrm{loc}}(\mathbb{R}, L^q)$, $q = p/(p-1)$, and therefore we obtain

$$\lim_{n\to\infty}\|\omega_n(0)\|^p_{L^p} = 2p\int_{-\infty}^0\int_{\mathbb{R}^2}\omega|\omega|^{p-2}\operatorname{curl} g\, e^{2rt}dxdt.$$

Multiplying by e^{2rpt} and integrating from $-\infty$ to 0 the equation (6.10) for $\omega = \operatorname{curl} U$, where U is the complete trajectory $U(t)$, $t \in \mathbb{R}$ we obtain

$$\|\omega(0)\|^p_{L^p} = 2p\int_{-\infty}^0\int_{\mathbb{R}^2}\omega|\omega|^{p-2}\operatorname{curl} g\, e^{2rt}dxdt,$$

which gives that

$$\lim_{n\to\infty}\|\omega_n(0)\|_{L^p} = \|\omega(0)\|_{L^p}. \tag{6.11}$$

The weak convergence (6.9) and (6.11) give (are equivalent to) the strong convergence

$$\|\omega_n(0) - \omega(0)\|_{L^p} \to 0,$$

see, for instance, [25, Theorem II.37]. The proof is complete.

□

7. L^∞-bound and uniqueness

Lemma 7.1. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_\infty$. Then on the attractor $\mathfrak{A}$*

$$\|\omega(t)\|_{L^\infty} \le \frac{1}{r}\|\operatorname{curl} g\|_{L^\infty}. \tag{7.1}$$

Proof. We have shown in Theorem 6.2 that

$$\|\omega(t)\|_{L^p} \le \frac{1}{r}\|\operatorname{curl} g\|_{L^p}$$

for every $p < \infty$. It remains to let $p \to \infty$ in this estimate.

In fact, since $\operatorname{curl} g \in L^2 \cap L^\infty$, in view of the elementary inequality

$$\|f\|_{L^p} \le \|f\|_{L^2}^{2/p}\|f\|_{L^\infty}^{1-2/p}, \quad p \ge 2, \tag{7.2}$$

we find that

$$\|\omega(t)\|_{L^p} \le \frac{1}{r}\|\operatorname{curl} g\|_{L^p} \le \frac{1}{r}\|\operatorname{curl} g\|_{L^2}^{2/p}\|\operatorname{curl} g\|_{L^\infty}^{1-2/p},$$

and

$$\limsup_{p\to\infty}\|\omega(t)\|_{L^p} \le \frac{1}{r}\|\operatorname{curl} g\|_{L^\infty}. \tag{7.3}$$

This gives (7.1) because $\|f\|_{L^\infty} = \lim_{p\to\infty}\|f\|_{L^p}$ if it is known that $\|f\|_{L^p} \le C$ for all $p \ge p_0$. □

Lemma 7.2. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_\infty$. Then the trajectory attractor $\mathfrak{A}$ belongs to the space $C_b(\mathbb{R}_+; \mathcal{W}^{1,p})$ for any $2 \le p < \infty$ and for every $u \in \mathfrak{A}$*

$$\|\nabla u\|_{C_b(\mathbb{R}_+;L^p)} \le Cp\|\operatorname{curl} g\|_{L^\infty}, \tag{7.4}$$

where the constant $C = c'/r$ is independent of p as $p \to \infty$.

Proof. We recall that the vector function u is recovered from the vorticity ω by the Biot-Savart law

$$u = \nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi),$$

where

$$\psi = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln|x-y|\omega(y)dy$$

solves in $\mathbb{R}^2$ the equation

$$\Delta\psi = \omega. \tag{7.5}$$

Next, we clearly have

$$|\nabla u(x)| < \sum_{i,j=1}^2 \left|\frac{\partial^2\psi(x)}{\partial x_i \partial x_j}\right|.$$

Therefore (7.4) follows from the Yudovich elliptic estimate for the equation (7.5):

$$\left\| \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} \le c_n \frac{p^2}{p-1} \|\omega\|_{L^p(\mathbb{R}^n)}, \tag{7.6}$$

where $1 < p < \infty$ (see [33]–[36]) and (7.3). □

Finally, we prove that if $g \in \mathcal{H}^1 \cap \mathcal{E}_\infty$, then the solutions of the dissipative 2D Euler system lying on the trajectory attractor $\mathfrak{A}$ are unique. The damping does not play a role here and the proof closely follows the celebrated Yudovich uniqueness theorem for the 2D Euler system (see [33, 34]).

Theorem 7.3. *Let $g \in \mathcal{H}^1 \cap \mathcal{E}_\infty$, and let $u_1(\cdot), u_2(\cdot)$ be two weak solutions of the damped 2D Euler system (2.1). Let the initial vorticities $\omega_i(0) = \operatorname{curl} u_i(0)$ be bounded: $\omega_1(0), \omega_2(0) \in L^\infty$. Then, $u_1(0) = u_2(0)$ implies $u_1(t) = u_2(t)$ for all $t \in [0, M]$.*

Proof. Let $u(t) = u_1(t) - u_2(t)$. We set $E(t) = \|u_1(t) - u_2(t)\|^2$. Since $u \in C([0, M]; \mathcal{H})$, we have $E \in C([0, M])$. Suppose that $E(t) > 0$ for $t \in (0, \delta)$ for some δ (otherwise, the theorem is proved). Then $u(t)$ solves the equation:

$$\partial_t u + (u_1, \nabla)u - (u, \nabla)u_2 + ru + \nabla(p_1 - p_2) = 0.$$

Multiplying this equation by $u(t)$ and integrating over $\mathbb{R}^2$, we have

$$\frac{d}{dt}E(t) + 2rE(t) = \int_{\mathbb{R}^2} ((u, \nabla)u_2, u)dx \le \|\nabla u_2(t)\|_{L^p} \|u\|^2_{L^{2q}},$$

where $p \ge 2$ is arbitrary and $1/p + 1/q = 1$. In view of (7.2),

$$\|u\|_{L^{2q}} \le \|u\|_{L^2}^{1-1/p} \|u\|_{L^\infty}^{1/p},$$

where $u(t)$ is bounded in L^∞, since $W^{1,p_0}(\mathbb{R}^2) \subset C_b(\mathbb{R}^2)$ for a $p_0 > 2$. Dropping the term $2rE(t)$ on the left-hand side and using the key estimate (7.4), we obtain

$$\frac{d}{dt}E(t) \le C_2 p [E(t)]^{1-1/p}, \tag{7.7}$$

where $C_2 = C_2(\|g\|_1, \|\operatorname{curl} g\|_{L^\infty})$ is independent of p as $p \to \infty$. We now pick the exponent p in (7.7) in an optimal way, namely, we set

$$p = \ln \frac{K}{E(t)},$$

where $K > 1$ is sufficiently large to guarantee that $p > 2$ for all $t \in (0, \delta)$ (recall that $E(t)$ is a bounded function). We derive from (7.7) the inequality

$$\frac{d}{dt}E(t) \le C_2 E(t) \ln \frac{K}{E(t)} \left(E(t)^{-1/\ln \frac{K}{E(t)}} \right) \le C_3 E(t) \ln \frac{K}{E(t)},$$

where we used the inequality $E^{-1/\ln \frac{K}{E}} \le e$ which (since $\ln(K/E) > 0$) is equivalent to the assumed inequality $K \ge 1$. We finally obtain

$$\frac{d}{dt} \ln \frac{E(t)}{K} \le -C_3 \ln \frac{E(t)}{K}.$$

Integrating this differential inequality over $[\varepsilon, t]$ we arrive at

$$\ln \frac{E(t)}{K} \le \ln \frac{E(\varepsilon)}{K} e^{-C_3(t-\varepsilon)},$$

which gives

$$E(t) \le K \left[\frac{E(\varepsilon)}{K} \right]^{e^{-C_3(t-\varepsilon)}}$$

for all sufficiently small $\varepsilon > 0$. Passing to the limit $\varepsilon \to 0+$ ($E(\varepsilon)$ is continuous in ε and $E(0) = 0$), we obtain $E(t) \equiv 0$. The theorem is proved. $\square$

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