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TRAJECTORY ATTRACTORS FOR NON-AUTONOMOUS DISSIPATIVE 2D EULER EQUATIONS

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Abstract. We construct the trajectory attractor $\mathfrak{A}_\Sigma$ for the non-autonomous dissipative 2d Euler systems with periodic boundary conditions that contain time dependent dissipation terms $-r(t)u$ such that $0 < \alpha \leqslant r(t) \leqslant \beta$, for $t \geqslant 0$. External forces $g(x,t), x \in \mathbb{T}^2, t \geqslant 0$, also depend on time. The corresponding non-autonomous dissipative 2d Navier–Stokes systems with the same terms $-r(t)u$ and $g(x,t)$ and with viscosity $\nu > 0$ also have the trajectory attractor $\mathfrak{A}_\Sigma^\nu$. Such systems model large-scale geophysical processes in atmosphere and ocean. We prove that $\mathfrak{A}_\Sigma^\nu \to \mathfrak{A}_\Sigma$ as viscosity $\nu \to 0+$ in the corresponding metric space. Moreover, we establish the existence of the minimal limit $\mathfrak{A}_\Sigma^{\min} \subseteq \mathfrak{A}_\Sigma$ of the trajectory attractors $\mathfrak{A}_\Sigma^\nu$ as $\nu \to 0+$. Every set $\mathfrak{A}_\Sigma^\nu$ is connected. We prove that $\mathfrak{A}_\Sigma^{\min}$ is a connected invariant subset of $\mathfrak{A}_\Sigma$. The problem of the connectedness of the trajectory attractor $\mathfrak{A}_\Sigma$ itself remains open.

Introduction. In the present paper, we construct the trajectory attractor $\mathfrak{A}_\Sigma$ for the non-autonomous dissipative 2d Euler systems

$$\partial_t u + (u, \nabla_x)u + r(t)u + \nabla_x p = g(x,t), \ \operatorname{div} u = 0,$$

that contain time dependent dissipative terms $-r(t)u$, such that $0 < \alpha \leqslant r(t) \leqslant \beta$, for $t \geqslant 0$. External forces $g(x,t)$ also depend on time and $g(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^1)$. The equations are equipped with periodic boundary conditions. In the system, $u = (u^1(x,t), u^2(x,t))$ denotes the unknown velocity vector field and $p = p(x,t)$ is the unknown pressure function. Such systems describe in geophysical hydrodynamics two-dimensional fluids moving on a rough surfaces and model large-scale processes in atmosphere and ocean. The term $-r(t)u$ parameterizes the main dissipation that occurs in the planetary boundary layer (see, for example, the book [21, Ch.4]).

The function $\sigma(t) = (r(t), g(x,t)), t \geqslant 0$, is called the time symbol of the system. We assume that $\sigma(t)$ is a translation compact function in the corresponding topological space (see Sec. 1). In fact, $\mathfrak{A}_\Sigma$ is the trajectory attractor of the entire family of non-autonomous dissipative 2d Euler systems with time symbols $\sigma \in \Sigma$, where $\Sigma = \mathcal{H}_+(\sigma_0)$ is the hull of a given translation compact symbol

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$\sigma_0(t) = (r_0(t), g_0(x,t))$ that satisfies inequalities $\alpha \leqslant r_0(t) \leqslant \beta$ for $t \geqslant 0$ and $g_0(\cdot) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^1)$ (see Sec. 1).

We also construct the trajectory attractor $\mathfrak{A}_\Sigma^\nu$ for the 2d Navier–Stokes systems with the same dissipation terms $-r(t)u$ and the external forces $g(x,t)$ and with viscosity coefficient $\nu > 0$:

$$\partial_t u + (u, \nabla_x)u + r(t)u + \nabla_x p = \nu\Delta u + g(x,t),\ \operatorname{div} u = 0.$$

In the mentioned above geophysical models, the viscosity term $\nu\Delta u$ is responsible for small-scale dissipation (note that $0 < \nu \ll \alpha$ in the physically relevant cases).

We prove that the Hausdorff deviation of the set $\mathfrak{A}_\Sigma^\nu$ form the set $\mathfrak{A}_\Sigma$ (measured in the corresponding metric $\rho(\cdot,\cdot)$) approaches zero as viscosity ν vanishes:

$$\operatorname{dist}_\rho(\mathfrak{A}_\Sigma^\nu, \mathfrak{A}_\Sigma) \to 0 \text{ as } \nu \to 0+\,.$$

We also study some important properties of the trajectory attractors $\mathfrak{A}_\Sigma$ and $\mathfrak{A}_\Sigma^\nu$ specifying below.

We note that autonomous 2d Euler and Navier–Stokes systems with dissipation was considered in a number of papers (see, for example, [4], [22],[15] for dissipative Euler systems and [16]–[18] for dissipative Navier–Stokes systems).

The methods of trajectory attractors for evolution partial differential equations was developed in [6]–[10] (see also the review [25]). This approach is highly fruitful in the study of the long time behaviour of solutions to evolution equations for which the uniqueness theorem of the corresponding initial-value problem is not proved yet (e.g., for the 3D Navier–Stokes system) or does not hold.

The trajectory attractors for autonomous 2d Euler system with dissipation (that is, when $r \equiv \alpha$ and $g \equiv g(x) \in H^1$ are independent of time) and for the corresponding autonomous dissipative 2d Navier-Stokes systems with vanishing viscosity have been studied in [11].

The paper is organized as follows. In Sec. 1, we study the non-autonomous dissipative 2d Euler system with periodic boundary conditions. Using the Galerkin method, we prove that the initial-value problem for this system has at least one weak distribution solution $u(x,t)$ such that $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ and $\partial_t u(x,t) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^{-1})$. Here H^1 denotes the space of periodic solenoidal vector fields with a finite Sobolev H^1-norm and the space $H^{-1} = \left(H^1\right)^*$ is dual for H^1. Moreover, the constructed solution $u(x,t)$ satisfies the corresponding energy inequality (see the next paragraph) that is important for the subsequent study. Note that the uniqueness theorem for weak solutions to the 2d Euler system in the class $L_\infty(\mathbb{R}_+; H^1)$ is not proved.

In Sec. 2, we construct the trajectory attractor for the non-autonomous dissipative 2d Euler systems with symbols $(r,g) = \sigma \in \Sigma = \mathcal{H}_+(\sigma_0)$. We define spaces $\mathcal{F}_+^{\mathrm{b}}$ and $\mathcal{F}_+^{\mathrm{loc}}$ ($\mathcal{F}_+^{\mathrm{b}} \subset \mathcal{F}_+^{\mathrm{loc}}$) that contain the weak solutions $u(x,t)$ constructed in Sec. 1. Then, we define the space of trajectories (solutions) $\mathcal{K}_\Sigma^+(N) \subset \mathcal{F}_+^{\mathrm{b}}$ depending on $N > 0$. The set $\mathcal{K}_\Sigma^+(N)$ consists of the weak solutions $u(x,t)$ of the system that satisfy the following energy inequality:

$$\|u(t)\|^2 \leqslant Ne^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)},\ \forall t \in \mathbb{R}_+,$$

where $\gamma = \gamma(\alpha) > 0$ and $\|\cdot\| := \|\cdot\|_{H^1}$ denotes the norm in H^1 and g_0 is the original external force in the dissipative Euler system. The space $\mathcal{F}_+^{\mathrm{loc}}$ is equipped with the local weak topology Θ_+^{loc} generated by the weak convergence of sequences $\{v_n(x,t)\} \subset \mathcal{F}_+^{\mathrm{loc}}$. We prove that the trajectory space $\mathcal{K}_\Sigma^+(N)$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and closed in the topology $\Theta_+^{\rm loc}$. These assertions are very important for the subsequent study. Consider the translation semigroup $\{T(h), h \geqslant 0\}$ acting on a trajectory (solution) $u(x,t)$ by the formula: $T(h)u(x,t) = u(x,h+t)$. It follows from the definition of the trajectory space that $\mathcal{K}_\Sigma^+(N)$ is invariant with respect to $\{T(h)\}$: $T(h)\mathcal{K}_\Sigma^+(N) \subseteq \mathcal{K}_\Sigma^+(N)$ for all $h \geqslant 0$. Using these facts and applying the theory of trajectory attractors, we prove that the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\Sigma^+(N)$ has the global attractor $\mathfrak{A}_\Sigma(N)$ which we call the *trajectory attractor* of the system. Recall that $T(h)\mathfrak{A}_\Sigma(N) = \mathfrak{A}_\Sigma(N)$ for all $h \geqslant 0$. We then prove that the set $\mathfrak{A}_\Sigma(N)$ is independent of N : $\mathfrak{A}_\Sigma(N) = \mathfrak{A}_\Sigma(0) =: \mathfrak{A}_\Sigma$ for all $N \geqslant 0$.

In Sec. 3 and 4, we study the non-autonomous dissipative 2d Navier–Stokes systems with periodic boundary conditions and with viscosity $\nu > 0$ that contains the dissipative terms $-r(t)u$ and the external forces $g(x,t)$ such that $(r,g) = \sigma \in \Sigma$. The corresponding initial-value problem is well-posed and we construct the trajectory attractor $\mathfrak{A}_\Sigma^\nu$ for these equations. We prove that $\mathrm{dist}_\rho(\mathfrak{A}_\Sigma^\nu, \mathfrak{A}_\Sigma) \to 0$ as viscosity $\nu \to 0+$. The trajectory attractor $\mathfrak{A}_\Sigma^\nu$ is a connected set in the topology $\Theta_+^{\rm loc}$ for all $\nu > 0$.

In Sec. 5, we prove the existence of the minimal limit $\mathfrak{A}_\Sigma^{\min}$ of the trajectory attractors $\mathfrak{A}_\Sigma^\nu$ as $\nu \to 0+$, that is, $\mathfrak{A}_\Sigma^{\min} \subseteq \mathfrak{A}_\Sigma$, $\mathfrak{A}_\Sigma^{\min}$ is closed in $\Theta_+^{\rm loc}$, $\mathrm{dist}_\rho(\mathfrak{A}_\Sigma^\nu, \mathfrak{A}_\Sigma^{\min}) \to 0$ as $\nu \to 0+$, and $\mathfrak{A}_\Sigma^{\min}$ is the minimal set that satisfies these properties. We prove that the set $\mathfrak{A}_\Sigma^{\min}$ is connected in the topology $\Theta_+^{\rm loc}$ and strictly invariant with respect to the translation semigroup. The question of whether or not the trajectory attractor $\mathfrak{A}_\Sigma$ by itself is a connected subset of $\Theta_+^{\rm loc}$ remains an open problem.

1. Non-autonomus 2d Euler systems with dissipation.

We consider the following equations:

$$\begin{gathered}\partial_t u + B(u,u) + r(t)u = g(x,t), \quad (\nabla, u) := \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0, \\ u = (u^1(x,t), u^2(x,t)), \quad x = (x_1, x_2) \in \mathbb{T}^2, \quad t \geqslant 0,\end{gathered} \tag{1.1}$$

where $\mathbb{T}^2 := [\mathbb{R} \mod 2\pi]^2$ is the two-dimensional torus, $B(u,v) := P(u^1\partial_{x_1} v + u^2 \partial_{x_2} v)$ and P is the orthogonal Leray operator, which projects the space $[L_2(\mathbb{T}^2)]^2$ onto the subspace

$$H := [\{v(x) \in [C^\infty(\mathbb{T}^2)]^2 \mid (\nabla, u) = 0]_{[L_2(\mathbb{T}^2)]^2}.$$

($[X]_E$ denotes the closure of the set X in the topological space E). We define similarly the space

$$H^1 := [\{v(x) \in [C^\infty(\mathbb{T}^2)]^2 \mid (\nabla, u) = 0]_{[H^1(\mathbb{T}^2)]^2},$$

which is embedded into the standard scale of Sobolev spaces $H^s,\ s \in \mathbb{R}$, where $H^0 = H,\ H^{-s} = (H^s)^*$ is the dual space of $H^s, s \geqslant 0$. The norms in H and H^1 are denoted by $|\cdot|$ and $\|\cdot\|$, respectively. Note that $H^1 \Subset H$.

Recall that for u satisfying $(\nabla, u) = 0$ we formally have

$$B(u,v) = P(\partial_{x_1}(u^1 v) + \partial_{x_2}(u^2 v)). \tag{1.2}$$

In particular by the Gagliardo-Nirenberg inequality, $B(u,v) \in H$ for $u, v \in H^2$ (see, for example, [23]). Moreover, the trilinear form $b(u,v,w) := (B(u,v)w)$ is continuous on $H^1 \times H^1 \times H^1$ and therefore $B(u,v) \in H^{-1}$ for $u,v \in H^1$.

The unknown pressure function $p(x,t)$ is eliminated form the first equation of the system (1.1) by applying the operator P to both sides.

In (1.1), the time-dependent dissipation coefficient $r(t)$ satisfies the inequalities

$$0 < \alpha \leqslant r(t) \leqslant \beta, \quad t \geqslant 0, \tag{1.3}$$

and, for simplicity, we assume that $r(t) \in C^{\mathrm{b}}(\mathbb{R}_+)$ where $\mathbb{R}_+ := \{0 \leqslant t < +\infty\}$. We also assume that the external force

$$g(x,t) = (g^1(x,t), g^2(x,t)) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^1), \tag{1.4}$$

that is,

$$\|g(\cdot,\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} := \sup_{t\geqslant 0} \int_t^{t+1} \|g(\cdot,s)\|^2 ds < +\infty. \tag{1.5}$$

The autonomous dissipative 2D Euler system (1.1), with $r(t) \equiv \alpha$ and $g(x,t) \equiv g_0(x)$ for $t \geqslant 0$, was considered in a number of papers (see, for example, [4], [22], [15]). Equations (1.1) describe large-scale geophysical processes in atmosphere and ocean when the main dissipation occurs in the planetary boundary layer and is parameterized by the term $-ru$ (see, for example, [21, Ch.4]).

We now consider the time dependent terms $r(t)$ and $g(x,t)$ in greater details. We denote the pair of function

$$\sigma(s) := (r(s), g(\cdot,s)), \quad s \geqslant 0,$$

and we call this function the *time symbol* (or *symbol*) of the system (1.1). Here, it is convenient to use the letter s as the time variable instead t.

We set $\Xi_+ := C^{\mathrm{loc}}(\mathbb{R}_+) \times L_2^{\mathrm{loc}}(\mathbb{R}_+; H^1)$. According to the assumptions made, the symbol $\sigma(\cdot) \in \Xi_+$. We endow the space Ξ_+ with the local convergence topology. By the definition, a sequence $\{\xi_n(\cdot)\} \subset \Xi_+$ converges to an element $\tilde{\xi}(\cdot) \in \Xi_+$ iff for every $\ell > 0$

$$\max_{s\in[0,\ell]} |r_n(s) - \tilde{r}(s)| \to 0 \quad \text{and} \quad \int_0^\ell \|g_n(\cdot,s) - \tilde{g}(\cdot,s)\|^2 ds \to 0 \text{ as } n \to \infty,$$

where $(r_n(s), g_n(\cdot,s)) = \xi_n(s)$ and $(\tilde{r}(s), \tilde{g}(\cdot,s)) = \tilde{\xi}(s)$. It is clear, that Ξ_+ is a complete metric space (see, for example, [10]).

We consider the translation operators $T(h), h \geqslant 0$, acting in the space Ξ_+ by the formula $T(h)\xi(s) := \xi(h+s), s \geqslant 0$. It is clear that the set of operators $\{T(h), h \geqslant 0\}$ forms a semigroup that maps Ξ_+ to itself.

For an element $\xi(\cdot) \in \Xi_+$, we define the *hull* $\mathcal{H}_+(\xi)$ of ξ in the space Ξ_+ by the following formula:

$$\mathcal{H}_+(\xi) := \left[\bigcup_{h\geqslant 0} T(h)\xi\right]_{\Xi_+} = [\{\xi(h+s) \mid h \geqslant 0\}]_{\Xi_+}.$$

Thus, $\tilde{\xi} \in \mathcal{H}_+(\xi)$ iff there exists a real sequence $\{h_n\} \subset \mathbb{R}_+$ such that $\xi(h_n + s) \to \tilde{\xi}(s)$ in the above topology as $n \to \infty$.

Definition 1.1. A function $\xi(\cdot) \in \Xi_+$ is said to be translation compact in Ξ_+ if its hull $\mathcal{H}_+(\xi)$ is compact in Ξ_+.

We now come back to the system (1.1). Let we be given an original time symbol $\sigma_0(s) := (r_0(s), g_0(\cdot,s)) \in \Xi_+$ that satisfies inequalities (1.3) and (1.5). Moreover, we assume that the function $\sigma_0(s)$ is translation compact in Ξ_+. (For translation compactness criteria see [10].) As examples of translation compact functions in Ξ_+, one can consider periodic, quasiperiodic or almost periodic functions $\sigma_0(s), s \geqslant 0$, with values in the space $\mathbb{R}_+ \times H^1$ (for more examples, see [10]).

We shall use for $\mathcal{H}_+(\sigma_0)$ a short notation $\Sigma = \mathcal{H}_+(\sigma_0)$. We note that every time symbol $\sigma(s) = (r(s), g(\cdot,s)) \in \Sigma$ has the following property. The function $\sigma(s)$ is also translation compact in Ξ_+ and $\mathcal{H}_+(\sigma) \subseteq \Sigma$. Moreover, the function $r(s)$ satisfies inequalities (1.3) and the function $g(s) := g(\cdot, s)$ satisfies the inequality

$$\|g(\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} \leqslant \|g_0(\cdot)\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}.$$

In what follows, we formulate several propositions concerning weak solutions of system (1.1) with a fixed time symbol that satisfies (1.3) and (1.5). It is clear that all these proposition hold for every symbol $\sigma \in \Sigma$.

We consider the initial condition for (1.1) at $t = 0$:

$$u|_{t=0} = u_0, \quad u_0 \in H_1. \tag{1.6}$$

Recall that a function $u = u(x,t), x \in \mathbb{T}^2, t \geqslant 0$, is said to be a weak distribution solution of (1.1) if $u \in L_\infty(0, M; H^1)$ for every $M > 0$ and $u = u(x,t)$ satisfies the system in the distribution sense of the space $\mathcal{D}(0, M; H^{-1})$.

We prove the existence of a weak distribution solution of the problem (1.1), (1.6) by the Galerkin method. Consider as a basis in H an orthogonal (in H) system of eigenfunctions $\{e_j(x) = (e_j^1(x), e_j^2(x)) \in H^2,\ j = 0, 1, 2, \dots\}$ of the Stokes operator

$$-P\Delta e_j(x) = \lambda_j e_j(x), \quad (\nabla, e_j(x)) = 0, \quad x \in \mathbb{T}^2, \quad j = 0, 1, 2, \dots$$

We point out that $P\Delta \equiv \Delta$ in the space H^2 with periodic boundary conditions (see, for example, [24]). Recall that $e_0(x) \equiv e_0$ is a constant vector and $0 = \lambda_0 < \lambda_1 < \lambda_2 \leqslant \cdots \lambda_j \to +\infty$ as $j \to \infty$. We seek a Galerkin approximation in the form

$$u_n(x,t) = \sum_{j=0}^{n} c_{jn}(t) e_j(x), \quad n = 1, \dots,$$

where the $c_{jn}(t)$ are unknown scalar functions, and the function $u_n(x,t)$ satisfies the equation

$$\partial_t u_n + \Pi_n B(u_n, u_n) + r(t) u_n = \Pi_n g(\cdot, t). \tag{1.7}$$

Here, Π_n denotes the orthogonal projector from H onto the finite-dimensional subspace $[e_0(x), \dots, e_n(x)]$. This equation is equivalent to the corresponding system of ODE with respect to the functions $c_{jn}(t),\ j = 0, 1, 2, \dots, n$. The initial condition

$$u_n|_{t=0} = u_n(0) = \Pi_n u_0 \tag{1.8}$$

is given at $t = 0$, where u_0 is the same as in (1.6).

Clearly, the problem (1.7), (1.8) has a unique solution $u_n(x,t) \in C^1([0, \tau_n); H^1)$ for some $\tau_n > 0$. We take the inner product in H of each side of equation (1.7) with $u_n(t) := u_n(\cdot, t)$ and use the classical identity

$$(B(u,u), u) = 0, \quad \forall\ u \in H^1 \tag{1.9}$$

(see [24], [13]). After elementary transformations, we get that

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 + r(t)|u_n(t)|^2 = (g(t), u_n(t)), \quad \forall t \in [0, \tau_n). \tag{1.10}$$

Here, recall, $|u_n(t)|^2 = \|u_n(t)\|_H^2$.

Taking the inner product in H of equation (1.7) with $-P\Delta u_n(t) = -\Delta u_n(t)$ and using the standard identities $-(u_n, \Delta u_n) = |\nabla u_n|^2$ and $-(g(t), \Delta u_n) = (\nabla g(t), \nabla u_n)$, we obtain the equality

$$\frac{1}{2}\frac{d}{dt}|\nabla u_n|^2 - (B(u_n,u_n), \Delta u_n) + r(t)|\nabla u_n|^2 = (\nabla g(t), \nabla u_n), \quad \forall t \in [0, \tau_n). \tag{1.11}$$

As is well known, in the case of periodic boundary conditions ($x \in \mathbb{T}^2$), we have the identity

$$(B(u,u), \Delta u) = 0, \quad \forall u \in H^2 \tag{1.12}$$

(see [24, Ch. VI, Lemma 3.1] and [15]), which plays a key role in our future reasonings. Equations (1.11) and (1.12) imply the equality

$$\frac{1}{2}\frac{d}{dt}\left|\nabla u_n(t)\right|^2 + r(t)\left|\nabla u_n(t)\right|^2 = (\nabla g(t), \nabla u_n(t)). \tag{1.13}$$

We now sum equalities (1.10) and (1.13)

$$\frac{1}{2}\frac{d}{dt}\left\|u_n(t)\right\|^2 + r(t)\left\|u_n(t)\right\|^2 = \left\langle g(t), u_n(t)\right\rangle_{H^1}, \quad \forall t \in [0, \tau_n), \tag{1.14}$$

where $\left\|u_n\right\|^2 = \left|u_n\right|^2 + \left|\nabla u_n\right|^2$ and $\left\langle g(t), u_n\right\rangle_{H^1} = (g(t), u_n) + (\nabla g(t), \nabla u_n)$.

Since $r(t) \geqslant \alpha > 0$, the differential equality (1.14) implies that

$$\frac{1}{2}\frac{d}{dt}\left\|u_n(t)\right\|^2 + \alpha\left\|u_n(t)\right\|^2 \leqslant \frac{1}{2\alpha}\left\|g(t)\right\|^2 + \frac{\alpha}{2}\left\|u_n(t)\right\|^2 \implies$$

$$\frac{d}{dt}\left\|u_n(t)\right\|^2 + \alpha\left\|u_n(t)\right\|^2 \leqslant \frac{1}{\alpha}\left\|g(t)\right\|^2. \tag{1.15}$$

Multiplying both sides of (1.15) by $e^{\alpha t}$ and integrating in t we obtain

$$e^{\alpha t}\left\|u_n(t)\right\|^2 - \left\|u_n(0)\right\|^2 \leqslant \frac{1}{\alpha}\int_0^t e^{\alpha t}\left\|g(s)\right\|^2 ds \implies$$

$$\left\|u_n(t)\right\|^2 \leqslant \left\|u_n(0)\right\|^2 e^{-\alpha t} + \frac{1}{\alpha}\int_0^t e^{-\alpha(t-s)}\left\|g(s)\right\|^2 ds. \tag{1.16}$$

Estimating the last integral, we have

$$\int_0^t e^{-\alpha(t-s)}\left\|g(s)\right\|^2 ds \leqslant \int_{t-1}^t e^{-\alpha(t-s)}\left\|g(s)\right\|^2 ds + \int_{t-2}^{t-1} e^{-\alpha(t-s)}\left\|g(s)\right\|^2 ds + \cdots$$

$$\leqslant \int_{t-1}^t \left\|g(s)\right\|^2 ds + e^{-\alpha}\int_{t-2}^{t-1}\left\|g(s)\right\|^2 ds + \cdots$$

$$\leqslant (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} = (1 - e^{-\alpha})^{-1}\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}. \tag{1.17}$$

Using (1.16) and (1.17), we arrive at the main estimate

$$\left\|u_n(t)\right\|^2 \leqslant \left\|u_n(0)\right\|^2 e^{-\alpha t} + \gamma^{-1}(\alpha)\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in [0, \tau_n), \tag{1.18}$$

where $\gamma(\alpha) = \alpha(1 - e^{-\alpha})$. Notice that $\gamma(\alpha) > \alpha^2/(1+\alpha)$.

It follows from the inequality (1.18) that the solution $u_n(t)$ of the problem (1.7), (1.8) can be extended to the entire half-line $\mathbb{R}_+$ (that is, $\tau_n = +\infty$ for every n), $u_n(t) \in C^1_{\mathrm{b}}(\mathbb{R}_+; H^1)$ and

$$\left\|u_n(t)\right\|^2 \leqslant \left\|u_n(0)\right\|^2 + \gamma^{-1}\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \geqslant 0,$$

for all $n \in \mathbb{N}$. In particular,

$$\|u_n\|^2_{L_\infty(\mathbb{R}_+;H^1)} := \operatorname{ess\,sup}\left\{\left\|u_n(t)\right\|^2 \;\middle|\; t \geqslant 0\right\} \leqslant \left\|u_0\right\|^2 + \gamma^{-1}\left\|g\right\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}. \tag{1.19}$$

Since $u_0 \in H^1$, the initial condition of the problem (1.7), (1.8) satisfies

$$u_n(0) = \Pi_n u_0 \to u_0 \ (n \to \infty) \quad \text{strongly in } H^1. \tag{1.20}$$

Inequality (1.19) implies the existence of a subsequence $\{n'\} \subset \{n\}$ such that

$$u_{n'}(\cdot, t) \rightharpoonup u(\cdot, t) \ (n' \to \infty) \quad \ast\text{-weakly in } L^{\mathrm{loc}}_\infty(\mathbb{R}_+; H^1) \tag{1.21}$$

for some function $u(\cdot,t) \in L_\infty(\mathbb{R}_+; H^1)$.

We assert that $u(x,t)$ is a weak solution of the problem (1.1), (1.6). Indeed, from equations (1.7), the assumption (1.3), and the estimate (1.19), we find that

$$\begin{aligned}
&\|\partial_t u_n(t)\|_{H^{-1}} \leqslant \|B(u_n(t), u_n(t))\|_{H^{-1}} + \beta \|u_n(t)\|_{H^{-1}} + \|g(t)\|_{H^{-1}} \\
&\leqslant C\left(\|u_n(t)\|^2_{L_4(\mathbb{T}^2)^2} + \|u_n(t)\|_{H^1} + \|g(t)\|_{H^1}\right) \leqslant C_1\left(\|u_n(t)\|^2 + \|g(t)\| + 1\right) \\
&\leqslant C_1\left(\|u_0\|^2 + \gamma^{-1}\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g(t)\| + 1\right), \quad t \geqslant 0. \qquad (1.22)
\end{aligned}$$

Here, we have used the inequalities

$$\|B(u_n, u_n)\|_{H^{-1}} \leqslant \|u_n\|^2_{L_4(\mathbb{T}^2)^2}, \quad \|u\|^2_{L_4(\mathbb{T}^2)^2} \leqslant c\|u\|^2, \quad \forall u \in H^1, \qquad (1.23)$$

that follow from the identity (1.2) and the embedding $H^1 \subset L_4(\mathbb{T}^2)^2$, respectively. Therefore,

$$\int_t^{t+1} \|\partial_t u_n(s)\|^2_{H^{-1}}\, ds \leqslant C_2(\alpha)\left(\|u_0\|^4 + \|g\|^4_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right), \quad t \geqslant 0. \qquad (1.24)$$

Relations (1.21) and (1.24) imply that

$$\partial_t u_{n'}(\cdot,t) \rightharpoonup \partial_t u(\cdot,t)\ (n' \to \infty) \quad \text{weakly in } L_2^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}). \qquad (1.25)$$

Using now (1.21), (1.25), and the Aubin compactness theorem (see [2, 14, 20]), we obtain that

$$u_{n'}(\cdot,t) \rightharpoonup u(\cdot,t)\ (n' \to \infty) \quad \text{strongly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H). \qquad (1.26)$$

It follows from (1.26) (by using the routine reasoning similar to [23, 20, 19]) that

$$B(u_{n'}, u_{n'}) \rightharpoonup B(u,u)\ (n' \to \infty) \quad *\text{-weakly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+; H^{-1}). \qquad (1.27)$$

Now, with regard to relations (1.21), (1.25), and (1.27), we can pass to the limit as $n' \to \infty$ in equation (1.7) in the space of distributions $D'(\mathbb{R}_+; H^{-1})$ (see [20]). We obtain that the function $u(x,t)$ is a weak distribution solution of equation (1.1) in the space $D'(\mathbb{R}_+; H^{-1})$ and $\partial_t u(\cdot,t)$ belongs to $L_2^{\mathrm{b}}(\mathbb{R}_+; H^{-1})$ while it follows from (1.20) that $u(x,t)$ satisfies the initial condition (1.8). Finally, from the inequality (1.18) we conclude that the constructed weak solution $u(x,t)$ satisfies the estimate

$$\|u(t)\|^2 \leqslant \|u(0)\|^2 e^{-\alpha t} + \gamma^{-1}(\alpha)\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in \mathbb{R}_+. \qquad (1.28)$$

We have proved the following assertion.

Proposition 1.1. *For any $u_0(x) \in H^1$, the problem (1.1), (1.6) has a weak solution $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ such that $\partial_t u(x,t) \in L_2^{\mathrm{b}}(\mathbb{R}_+; H^{-1})$ and the inequality (1.28) holds.*

Remark 1.1. Any weak solution $u(x,t) \in L_\infty(\mathbb{R}_+; H^1)$ of equation (1.1) satisfies the energy identity

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + r(t)|u(t)|^2 = (g, u(t)), \quad \forall t \geqslant 0,$$

where the function $|u(t)|^2$ is absolutely continuous (cf. (1.10)). However, an analogous identity (see (1.13)) for the function $|\nabla u(t)|^2$, $t \geqslant 0$, does not hold since a weak solution is no longer smooth enough.

Remark 1.2. The uniqueness question for the problem (1.1), (1.6) in the weak space $L_\infty(\mathbb{R}_+;H^1)$ remains open. The analogous obstacle is known for the classical conservative 2d Euler system (1.1) with $r=0$, for which existence and uniqueness theorems have been proved in the class of functions $u(x,t)$ with vortex $\nabla\times u := \partial_{x_1}u^2-\partial_{x_2}u^1 \in L_\infty(\mathbb{R}_+;L_\infty(\mathbb{T}^2))$ under the condition that $\nabla\times u_0$ and $\nabla\times g$ belong to $L_\infty(\mathbb{T}^2)$ (see [26], [27], [5]). These results can be extended to equations (1.1). However, it will be shown in the next section that no uniqueness theorem is required in the study of trajectory attractors for non-autonomous 2d Euler system with dissipation (1.1).

2. **Trajectory attractor of non-autonomous 2d Euler equations with dissipation .** We introduce the spaces $\mathcal{F}_+^{\mathrm{b}}$ and $\mathcal{F}_+^{\mathrm{loc}}$:

$$\mathcal{F}_+^{\mathrm{b}} := \left\{v(x,s),\ s\in\mathbb{R}_+ \mid v\in L_\infty(\mathbb{R}_+;H^1),\ \partial_t v\in L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})\right\},$$
$$\mathcal{F}_+^{\mathrm{loc}} := \left\{v(x,s),\ s\in\mathbb{R}_+ \mid v\in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^1),\ \partial_t v\in L_2^{\mathrm{loc}}(\mathbb{R}_+;H^{-1})\right\}.$$

Recall that $z(\cdot)\in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;E)$, where E is a Banach space, if and only if $z(\cdot)\in L_\infty(0,M;E)$ for every $M>0$. Obviously, $\mathcal{F}_+^{\mathrm{b}}\subset\mathcal{F}_+^{\mathrm{loc}}$ and $\mathcal{F}_+^{\mathrm{b}}$ is a Banach space with norm defined by

$$\|v\|_{\mathcal{F}_+^{\mathrm{b}}} := \|v\|_{L_\infty(\mathbb{R}_+;H^1)} + \|\partial_t v\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})}.$$

The space $\mathcal{F}_+^{\mathrm{loc}}$ is equipped with the topology Θ_+^{loc} generated by the following weak convergence of sequences $\{v_n(s)\}\subset\mathcal{F}_+^{\mathrm{loc}}$: by definition, $v_n(\cdot)\rightharpoonup v(\cdot)$ $(n\to\infty)$ in the topology Θ_+^{loc} if for any $\ell>0$

1. $v_n(\cdot,s)\rightharpoonup v(\cdot,s)$ $(n\to\infty)$ $\quad *$-weakly in $L_\infty(0,\ell;H^1)$ and
2. $\partial_t v_n(\cdot,s)\rightharpoonup \partial_t v(\cdot,s)$ $(n\to\infty)$ $\quad$ weakly in $L_2(0,\ell;H^{-1})$.

The topology Θ_+^{loc} is metrizable on any ball

$$\mathcal{B}_R := \{v\in\mathcal{F}_+^{\mathrm{b}} \mid \|v\|_{\mathcal{F}_+^{\mathrm{b}}}\leqslant R\}$$

in the space $\mathcal{F}_+^{\mathrm{b}}$. The corresponding metric for $\mathcal{B}_R$ is denoted by $\rho(\cdot,\cdot)$. Moreover, any ball $\mathcal{B}_R$ is a compact set in the topology Θ_+^{loc} and thus $\mathcal{B}_R$ is a compact metric space (see, for example, [10], [1]).

We consider the non-autonomous dissipative 2d Euler system (1.1) with symbols $\sigma(s)=(r(s),g(s)), s\geqslant 0$, belonging to the hull $\Sigma := \mathcal{H}_+(\sigma_0)$ in Ξ_+ of the initial symbol $\sigma_0(\cdot) := (r_0(\cdot),g_0(\cdot))\in\Xi_+$. Recall that $r_0(s)$ satisfies (1.3), $g_0(\cdot,s)\in L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)$, and $\sigma_0(s)$ is a translation compact function in Ξ_+.

We now define the trajectory set $\mathcal{K}_\sigma^+(N)$ of the system (1.1) with symbol σ that depend on a number $N\geqslant 0$.

Definition 2.1. A function $u(s)\in\mathcal{F}_+^{\mathrm{b}}$ belongs to $\mathcal{K}_\sigma^+(N)$ if

1. $u(\cdot)\in L_\infty(\mathbb{R}_+;H^1)$ and $u(\cdot)$ is a weak solution of (1.1) with symbol $\sigma\in\mathcal{H}_+(\sigma_0)$ in the space $D'(\mathbb{R}_+;H^{-1})$;
2. $u(\cdot)$ satisfies the inequality
$$\|u(t)\|^2\leqslant Ne^{-\alpha t}+\gamma^{-1}(\alpha)\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)},\quad \forall t\in\mathbb{R}_+. \tag{2.1}$$

We note that the set $\mathcal{K}_\sigma^+(N)$ is non-empty for all $\sigma\in\Sigma$ and for any $N\geqslant 0$. Indeed, if $u_0\in H^1$ and $\|u_0\|^2\leqslant N$, then the solution $u(\cdot,t)$ of (1.1), (1.6) with specified initial data u_0 obtained by the Galerkin method (see Proposition 1.1) is a weak solution of (1.1) in $D'(\mathbb{R}_+; H^{-1})$ and satisfies inequality (2.1) (see (1.28) for $\|u_0\|^2 \leqslant N$). Therefore, $u(x,t)$ belongs to $\mathcal{K}_\sigma^+(N)$.

Consider the translation semigroup $\{T(t)\} := \{T(t), t \geqslant 0\}$ acting in the spaces $\mathcal{F}_+^{\mathrm{b}}$ and $\mathcal{F}_+^{\mathrm{loc}}$ by the formula $T(t)v(s) = v(t+s)$. The semigroup $\{T(t)\}$ is continuous in the topology Θ_+^{loc}. This semigroup also acts in $\mathcal{K}_\sigma^+(N)$ and the following translation identity holds:

$$T(t)\mathcal{K}_\sigma^+(N) \subseteq \mathcal{K}_{T(t)\sigma}^+(N) \quad \text{for all } \sigma \in \Sigma, \quad t \geqslant 0. \tag{2.2}$$

Indeed, if $u(\cdot) \in \mathcal{K}_\sigma^+(N)$ is a weak solution of (1.1) with symbol $\sigma(s)$, then the function $T(h)u(s) = u(h+s)$ is a weak solution of (1.1) with the shifted symbol $\sigma(h+s) = T(h)\sigma(s)$. Moreover, since $u(\cdot)$ satisfies inequality (2.1), we see that, for all $h \geqslant 0$,

$$\begin{aligned} \|T(h)u(t)\|^2 = \|u(h+t)\|^2 &\leqslant Ne^{-\alpha(t+h)} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} \\ &\leqslant Ne^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in \mathbb{R}_+. \end{aligned}$$

Therefore, $T(h)u(t)$ also satisfies (2.1), that is, $T(h)u(\cdot) \in \mathcal{K}_{T(h)\sigma}^+(N)$ for all $u \in \mathcal{K}_\sigma^+(N)$.

The set

$$\mathcal{K}_\Sigma^+(N) := \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma^+(N) \tag{2.3}$$

is called the *trajectory space* of the system (1.1). It is clear that $\mathcal{K}_\Sigma^+(N) \subseteq \mathcal{F}_+^{\mathrm{loc}}$.

Proposition 2.1. *For any fixed $N \geqslant 0$, the space $\mathcal{K}_\Sigma^+(N)$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and closed in the topology Θ_+^{loc}.*

Proof. The boundedness of $\mathcal{K}_\Sigma^+(N)$ in $\mathcal{F}_+^{\mathrm{b}}$ follows from the inequality (2.1) and the estimate similar to (1.22):

$$\begin{aligned} \|\partial_t u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} &\leqslant \|B(u,u)\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} + \beta\,\|u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} + \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \\ &\leqslant C_1\left(\|u\|^2_{L_\infty(\mathbb{R}_+;H^1)} + \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right) \qquad (2.4) \\ &\leqslant C_1\left(N + \|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right) = C_1N + R_1. \qquad (2.5) \end{aligned}$$

Recall that $\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} \leqslant \|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}$ for all $\sigma = (r,g) \in \Sigma$. Combining (2.1) and (2.5) for $t = 0$, we see that $\mathcal{K}_\Sigma^+(N)$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$.

We claim that $\mathcal{K}_\Sigma^+(N)$ is closed in Θ_+^{loc}. Let $\{u_n(x,t)\} \subset \mathcal{K}_\Sigma^+(N)$ be a sequence that converges to a function $w(\cdot) \in \mathcal{F}_+^{\mathrm{b}}$ in Θ_+^{loc}, that is,

$$u_n(\cdot,t) \rightharpoonup w(\cdot,t)\ (n \to \infty) \quad *\text{-weakly in } L_\infty(0,\ell;H^1), \tag{2.6}$$
$$\partial_t u_n(\cdot,t) \rightharpoonup \partial_t w(\cdot,t)\ (n \to \infty) \quad \text{weakly in } L_2(0,\ell;H^{-1}), \quad \forall \ell > 0. \tag{2.7}$$

It is clear that $u_n \in \mathcal{K}_{\sigma_n}^+(N)$ for some $\sigma_n \in \Sigma$. Since Σ is compact in Ξ_+, there exists a subsequence of $\{\sigma_n\}$ (for which we preserve the notation $\{\sigma_n\}$) such that

$$\sigma_n \to \sigma \text{ as } n \to \infty \quad \text{in the space } \Xi_+$$

for some $\sigma \in \Sigma$. Let $\sigma_n(t) = (r_n(t), g_n(t))$ and $\sigma(t) = (r(t), g(t))$. Therefore,

$$r_n(t) \to r_n(t)\ (n \to \infty) \quad \text{in } C([0,\ell]) \quad \text{and} \tag{2.8}$$

$$g_n(t) \to g(t)\ (n \to \infty) \quad \text{in } L_2(0,\ell;H^1), \quad \forall \ell > 0. \tag{2.9}$$

The sequence $\{u_n(t)\}$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ since $\mathcal{K}_\Sigma^+(N)$ is a bounded set. We state that $w(\cdot) \in \mathcal{K}_\Sigma^+(N)$. The functions $u_n(x,t)$ satisfy the equations

$$\partial_t u_n + B(u_n,u_n) + r_n(t)u_n = g_n(x,t), \quad (\nabla, u_n) = 0. \tag{2.10}$$

First, we show that $w(t)$ is a weak solution of the system (1.1) with symbol $\sigma(t) = (r(t), g(t))$. We fix an arbitrary $\ell > 0$. Using (2.6), (2.7), and applying the Aubin compactness theorem (see [2, 14, 20]), we obtain that, passing to a subsequence of $\{u_n(t)\}$ (for which we preserve the notation $\{u_n(t)\}$), we may assume that

$$u_n(t) \to w(t)\ (n \to \infty) \quad \text{strongly in } L_2(0,\ell;H).$$

Then, using (2.8), we clearly have

$$r_n(t)u_n(t) \to r(t)w(t)\ (n \to \infty) \quad \text{strongly in } L_2(0,\ell;H). \tag{2.11}$$

Recall that $L_2(0,\ell;H) \subset L_2(\mathbb{T}^2 \times [0,\ell])^2$ and therefore we may assume that

$$u_n(x,t) \to w(x,t)\ (n \to \infty) \quad \text{for a.e. } (x,t) \in \mathbb{T}^2 \times [0,\ell]. \tag{2.12}$$

We now study the behavior of the nonlinear term $B(u_n,u_n)$ in (2.10). Identity (1.2) implies that

$$B(u_n,u_n) = P\left[\partial_{x_1}\left(u_n^1 u_n\right) + \partial_{x_2}\left(u_n^2 u_n\right)\right]. \tag{2.13}$$

It follows from (2.12) that, for $j = 1, 2$,

$$u_n^j(x,t)u_n(x,t) \to w^j(x,t)w(x,t)\ (n \to \infty) \quad \text{for a.e. } (x,t) \in \mathbb{T}^2 \times [0,\ell]. \tag{2.14}$$

Recall that $\{u_n\}$ is bounded in $L_\infty(0,M;H^1)$. Hence, the second inequality in (1.23) implies that

$$\{u_n^j u_n\} \quad \text{is bounded in } L_\infty(0,\ell;H) \tag{2.15}$$

and in $L_2(\mathbb{T}^2 \times [0,\ell])^2$ as well. Applying the known lemma on the weak convergence from [20], we conclude from (2.14) and (2.15) that

$$u_n^j(t)u_n(t) \rightharpoonup w^j(t)w(t)\ (n \to \infty)$$

weakly in $L_2(\mathbb{T}^2 \times [0,\ell])^2$ and $*$-weakly in $L_\infty(0,\ell;H)$ since (2.15) holds. Therefore, due to (2.13),

$$B(u_n(t),u_n(t)) \rightharpoonup B(w(t),w(t))\ (n \to \infty) \tag{2.16}$$

$*$-weakly in $L_\infty(0,\ell;H^{-1})$.

We now observe that, using (2.7), (2.16), (2.11), and (2.9), we can pass to the limit as $n \to \infty$ in every term of equation (2.10) in the distribution space $D'(0,\ell;H^{-1})$ and find that the function $w(x,t)$ satisfies the equation

$$\partial_t w + B(w,w) + r(t)w = g(x,t), \quad (\nabla, w) = 0.$$

Since the number ℓ was arbitrary, the function $w(x,t)$ is a weak solution of (1.1) in the space $D'(\mathbb{R}_+;H^{-1})$.

Second, let us prove that $w(x,t)$ satisfies inequality (2.1). Recall that $u_n(x,t)$ belongs to $\mathcal{K}_{\sigma_n}^+(N)$ and, thus, $u_n(x,t)$ satisfies the inequality:

$$\|u_n(t)\|^2 = Ne^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in \mathbb{R}_+. \tag{2.17}$$

It follows from (2.6) that, for all $t \geqslant 0$,

$$\|w(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \liminf_{n\to\infty} \|u_n(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)}$$

and hence (with regard to (2.17))

$$\|w(t)\|^2 \leqslant Ne^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in \mathbb{R}_+.$$

We have established that $w \in \mathcal{K}^+_\sigma(N)$ and, thereby, $\mathcal{K}^+_\Sigma(N)$ is closed in Θ^{loc}_+. Proposition 2.1 is proved. □

Consider the action of the translation semigroup $\{T(t)\}$ in the trajectory space $\mathcal{K}^+_\Sigma(N)$. We conclude from (2.2) that

$$T(t)\mathcal{K}^+_\Sigma(N) \subseteq \mathcal{K}^+_\Sigma(N), \quad \forall t \geqslant 0. \tag{2.18}$$

Proposition 2.1 and formula (2.18) imply that the semigroup $\{T(t)\}$ acts on the compact metric space $\mathcal{K}^+_\Sigma(N)$ since $\mathcal{K}^+_\Sigma(N) \subset \mathcal{B}_R$, where $\mathcal{B}_R$ is a ball in $\mathcal{F}^{\mathrm{b}}_+$ with sufficiently large radius R. We denote by $\rho(\cdot,\cdot)$ the corresponding metric in $\mathcal{B}_R$ (and in its metric subspace $\mathcal{K}^+_\Sigma(N)$). Recall that the semigroup $\{T(t)\}$ is continuous in this metric space. These facts imply that the semigroup $\{T(t)\}|_{\mathcal{K}^+_\Sigma(N)}$ has a global attractor $\mathfrak{A}_\Sigma(N) \subseteq \mathcal{K}^+_\Sigma(N)$, called the *trajectory attractor* of equations (1.1) (see [8, 10, 24, 3]). Recall that

$$\mathfrak{A}_\Sigma(N) = \bigcap_{t\geqslant 0}\left[\bigcup_{\theta\geqslant t} T(\theta)\mathcal{K}^+_\Sigma(N)\right]_{\Theta^{\mathrm{loc}}_+}, \tag{2.19}$$

the set $\mathfrak{A}_\Sigma(N)$ is strictly invariant with respect to $\{T(t)\}$:

$$T(t)\mathfrak{A}_\Sigma(N) = \mathfrak{A}_\Sigma(N), \quad \forall t \geqslant 0,$$

and, for any trajectory set $\mathfrak{B} \subseteq \mathcal{K}^+_\Sigma(N)$,

$$\mathrm{dist}_\rho(T(t)\mathfrak{B}, \mathfrak{A}_\Sigma(N)) \to 0 \ (t \to +\infty)$$

(see, for example, [10, 24, 3]). Here

$$\mathrm{dist}_\mu(X,Y) := \sup_{x\in X} \mathrm{dist}_\mu(x,Y) = \sup_{x\in X}\inf_{y\in Y} \mu(x,y)$$

is the Hausdorff deviation of a set X from a set Y in a metric space $\mathcal{M}$ with metric μ.

Proposition 2.2. *The trajectory attractor is independent of N: $\mathfrak{A}_\Sigma(N) = \mathfrak{A}_\Sigma(0) =: \mathfrak{A}_\Sigma$ for all $N \geqslant 0$ and*

$$\mathrm{dist}_\rho(T(t)\mathfrak{B}, \mathfrak{A}_\Sigma) \to 0 \ (t \to +\infty), \quad \forall \mathfrak{B} \subseteq \mathcal{K}^+_\Sigma(N). \tag{2.20}$$

Moreover, $\mathfrak{A}_\Sigma \subseteq \mathcal{K}^+_\Sigma(0)$, that is,

$$\|u(\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \gamma^{-1}(\alpha)\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall u \in \mathfrak{A}_\Sigma. \tag{2.21}$$

Proof. It follows from the definition of $\mathcal{K}^+_\Sigma(N)$, that $\mathcal{K}^+_\Sigma(N) \subseteq \mathcal{K}^+_\Sigma(N_1)$ for $N_1 \geqslant N$. Therefore, formula (2.19) implies that $\mathfrak{A}_\Sigma(N) \subseteq \mathfrak{A}_\Sigma(N_1)$ for $N_1 \geqslant N$. Moreover, we see from inequality (2.1) that $T(t)\mathcal{K}^+_\Sigma(N_1) \subseteq \mathcal{K}^+_\Sigma(N)$ for all $t \geqslant \alpha^{-1}\ln(N_1/N)$, $N_1 \geqslant N > 0$. Using (2.19) once again, we obtain that $\mathfrak{A}_\Sigma(N_1) \subseteq \mathfrak{A}_\Sigma(N)$ for all $N_1 \geqslant N > 0$. We conclude that $\mathfrak{A}_\Sigma(N_1) = \mathfrak{A}_\Sigma(N)$ for all $N_1 \geqslant N > 0$. In particular, owing to (2.1),

$$\|u(\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall u \in \mathfrak{A}_\Sigma(N),$$

that is $\mathfrak{A}_\Sigma(N) \subseteq \mathcal{K}_\Sigma^+(0)$. Finally, since $\mathfrak{A}_\Sigma(N)$ is strictly invariant with respect to $\{S(t)\}$, we see that $\mathfrak{A}_\Sigma(N)$ is the global attractor of $\{S(t)\}$ acting on $\mathcal{K}_\Sigma^+(0)$. Therefore, $\mathfrak{A}_\Sigma(N) = \mathfrak{A}_\Sigma(0) \subseteq \mathcal{K}_\Sigma^+(0)$ for all $N \geqslant 0$ and (2.21) is also established. □

Using (2.21) and (2.4), we have

Corollary 2.1. *For all $u \in \mathfrak{A}_\Sigma$, the following inequalities hold:*

$$\|\partial_t u(\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \leqslant C_1 \left(\gamma^{-1}(\alpha)\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right).$$

Recall that $\Sigma = \mathcal{H}_+(\sigma_0)$ and $\sigma_0(s)$ is a translation compact function in Ξ_+, that is, the set $\mathcal{H}_+(\sigma_0) = [\{\sigma_0(h+s) \mid h \geqslant 0\}]_{\Xi_+}$ is compact in the metric space Ξ_+. Consider the translation semigroup $\{T(t)\}$ acting on $\Sigma = \mathcal{H}_+(\sigma_0) : T(t)\sigma(s) = \sigma(t+s)$ for all $\sigma \in \Sigma$, $T(t) : \Sigma \mapsto \Sigma$. This semigroup is continuous and Σ is a compact metric space. Hence, the semigroup $\{T(t)\}$ possesses the global attractor in Σ which coincides with the ω-limit set of the whole space Σ :

$$\omega(\Sigma) := \bigcap_{t\geqslant 0}\left[\bigcup_{\theta\geqslant t} T(\theta)\Sigma\right]_{\Xi_+}$$

(see, for example, [24, 3]). Moreover,

$$T(t)\omega(\Sigma) = \omega(\Sigma), \quad \forall t \geqslant 0.$$

Together with the trajectory space $\mathcal{K}_\Sigma^+(N)$ of equations (1.1) with the symbol space $\Sigma = \mathcal{H}_+(\sigma_0)$, we consider a smaller trajectory space $\mathcal{K}_{\omega(\Sigma)}^+(N)$ of this equations which corresponds to strictly invariant subset $\omega(\Sigma) \subseteq \Sigma$. We clearly have that

$$\begin{gathered} \mathcal{K}_{\omega(\Sigma)}^+(N) \subseteq \mathcal{K}_\Sigma^+(N), \\ T(t)\mathcal{K}_{\omega(\Sigma)}^+(N) \subseteq \mathcal{K}_{\omega(\Sigma)}^+(N), \quad \forall t \geqslant 0. \end{gathered} \tag{2.22}$$

Moreover, $\mathcal{K}_{\omega(\Sigma)}^+(N)$ is bounded in $\mathcal{F}_+^{\mathrm{b}}$ and closed in the topology Θ_+^{loc}. The proof of this assertion is similar to the proof of Proposition 2.1. Thus, the semigroup $\{T(t)\}|_{\mathcal{K}_{\omega(\Sigma)}^+(N)}$ also has the trajectory attractor $\mathfrak{A}_{\omega(\Sigma)} \subseteq \mathcal{K}_{\omega(\Sigma)}^+(N)$, which is independent of N (see the proof of Proposition 2.2). Inclusion (2.22) implies that $\mathfrak{A}_{\omega(\Sigma)} \subseteq \mathfrak{A}_\Sigma$. What is more, we have the following result.

Theorem 2.1. *The trajectory attractor $\mathfrak{A}_\Sigma$ of equations (1.1) with the symbol space $\Sigma = \mathcal{H}_+(\sigma_0)$ coincides with the trajectory attractor $\mathfrak{A}_{\omega(\Sigma)}$ of these equation with the symbol space $\omega(\Sigma)$:*

$$\mathfrak{A}_\Sigma = \mathfrak{A}_{\omega(\Sigma)}.$$

We leave the proof for the reader. It is sufficient to establish that $\mathfrak{A}_{\omega(\Sigma)}$ is an attracting set for the semigroup $\{T(t)\}|_{\mathcal{K}_\Sigma^+(N)}$. More general theorem has been proved in [10, Chapter XIV] (see also [8]).

Remark 2.1. The following embedding is continuous:

$$\Theta_+^{\mathrm{loc}} \subseteq C^{\mathrm{loc}}(\mathbb{R}_+;H^\delta), \quad \forall \delta \in [0,1[,$$

(see [20, 10]). The trajectory attractor $\mathfrak{A}_\Sigma$ of equations (1.1) satisfies (2.20) and hence, for any set $\mathfrak{B} \in \mathcal{K}_\Sigma^+(N)$,

$$\mathrm{dist}_{C([0,\ell];H^\delta)}(T(h)B, \mathfrak{A}_\Sigma) \to 0 \ (h \to +\infty), \quad \forall \ell > 0. \tag{2.23}$$

Remark 2.2. We note that we cannot put $\delta = 1$ in formula (2.23). We have constructed the trajectory attractor $\mathfrak{A}_\Sigma$ in the "weak" topology $\Theta_+^{\mathrm{w,loc}}$ of the space $\mathcal{F}_+^{\mathrm{loc}}$. It seems natural to consider also the "strong" topology $\Theta_+^{\mathrm{s,loc}}$ in $\mathcal{F}_+^{\mathrm{loc}}$. However, the attraction to the trajectory attractor $\mathfrak{A}_\Sigma$ in the topology $\Theta_+^{\mathrm{s,loc}}$ could not be proved without additional assumptions. Dealing with autonomous equations (1.1), this fact has been proved in [12] under the assumption that the curl of the external force $\nabla \times g$ belongs to $L_\infty(\mathbb{T}^2)$. The proof is based on the technique of Yudovich (see [26, 27]). In the more general case considered in this paper, where $\nabla \times g \in L_2(\mathbb{T}^2)$, the question remains open.

3. **Non-autonomous 2d Navier-Stokes systems with dissipation .** Consider the following non-autonomous 2d Navier–Stokes system with dissipation:

$$\partial_t u + B(u,u) - \nu\Delta u + r(t)u = g(x,t), \quad (\nabla, u) = 0, \quad x = (x_1, x_2) \in \mathbb{T}^2,\ t \geqslant 0. \tag{3.1}$$

We use the same notations as in the Euler system (1.1) with dissipation. In equations (3.1), $\nu > 0$ denotes the kinematic viscosity. The pressure $p(x,t)$ is eliminated from the system by applying the Leray operator P to both sides.

The system (3.1) also has a geophysical interpretation (see [21]). The main dissipation acts in the planetary boundary layer which is described by the time dependent term $-r(t)u$, while a viscosity term $\nu\Delta u$ is responsible for small-scale dissipation (note that in physically relevant cases $0 < \nu \ll \alpha$).

Remark 3.1. Studying the classical 2d Navier–Stokes system ($r = 0$) with periodic boundary conditions, one usually assumes that the functions $u(x,t)$ and $g(x,t)$ have zero means in x over the torus $\mathbb{T}^2$, in order to avoid the linear growth of the solutions. When $r > 0$, this assumption can be dropped since the term $-r(t)u$ introduces additional dissipation.

The non-autonomous system (1.1) has the time symbol $\sigma(s) = (r(s), g(\cdot, s))$, $s \geqslant 0$. As before, we assume that $\sigma(\cdot) \in \Sigma := \mathcal{H}_+(\sigma_0)$ and $\sigma_0(s) = (r_0(s), g_0(s))$ is the original symbol that generates the corresponding family of the 2d Euler systems (1.1) with dissipation. Clearly, the functions $r(t)$ and $g(x,t)$ satisfy conditions (1.3)–(1.5).

The Cauchy problem (3.1), (1.6) is uniquely solvable: for all $u_0 \in H^1$, the solution $u(x,t)$ belongs to the class $C_{\mathrm{b}}(\mathbb{R}_+; H^1) \cap L_2^{\mathrm{b}}(\mathbb{R}_+; H^2)$ and the time derivation $\partial_t u \in L_2^{\mathrm{b}}(\mathbb{R}_+; H)$. Such solutions are called strong solution (see [16]–[18],[12], the case $r = 0$ has been considered, for example, in [7, 8, 10, 24, 19, 3]). Notice that the dissipative term $-r(t)u$ has no effect on this result. Any solution $u(x,t)$ of the problem (3.1), (1.6) satisfies the following identities:

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu|\nabla u(t)|^2 + r(t)|u(t)|^2 = (g(t), u(t)), \tag{3.2}$$

$$\frac{1}{2}\frac{d}{dt}|\nabla u(t)|^2 + \nu|\Delta u(t)|^2 + r(t)|\nabla u(t)|^2 = (\nabla g(t), \nabla u(t)), \tag{3.3}$$

that are analogous to the corresponding identities for strong solutions of the classical 2d Navier–Stokes system with $r = 0$ (see, e.g., [7, 10, 24, 3]). Identities (1.9) and (1.12) are used in the proof of (3.2) and (3.3).

The identities (3.2) and (3.3) imply the following inequality:

$$\frac{d}{dt}\|u(t)\|^2 + \alpha\|u(t)\|^2 + 2\nu|\nabla u(t)|^2 + 2\nu|\Delta u(t)|^2 \leqslant \alpha^{-1}\|g(t)\|^2. \tag{3.4}$$

Omitting the positive terms containing the coefficient ν, we deduce from (3.4) the inequality

$$\|u(t)\|^2 \leqslant \|u_0\|^2 e^{-\alpha t} + \gamma^{-1}(\alpha)\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \tag{3.5}$$

that is similar to inequality (1.18). Recall that $\|v\|^2 = |\nabla v|^2 + |v|^2 = \|v\|^2_{H^1}$ and $\gamma(\alpha) = \alpha(1 - e^{-\alpha})$.

Integrating (3.4) over $[t, t+1]$ and using (3.5), we obtain

$$2\nu \int_t^{t+1} |\Delta u(s)|^2 \, ds \leqslant \|u_0\|^2 e^{-\alpha t} + (\alpha^{-1} + \gamma^{-1})\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}. \tag{3.6}$$

Equation (3.1) implies that

$$\begin{aligned} \|\partial_t u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} &\leqslant \|B(u,u)\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} + \beta \|u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} + \nu \|u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} \\ &+ \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \leqslant C_1 \left(\|u\|^2_{L_\infty(\mathbb{R}_+;H^1)} + \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1 \right) \end{aligned} \tag{3.7}$$

where the constant C_1 depends on α and is independent of ν, $0 < \nu \leqslant 1$. The proof of (3.7) is similar to that of (1.22) and (2.4) and use the inequalities (1.23). Then from (3.5), we conclude that

$$\|\partial_t u\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \leqslant C_1 \left(\|u_0\|^2 e^{-\alpha t} + \gamma^{-1} \|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1 \right). \tag{3.8}$$

Remark 3.2. Note that the constants in the right-hand sides of inequalities (3.5), (3.6), and (3.8) are independent of ν, $0 < \nu \leqslant 1$. These estimates are similar to the inequalities that were established in Sec. 1 and 2 for weak solutions of the 2d Euler equation with dissipation.

We now study the behaviour of solutions of system (3.1) as $\nu \to 0+$.

Let $\{u_\nu(x,t),\ 0 < \nu \leqslant 1\}$ be a family of solutions to the system (3.1) with symbols $\sigma_\nu(\cdot) \in \mathcal{H}_+(\sigma_0)$ such that

$$\|u_\nu(\cdot)\|_{L_\infty(\mathbb{R}_+;H^1)} \leqslant M, \quad \|\partial_t u_\nu(\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \leqslant M. \tag{3.9}$$

Estimates (3.5) and (3.8) imply that property (3.9) holds if the initial data for (3.1) satisfies the inequality

$$\|u_\nu(0)\| \leqslant m,$$

and $M = M(m)$ is sufficiently large.

Theorem 3.1. *Consider a sequence of solutions $\{u_{\nu_n}(x,t),\ 0 < \nu_n \leqslant 1\}$ of equations (3.1) with viscosities $\nu_n > 0$ and with symbols $\sigma_{\nu_n}(s) \in \mathcal{H}_+(\sigma_0)$ that satisfy (3.9). We assume that $\sigma_{\nu_n}(s) \to \xi(s)$ in Ξ_+ as $n \to \infty$ for some $\xi(\cdot) \in \mathcal{H}_+(\sigma_0)$. We also assume that $\nu_n \to 0+$ as $n \to \infty$. Then there exists a subsequence $\{\nu_{n'}\} \subset \{\nu_n\}$ such that*

$$u_{\nu_{n'}}(\cdot) \to w(\cdot) \text{ as } n' \to \infty \quad \text{in } \Theta_+^{\mathrm{loc}},$$

where $w(x,t)$ is a weak solution of the dissipative 2d Euler equation (1.1) with the symbol $\xi(\cdot)$ such that $w(\cdot) \in \mathcal{K}_\xi^+(M)$.

Proof. The function $u_{\nu_n}(x,t)$ satisfies the equation

$$\partial_t u_{\nu_n} + B(u_{\nu_n}, u_{\nu_n}) - \nu_n \Delta u_{\nu_n} + r_{\nu_n}(t) u_{\nu_n} = g_{\nu_n}(x,t), \tag{3.10}$$

where $\sigma_{\nu_n}(t) = (r_{\nu_n}(t), g_{\nu_n}(\cdot,t))$. Let $\xi(s) = (r(s), g(\cdot,s))$. Since $\sigma_{\nu_n}(s) \to \xi(s)$ in Ξ_+ as $n \to \infty$ we have that

$$r_{\nu_n}(t) \to r(t) \quad \text{in } C^{\mathrm{loc}}(\mathbb{R}_+) \quad \text{and} \tag{3.11}$$

$$g_{\nu_n}(\cdot,t) \to g(\cdot,t) \quad \text{in } L_2^{\mathrm{loc}}(\mathbb{R}_+;H^1) \text{ as } n \to \infty, \tag{3.12}$$

The sequence $\{u_{\nu_n}(\cdot,t)\}$ is weakly compact in Θ_+^{loc} since it satisfies (3.9) and, therefore, it contains a convergent subsequence $\{u_{\nu_{n'}}(\cdot,t)\}$:

$$u_{\nu_{n'}}(\cdot,t) \to w(\cdot,t) \text{ as } n' \to \infty \text{ in } \Theta_+^{\mathrm{loc}}$$

for some $w(\cdot) \in \mathcal{F}_+^{\mathrm{b}}$, i.e.,

$$u_{\nu_{n'}}(\cdot,t) \rightharpoonup w(\cdot,t) \quad * \text{ -weakly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^1), \tag{3.13}$$

$$\partial_t u_{\nu_{n'}}(\cdot,t) \rightharpoonup \partial_t w(\cdot,t) \quad \text{weakly in } L_2^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.14}$$

Following the reasoning in the proof of Proposition 2.1 (from formula (2.6) to formula (2.16)), we obtain that

$$u_{\nu_{n'}}(t) \to w(t) \quad \text{strongly in } L_2^{\mathrm{loc}}(\mathbb{R}_+;H) \quad \text{and} \tag{3.15}$$

$$B(u_{\nu_{n'}}(t), u_{\nu_{n'}}(t)) \rightharpoonup B(w(t),w(t)) \quad *\text{-weakly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.16}$$

Combining (3.11) and (3.15) we obtain

$$r_{\nu_{n'}}(t)u_{\nu_{n'}}(t) \to r(t)w(t) \quad \text{strongly in } L_2^{\mathrm{loc}}(\mathbb{R}_+;H) \text{ as } n' \to \infty. \tag{3.17}$$

Consider the term $\nu_n \Delta u_{\nu_n}$ of equation (3.10). Owing to estimate (3.9)

$$\|\nu_n \Delta u_{\nu_n}(\cdot)\|_{L_\infty(\mathbb{R}_+;H^{-1})} \leqslant \nu_n C \|u_{\nu_n}(\cdot)\|_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \nu_n CM \to 0,$$

since $\nu_n \to 0+$, that is,

$$\nu_n \Delta u_{\nu_n}(\cdot) \to 0 \quad \text{strongly in } L_\infty^{\mathrm{loc}}(\mathbb{R}_+;H^{-1}). \tag{3.18}$$

Relations (3.14), (3.16), (3.18), (3.17), and (3.12), yields that we can pass to the limit in the equations (3.10) as $n' \to \infty$ in the distribution space $D'(\mathbb{R}_+;H^{-1})$ and obtain that the function $w(x,t)$ satisfies the equation

$$\partial_t w + B(w,w) + r(t)w = g(x,t)$$

in the distribution sense, that is, w is a weak solution to (1.1) with the symbol ξ.

It remains to show that $w(x,t) \in \mathcal{K}_\xi^+(M)$. Since $u_{\nu_n}(\cdot) \in C^{\mathrm{b}}(\mathbb{R}_+;H^1)$, it follows from (3.9) that

$$\|u_{\nu_n}(0)\| \leqslant M.$$

Each function $u_{\nu_n}(x,t)$ satisfies inequality (3.5), therefore,

$$\sup\left\{\|u_{\nu_n}(t+s)\|^2 \ \mid s \geqslant 0\right\} \leqslant Me^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}.$$

Recall $\|g\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} \leqslant \|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}$, see also (3.9). Since

$$\|w(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \liminf_{n\to\infty} \|u_{\nu_n}(t+\cdot)\|^2_{L_\infty(\mathbb{R}_+;H^1)}$$

(see (3.13)) we have

$$\|w(t)\|^2 \leqslant \sup\left\{\|w(t+s)\|^2 \ \mid s \geqslant 0\right\} \leqslant Me^{-\alpha t} + \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, \quad \forall t \in \mathbb{R}_+$$

and therefore $w \in \mathcal{K}_\xi^+(M)$. The theorem is proved. □

4. **Trajectory attractors of dissipative 2d Navier-Stokes systems with vanishing viscosity .** In this section, we study the convergence of the trajectory attractors $\mathfrak{A}_\Sigma^\nu$ of the non-autonomous dissipative 2d Navier–Stokes systems (3.1) to the trajectory attractor $\mathfrak{A}_\Sigma$ of the non-autonomous dissipative 2d Euler equations (1.1) as viscosity ν vanishes. The spaces $\mathcal{F}_+^{\mathrm{b}}, \mathcal{F}_+^{\mathrm{loc}}$, and Θ_+^{loc} were defined in Sec. 2.

Consider an arbitrary symbol $\sigma \in \Sigma$. Similarly to Sec. 2, we define the trajectory set

$$\mathcal{K}_\sigma^+(\nu) \subset C_{\mathrm{b}}(\mathbb{R}_+; H^1) \cap L_2^{\mathrm{b}}(\mathbb{R}_+; H^2) \subset \mathcal{F}_+^{\mathrm{b}}$$

of the system (3.1) having the symbol $\sigma = (r, g)$ and with a fixed viscosity coefficient $\nu > 0$. By definition, $\mathcal{K}_\sigma^+(\nu)$ consists of all strong solutions $u(s) := u(\cdot, s), s \geqslant 0$, of this system. Here, we denote the time variable s instead t. We have proved in Sec. 3 that, for any data $u_0 = u(0) \in H^1$, there exists a unique function $u(\cdot) \in \mathcal{K}_\sigma^+(\nu)$ such that $u(0) = u_0$. We clearly have the translation identity

$$T(h)\mathcal{K}_\sigma^+(\nu) \subseteq \mathcal{K}_{T(h)\sigma}^+(\nu) \quad \text{for all } \sigma \in \Sigma,\ h \geqslant 0, \tag{4.1}$$

that is similar to the identity (2.2). Analogously to (2.3), we define the *trajectory space* of the system (3.1) with fixed viscosity $\nu > 0$ by the formula

$$\mathcal{K}_\Sigma^+(\nu) := \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma^+(\nu) \tag{4.2}$$

It is clear that $\mathcal{K}_\Sigma^+(\nu) \subseteq \mathcal{F}_+^{\mathrm{b}}$ and it is easy to prove that the trajectory space $\mathcal{K}_\Sigma^+(\nu)$ is closed in Θ_+^{loc}.

The translation semigroup $\{T(t)\}$ acts on the trajectory space $\mathcal{K}_\Sigma^+(\nu)$ by the formula $T(t)u(s) = u(t + s)$. It is clear that $T(t)\mathcal{K}_\Sigma^+(\nu) \subseteq \mathcal{K}_\Sigma^+(\nu)$ for all $t \geqslant 0$. We claim that the translation semigroup $\{T(t)\}$ acting on $\mathcal{K}_\Sigma^+(\nu)$ has the trajectory attractor $\mathfrak{A}_\Sigma^\nu \subset \mathcal{K}_\Sigma^+(\nu)$ which attracts bounded (in $\mathcal{F}_+^{\mathrm{b}}$) families of solutions to system (3.1) in the topology Θ_+^{loc}. (See the similar proof in [7, 10] for the case $r = 0$ and Sec. 2 for $\nu = 0$).

Recall that the set $\mathfrak{A}_\Sigma^\nu$ is strictly invariant with respect to $\{T(t)\}$:

$$T(t)\mathfrak{A}_\Sigma^\nu = \mathfrak{A}_\Sigma^\nu, \quad \forall t \geqslant 0, \tag{4.3}$$

and, for any bounded (in $\mathcal{F}_+^{\mathrm{b}}$) set $\mathfrak{B}_\nu \subseteq \mathcal{K}_\Sigma^+(\nu)$, we have

$$\mathrm{dist}_\rho(T(t)\mathfrak{B}^\nu, \mathfrak{A}_\Sigma^\nu) \to 0 \ (t \to +\infty),$$

where ρ is the metric generating the topology Θ_+^{loc} on a ball in $\mathcal{F}_+^{\mathrm{b}}$ containing $\mathfrak{B}^\nu$ (see Sec. 2).

Using inequalities (3.5), (3.6), and (3.8), we have

Proposition 4.1. *The trajectory attractors $\mathfrak{A}_\Sigma^\nu$ are uniformly bounded for $0 < \nu \leqslant 1$ in the space $\mathcal{F}_+^{\mathrm{b}}$ and for all $u_\nu \in \mathfrak{A}_\Sigma^\nu$*

$$\|u_\nu(\cdot)\|_{L_\infty(\mathbb{R}_+;H^1)} \leqslant \gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)},$$

$$\|\partial_t u_\nu(\cdot)\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^{-1})} \leqslant C_1\left(\gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right),$$

$$2\nu\int_t^{t+1} |\Delta u_\nu(s)|^2\, ds \leqslant (\alpha^{-1} + \gamma^{-1})\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}.$$

Comparing Corollary 2.1 and Proposition 4.1, we observe that the trajectory attractor $\mathfrak{A}_\Sigma$ of the dissipative 2d Euler system (1.1) and the trajectory attractors $\mathfrak{A}_\Sigma^\nu$

of the 2d Navier–Stokes system (3.1) belong to the same ball $\mathcal{B}_{R_0}$ in $\mathcal{F}_+^{\mathrm{b}}$ having the radius R_0, $R_0^2 = \max\left\{\gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)}, C_1\left(\gamma^{-1}\|g_0\|^2_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + \|g_0\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;H^1)} + 1\right)\right\}$:

$$\mathfrak{A}_\Sigma \subset \mathcal{B}_{R_0} \quad \text{and} \quad \mathfrak{A}_\Sigma^\nu \subset \mathcal{B}_{R_0}, \quad \forall \nu \in]0,1]. \tag{4.4}$$

We now study the Hausdorff deviation of $\mathfrak{A}_\Sigma^\nu$ from $\mathfrak{A}_\Sigma$ as ν approaches zero in the topology Θ_+^{loc} generated by the metric ρ in $\mathcal{B}_{R_0}$ described in Sec. 2.

The main result of this section is the following theorem.

Theorem 4.1. *The trajectory attractors $\mathfrak{A}_\Sigma^\nu$ of system (3.1) converge to the trajectory attractor $\mathfrak{A}_\Sigma$ of (1.1):*

$$\mathrm{dist}_\rho(\mathfrak{A}_\Sigma^\nu, \mathfrak{A}_\Sigma) \to 0+ \ \text{ as } \nu \to 0+\,. \tag{4.5}$$

Let $\mathfrak{B}^\nu \subseteq \mathcal{K}_\Sigma^+(\nu)$ be bounded (in $\mathcal{F}_+^{\mathrm{b}}$) sets of trajectories of the system (3.1):

$$\|\mathfrak{B}^\nu\|_{\mathcal{F}_+^{\mathrm{b}}} \leqslant M \ (0 < \nu \leqslant 1). \tag{4.6}$$

Then

$$\mathrm{dist}_\rho(T(t)\mathfrak{B}^\nu, \mathfrak{A}_\Sigma) \to 0 \quad \text{ as } \nu \to 0+ \ \text{ and } \ t \to +\infty. \tag{4.7}$$

Proof. It suffices to prove (4.7). Indeed, taking in (4.7) the set $\mathfrak{B}^\nu = \mathfrak{A}_\Sigma^\nu$ and using the invariance property (4.3), we obtain (4.5). Assume that the relation (4.7) fails to hold. Then there is a neighbourhood $\mathcal{O}(\mathfrak{A}_\Sigma)$ in Θ_+^{loc}, two sequences $\nu_n \to 0+$ and $h_n \to +\infty$ as $n \to \infty$ such that

$$T(h_n)\mathfrak{B}^{\nu_n} \not\subset \mathcal{O}(\mathfrak{A}_\Sigma)$$

for some sets $\mathfrak{B}^{\nu_n} \subseteq \mathcal{K}_\Sigma^+(\nu_n)$ that satisfy (4.6). So, there are solutions of system (3.1) $w_{\nu_n}(\cdot) \in \mathfrak{B}^{\nu_n}$ such that the functions

$$W_{\nu_n}(t) := T(h_n)w_{\nu_n}(t) = w_{\nu_n}(h_n + t) \tag{4.8}$$

do not belong to $\mathcal{O}(\mathfrak{A}_\Sigma)$:

$$W_{\nu_n}(t) \notin \mathcal{O}(\mathfrak{A}_\Sigma), \quad \forall n \in \mathbb{N}. \tag{4.9}$$

Apparently for every $n \in \mathbb{N}$, the function $w_{\nu_n}(\cdot) \in \mathcal{K}_{\sigma_n}^+(\nu)$ for some $\sigma_n \in \Sigma$. Therefore, due to translation identity (4.1),

$$W_{\nu_n}(t) = T(h_n)w_{\nu_n}(t) \in \mathcal{K}_{T(h_n)\sigma_n}^+(\nu),$$

where $T(h_n)\sigma_n(t) = \sigma_n(h_n + t)$. We denote $\xi_n(t) := \sigma_n(h_n + t)$. The function $\xi_n(t)$ is defined for all $t \in [-h_n, +\infty)$.

We note that, for every fixed $\ell \geqslant 0$, the sequence $\{\xi_n(t)\}$ is precompact in the space $\Xi_{-\ell,+\infty} := C^{\mathrm{loc}}([-\ell,+\infty)) \times L_2^{\mathrm{loc}}(-\ell,+\infty;H^1)$ (we consider ξ_n with indices n such that $h_n \geqslant \ell$). This assertion follows from the fact that the sequence $\{\sigma_n(t)\}$ is precompact in $\Xi_+ := C^{\mathrm{loc}}([0,+\infty) \times L_2^{\mathrm{loc}}(0,+\infty;H^1)$ because it belongs to a compact (in Ξ_+) set $\Sigma = \mathcal{H}_+(\sigma_0)$. Therefore, for every fixed $\ell > 0$, we can find a subsequence of indices $\{n'\} \subset \{n\}$ such that $\xi_{n'}(t)$ converges in $\Xi_{-\ell,+\infty}$ as $n' \to \infty$. Applying now the standard Cantor diagonal procedure, we can construct a function $\xi(t), t \in \mathbb{R}$, and a subsequence $\{n''\} \subset \{n\}$ such that

$$\xi_{n''}(t) \to \xi(t) \text{ in } \Xi_{-\ell,+\infty} \quad \text{for any } \ell \geqslant 0.$$

Moreover the function $\xi(t+h), t \geqslant 0$, belongs to $\Sigma = \mathcal{H}_+(\sigma_0)$ for all $h \in \mathbb{R}$, since the set $\mathcal{H}_+(\sigma_0)$ is closed in Ξ_+. In particular, (1.3) and (1.5) imply the following inequalities:

$$\alpha \leqslant r(t) \leqslant \beta, \quad \forall t \in \mathbb{R},$$

$$\|g\|^2_{L^{\mathrm{b}}_2(\mathbb{R};H^1)} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(\cdot,s)\|^2 ds \leqslant \|g_0\|^2_{L^{\mathrm{b}}_2(\mathbb{R}_+;H^1)},$$

where $\xi(t) = (r(t), g(x,t)), t \in \mathbb{R}$. Thus, we have constructed the time symbol $\xi(t)$ of the non-autonomous 2d Euler system (1.1) defined on the entire time axis $\mathbb{R}$.

We now observe that the function $W_{\nu_n}(t)$ (see (4.8)) is a solution of equation (3.1) with viscosity $\nu = \nu_n$ and with the symbol $\xi_n(t) = \sigma_n(h_n + t)$ on the semiaxis $[-h_n, +\infty)$, since $w_{\nu_n}(t)$ is a solution of equation (3.1) with symbol $\sigma_n(t)$ on $[0, +\infty)$.

It follows from (4.6) that

$$\sup_{s\geqslant 0}\|w_{\nu_n}(s)\|_{H^1} + \left(\sup_{s\geqslant 0}\int_s^{s+1} \|\partial_t w_{\nu_n}(\theta)\|^2_{H^{-1}} d\theta\right)^{1/2} \leqslant M.$$

Therefore,

$$\sup_{s\geqslant -h_n}\|W_{\nu_n}(s)\|_{H^1} + \left(\sup_{s\geqslant -h_n}\int_s^{s+1} \|\partial_t W_{\nu_n}(\theta)\|^2_{H^{-1}} d\theta\right)^{1/2} \leqslant M. \tag{4.10}$$

Besides, inequality (3.5) yields

$$\begin{aligned}\sup_{s\geqslant -h_n+\tau}\|W_{\nu_n}(s)\|^2_{H^1} &= \sup_{s\geqslant \tau}\|w_{\nu_n}(s)\|^2_{H^1} \leqslant \|w_{\nu_n}(0)\|^2 e^{-\alpha\tau} \\ &+ \gamma^{-1}\|g_0\|^2_{L^{\mathrm{b}}_2(\mathbb{R}_+;H^1)} \leqslant M^2 e^{-\alpha\tau} + \gamma^{-1}\|g_0\|^2_{L^{\mathrm{b}}_2(\mathbb{R}_+;H^1)}, \quad \forall \tau \geqslant 0. \end{aligned}\tag{4.11}$$

Inequality (4.10) implies that the sequence $\{W_{\nu_n}(\cdot)\}$ is compact in the weak topology of the space $\Theta^{\mathrm{loc}}_{-\ell,+\infty} = L^{\mathrm{loc}}_{\infty,*w}(-\ell,+\infty;H^1) \cap \{v \mid \partial_t v \in L^{\mathrm{loc}}_{2,w}(-\ell,+\infty;H^{-1}\}$ for every fixed $\ell \geqslant 0$, if we consider ν_n with indices n such that $h_n \geqslant \ell$. Therefore, for every fixed $\ell \geqslant 0$, we can choose a subsequence $\{\nu_{n'}\} \subset \{\nu_n\}$ such that $\{W_{\nu_{n'}}(\cdot)\}$ converges weakly in $\Theta_{-\ell,+\infty}$. Using again the Cantor diagonal procedure, we can construct a function $W(t), t \in \mathbb{R}$, and a subsequence $\{\nu_{n''}\} \subset \{\nu_n\}$ such that

$$W_{\nu_{n''}}(\cdot) \to W(\cdot) \text{ in the topology } \Theta^{\mathrm{loc}}_{-\ell,+\infty} \text{ as } n'' \to \infty \quad \text{for any } \ell \geqslant 0. \tag{4.12}$$

For the limit function $W(t), t \in \mathbb{R}$, it follows from (4.10) and (4.11) that

$$\begin{aligned}&\|W(\cdot)\|_{L_\infty(\mathbb{R};H^1)} + \|\partial_t W(\cdot)\|_{L^{\mathrm{b}}_2(\mathbb{R};H^{-1})} \leqslant M, \\ &\|W(\cdot)\|^2_{L_\infty(\mathbb{R};H^1)} \leqslant M^2 e^{-\alpha\tau} + \gamma^{-1}\|g_0\|^2_{L^{\mathrm{b}}_2(\mathbb{R}_+;H^1)}, \quad \forall \tau \geqslant 0.\end{aligned} \tag{4.13}$$

Passing to the limit in (4.13) as $\tau \to \infty$, we obtain

$$\|W(\cdot)\|^2_{L_\infty(\mathbb{R};H^1)} \leqslant \gamma^{-1}\|g_0\|^2_{L^{\mathrm{b}}_2(\mathbb{R}_+;H^1)}. \tag{4.14}$$

We now apply Theorem 3.1, where we can assume that all the time symbols $\sigma_{\nu_n}(s)$ and solutions $u_{\nu_n}(x,t)$ are defined on the semiaxis $[-\ell, +\infty)$ instead of $[0, +\infty)$. Then, from (4.12), we conclude that $W(x,t), t \in \mathbb{R}$, is a weak solution of the 2d Euler system (1.1) on the entire time axis $\mathbb{R}$ with the symbol $\xi(t), t \in \mathbb{R}$, and, due to (4.14), $W(x,t)$ is bounded in the space

$$\mathcal{F}^{\mathrm{b}} := \left\{v(x,s),\ s \in \mathbb{R} \mid v \in L_\infty(\mathbb{R};H^1),\ \partial_t v \in L^{\mathrm{b}}_2(\mathbb{R};H^{-1})\right\}.$$

Let Π_+ be the operator of restriction to the semiaxis $\mathbb{R}_+$ of functions that are defined on the entire $\mathbb{R}$. Then by the construction of the functions $W(\cdot)$ and $\xi(\cdot)$, we clearly have

$$\Pi_+ W(\cdot) \in \mathcal{K}^+_{\Pi_+\xi}(0), \quad \Pi_+\xi \in \Sigma,$$

and moreover

$$\Pi_+ T(h)W(\cdot) \in \mathcal{K}^+_{\Pi_+ T(h)\xi}(0), \quad \Pi_+ T(h)\xi \in \Sigma, \quad \forall h \in \mathbb{R}. \tag{4.15}$$

We claim that $\Pi_+ W \in \mathfrak{A}_\Sigma$. Consider the following set

$$\mathfrak{B}_W := \{\Pi_+ W(h+s), s \geqslant 0 \mid h \in \mathbb{R}\}.$$

It follows from (4.15) that $\mathfrak{B}_W \subseteq \mathcal{K}^+_\Sigma(0)$. Therefore, $\operatorname{dist}_\rho(T(t)\mathfrak{B}_W, \mathfrak{A}_\Sigma) \to 0$ $(t \to +\infty)$. However, $\mathfrak{B}_W$ is strictly invariant, that is, $T(t)\mathfrak{B}_W = \mathfrak{B}_W$ for all $t \geqslant 0$. Hence, $\operatorname{dist}_\rho(\mathfrak{B}_W, \mathfrak{A}_\Sigma) = 0$ and $\mathfrak{B}_W \subseteq \mathfrak{A}_\Sigma$ since $\mathfrak{A}_\Sigma$ is a closed set. Therefore, $\Pi_+ W \in \mathfrak{A}_\Sigma$.

At the same time, we have established that

$$W_{\nu_{n''}} \to \Pi_+ W \text{ in } \Theta^{\text{loc}}_+ \quad \text{as } n \to \infty$$

(see (4.12) for $\ell = 0$ and $\Theta^{\text{loc}}_{0,+\infty} = \Theta^{\text{loc}}_+$). In particular for a large n''

$$W_{\nu_{n''}} \in \mathcal{O}(\Pi_+ W) \subseteq \mathcal{O}(\mathfrak{A}_\Sigma).$$

This contradicts (4.9). Therefore, (4.7) is true. Finally to prove (4.5), we apply (4.7) for $\mathfrak{B}^\nu = \mathfrak{A}^\nu_\Sigma$. □

Remark 4.1. Recall that $\Theta^{\text{loc}}_+ \subset C^{\text{loc}}(\mathbb{R}_+; H^\delta), 0 \leqslant \delta < 1$, and the convergencies (4.5) and (4.7) also hold in the strong metric of $C([0, M]; H^\delta)$ for every $M > 0$ (see Remark 2.1):

$$\operatorname{dist}_{C([0,M];H^\delta)}(\mathfrak{A}^\nu_\Sigma, \mathfrak{A}_\Sigma) \to 0 \ (\nu \to 0+), \tag{4.16}$$

$$\operatorname{dist}_{C([0,M];H^\delta)}(T(t)\mathfrak{B}^\nu, \mathfrak{A}_\Sigma) \to 0 \ (\nu \to 0+, t \to +\infty).$$

In conclusion, we formulate two assertions that follow from the well-posedness of the Cauchy problem for the dissipative 2d Navier–Stokes system (see, e.g., [24, 3]). We shall use these facts in the next section.

Proposition 4.2. *For any $\nu > 0$, the trajectory attractor $\mathfrak{A}^\nu_\Sigma$ of equation (3.1) is connected in the topological space Θ^{loc}_+.*

Proposition 4.3. *The family of sets $\{\mathfrak{A}^\nu_\Sigma, 0 < \nu \leqslant 1\}$ is upper semicontinuous in Θ^{loc}_+, i.e., for every $\nu, 0 < \nu \leqslant 1$, and for any neighborhood $\mathcal{O}(\mathfrak{A}^\nu_\Sigma)$ there is a $\delta = \delta(\nu, \mathcal{O}) > 0$ such that*

$$\mathfrak{A}^{\nu'}_\Sigma \subseteq \mathcal{O}(\mathfrak{A}^\nu_\Sigma), \quad \forall \nu' > 0, \quad |\nu' - \nu| < \delta.$$

5. **On minimal zero viscosity limit of trajectory attractors $\mathfrak{A}^\nu_\Sigma$.** Let $\mathfrak{A}_\Sigma$ be the trajectory attractor of the non-autonomous Euler system (1.1) with dissipation and $\mathfrak{A}^\nu_\Sigma$ be the trajectory attractor of the dissipative 2d Navier–Stokes system (3.1) for $\nu > 0$. We have proved in Sec. 4 that $\mathfrak{A}_\Sigma \subset \mathcal{B}_{R_0}$ and $\mathfrak{A}^\nu_\Sigma \subset \mathcal{B}_{R_0}$ for all $\nu \in (0, 1]$, where $\mathcal{B}_{R_0}$ is the ball in $\mathcal{F}^{\text{b}}_+$ (see (4.4)) whose radius R_0 is independent of ν:

$$\|\mathfrak{A}_\Sigma\|_{\mathcal{F}^{\text{b}}_+} \leqslant R_0 \quad \text{and} \quad \|\mathfrak{A}^\nu_\Sigma\|_{\mathcal{F}^{\text{b}}_+} \leqslant R_0, \quad \forall \nu, \ 0 < \nu \leqslant 1.$$

Recall that the ball $\mathcal{B}_0$ with topology Θ^{loc}_+ is a compact metric space with the metric $\rho(\cdot, \cdot)$. It was proved in Theorem 4.1 that

$$\operatorname{dist}_\rho(\mathfrak{A}^\nu_\Sigma, \mathfrak{A}_\Sigma) \to 0 \text{ as } \nu \to 0+. \tag{5.1}$$

Note that, in fact, the limit relation (5.1) is stronger than that of (4.16).

Recall that the set $\mathfrak{A}_\Sigma \subset \mathcal{B}_{R_0}$ is closed in $\mathcal{B}_{R_0}$. Let $\mathfrak{A}^{\min}_\Sigma$ be the minimal closed subset of $\mathfrak{A}_\Sigma$ which satisfies the attracting property (5.1), that is, by definition,

$$\lim_{\nu \to 0+} \operatorname{dist}_\rho\left(\mathfrak{A}^\nu_\Sigma, \mathfrak{A}^{\min}_\Sigma\right) = 0$$

and $\mathfrak{A}_\Sigma^{\min}$ belongs to every closed subset $\mathfrak{A}' \subseteq \mathfrak{A}_\Sigma$ for which

$$\lim_{\nu\to 0+} \operatorname{dist}_\rho(\mathfrak{A}_\Sigma^\nu, \mathfrak{A}') = 0.$$

We refer to the set $\mathfrak{A}_\Sigma^{\min}$ as the *minimal limit of the trajectory attractors* $\mathfrak{A}_\Sigma^\nu$ *as* $\nu \to 0+$.

We state that the set $\mathfrak{A}_\Sigma^{\min}$ exists and is unique. We have just to prove that

$$\mathfrak{A}_\Sigma^{\min} = \bigcap_{0<\delta\leqslant 1} \left[\bigcup_{0<\nu\leqslant\delta} \mathfrak{A}_\Sigma^\nu \right]_{\mathcal{B}_{R_0}}. \tag{5.2}$$

The set on the right-hand side of (5.2) is clearly nonempty. It is easy to prove that a point w belongs to the right-hand side of (5.2) if and only if there are points $w_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$, $n = 1, 2, \dots$, $\nu_n \to 0+$ as $n \to \infty$ such that $\rho(w_{\nu_n}, w) \to 0$ as $n \to \infty$. Due to (5.1), such a limit point w always belongs to $\mathfrak{A}_\Sigma$ and, moreover, it belongs to *every* closed attracting set $\mathfrak{A}'$. The set (5.2) is attracting for $\mathfrak{A}_\Sigma^\nu$ as $\nu \to 0+$. Indeed, assuming the converse, we see that there is a sequence $w_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$, such that $\nu_n \to 0+$ and

$$\operatorname{dist}_\rho\left(w_{\nu_n}, \mathfrak{A}_\Sigma^{\min}\right) \geqslant \varepsilon \tag{5.3}$$

for some $\varepsilon > 0$. Recall that $w_{\nu_n} \in \mathcal{B}_{R_0}$ and $\mathcal{B}_{R_0}$ is a compact metric space. Then, passing to a subsequence $\{w_{\nu_{n'}}\} \subset \{w_{\nu_n}\}$, we may assume that $\rho(w_{\nu_{n'}}, w') \to 0$ as $\nu_{n'} \to 0$ for some $w' \in \mathcal{B}_{R_0}$. Thus, by the above property, $w' \in \mathfrak{A}_\Sigma^{\min}$, that contradicts (5.3). We have proved that the set $\mathfrak{A}_\Sigma^{\min}$ defined in (5.2) is a minimal closed attracting subset of $\mathfrak{A}_\Sigma$.

Proposition 5.1. *The minimal limit* $\mathfrak{A}_\Sigma^{\min}$ *of trajectory attractors* $\mathfrak{A}_\Sigma^\nu$ *as* $\nu \to 0+$ *is a connected subset of* $\mathfrak{A}_\Sigma$ *in* $\mathcal{B}_{R_0}$.

Proof. Assume the converse. In this case, the set $\mathfrak{A}_\Sigma^{\min}$ is the union of two closed disjoint subsets $\mathfrak{A}_1$ and $\mathfrak{A}_2$, that is,

$$\mathfrak{A}_\Sigma^{\min} = \mathfrak{A}_1 \cup \mathfrak{A}_2 \quad \text{and} \quad \mathfrak{A}_1 \cap \mathfrak{A}_2 = \varnothing.$$

Since the metric space $\mathcal{B}_{R_0}$ is compact, there are two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathcal{B}_{R_0}$ such that $\mathfrak{A}_1 \subset \mathcal{O}_1$, $\mathfrak{A}_2 \subset \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$. Clearly, $\mathfrak{A}_\Sigma^{\min} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. Therefore, by (5.1), there is a number $\nu_0 > 0$ for which

$$\mathfrak{A}_\Sigma^\nu \subset \mathcal{O}_1 \cup \mathcal{O}_2, \quad \forall \nu,\ 0 < \nu \leqslant \nu_0. \tag{5.4}$$

Note that each set $\mathfrak{A}_\Sigma^\nu$ is connected (see Proposition 4.2), that is, $\mathfrak{A}_\Sigma^\nu \subset \mathcal{O}_1$ or $\mathfrak{A}_\Sigma^\nu \subset \mathcal{O}_2$ for all $\nu < \nu_0$. At the same time, since $\mathfrak{A}_\Sigma^{\min}$ is the *minimal* limit of $\mathfrak{A}_\Sigma^\nu$, we can find ν_1 and ν_2 such that

$$\mathfrak{A}_\Sigma^{\nu_1} \subset \mathcal{O}_1 \quad \text{and} \quad \mathfrak{A}_\Sigma^{\nu_2} \subset \mathcal{O}_2. \tag{5.5}$$

(otherwise, we can diminish $\mathfrak{A}_\Sigma^{\min}$). To be definite, assume that $0 < \nu_2 < \nu_1 < \nu_0$. Denote

$$\delta^* = \sup\{\delta : \mathfrak{A}_\Sigma^\nu \subset \mathcal{O}_2,\ \nu_2 \leqslant \nu < \nu_2 + \delta\}. \tag{5.6}$$

Note that $\nu_2 + \delta^* \leqslant \nu_1 < \nu_0$, (see (5.5)) and $\mathfrak{A}_\Sigma^{\nu_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$ since $\nu_2 + \delta^* < \nu_0$ (see (5.4)).

We now claim that $\mathfrak{A}_\Sigma^{\nu_2+\delta^*}$ cannot belong to $\mathcal{O}_2$. Indeed, if $\mathfrak{A}_\Sigma^{\nu_2+\delta^*} \subset \mathcal{O}_2$ then, by Proposition 4.3, there is a small $\delta_2 > 0$ such that $\mathfrak{A}_\Sigma^{\nu_2+\delta^*+\delta_2} \subset \mathcal{O}_2$. This contradicts the definition of δ^* in (5.6). At the same time, $\mathfrak{A}_\Sigma^{\nu_2+\delta^*}$ cannot belong to $\mathcal{O}_1$ neither. Indeed, if $\mathfrak{A}_\Sigma^{\nu_2+\delta^*} \subset \mathcal{O}_1$ then, by Proposition 4.3 again, there is a small $\delta_1 > 0$

such that $\mathfrak{A}_\Sigma^{\nu_2+\delta^*-\delta_1} \subset \mathcal{O}_1$, which contradicts the definition of δ^*. However, all this contradicts the inclusion $\mathfrak{A}_\Sigma^{\nu_2+\delta^*} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. This completes the proof. □

Recall that the set $\mathfrak{A}_\Sigma^{\min}$ is compact. In conclusion, we prove the following assertion.

Proposition 5.2. *The minimal limit $\mathfrak{A}_\Sigma^{\min}$ of trajectory attractors $\mathfrak{A}_\Sigma^{\nu}$ as $\nu \to 0+$ is strictly invariant with respect to the translation semigroup $\{T(t)\}$, that is*

$$T(t)\mathfrak{A}_\Sigma^{\min} = \mathfrak{A}_\Sigma^{\min}, \quad \forall t \geqslant 0. \tag{5.7}$$

Proof. Consider an arbitrary point $w \in \mathfrak{A}_\Sigma^{\min}$. By definition, there is a sequence $w_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$ such that $\rho(w_{\nu_n}, w) \to 0$ as $\nu_n \to 0+$. The translation semigroup $\{T(t)\}$ is continuous in $\Theta_+^{\rm loc}$ and, therefore, $\rho(T(t)w_{\nu_n}, T(t)w) \to 0$ as $\nu_n \to 0+$. Since every set $\mathfrak{A}_\Sigma^{\nu_n}$ is strictly invariant, we obtain that $T(h)w_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$. Thus, $T(t)w \in \mathfrak{A}_\Sigma^{\min}$ and we have proved that

$$T(t)\mathfrak{A}_\Sigma^{\min} \subseteq \mathfrak{A}_\Sigma^{\min}, \quad \forall h \geqslant 0.$$

Let us check the inverse inclusion. For any $t \geqslant 0$ and for an arbitrary point $w \in \mathfrak{A}_\Sigma^{\min}$ with corresponding $w_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$ such that $\rho(w_{\nu_n}, w) \to 0$ $(\nu_n \to 0+)$, we must find a point $W \in \mathfrak{A}_\Sigma^{\min}$ such that $T(t)W = w$. Since $\mathfrak{A}_\Sigma^{\nu_n}$ is strictly invariant, there is a point $W_{\nu_n} \in \mathfrak{A}_\Sigma^{\nu_n}$ such that $T(t)W_{\nu_n} = w_{\nu_n}$. The sequence $\{W_{\nu_n}\}$ belongs to the compact set $\mathcal{B}_{R_0}$. Passing to a subsequence $\{\nu_{n'}\}$, we see that $W_{\nu_{n'}} \to W$ $(n' \to \infty)$ for some $W \in \mathcal{B}_{R_0}$. Then $W \in \mathfrak{A}_\Sigma^{\min}$. Since $\{T(t)\}$ is continuous, we obtain $T(t)W_{\nu_{n'}} \to T(t)W$ $(n' \to \infty)$. However, $T(t)W_{\nu_{n'}} = w_{\nu_{n'}}$, and thus we have $w_{\nu_{n'}} \to T(t)W$ $(n' \to \infty)$. Recall that $w_{\nu_n} \to w$ $(n \to \infty)$. Hence, $T(t)W = w$ and we have proved that

$$\mathfrak{A}_\Sigma^{\min} \subseteq T(t)\mathfrak{A}_\Sigma^{\min}, \quad \forall h \geqslant 0.$$

We thus obtain (5.7). □

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