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FURTHER RESULTS ON MULTIPLE COVERINGS OF THE FARTHEST-OFF POINTS

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Abstract. Multiple coverings of the farthest-off points ((R, μ)-MCF codes) and the corresponding (ρ, μ)-saturating sets in projective spaces $\mathrm{PG}(N, q)$ are considered. We propose some methods which allow us to obtain new small $(1, \mu)$-saturating sets and short $(2, \mu)$-MCF codes with μ-density either equal to 1 (optimal saturating sets and almost perfect MCF-codes) or close to 1 (roughly $1+1/cq$, $c \geq 1$). In particular, we provide some algebraic constructions and bounds. Also, we classify minimal and optimal $(1, \mu)$-saturating sets in $\mathrm{PG}(2, q)$, q small.

1. Introduction

A code C is called (R, μ)-*multiple covering of the farthest-off points* (or (R, μ)-MCF for short) if for every word x at distance R from C there are at least μ codewords in the Hamming sphere $S(x, R)$, where R is the covering radius of C.

Multiple coverings can be viewed as a generalization of covering codes, see [5, 6]. Motivations for studying MCF codes come from the generalized football pool problem (see e.g. [20, 21, 25] and the references therein) and list decoding (see e.g. [31]).

In [6, 18, 19, 23, 24, 28, 29] results on MCF codes, mostly concerning the binary and the ternary cases, can be found. The development of this topic for arbitrary q was presented in [1, 15, 27] and in the recent paper [2]. In particular, important parameters of (R, μ)-MCF codes such as the μ-*density* and the μ-length function

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have been introduced in [2]. In the same paper the notion of a (ρ, μ)-saturating set as the geometrical counterpart of $(\rho + 1, \mu)$-MCF codes was proposed. Many useful results and constructions of MCF codes were obtained in [2] by geometrical methods. For an introduction to projective spaces over finite fields see [22].

The *μ-density* of an (R, μ)-MCF code C is the average value of $\frac{1}{\mu}\#(S(x, R) \cap C)$, where x is a word at distance R from C. The *μ-density* is greater than or equal to 1. If the minimum distance d of C is at least $2R - 1$, then the best μ-density among linear q-ary codes with same codimension r and covering radius R is achieved by the shortest ones. An important class of MCF codes are *almost perfect* and *perfect* MCF codes which correspond to *optimal* saturating sets. For these codes each word at distance R from the code belongs to exactly μ spheres centered in codewords; they have the best possible μ-density, i.e. equal to 1. The μ-length function $\ell_\mu(R, r, q)$ is defined as the smallest length n of a linear (R, μ)-MCF code with parameters $[n, n - r, d]_q R$, $d \geq 3$.

In this paper, we continue and develop the geometrical approach of [2] for constructing MCF codes with small μ-density. We present a number of $(1, \mu)$-saturating sets (and the corresponding $(2, \mu)$-MCF codes) with good parameters.

In the space $\mathrm{PG}(N, q)$, $q > 2$ even, we obtain $(1, \mu)$-saturating sets with $\mu = \frac{q-2}{2}$ such that the μ-density of the corresponding $(2, \mu)$-MCF code tends to 1 when N is fixed and q tends to infinity, see Section 4.

New results concerning $(1, \mu)$-saturating sets in planes $\mathrm{PG}(2, q)$ are presented in Section 6. We give some upper and lower bounds on the size of $(1, \mu)$-saturating sets, see Subsection 6.1. Also, we present many examples of optimal saturating sets using classical geometrical objects such as partitions of $\mathrm{PG}(2, q)$ in Singer point-orbits and sets of convenient lines, see Sections 5 and 7 and Subsection 6.3. Unfortunately, it is not always possible to construct almost perfect codes; in some cases we construct examples of $(1, \mu)$-saturating sets with μ-density roughly of the same order of magnitude of $1 + 1/cq$, $c > 1$. In general, we give families of μ-saturating sets of size less than $\mu\bar{\ell}(2, 3, q)$, where $\bar{\ell}(2, 3, q)$ is the minimum known size of a 1-saturating set in $\mathrm{PG}(2, q)$, see e.g. [3, 11, 13] and the references therein.

Another achievement of this paper is the classification of minimal and optimal $(1, \mu)$-saturating sets in $\mathrm{PG}(2, q)$ for small q, see Section 7.

The paper is organized as follows. In Section 2 we recall some definitions and results from [2] concerning MCF codes and (ρ, μ)-saturating sets; in Section 3 we focus on $(1, \mu)$-saturating sets. In Section 4 we deal with $(1, \mu)$-saturating sets in $\mathrm{PG}(N, q)$, q even, having small size. In Section 5 perfect and almost perfect $(2, \mu)$-MCF codes are constructed from classical geometrical objects. In Section 6 we present some constructions and bounds on $(1, \mu)$-saturating sets in $\mathrm{PG}(2, q)$. Finally, in Section 7, computational results on the classification of minimal and optimal $(1, \mu)$-saturating sets in $\mathrm{PG}(2, q)$ are presented.

2. Multiple coverings and (ρ, μ)-saturating sets

In the following we use the same notation as in [2]. An $(n, M, d)_q R$ code C is a code of length n, cardinality M, minimum distance d, and covering radius R, over the finite field $\mathbb{F}_q$ with q elements. If C is linear of dimension k over $\mathbb{F}_q$, then C is also said to be an $[n, k, d]_q R$ code. When either d or R are not relevant or unknown they can be omitted in the above notation. Let $\mathbb{F}_q^n$ be the linear space of dimension n over $\mathbb{F}_q$, equipped with the Hamming distance. The Hamming sphere of radius j

centered at $x \in \mathbb{F}_q^n$ is denoted by $S(x,j)$. The size $V_q(n,j)$ of such a sphere is

$$V_q(n,j) = \sum_{i=0}^{j} \binom{n}{i} (q-1)^i.$$

Let $\overline{S}(x,R)$ be the surface of the sphere $S(x,R)$. For an $(n,M)_q R$ code C, $A_w(C)$ denotes the number of codewords in C of weight w, and $f_\theta(e,C)$ denotes the number of codewords at distance θ from a vector e in $\mathbb{F}_q^n$; equivalently, $f_\theta(e,C) = \#(\overline{S}(e,\theta) \cap C)$. Let

$$\delta(C,R) = \frac{\sum_{x \in \mathbb{F}_q^n} \#\{c \in C \mid d(c,x) \le R\}}{q^n} = \frac{M \cdot V_q(n,R)}{q^n}$$

be the *density* of an $(n,M)_q R$ covering code C. In general, $\delta(C,R) \ge 1$, and equality holds if and only if C is a perfect code.

Definition 1 ([2, 6, 19, 20]).

1. An $(n,M)_q R$ code C is said to be an (R,μ) *multiple covering of the farthest-off points* ((R,μ)-MCF code for short) if for all $x \in \mathbb{F}_q^n$ such that $d(x,C) = R$ the number of codewords c such that $d(x,c) = R$ is at least μ.
2. An $(n,M,d(C))_q R$ code C is said to be an (R,μ) *almost perfect multiple covering of the farthest-off points* ((R,μ)-APMCF code for short) if for all $x \in \mathbb{F}_q^n$ such that $d(x,C) = R$ the number of codewords c such that $d(x,c) = R$ is exactly μ. If, in addition, $d(C) \ge 2R$ holds, then the code is called (R,μ) *perfect multiple covering of the farthest-off points* ((R,μ)-PMCF code for short).

In the literature, MCF codes are also called multiple coverings of deep holes, see e.g. [6, Chapter 14].

As already pointed out in [2, Sect. 2], there exists a connection between (R,μ)-MCF and (R,μ)-PMCF with weighted **m**-coverings; see e.g. [6, Sect. 13.1].

Definition 2 ([2]). Let C be an (R,μ)-MCF code. Let $\{x_1,\dots,x_{N_R(C)}\}$ be the set of vectors in $\mathbb{F}_q^n$ with distance R from C. The μ-*density* of C is

$$\gamma_\mu(C,R) = \frac{\sum_{i=1}^{N_R(C)} f_R(x_i,C)}{\mu N_R(C)}. \tag{1}$$

It is easily seen that $\gamma_\mu(C,R) \ge 1$, and that C is an (R,μ) APMCF code precisely when equality holds. In general, this parameter is a measure of the quality of an (R,μ)-MCF code.

The goal of this paper is the construction of (R,μ)-MCF codes with small μ-density.

We recall the following proposition from [2] concerning the μ-density of an (R,μ)-MCF code.

Proposition 1. *Let C be a linear $[n,k,d(C)]_q R$ code with $d(C) \ge 2R-1$. If C is (R,μ)-MCF, then*

$$\gamma_\mu(C,R) = \frac{\binom{n}{R} \cdot (q-1)^R - \binom{2R-1}{R-1} \cdot A_{2R-1}(C)}{\mu \cdot (q^{n-k} - V_q(n,R-1))}.$$

In the rest of the paper we will assume that

$$d(C) \ge 2R-1. \tag{2}$$

Let $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ be the number of errors that can be corrected by a code with minimum distance d. Note that under Condition (2), $R = t + 1$; equivalently, C is a quasi-perfect code in the classical sense.

The following definition of a (ρ, μ)-saturating set in $\mathrm{PG}(N, q)$ is given as in [2].

Definition 3. Let $S = \{P_1, \ldots, P_n\}$ be a subset of points of $\mathrm{PG}(N, q)$. Then S is said to be (ρ, μ)-saturating if:

(M1) S generates $\mathrm{PG}(N, q)$;
(M2) there exists a point Q in $\mathrm{PG}(N, q)$ which does not belong to any subspace of dimension $\rho - 1$ generated by the points of S;
(M3) every point Q in $\mathrm{PG}(N, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S, is such that the number of subspaces of dimension ρ generated by the points of S and containing Q, counted with multiplicity, is at least μ. The multiplicity m_T of a subspace T is computed as the number of distinct sets of $\rho + 1$ independent points contained in $T \cap S$.

Note that if any $\rho+1$ points of S are linearly independent (that is, the minimum distance of the corresponding code is at least $\rho + 2$), then

$$m_T = \binom{\#(T \cap S)}{\rho + 1}.$$

A (ρ, μ)-saturating n-set in $\mathrm{PG}(N, q)$ is called *minimal* if it does not contain a (ρ, μ)-saturating $(n - 1)$-set in $\mathrm{PG}(N, q)$.

Let S be a (ρ, μ)-saturating n-set in $\mathrm{PG}(n-k-1, q)$. The set S is called *optimal* (ρ, μ)*-saturating set* ((ρ, μ)-OS set for short) if every point Q in $\mathrm{PG}(n - k - 1, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S, is such that the number of subspaces of dimension ρ generated by the points of S and containing Q, counted with multiplicity, is exactly μ.

An $[n, k]_q R$ code C with $R = \rho + 1$ *corresponds* to a (ρ, μ)-saturating n-set S in $\mathrm{PG}(n-k-1, q)$ if C admits a parity-check matrix whose columns are homogeneous coordinates of the points in S.

By [2, Proposition 3.6], a linear $[n, k]_q R$ code C corresponding to a (ρ, μ)-saturating n-set S in $\mathrm{PG}(n - k - 1, q)$ is a $(\rho + 1, \mu)$-MCF code. Also, if S is a (ρ, μ)-OS set, the corresponding linear $[n, k]_q R$ code C is a $(\rho + 1, \mu)$ APMCF code with $\gamma_\mu(C, \rho + 1) = 1$. If in addition $d(C) = 2R$, then it is a $(\rho + 1, \mu)$ PMCF code.

3. $(1, \mu)$-saturating sets

For $\rho = 1$ Conditions (M1)-(M3) read as follows:

(M1) S generates $\mathrm{PG}(N, q)$;
(M2) S is not the whole $\mathrm{PG}(N, q)$;
(M3) every point Q in $\mathrm{PG}(N, q)$ not belonging to S is such that the number of secants of S through Q is at least μ, counted with multiplicity. The multiplicity m_ℓ of a secant ℓ is computed as

$$m_\ell = \binom{\#(\ell \cap S)}{2}.$$

As already observed in [2], from Conditions (M1)-(M3) the following holds.

Proposition 2. (i) *Let C be the linear $[n, n-N-1]_q 2$ code corresponding to a $(1,\mu)$-saturating n-set S. Then $\mu\gamma_\mu(C,2)$ is equal to the average number of secants of S, counted with multiplicity, through a fixed point $Q \in \mathrm{PG}(N,q) \setminus S$.*

(ii) *Let S be a $(1,\mu)$-saturating set in $\mathrm{PG}(N,q)$. Then S is a $(1,\mu)$-*OS set *precisely when each point $Q \in \mathrm{PG}(N,q) \setminus S$ belongs to exactly μ secants of S, counted with multiplicity.*

Note that as $R = 2$, the condition $d(C) > 2R-1$ reads as $d(C) > 3$.

Let $B_3(S)$ denote the number of triples of collinear points in S. The following is a characterization of $(1,\mu)$-*OS sets* in $\mathrm{PG}(N,q)$; see [2].

Proposition 3. *Let S be a $(1,\mu)$-saturating set in $\mathrm{PG}(N,q)$. Let C_S be the $[n, n-N-1]_q 2$ code corresponding to S.*

(i) *For the μ-density of C_S it holds that*

$$\gamma_\mu(C_S,2) = \frac{\frac{n-1}{2}(q-1) - \frac{3}{n}B_3(S)}{\mu \cdot \left(\frac{\#\mathrm{PG}(N,q)}{n} - 1\right)}. \tag{3}$$

(ii) *The set S is a $(1,\mu)$-*OS set *if and only if*

$$\frac{n-1}{2}(q-1) - \frac{3}{n}B_3(S) = \mu \cdot \left(\frac{\#\mathrm{PG}(N,q)}{n} - 1\right).$$

It is clear that if q, N, μ are fixed, then the best μ-density is achieved for small n and therefore, the following parameter seems to be relevant in this context.

Definition 4. The μ-length function $\ell_\mu(2,r,q)$ is the smallest length n of a linear $(2,\mu)$-MCF code with parameters $[n, n-r, d]_q 2$, $d \geq 3$, or equivalently the smallest cardinality of a $(1,\mu)$-saturating set in $\mathrm{PG}(r-1,q)$. For $\mu = 1$, $\ell_1(2,r,q)$ is the usual length function $\ell(2,r,q)$ [5,6,11] for 1-fold coverings.

Remark 1 ([2])**.** A number μ of disjoint copies of a 1 -saturating set in $\mathrm{PG}(N,q)$ give rise to a $(1,\mu)$-saturating set in $\mathrm{PG}(N,q)$. Therefore,

$$\ell_\mu(2,r,q) \leq \mu\ell(2,r,q). \tag{4}$$

Denote by $\gamma_\mu(2,r,q)$ the minimum μ-density of a linear $(2,\mu)$-MCF code of codimension r over $\mathbb{F}_q$. Let $\delta(2,r,q)$ be the minimum density of a linear code with covering radius 2 and codimension r over $\mathbb{F}_q$, then

$$\gamma_\mu(2,r,q) \leq \frac{\frac{1}{2}(\mu\ell(2,r,q)-1)(q-1)}{\mu \cdot (\frac{\#\mathrm{PG}(r-1,q)}{\mu\ell(2,r,q)} - 1)} - 1 \sim \mu\delta(2,r,q). \tag{5}$$

The same inequalities clearly hold for the best known lengths and densities, denoted, respectively, by $\overline{\ell}_\mu(2,r,q)$, $\overline{\ell}(2,r,q)$, $\overline{\gamma}_\mu(2,r,q)$, and $\overline{\delta}(2,r,q)$:

$$\overline{\ell}_\mu(2,r,q) \leq \mu\overline{\ell}(2,r,q). \tag{6}$$

$$\overline{\gamma}_\mu(2,r,q) \lesssim \mu\overline{\delta}(2,r,q). \tag{7}$$

From Equations (4)–(7), results for parameters $\ell_\mu(2,r,q)$, $\overline{\ell}_\mu(2,r,q)$, $\gamma_\mu(2,r,q)$, and $\overline{\gamma}_\mu(2,r,q)$, can be immediately obtained from the vast body of literature on 1-saturating sets in finite projective spaces; see e.g. [3–7, 10, 11, 13–17, 26, 30].

The aim of the present paper is to construct $(1,\mu)$-saturating sets in $\mathrm{PG}(N,q)$ giving rise to $(2,\mu)$-MCF codes with cardinality and density smaller to those in Inequalities (4)–(7).

4. Small $(1,\mu)$-saturating sets in $\mathrm{PG}(N,q)$, q even

In $\mathrm{PG}(N,q)$, q even, small 1-saturating sets have been constructed; see [10, 14]. Roughly speaking, if N is odd, then in $\mathrm{PG}(N,q)$, q even, there are 1-saturating sets whose size is of the same order of magnitude as $2q^{(N-1)/2}$. If N is even, then there exist 1-saturating sets of size about $t_2(q)q^{(N-2)/2}$, where $t_2(q)$ denotes the size of the smallest saturating set in $\mathrm{PG}(2,q)$. When q is a square, $t_2(q) \le 3\sqrt{q}-1$ holds [7]. Therefore, for q a square,

$$\bar{\ell}(2,r,q) \sim c(r)q^{(r-2)/2}, \text{ with } c(r) = \begin{cases} 2 & \text{for even } r \\ 3 & \text{for odd } r \end{cases},$$

and the following results on density about the order of magnitude of $\overline{\gamma}_\mu(2,r,q)$ can be easily obtained, see (7):

$$\overline{\gamma}_\mu(2,r,q) \sim \frac{1}{2}\mu c(r)^2. \tag{8}$$

In this section we significantly improve (8) for the case $\mu = \frac{q-2}{2}$, see (9) and (10).

For $i = 0,\ldots,N$, let π_i be the subset of $\mathrm{PG}(N,q)$ defined as follows:

$$\pi_i := \{(x_0,\ldots,x_N) \mid x_0 = x_1 = \ldots = x_{i-1} = 0, x_i \ne 0\},$$

where $x_0,\ldots,x_N$ are the homogeneous coordinates of a point in $\mathrm{PG}(N,q)$. Clearly, $\mathrm{PG}(N,q)$ is the disjoint union of $\pi_0 \cup \pi_1 \cup \ldots \cup \pi_N$; also, each π_i with $i < N$ can be viewed as an affine space $\mathrm{AG}(N-i,q)$, whereas π_N consists of a single point, namely $(0,\ldots,0,1)$.

Lemma 1. *In an affine space $\mathrm{AG}(N,q)$ with q even, $q>2$, there exists a subset K of size less than or equal to*

$$\left\lceil \frac{1+\sqrt{4q-7}}{2} \right\rceil q^{(N-1)/2}$$

such that every point of $\mathrm{AG}(N,q)\setminus K$ belongs to at least $(q-2)/2$ distinct secants of K.

Proof. Assume that N is even. Then there exists a translation cap K in $\mathrm{AG}(N,q)$ of size $q^{N/2}$ (see (2.6) in [14]). By Proposition 2.5 in [14], every point of $\mathrm{AG}(N,q)\setminus K$ belongs to $(q-2)/2$ distinct secants of K. Then the assertion follows from

$$\sqrt{q} \le \left\lceil \frac{1+\sqrt{4q-7}}{2} \right\rceil.$$

To deal with the case of odd dimension N, we set $s = \left\lceil \frac{1+\sqrt{4q-7}}{2} \right\rceil$ and we fix s distinct elements $a_1,\ldots,a_s$ in $\mathbb{F}_q$. Note that

$$\binom{s}{2} \ge \frac{q-2}{2}.$$

Let K' be a translation cap in $\mathrm{AG}(N-1,q)$ of size $q^{(N-1)/2}$. Let

$$K = \{(P,a_i) \mid P \in K', i = 1,\ldots,s\} \subset \mathrm{AG}(N,q).$$

By the doubling construction (see e.g. Remark 2.14 in [14]), we have that for each pair of integers i,j with $1 \le i < j \le s$, the subset of K

$$K_{i,j} = \{(P,a_i),(P,a_j) \mid P \in K'\}$$

is a complete cap of size $q^{(N-1)/2}$ in $\mathrm{AG}(N,q)$.

Therefore, every point of $\mathrm{AG}(N,q)\setminus K$ is covered by at least $\binom{s}{2}\geq (q-2)/2$ secants of K. □

A slight improvement of Lemma 1 can be obtained when q is a square and there exists a translation cap of size $q^{3/2}$ in $\mathrm{AG}(3,q)$.

Lemma 2. *Assume that q is a square and there exists a translation cap of size $q^{3/2}$ in $\mathrm{AG}(3,q)$. Then in an affine space $\mathrm{AG}(N,q)$ with q even, $N>2$, there exists a subset K of size less than or equal to $q^{N/2}$ such that every point of $\mathrm{AG}(N,q)\setminus K$ belongs to at least $(q-2)/2$ distinct secants of K.*

Proof. The assertion for N even follows from the proof of Lemma 1. Here it is possible to construct a translation cap in $\mathrm{AG}(N,q)$ of size $q^{N/2}$ also for odd $N>1$ by using Proposition 2.8 in [14]. □

Remark 2. The hypothesis of Lemma 2 are satisfied for instance for $q=16$ (see [16, Lemma 3.1]).

Consider the partition

$$\mathrm{PG}(N,q)=\pi_0\cup\pi_1\cup\ldots\cup\pi_{N-1}\cup\{(0,\ldots,0,1)\}.$$

As each π_i is an $\mathrm{AG}(N-i,q)$, by Lemma 1 we are able to find sets K_i contained in π_i, of size at most

$$\left\lceil\frac{1+\sqrt{4q-7}}{2}\right\rceil q^{(N-i-1)/2},$$

and such that each point of $\pi_i\setminus K_i$ belongs at least $(q-2)/2$ distinct secants of K_i.

Then the following result is obtained by considering the union of the K_i's, together with the point $\{(0,\ldots,0,1)\}$.

Theorem 1. *In a projective space $\mathrm{PG}(N,q)$ with q even, $q>2$, there exists a subset S of size equal to*

$$1+\left\lceil\frac{1+\sqrt{4q-7}}{2}\right\rceil(q^{\frac{N-1}{2}}+q^{\frac{N-2}{2}}+\ldots+q^{\frac{1}{2}}+1)$$

such that every point of $\mathrm{PG}(N,q)\setminus S$ belongs to at least $(q-2)/2$ distinct secants of S.

If in addition q is a square and there exists a translation cap of size $q^{3/2}$ in $\mathrm{AG}(3,q)$, then S can be chosen in such a way that

$$|S|=q^{\frac{N}{2}}+q^{\frac{N-1}{2}}+q^{\frac{N-2}{2}}+\ldots+q+q^{\frac{1}{2}}+1=\frac{q^{(N+1)/2}-1}{q^{1/2}-1}.$$

As for the density, we have that if $C(N,q)$ is the code corresponding to the set S of Theorem 1, then by Proposition 3(i),

$$\gamma_{\frac{q-2}{2}}(C(N,q),2)\leq\frac{\frac{|S|-1}{2}(q-1)}{\frac{q-2}{2}(\frac{|\mathrm{PG}(N,q)|}{|S|}-1)}=\frac{q-1}{q-2}\cdot\frac{|S|(|S|-1)}{|\mathrm{PG}(N,q)|-|S|},$$

whence

$$\gamma_{\frac{q-2}{2}}(C(N,q),2)<\frac{q-1}{q-2}\cdot\frac{q^N+10q^{N-\frac{1}{2}}+25q^{N-1}}{q^N+q^{N-1}+\ldots+q+1-q^{\frac{N}{2}}-5q^{\frac{N-1}{2}}}. \tag{9}$$

Interestingly, if we fix codimension N and let q vary, we get that

$$\lim_{q\to\infty}\gamma_{\frac{q-2}{2}}(C(N,q),2)=1.$$

The situation is even more interesting if the hypothesis of Lemma 2 are satisfied. In this case we obtain the following very nice formula for $\gamma_{\frac{q-2}{2}}(C(N,q),2)$, which is independent of N:

$$\gamma_{\frac{q-2}{2}}(C(N,q),2) = \frac{q-1}{q-2} \cdot \frac{\frac{q^{(N+1)/2}-1}{q^{1/2}-1} - 1}{\frac{q^{(N+1)/2}+1}{q^{1/2}+1} - 1} = \frac{q-1}{q-2} \cdot \frac{\sqrt{q}+1}{\sqrt{q}-1} = \frac{(\sqrt{q}+1)^2}{q-2}. \tag{10}$$

In particular, for $q = 16$ we have $|S| = \frac{4^{N+1}-1}{3}$, and we get that $\gamma_{\frac{q-2}{2}}(C(N,q),2)$ is equal to $25/14$ independently of N.

5. Perfect and almost perfect $(2,\mu)$-MCF codes from classical geometrical objects in $\mathrm{PG}(N,q)$

In this section we construct optimal $(1,\mu)$-saturating n-sets ($(1,\mu)$-OS n-sets) in $\mathrm{PG}(N,q)$. Recall that an $[n, n-(N+1), d(C)]$ code C corresponding to a $(1,\mu)$-OS set is an almost perfect $(2,\mu)$-MCF code (APMCF code) if $d(C) = 3$ or perfect $(2,\mu)$-MCF code (PMCF code) if $d(C) = 4$, see Sections 2 and 3. The μ-density of any APMCF or PMCF code C is $\gamma_\mu(C,2) = 1$.

In Proposition 9 we obtain MCF codes C with μ-density $\gamma_\mu(C,2) = 1 + \frac{1}{q}$.

Proposition 4. *Let $q = 2^v$ be even. Let $s = 2^k$, $1 \le k \le v$. Finally, let $n = (s-1)q + s$ and let $\mathcal{K}$ be a maximal (n,s)-arc in $\mathrm{PG}(2,q)$. Then $\mathcal{K}$ is a $(1,\mu)$-OS n-set with parameters*

$$n = (s-1)q + s, \;\; \mu = \frac{1}{2}(s-1)n.$$

An $[n, n-3, d(C_\mathcal{K})]_q 2$ code $C_\mathcal{K}$ corresponding to $\mathcal{K}$ is a $(2,\mu)$-PMCF code if $s = 2$ and a $(2,\mu)$-APMCF code if $s \ge 4$.

Proof. Every line of $\mathrm{PG}(2,q)$ meets a maximal (n,s)-arc either in zero or in s points whence it follows that every point of $\mathrm{PG}(2,q)$ outside the arc lies on $\frac{n}{s}$ s-secant. Therefore, $R = 2$ and $\mu = \binom{s}{2}\frac{n}{s} = \frac{1}{2}(s-1)n$. For $s = 2$ the minimum distance $d(C_\mathcal{K})$ is 4 as in this case the arc $\mathcal{K}$ is a hyperoval. □

Proposition 5. *An elliptic quadric $\mathcal{Q}$ in $\mathrm{PG}(3,q)$ is a $(1,\mu)$-OS n-set with parameters*

$$n = q^2 + 1, \;\; \mu = \frac{1}{2}(q^2 - q).$$

A $[n, n-4, 4]_q 2$ code $C_\mathcal{Q}$ corresponding to $\mathcal{Q}$ is a $(2,\mu)$-PMCF code.

Proof. Every point outside of the elliptic quadric $\mathcal{Q}$ lies on $\frac{1}{2}(q^2-q)$ bisecants. □

Proposition 6. *Let q be square. A Hermitian curve $\mathcal{H}$ in $\mathrm{PG}(2,q)$ is a $(1,\mu)$-OS n-set with parameters*

$$n = q\sqrt{q} + 1, \;\; \mu = \frac{1}{2}(q^2 - q).$$

An $[n, n-3, 3]_q 2$ code $C_\mathcal{H}$ corresponding to $\mathcal{H}$ is a $(2,\mu)$-APMCF code.

Proof. Every point outside of the Hermitian curve lies on $q - \sqrt{q}$ lines that are $(\sqrt{q}+1)$-secants and on $\sqrt{q}+1$ tangent lines to $\mathcal{H}$. Therefore $\mu = (q-\sqrt{q})\binom{\sqrt{q}+1}{2} = \frac{1}{2}(q^2-q)$. □

Proposition 7. *Let q be square. A Baer subplane $\mathcal{B}$ in* $\mathrm{PG}(2,q)$ *is a* $(1,\mu)$*-OS* n*-set with parameters*

$$n = q + \sqrt{q} + 1,\ \ \mu = \frac{1}{2}(q + \sqrt{q}).$$

An $[n, n-3, 3]_q 2$ *code* $C_{\mathcal{B}}$ *corresponding to* $\mathcal{B}$ *is a* $(2,\mu)$*-APMCF code.*

Proof. Here $\mu = \binom{\sqrt{q}+1}{2} = \frac{1}{2}(q + \sqrt{q})$ as every point outside the Baer subplane lies exactly on one $(\sqrt{q}+1)$-secant of the subplane and on q tangents to $\mathcal{B}$. □

Proposition 8. *Let* $\mathcal{S} \subset \mathrm{PG}(N,q)$ *be a set such that through each point* $P \in \mathcal{S}$ *the number of* i*-secants of* $\mathcal{S}$ *is a fixed integer* x_i*. Then* $\mathrm{PG}(N,q) \setminus \mathcal{S}$ *is a* $(1,\mu)$*-OS* n*-set with parameters*

$$n = \frac{q^{N+1}-1}{q-1} - |\mathcal{S}|,\ \ \mu = \sum_{i=1}^{\frac{q^N-1}{q-1}} x_i \binom{q+1-i}{2}.$$

Proof. It is enough to observe that each i-secant of $\mathcal{S}$ becomes a $(q+1-i)$-secant of $\mathrm{PG}(N,q) \setminus \mathcal{S}$. □

Corollary 1. (i) *Let* $q = 2^v$ *be even and* $s = 2^k$, $1 \le k \le v-1$*. Consider* $n = (s-1)q + s$ *and let* $\mathcal{K}$ *be a maximal* (n,s)*-arc in* $\mathrm{PG}(2,q)$*. The set* $\mathcal{S} = \mathrm{PG}(2,q) \setminus \mathcal{K}$ *is a* $(1,\mu)$*-OS* n*-set with*

$$n = q^2 + q + 1 - (s-1)q - s,\ \ \mu = (q+1)\binom{q+1-s}{2}.$$

An $[n, n-3, 3]_q 2$ *code* $C_{\mathcal{S}}$ *corresponding to* $\mathcal{S}$ *is a* $(2,\mu)$*-APMCF code.*

(ii) *Let q be square and let $\mathcal{H}$ be a Hermitian curve in* $\mathrm{PG}(2,q)$*. The set* $\mathcal{S} = \mathrm{PG}(2,q) \setminus \mathcal{H}$ *is a* $(1,\mu)$*-OS* n*-set with parameters*

$$n = q^2 - q\sqrt{q} + q,\ \ \mu = \binom{q}{2} + q\binom{q-\sqrt{q}}{2}.$$

An $[n, n-3, 3]_q 2$ *code* $C_{\mathcal{S}}$ *corresponding to* $\mathcal{S}$ *is a* $(2,\mu)$*-APMCF code.*

(iii) *Let q be square and consider a Baer subplane $\mathcal{B}$ in* $\mathrm{PG}(2,q)$*. The set* $\mathcal{S} = \mathrm{PG}(2,q) \setminus \mathcal{B}$ *is a* $(1,\mu)$*-OS* n*-set with parameters*

$$n = q^2 - \sqrt{q},\ \ \mu = (\sqrt{q}+1)\binom{q-\sqrt{q}}{2} + (q-\sqrt{q})\binom{q}{2}.$$

An $[n, n-3, 3]_q 2$ *code* $C_{\mathcal{S}}$ *corresponding to* $\mathcal{S}$ *is a* $(2,\mu)$*-APMCF code.*

(iv) *Let* $\mathcal{S} \subset \mathrm{PG}(N,q)$ *be a set such that* $\mathrm{PG}(N,q) \setminus \mathcal{S}$ *is a* k*-cap,* $N \ge 2$, $k \ge 2$*. Then* $\mathcal{S}$ *is a* $(1,\mu)$*-OS* n*-set with parameters*

$$n = \frac{q^{N+1}-1}{q-1} - k,\ \ \mu = (k-1)\binom{q-1}{2} + \left(\frac{q^N-1}{q-1} - k + 1\right)\binom{q}{2}.$$

An $[n, n-N-1, 3]_q 2$ *code* $C_{\mathcal{S}}$ *corresponding to* $\mathcal{S}$ *is a* $(2,\mu)$*-APMCF code.*

Proof. The claims follow directly from Proposition 8 and properties of the geometrical objects as pointed out in the previous propositions. □

Proposition 9. *Let q be odd. An oval $\mathcal{O}$ in* $\mathrm{PG}(2,q)$ *is a* $(1,\mu)$*-saturating* n*-set with parameters*

$$n = q + 1,\ \ \mu = \frac{1}{2}(q-1).$$

An $[n, n-3, 4]_q 2$ *code* $C_{\mathcal{O}}$ *corresponding to* $\mathcal{O}$ *is a* $(2, \mu)$*-MCF code with* μ*-density* $\gamma_\mu(C_{\mathcal{O}}, 2) = 1 + \frac{1}{q}$.

Proof. Each internal point of the oval lies on $\frac{1}{2}(q+1)$ bisecants, whereas every external point lies on $\frac{1}{2}(q-1)$ bisecants. Therefore, $\mu = \frac{1}{2}(q-1)$. By Proposition 2(i),

$$\mu\gamma_\mu(C_{\mathcal{O}}, 2) = \frac{\frac{1}{2}q(q-1)\cdot\frac{1}{2}(q+1) + \frac{1}{2}q(q+1)\cdot\frac{1}{2}(q-1)}{\frac{1}{2}q(q-1) + \frac{1}{2}q(q+1)} = \frac{(q+1)(q-1)}{2q}. \qquad \square$$

6. Constructions of small $(1, \mu)$-saturating sets in PG$(2, q)$

In this section, we summarize some results concerning $(1, \mu)$-saturating sets in projective planes PG$(2, q)$.

6.1. Bounds.

In this subsection we present some upper and lower bounds on the size of minimal $(1, \mu)$-saturating sets in PG$(2, q)$.

Proposition 10. *In* PG$(2, q)$*, for a minimal* $(1, \mu)$*-saturated set* S *the following holds.*

(i)

$$\mu \leq (q+1)\binom{q}{2}.$$

(ii)

$$|S| \leq \begin{cases} q+\mu+1 & \text{if} \quad \mu \leq q+2 \\ \min\{q+\mu,\ q^2+q\} & \text{if} \quad \mu \geq q+3 \end{cases}.$$

Proof. (i) Any (q^2+q)-set in PG$(2, q)$ is a $\left((q+1)\binom{q}{2}\right)$-saturating set.

(ii) Let S be $(q+\mu+1)$-set in PG$(2, q)$, $q > 2$, $\mu \leq q+2$ and consider a point $P \in \mathrm{PG}(2, q) \setminus S$. On the $q+1$ lines through P there are at least μ pairs of points of S and therefore S is a $(1, \mu)$-saturating set, possibly not minimal.

If $\mu \geq q+3$ and $|S| = q+\mu$, then at least one triple of points of S lies on the same line through P. So, there are at least $\binom{3}{2} + (\mu - 3) = \mu$ pairs of points of S. The bound $|S| \leq q^2 + q$ holds due to Condition (M2). $\square$

We recall the following proposition from [2] concerning bounds on the smallest possible size of $(1, \mu)$-saturating sets.

Proposition 11. *For the length function* $\ell_\mu(2, 3, q)$*, the following relations hold.*

(i) *Trivial bound:*

$$\ell_\mu(2, 3, q) \geq \sqrt{2\mu q}. \tag{11}$$

(ii) *Probabilistic bound:*

$$\ell_\mu(2, 3, q) < 66\sqrt{\mu q \ln q}, \quad \text{if } \mu < 121 q \log q. \tag{12}$$

(iii) *Baer bound for* q *a square:*

$$\ell_\mu(2, 3, q) \leq \mu(3\sqrt{q} - 1).$$

In some cases we can do better than the trivial lower bound mentioned above.

Proposition 12. *Let A be a $(1,\mu)$-saturating set in $\mathrm{PG}(2,q)$ of size k. Suppose that ℓ and ℓ' are an r-secant and an s-secant of A respectively, with $s \geq r$. Then*

$$k \geq \min \left\{ \begin{array}{l} r+\frac{1}{2}+\sqrt{(s-r)(s+r-2)+2\mu(q-r+1)+\frac{5}{4}}, \\ r+\frac{1}{2}+\sqrt{(s-r)(s+r-1)+2\mu(q-r)+\frac{1}{4}} \end{array} \right\}.$$

Proof. There are $k-r-s+1$ or $k-r-s$ points of A not contained in $\ell \cup \ell'$, depending on the point $\ell \cap \ell'$ belonging or not to A. Since A is a $(1,\mu)$-saturating set, then each point of ℓ not belonging to A is covered at least μ times. Therefore, in the first case we obtain

$$(k-r-s+1)(s-1)+\binom{k-r-s+1}{2}+\binom{r}{2} \geq \mu(q+1-r)$$

which implies

$$k \geq r+\frac{1}{2}+\sqrt{(s-r)(s+r-2)+2\mu(q-r+1)+\frac{5}{4}},$$

whereas in the second case

$$(k-r-s)s+\binom{k-r-s}{2}+\binom{r}{2} \geq \mu(q-r)$$

implies

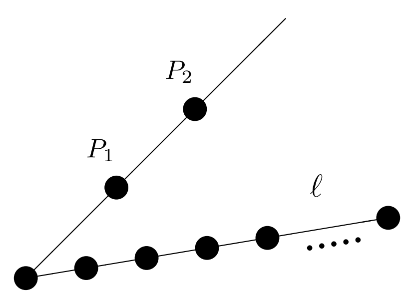
$$k \geq r+\frac{1}{2}+\sqrt{(s-r)(s+r-1)+2\mu(q-r)+\frac{1}{4}}.$$

Note that in the second case we do not consider how many times the intersection point of ℓ and ℓ' is covered. □

6.2. Constructions. In this subsection we explicitly construct examples of $(1,2)$-saturating sets in $\mathrm{PG}(2,q)$ of sizes $q+2$ and $q+3$.

Theorem 2. *There exists a minimal $(1,2)$-saturating set of size $q+3$ in $\mathrm{PG}(2,q)$, with $q=p^h$, p prime. Its stabilizer in $\mathrm{P\Gamma L}(3,q)$ has size $hq(q-1)$. The corresponding $[q+3,q]_q 2$ code C is a $(2,2)$-MCF with μ-density $\gamma_2(C,2) \approx 1+\frac{1}{2q}$.*

Proof. Let ℓ be a line and consider two points $P_1, P_2 \notin \ell$. It is straightforward to check that $A=\ell \cup \{P_1,P_2\}$ is a minimal $(1,2)$-saturating set. Consider now two of such sets $A_1=\ell_1 \cup \{P_1,Q_1\}$ and $A_2=\ell_2 \cup \{P_2,Q_2\}$, with $P_i,Q_i \notin \ell_i$ and let $X_i,Y_i \in \ell_i$. Let $\varphi : \mathrm{PG}(2,q) \to \mathrm{PG}(2,q)$ be the collineation such that $\varphi(P_1)=P_2$, $\varphi(Q_1)=Q_2$, $\varphi(X_1)=X_2$, and $\varphi(Y_1)=Y_2$. It sends A_1 in A_2 and therefore A_1 and A_2 are projective equivalent.



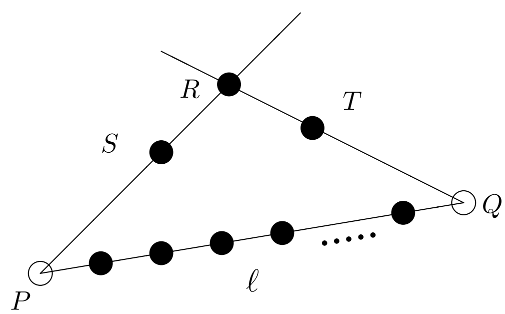
The line ℓ can be chosen in q^2+q+1 different ways. P_1 and P_2 can be taken in an arbitrary way in $\mathrm{PG}(2,q)\setminus\ell$. Therefore A can be chosen in $q^2(q^2-1)(q^2+q+1)$. Hence, its stabilizer in $\mathrm{PG}L(3,q)$ has size

$$\frac{|\mathrm{P\Gamma L}(3,q)|}{hq^2(q^2-1)(q^2+q+1)} = \frac{hq^3(q^2-1)(q^3-1)}{q^2(q^2-1)(q^2+q+1)} = hq(q-1).$$

The μ-density of the code C can be calculated by (3) where clearly $n=q+3$, $B_3(S)=1+\binom{q+1}{3}$, $\#\mathrm{PG}(N,q)=q^2+q+1$. □

Theorem 3. *Let ℓ be a line and $P,Q,R,S,$ and T be points such that P,R,S and Q,R,T are collinear, and $P,Q\in\ell$. Then $A=(\ell\setminus\{P,Q\})\cup\{R,S,T\}\subset \mathrm{PG}(2,q)$ is a minimal $(1,2)$-saturating $(q+2)$-set for all $q\geq 4$.*

Proof. All the points on the line PR are covered once by RS and once by the lines joining T and the points of $\ell\setminus\{P\}$. The points of the line PT are covered twice by the lines through R and S and the points of $\ell\setminus\{P\}$. A similar argument holds for the points on the lines QR and QS. The points on the lines through P distinct from ℓ, PR, and PT are covered at least two times by the lines through T,R,S and the points of $\ell\setminus\{P\}$. A similar argument holds for the points on the lines through Q distinct from ℓ, QR, and QS.

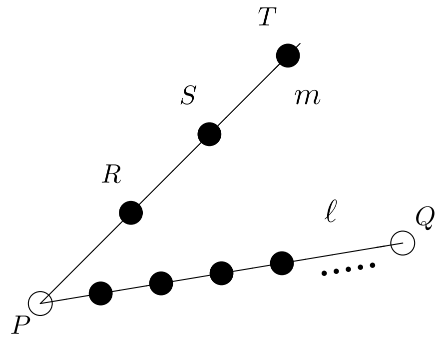


This example is minimal. In fact it is not possible to delete T (resp. S) since the points on the line SR (resp. RT) would not be covered twice. $A\setminus\{R\}$ does not cover R. Let $X\in\ell\setminus\{P,Q\}$, then $A\setminus\{X\}$ does not cover twice the point $TX\cap m$. □

Theorem 4. *Let ℓ be a line and $P,Q,R,S,$ and T be points such that P,R,S,T are collinear, and $P,Q\in\ell$. Then $A=(\ell\setminus\{P,Q\})\cup\{R,S,T\}$ is a minimal $(1,2)$-saturating $(q+2)$-set for all $q\geq 4$.*

Proof. Let s be a line through Q different from ℓ and let m be the line containing R,S,T.

Let $X=m\cap s$. If $X\in\{R,S,T\}$, then every point of $s\setminus\{Q\}$ is covered twice by the lines through the points $\{R,S,T\}\setminus\{X\}$ and the $q-1$ points of ℓ. If $X\notin\{R,S,T\}$, then every point of $s\setminus\{Q,X\}$ is covered three times by the lines through R,S,T and the $q-1$ points of ℓ.



It is not possible to delete R, S, T since in this case the points on m are covered only once. Also, it is not possible to delete a point $X \in \ell$, since in this case in the line XT only $(q-2)$ points are covered twice. Hence A is a minimal $(1,2)$-saturating set of size $(q+2)$. □

6.3. $(1,\mu)$-SATURATING SETS AND PARTITIONS OF $\mathrm{PG}(2,q)$ IN SINGER POINT-ORBITS. In [2, 8, 9, 12], partitions of $\mathrm{PG}(2,q)$ by Singer subgroups are considered. Methods of [8, 12], allow us to represent an incidence matrix of the plane $\mathrm{PG}(2,q)$ as a BDC matrix defined below. We present some new results; see [12, Sec. 7.3] for comparison. We recall the following definition.

Definition 5 ([8])**.** Let $v = td$. A $v \times v$ matrix $\mathbf{A}$ is said to be *block double-circulant matrix* (or *BDC matrix*) if

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_{0,0} & \mathbf{C}_{0,1} & \dots & \mathbf{C}_{0,t-1} \\ \mathbf{C}_{1,0} & \mathbf{C}_{1,1} & \dots & \mathbf{C}_{1,t-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{C}_{t-1,0} & \mathbf{C}_{t-1,1} & \dots & \mathbf{C}_{t-1,t-1} \end{bmatrix}, \tag{13}$$

$$\mathbf{W}(\mathbf{A}) = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & \dots & w_{t-2} & w_{t-1} \\ w_{t-1} & w_0 & w_1 & w_2 & \dots & w_{t-3} & w_{t-2} \\ w_{t-2} & w_{t-1} & w_0 & w_1 & \dots & w_{t-4} & w_{t-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_1 & w_2 & w_3 & w_4 & \dots & w_{t-1} & w_0 \end{bmatrix}, \tag{14}$$

where $\mathbf{C}_{i,j}$ is a *circulant* $d \times d$ 0,1-matrix for all i, j; submatrices $\mathbf{C}_{i,j}$ and $\mathbf{C}_{l,m}$ with $j - i \equiv m - l \pmod t$ have equal weights; $\mathbf{W}(\mathbf{A})$ is a *circulant* $t \times t$ matrix whose entry in a position i, j is the *weight* of $\mathbf{C}_{i,j}$. $\mathbf{W}(\mathbf{A})$ is called a *weight matrix* of $\mathbf{A}$. The vector $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, w_1, \dots, w_{t-1})$ is called a *weight vector* of $\mathbf{A}$.

Let $q^2 + q + 1 = dt$. Then, by using the cyclic Singer group of $\mathrm{PG}(2,q)$, the incidence matrix of the plane $\mathrm{PG}(2,q)$ can be constructed as a BDC matrix $\mathbf{A}$ of the form (13). In this plane, we number the points $P_1, \dots, P_{q^2+q+1}$ and the lines $\ell_1, \dots, \ell_{q^2+q+1}$ so that P_i corresponds to the ith column of $\mathbf{A}$ and ℓ_i corresponds to the ith row of $\mathbf{A}$. Denote by $\mathbf{P}_v = \{P_{dv+1}, \dots, P_{dv+d}\}$, $0 \le v \le t-1$, the point set corresponding to the $(v+1)$th block column of $\mathbf{A}$. Let $\mathbf{L}_u = \{\ell_{du+1}, \dots, \ell_{du+d}\}$, $0 \le u \le t-1$, be the line set corresponding to the $(u+1)$th block row of $\mathbf{A}$. Here and further addition and subtraction of subscripts are expressed modulo t.

We give a development of [2, Lemma 7.7].

Lemma 3. *Let $1 \le m \le t-1$. An md-set*

$$\mathbf{P}^{(m)} = \mathbf{P}_0 \cup \mathbf{P}_1 \cup \ldots \cup \mathbf{P}_{m-1}$$

corresponding to the first md columns of $\mathbf{A}$ *in* (13) *is a* $(1,\mu)$*-saturating set* S *in* $\mathrm{PG}(2,q)$ *with*

$$\mu = \min_v N_v^{(m)},\ 1 \le m \le v \le t-1, \tag{15}$$

$$N_v^{(m)} = \sum_{u=0}^{t-1} w_{t-u+v} \binom{w_u^{(m)}}{2} \ge 0,\ w_u^{(m)} = \sum_{j=0}^{m-1} w_{t-u+j}.$$

Moreover, an $[md, md-3, 3]_q 2$ *code* C_S *corresponding to* S *is a* $(2,\mu)$*-MCF code with minimum distance* $d = 3$ *and* μ*-density*

$$\gamma_\mu(C_S, 2) = \frac{1}{\mu} \cdot \frac{\sum\limits_{v=m}^{t-1} N_v^{(m)}}{t-m}. \tag{16}$$

Proof. Every line of $\mathbf{L}_u$ is a $w_u^{(m)}$-secant of $\mathbf{P}^{(m)}$. Let $v \ge m$. Every point of $\mathbf{P}_v$ is covered by w_{t-u+v} specimens of $w_u^{(m)}$-secants of $\mathbf{P}^{(m)}$ with multiplicity $\binom{w_u^{(m)}}{2}$ for $0 \le u \le t-1$. So, every point of $\mathbf{P}_v$, $m \le v \le t-1$, is covered by $N_v^{(m)}$ secants of $\mathbf{P}^{(m)}$. This implies (15) and, together with Proposition 2, gives rise to (16). □

By [2, Th. 7.8], the Singer partition of $\mathrm{PG}(2,q)$ gives a $(1,\mu)$-OS set and the corresponding $(1,\mu)$-APMCF code in the following cases:

- $m = t-1$ for an arbitrary weight vector;
- $1 \le m \le t-1$ and the weight vector has the form $\overline{\mathbf{W}}(\mathbf{A}) = (w_0, w, \ldots, w)$;
- $m = 1$ and the weight vector contains exactly two distinct weights.

Example 1. All the $(1,\mu)$-saturating md-sets below are optimal by [2, Th. 7.8]. The corresponding $[md, md-3, 3]_q 2$ codes C are $(1,\mu)$-APMCF with $\gamma_\mu(C,2) = 1$. The multiplicity μ has been calculated by (15) or by [2, Th. 7.8].

(i) Let q be square. Then $(q^2+q+1) = (q+\sqrt{q}+1)(q-\sqrt{q}+1)$. Let $d = q+\sqrt{q}+1$, $t = q-\sqrt{q}+1$. There is a partition of $\mathrm{PG}(2,q)$ such that all the subsets $\mathbf{P}_v$ are disjoint Baer subplanes. We have $\overline{\mathbf{W}}(\mathbf{A}) = (\sqrt{q}+1, 1, \ldots, 1)$ whence

$$\mu = m\binom{\sqrt{q}+m}{2} + (q+1-m)\binom{m}{2},\ 1 \le m \le t-1.$$

The case $m = 1$ coincides with code of Proposition 7.

(ii) Let $q = p^{4v+2}$, $p \equiv 2 \pmod 3$. Then by [12, Prop. 4],

$$t = 3,\ d = \frac{q^2+q+1}{3},\ w_0 = \frac{q+2\sqrt{q}+1}{3},\ w_1 = w_2 = w = \frac{q-\sqrt{q}+1}{3}.$$

For $m = 1$ we have

$$\mu = w\binom{w_0}{2} + \binom{w}{2}(w_0+w) = \frac{1}{18}(q^3 - q\sqrt{q} - 2).$$

(iii) Let $q = p^{4v}$, $p \equiv 2 \pmod 3$. Then by [12, Prop. 4],

$$t = 3,\ d = \frac{q^2+q+1}{3},\ w_0 = \frac{q-2\sqrt{q}+1}{3},\ w_1 = w_2 = w = \frac{q+\sqrt{q}+1}{3}.$$

For $m = 1$ we have

$$\mu = w\binom{w_0}{2} + \binom{w}{2}(w_0+w) = \frac{1}{18}(q^3 + q\sqrt{q} - 1).$$

(iv) Let $q = p^{2c}$. Let t be a prime divisor of q^2+q+1. Then t divides either $q+\sqrt{q}+1$ or $q-\sqrt{q}+1$. Assume that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$. By [12, Prop. 6], in this case $w_0 = (q+1\pm(1-t)\sqrt{q})/t$, $w_1 = \ldots = w_{t-1} = w = (q+1\pm\sqrt{q})/t$. For $m = 1$ we have

$$\mu = w\binom{w_0}{2} + \binom{w}{2}(w_0 + w(t-2)) = \frac{q^3 \pm (t-2)q\sqrt{q} - t + 1}{2t^2}.$$

Note that the hypothesis that $p \pmod t$ is a generator of the multiplicative group of $\mathbb{Z}_t$ holds, e.g. in the following cases: $q = 3^4$, $t = 7$; $q = 2^8$, $t = 13$; $q = 5^4$, $t = 7$; $q = 2^{12}$, $t = 19$; $q = 3^8$, $t = 7$; $q = 2^{16}$, $t = 13$; $q = 17^4$, $t = 7$; $p \equiv 2 \pmod t$, $t = 3$.

(v) Let $q = 125$. By [12, Tab. 1], there is the partition with $t = 19$, $d = 829$, $\overline{\mathbf{W}}(\mathbf{A}) = (4,9,9,9,9,4,4,9,9,4,9,9,4,4,4,4,9,4,9)$. For $m = 1$ we have $\mu = 2706$.

Partitions providing $(1,\mu)$-OS sets are not always possible. But, as rule, the partitions provide "good" $(1,\mu)$-saturating sets such that the corresponding $(1,\mu)$-MCF codes have μ-density $\gamma_\mu(C,2)$ of order of magnitude less than $1+\frac{1}{cq}, c \geq 1$. In the following we give examples of "good" $(1,\mu)$-saturating sets.

Example 2. Using the approach of [12], we obtain by computer search partitions with $t = 3$ and with three distinct values of w_i, see also [12, Table 1]. We take $m = 1$ and $n = d$. The values of q, w_i, n, μ, and $\gamma_\mu(C,2)$ are given in Table 1. The values of μ and $\gamma_\mu(C,2)$ are obtained by (15) and (16) respectively. In the last column we write relation of the form $1+\frac{1}{cq}$ such that $\gamma_\mu(C,2) < 1+\frac{1}{cq}$. One can see that $1 \leq c \leq 35$.

7. Classification of minimal and optimal $(1,\mu)$-saturating sets in PG$(2,q)$

We performed a computer based search for minimal $(1,2)$-saturating sets. The results are collected in Table 2. In the 2nd column, the values $\overline{\ell}(2,3,q)$ of the smallest cardinality of a 1-saturating set in PG$(2,q)$, taken from [3, 11], are given. The cases when $\ell(2,3,q) = \overline{\ell}(2,3,q)$ are marked by the dot "**.**". In the 5th column, we give some values of n for which minimal $(1,2)$-saturating n-sets in PG$(2,q)$ exist. For $3 \leq q \leq 17$, we have found the *complete spectrum* of sizes n. This situation is marked by the dot "**.** ". In the 2nd and the 5th columns, the superscript notes the numbers of nonequivalent sets of the corresponding size. For $3 \leq q \leq 9$, we obtain the *complete classification* of the spectrum of sizes n of minimal $(1,2)$-saturating n-sets in PG$(2,q)$. This situation is marked by the asterisk *. In the 3-rd column the trivial lower bound (11) is given. Finally, the size $q+\mu+1 = q+3$ of the largest minimal (1,2)-saturating set in PG$(2,q)$, see Proposition 10(ii), is written in the 4th column.

Using different constructions we obtained (1,2)-saturating n-sets in PG$(2,q)$ having several points on a conic, with sizes described in Table 3.

The smallest cardinalities of $(1,2)$-saturating sets for each q in Table 2 and sizes n in Table 3 are smaller than $2\overline{\ell}(2,3,q)$. So, for $\mu = 2$, $r = 3$, and q from Tables 2 and 3, the goal formulated at the end of Section 3 is achieved.

Table 1. Values of μ and μ-density for partitions with three distinct values of w_i

q	w_0	w_1	w_2	$n=d$	μ	$\gamma_\mu(C,2)$	$<$
7	1	4	3	19	18	1.0833	$1+1/q$
13	4	7	3	61	117	1.0256	$1+1/3q$
19	4	7	9	127	375	1.0200	$1+1/2q$
31	7	12	13	331	1656	1.0045	$1+1/7q$
37	13	9	16	469	2796	1.0075	$1+1/3q$
43	19	13	12	631	4392	1.0024	$1+1/9q$
49	13	21	16	817	6498	1.0046	$1+1/4q$
61	16	25	21	1261	12570	1.0036	$1+1/4q$
67	28	19	21	1519	16653	1.0019	$1+1/7q$
73	19	28	27	1801	21627	1.0008	$1+1/16q$
79	31	21	28	2107	27363	1.0019	$1+1/6q$
97	28	31	39	3169	50601	1.0013	$1+1/7q$
103	28	37	39	3571	60708	1.0008	$1+1/11q$
127	36	49	43	5419	113673	1.0012	$1+1/6q$
139	39	49	52	6487	149175	1.0007	$1+1/11q$
157	61	48	49	8269	214848	1.0002	$1+1/35q$
163	63	49	52	8911	240387	1.0005	$1+1/12q$

Table 2. The number of nonequivalent minimal (1,2)-saturating n-sets in $\mathrm{PG}(2,q)$ and the spectrum of sizes n

q	$\bar{\ell}(2,3,q)$	$\lceil 2\sqrt{q}\rceil$	$q+\mu+1$	Spectrum of n
3	4^{1} ▪	4	6	6^{4} ▪ $*$
4	5^{1} ▪	4	7	$6^{2}7^{5}$ ▪ $*$
5	6^{6} ▪	5	8	$6^{1}7^{4}8^{18}$ ▪ $*$
7	6^{3} ▪	6	10	$8^{13}9^{564}10^{424}$ ▪ $*$
8	6^{1} ▪	6	11	$8^{2}9^{154}10^{3372}11^{611}$ ▪ $*$
9	6^{1} ▪	6	12	$8^{1}9^{57}10^{12145}11^{76749}12^{3049}$ ▪ $*$
11	7^{1} ▪	7	14	$10^{1348}[11-14]$ ▪
13	8^{2} ▪	8	16	$10^{2}11^{50794}[12-16]$ ▪
16	9^{4} ▪	8	19	$11^{52}[12-19]$ ▪
17	10^{3640} ▪	9	20	$[12-20]$ ▪
19	10^{36} ▪	9	22	$[13-22]$
23	10^{1} ▪	10	26	$[15-26]$
25	12	10	28	$[17-28]$
27	12	11	30	$[17-30]$
29	13	11	32	$[19-32]$
31	14	12	34	$[19,21-34]$
32	13	12	35	$[20-35]$
37	15	13	38	$[23,26-40]$
41	16	13	44	$[25,29-44]$
43	16	14	46	$[25,30-46]$
47	18	14	50	$[27,34-50]$
49	18	14	52	$[29,34-52]$

TABLE 3. Sizes of small (1,2)-saturating n-sets in $\mathrm{PG}(2,q)$

q	53	59	61	67	71	73	79	81	83	89	97	101	103	107	109	113	125	127	131	139
n	31	33	35	37	39	41	39	45	45	49	53	55	55	57	59	61	67	67	69	73

In Table 4 a classification of minimal m-saturating sets for some values of $\mu \le (q+1)\binom{q}{2}$, $q \le 11$, is presented; the superscript over a size indicates the number of distinct μ-saturating sets of that size (up to collineations). If the subscript is absent, then there exists at least a $(1,\mu)$-saturating set of that size. The classification is complete for $q \le 5$ and every multiplicity, while for $7 \le q \le 9$ and $\mu \le 10$ the spectrum of the sizes of $(1,\mu)$-saturating sets is definitive.

Remark 3. By property (M3) of Definition 3, see also (M3) in Section 3, a $(1,\mu)$-saturating set is also a $(1,\mu-k)$-saturating set for $1 \le k \le \mu-1$. Moreover, a minimal $(1,\mu)$-saturating set can also be a minimal $(1,\mu-k)$-saturating set for $k = 1,2,\dots,\delta$, $\delta \ge 1$. This happens when removing any point from this set we obtain a $(1,\mu-\delta-1)$-saturating set. For example, let $q = 3$ and consider a line ℓ. Then $S = \mathrm{PG}(2,3) \setminus \ell$ is a $(1,9)$-saturating set and removing any point from S we obtain a $(1,4)$-saturating set. So, S is a minimal $(1,9)$-, $(1,8)$-, $(1,7)$-, $(1,6)$-, and $(1,5)$-saturating set. This example and many other such situations are written in Table 4.

In Table 5 we give the classification of optimal $(1,\mu)$-saturating sets ($(1,\mu)$-OS) in $\mathrm{PG}(2,q)$. For such sets S, every point of $\mathrm{PG}(2,q) \setminus S$ is covered exactly μ times. An entry of the form n^t_μ means that there exist t projectively distinct optimal $(1,\mu)$-saturating sets with size n. Entries provided by Propositions 4 – 7 and 13, Example 1, and Corollary 1 are written in bold font.

Observation 1. For $q = 3$, the (1,3)-OS of size 7 is 2 concurrent lines; the (1,4)-OS of size 7 is contained in 3 concurrent lines. For $q = 4$ the (1,3)-OS of size 6 is the hyperoval, the (1,4)-OS of size 7 is a complete 3-arc, cf. with Proposition 4.

In all these four cases, the points external to the $(1,\mu)$-OS, say S, form a unique orbit of the stabilizer group of S.

In the following we give some constructions of optimal $(1,\mu)$-OS in $\mathrm{PG}(2,q)$ providing many entries in Table 5.

Proposition 13. *The following sets $\mathcal{S}$ are $(1,\mu)$-OS in* $\mathrm{PG}(2,q)$.

(i) *The set $\mathcal{S}$ is the union of L concurrent lines, $2 \le L \le q$. It holds that*

$$|\mathcal{S}| = 1 + Lq,\ \ \mu = \binom{L}{2}q.$$

(ii) *The set $\mathcal{S}$ is the union of q concurrent lines and b other points on the $(q+1)$th one, $1 \le b \le q-1$. It holds that*

$$|\mathcal{S}| = 1 + q^2 + b,\ \ \mu = \binom{b+1}{2} + \binom{q}{2}q.$$

(iii) *The set $\mathcal{S}$ is a triangle. It holds that*

$$|\mathcal{S}| = 3q,\ \ \mu = 3 + (q-2)\binom{3}{2} = 3(q-1).$$

Table 4. Classification of minimal $(1, \mu)$-saturating sets in $\mathrm{PG}(2, q)$

μ	3	4	5	6	7	8	9	10
$q=3$	$6^1 7^2$	$7^1 8^2$	$8^1 9^1$	9^3	$9^1 10^1$	$9^1 10^1$	9^1	11^1
$q=4$	$6^1 7^2 8^7$	$8^3 9^{10}$	$9^6 10^7$	$9^2 10^8 11^1$	$10^2 11^{13}$	$11^7 12^5$	$11^1 12^{16}$	$12^5 13^5$
$q=5$	$8^5 96^5$	$9^7 10^{133}$	10^{26} 11^{162}	11^{121} 12^{102}	$11^3 12^{361}$	12^{40} 13^{437}	$12^3 13^{301}$ 14^{28}	13^{14} 14^{759}
$q=7$	$8^1 9^{10}$ 10^{1506} 11^{10014}	10^2 11^{3167} 12^{67454}	11^2 12^{11301} 13^{239140}	12^{59} 13^{83378} 14^{483925}	13^{430} 14^{613065} 15^{518711}	14^{7418} $15^{2860573}$ 16^{372495}	14^8 15^{247080} $16^{6331998}$	15^{384} $16^{3676357}$ 17
$q=8$	10^{137} 11^{19606} 12^{40514}	10^1 11^{89} 12^{63582} 13^{522250}	10^1 12^{107} 13^{239774} $14^{2910961}$	12^4 13^{820} $14^{3714769}$ 15	13^3 $14^{2267544}$ 15 16	14^2 15^{150981} 16 17	15^{69} 16 17 18	16^{5629} 17 18 19
$q=9$	9^1 10^{15} 11^{15351} $12^{1249084}$ 13^{596493}	10^1 11^3 12^{13809} $13^{6382440}$ 14	12^3 13^{23805} 14 15	13^{12} 14^{157518} 15 16	14^{11} 15 16 17	14^1 15^{74} 16 17 18	15^1 16^{5185} 17 18 19	16^6 17 18 19 20
$q=11$	$10^1\ 11^{36}$ 12^{697413} 13 14 15	12^4 13^{186404} 14 15 16	$12^1\ 13^2$ 14 15 16 17	15 16 17 18	15 16 17 18 19	17 18 19 20	17 18 19 20 21	18 19 20 21 22

μ	11	12	13	14	15	16	17	18
$q=3$	12^1	12^1	-	-	-	-	-	-
$q=4$	$12^1 13^{11}$	$12^1 13^3$ 14^3	14^{10}	$14^4 15^4$	$14^2 15^5$	$15^4 16^2$	$15^1 16^4$	16^5
$q=5$	$13^1 14^{198}$ 15^{171}	14^3 15^{933}	15^{159} 16^{309}	15^7 16^{907}	$15^2 16^{239}$ 17^{210}	16^{17} 17^{741}	$16^2 17^{436}$ 18^{104}	$16^2 17^{22}$ 18^{535}
$q=7$	15^1 16^{45704} 17 18	15^1 16^{115} 17 18	16^1 17^{12630} 18 19	17^{21} 18 19	18^{13730} 19 20	18^9 19 20	19^{21111} 20 21	19^9 20 21

μ	19	20	21	22	23	24	25	26
$q=4$	16^2	$16^1 17^2$	$16^1 17^2$	$16^1 18^1$	$16^1 18^1$	$16^1 18^1$	18^1	19^1
$q=5$	17^2 18^{558}	18^{99} 19^{219}	18^7 19^{447}	$19^{268} 20^6$	19^{18} 20^{239}	19^3 20^{289}	$20^{96} 21^{14}$	$20^4 21^{161}$

μ	27	28	29	30	31	32	33	34
$q=4$	19^1	20^1	20^1	20^1	-	-	-	-
$q=5$	20^1 21^{172}	$20^1 21^{39}$ 22^{14}	$20^1 21^4$ 22^{77}	$20^1 21^1$ 22^{88}	22^{29} 23^8	22^5 23^{32}	22^2 23^{38}	$22^1 23^{18}$ 24^6

μ	35	36	37	38	39	40	41	42
$q=5$	$23^4 24^{16}$	24^{22}	$24^{16} 25^1$	$24^4 25^7$	$24^1 25^{10}$	25^{12}	$25^6 26^1$	$25^2 26^3$

μ	43	44	45	46	47	48	49	50
$q=5$	$25^1 26^4$	$25^1 26^4$	$25^1 26^1$	$25^1 27^2$	$25^1 27^2$	$25^1 27^2$	$25^1 28^1$	$25^1 28^1$

μ	51	52	53	54	55	56	$57 \mid 58$	$59 \mid 60$
$q=5$	$27^1 28^1$	28^2	28^1	29^1	29^1	29^1	$30^1 \mid 30^1$	$30^1 \mid 30^1$

(iv) *The set* $\mathcal{S} = \mathrm{PG}(2,q) \setminus T$ *where* T *is a vertex-less triangle. It holds that*

$$|\mathcal{S}| = q^2 - 2q + 4,\ \ \mu = 1 + \binom{q}{2} + (q-1)\binom{q-2}{2}.$$

TABLE 5. Classification of optimal $(1, \mu)$-saturating n-sets in $\mathrm{PG}(2, q)$

q	n_μ^t
3	$\mathbf{7_3^1}$ $\mathbf{7_4^1}$ $\mathbf{9_6^1}$ $\mathbf{10_8^1}$ 9_9^1 $\mathbf{10_9^1}$ $\mathbf{11_{10}^1}$ $\mathbf{12_{12}^1}$
4	$\mathbf{6_3^1}$ $\mathbf{7_3^1}$ $\mathbf{9_4^1}$ 9_5^1 $\mathbf{9_6^1}$ 11_9^1 $\mathbf{12_9^1}$ $\mathbf{12_{10}^2}$ $\mathbf{13_{12}^2}$ $\mathbf{14_{15}^1}$ $\mathbf{15_{15}^1}$ 15_{16}^1 15_{17}^1 $\mathbf{16_{18}^1}$ $\mathbf{17_{21}^1}$ $\mathbf{16_{24}^1}$ $\mathbf{17_{24}^1}$ $\mathbf{18_{24}^1}$ $\mathbf{18_{25}^1}$ $\mathbf{19_{27}^1}$ $\mathbf{20_{30}^1}$
5	$\mathbf{11_5^1}$ 14_{11}^1 $\mathbf{15_{12}^1}$ $\mathbf{16_{15}^1}$ 16_{18}^1 19_{22}^1 $\mathbf{19_{23}^2}$ 21_{27}^1 $\mathbf{21_{30}^2}$ 22_{31}^2 22_{32}^1 23_{35}^1 $\mathbf{25_{40}^1}$ 25_{41}^1 25_{42}^1 $\mathbf{26_{44}^1}$ $\mathbf{27_{48}^1}$ 25_{50}^1 $\mathbf{26_{50}^1}$ $\mathbf{27_{51}^1}$ $\mathbf{28_{52}^1}$ $\mathbf{28_{53}^1}$ $\mathbf{29_{56}^1}$ $\mathbf{30_{60}^1}$

Proof. The sizes of $\mathcal{S}$ are obvious. Let P be a point of $\mathrm{PG}(2, q) \setminus \mathcal{S}$. Let G be the intersection point of concurrent lines.

(i) Every line through P distinct from PG intersects $\mathcal{S}$ in L points.

(ii) Every line through P distinct from PG intersects $\mathcal{S}$ in q points. The line PG provides is a $(b + 1)$-secant of $\mathcal{S}$.

(iii) Three lines through P and one vertex of the triangle are bisecants of $\mathcal{S}$, whereas every other line is a 3-secant.

(iv) This follows directly from Proposition 8. □

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