New Upper Bounds on the Smallest Size of a Saturating Set in a Projective Plane

Daniele Bartoli
Perugia University
Perugia, Italy
daniele.bartoli@unipg.it

Alexander A. Davydov
Institute for Information Transmission Problems
(Kharkevich institute), Russian Academy of Sciences
Moscow, Russian Federation
adav@iitp.ru

Massimo Giulietti, Stefano Marcugini, Fernanda Pambianco
Perugia University
Perugia, Italy
{massimo.giulietti, stefano.marcugini, fernanda.pambianco}@unipg.it

***Abstract*—In a projective plane Π_q (not necessarily Desarguesian) of order q, a point subset S is saturating (or dense) if any point of $\Pi_q \setminus S$ is collinear with two points in S. Using probabilistic methods, more general than those previously used for saturating sets, the following upper bound on the smallest size $s(2,q)$ of a saturating set in Π_q is proved:**

$$s(2,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}.$$

We also show that for any constant $c \geq 1$ a random point set of size k in Π_q with $2c\sqrt{(q+1)\ln(q+1)} + 2 \leq k < \frac{q^2-1}{q+2} \sim q$ is a saturating set with probability greater than $1 - 1/(q+1)^{2c^2-2}$.

Our probabilistic approach is also applied to multiple saturating sets. A point set $S \subset \Pi_q$ is $(1,\mu)$-saturating if for every point Q of $\Pi_q \setminus S$ the number of secants of S through Q is at least μ, counted with multiplicity. The multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$. The following upper bound on the smallest size $s_\mu(2,q)$ of a $(1,\mu)$-saturating set in Π_q is proved:

$$s_\mu(2,q) \leq 2(\mu+1)\sqrt{(q+1)\ln(q+1)} + 2 \text{ for } 2 \leq \mu \leq \sqrt{q}.$$

By using inductive constructions, upper bounds on the smallest size of a saturating set (as well as on a $(1,\mu)$-saturating set) in the projective space $PG(N,q)$ are obtained.

All the results are also stated in terms of linear covering codes.

I. Introduction

We denote by Π_q a projective plane (not necessarily Desarguesian) of order q and by $PG(2,q)$ the projective plane over the Galois field with q elements.

Definition 1. A point set $S \subset \Pi_q$ is *saturating* if any point of $\Pi_q \setminus S$ is collinear with two points in S.

Saturating sets are considered, for example, in [3], [4], [6], [8]–[15], [18], [20]. Saturating sets are also called "saturated sets" [8], [15], [20], "spanning sets" [6], "dense sets" [4], [10]–[13], and "1-saturating sets" [1], [2], [9], [18].

The homogeneous coordinates of the points of a saturating set of size k in $PG(2,q)$ form a parity check matrix of a q-ary linear code with length k, codimension 3, and covering radius 2. For an introduction to covering codes see [5], [7]. An online bibliography on covering codes is given in [17].

The main problem in this context is to find small saturating sets (i.e. short covering codes). Let $s(2,q)$ denote the smallest size of a saturating set in Π_q. In [4], by using the probabilistic approach previously introduced in [15], it is proved that

$$s(2,q) < 3\sqrt{2}\sqrt{q \ln q} < 5\sqrt{q \ln q}. \quad (1)$$

Surveys on random constructions for geometrical objects can be found in [4], [11], [15], [16]. Saturating sets in $PG(2,q)$ obtained by algebraic constructions or computer search can be found in [3], [6], [8]–[10], [12]–[14], [18], [20].

In this paper, we use probabilistic methods to obtain new upper bounds on $s(2,q)$. Our main results are as follows.

Theorem 2. *For the smallest size $s(2,q)$ of a saturating set in a projective plane of order q the following upper bound holds:*

$$s(2,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}. \quad (2)$$

Theorem 3. *Let c be a real number greater than or equal to 1 and let k be an integer such that*

$$2c\sqrt{(q+1)\ln(q+1)} + 2 \leq k < \frac{q^2-1}{q+2} \sim q.$$

Then in a projective plane of order q, a random point set of size k is a saturating set with probability greater than

$$1 - \frac{1}{(q+1)^{2c^2-2}} \ . \quad (3)$$

Theorem 2 improves the constant term of (1). It should be noted that our approach is different from those in [4], [15], where random sets lying on two or three lines are considered; in this paper arbitrary random sets are dealt with.

Theorem 2 can be expressed in terms of *covering codes*. The *length function* $\ell(R,r,q)$ denotes the smallest length of a q-ary linear code with covering radius R and codimension r [5]–[7]. Theorem 2 can be read as follows.

Corollary 4. *The following upper bound on the length function holds.*

$$\ell(2,3,q) \leq 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}.$$

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Our probabilistic approach can also be applied to *multiple saturating* sets.

Definition 5. A point set $S \subset \Pi_q$ is $(1,\mu)$-*saturating* if for every point Q of $\Pi_q \setminus S$ the number of secants of S through Q is at least μ, counted with multiplicity. Here the multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$.

For $\mu = 1$, a $(1,\mu)$-saturating set is a saturating set as in Definition 1.

The homogeneous coordinates of the points of a $(1,\mu)$-saturating set of size k in $PG(2,q)$ form a parity check matrix of a q-ary linear code with length k, codimension 3, covering radius 2. This code is a $(2,\mu)$-multiple covering of the farthest-off-points ($(2,\mu)$-MCF code or simply MCF code, for short). For an introduction to multiple saturating sets and MCF codes see [1], [2], [7, Chapters 13, 14], [14], [19].

The main problem in this context is to find small $(1,\mu)$-saturating sets (i.e. short MCF codes). Let $s_\mu(2,q)$ be the smallest size of a $(1,\mu)$-saturating set in Π_q. Our main results on $(1,\mu)$-saturating sets in Π_q are the following.

Theorem 6. *Let $\mu \geq 2$. For the smallest size $s_\mu(2,q)$ of a $(1,\mu)$-saturating set in a projective plane of order q the following upper bounds hold.*

$$s_\mu(2,q) \leq 2\Omega_\mu\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\Omega_\mu\sqrt{q\ln q}, \quad (4)$$

where

$$\Omega_\mu \leq \begin{cases} 2.4 & \text{for} \quad \mu = 2, \quad q \geq 97 \\ 2.6 & \text{for} \quad \mu = 3, \quad q \geq 181 \\ 2.8 & \text{for} \quad \mu = 4, \quad q \geq 125 \\ \mu+1 & \text{for} \quad \mu \leq \sqrt{q}, \quad q \geq 4 \\ 2\mu-1 & \text{for} \quad \sqrt{q} < \mu \leq M, \quad q \geq 3 \end{cases}, \quad (5)$$

$$M = \frac{(1-\delta)(q+1)}{2} + 1, \quad \delta = \frac{1}{\sqrt{(q+1)\ln(q+1)}}. \quad (6)$$

The μ-*length function* $\ell_\mu(R,r,q)$ denotes the smallest length of a linear q-ary (R,μ)-MCF code with covering radius R and codimension r [1], [2], [14]. For $\mu = 1$, $\ell_1(R,r,q)$ is the usual length function $\ell(R,r,q)$ for 1-fold coverings. In the *covering code language*, Theorem 6 can be read as follows.

Corollary 7. *Let Ω_μ be as in* (5). *The following upper bound on the μ-length function holds.*

$$\ell_\mu(2,3,q) \leq 2\Omega_\mu\sqrt{(q+1)\ln(q+1)} + 2. \quad (7)$$

In [1, Prop. 5.2] the following upper bounds on the μ-length function were obtained by adapting the probabilistic approach in [4], [15]:

$$s_\mu(2,q) \leq \ell_\mu(2,3,q) < 66\sqrt{\mu q \ln q} \text{ for } \mu < 121 q\ln q. \quad (8)$$

The bounds (4) and (7) improve (8) if $\Omega_\mu < 33\sqrt{\mu}$. This actually happens for a wide region of μ, see (5).

Let $PG(N,q)$ be the N-dimensional projective space over the Galois field of q elements.

From (2) and (4), by using inductive constructions from [1], [8], [9], upper bounds on the smallest size of a saturating set in the N-dimensional projective space $PG(N,q)$ can be obtained; see Section V. In many cases these bounds are better than the known ones.

II. Upper bound on the smallest size of a saturating set in a projective plane

Let $w > 0$ be a fixed integer. Consider a random $(w+1)$-point subset $\mathcal{K}_{w+1}$ of Π_q. The total number of such subsets is $\binom{q^2+q+1}{w+1}$. A fixed point A of Π_q is *covered* by $\mathcal{K}_{w+1}$ if it belongs to an r-secant of $\mathcal{K}_{w+1}$ with $r \geq 2$. We denote by $\mathrm{Prob}(\diamond)$ the probability of some event $\diamond$.

We estimate

$$\pi := \mathrm{Prob}(A \text{ not covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w+1)$-point subsets not covering A over the total number of subsets of size $(w+1)$. Since a set $\mathcal{K}_{w+1}$ does not cover A if and only if every line through A contains at most one point of $\mathcal{K}_{w+1}$, we have

$$\pi = \frac{q^{w+1}\binom{q+1}{w+1}}{\binom{q^2+q+1}{w+1}}. \quad (9)$$

By straightforward calculations,

$$\pi = \frac{(q^2+q)(q^2)\cdots(q^2+q-iq)\cdots(q^2+q-wq)}{(q^2+q+1)\cdots(q^2+q-i)\cdots(q^2+q+1-w)} =$$
$$= \prod_{i=0}^{w}\left(1 - \frac{iq-i+1}{q^2+q+1-i}\right) < \prod_{i=0}^{w}\left(1 - \frac{i(q-1)}{q^2+q+1}\right).$$

Using the inequality $1 - x \leq e^{-x}$ we obtain that

$$\pi < e^{-\sum_{i=0}^{w}\frac{i(q-1)}{q^2+q+1}} = e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}},$$

which implies

$$\pi < e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}} < e^{-\frac{w^2}{2q+2}}, \quad (10)$$

provided that

$$\frac{(w+1)(q^2-1)}{(q^2+q+1)} > w$$

that is

$$w < \frac{q^2-1}{q+2} \sim q. \quad (11)$$

The set $\mathcal{K}_{w+1}$ is not saturating if at least one point $A \in \Pi_q$ is not covered by $\mathcal{K}_{w+1}$. Similarly to [4, Prop. 4.1], we have

$$\mathrm{Prob}\left(\mathcal{K}_{w+1} \text{ is not saturating}\right) \leq \sum_{A \in \Pi_q} \mathrm{Prob}(A \text{ is not covered}). \quad (12)$$

Now, using (10), we obtain that

$$\mathrm{Prob}\left(\mathcal{K}_{w+1} \text{ is not saturating}\right) \leq (q^2+q+1)\pi < (q+1)^2 e^{-\frac{w^2}{2q+2}}. \quad (13)$$

So, the probability that all the points of Π_q are covered is

$$\mathrm{Prob}\left(\mathcal{K}_{w+1} \text{ is saturating}\right) > 1 - (q+1)^2 e^{-\frac{w^2}{2q+2}}. \quad (14)$$

This quantity is larger than 0 taking, for instance,

$$w = \left\lceil \sqrt{(2q+2)\ln\left((q+1)^2\right)} \right\rceil.$$

This shows that in Π_q there exists a saturating set with size

$$k \le 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q\ln q}.$$

and therefore Theorem 2 is proved.

In conclusion, we note that any value

$$w = \left\lceil c\sqrt{(2q+2)\ln\left((q+1)^2\right)} \right\rceil < \frac{q^2-1}{q+2},$$

where the parameter $c \ge 1$ is independent of q, provides in (14) a positive probability greater than $1 - 1/(q+1)^{2c^2-2}$; therefore Theorem 3 holds.

It is worth noting that in (3) for q large enough, choosing $c = 1+\varepsilon$, with $\varepsilon = o(1) > 0$, the probability is close to 1.

III. Upper bounds on the smallest size of a $(1,\mu)$-saturating set in a projective plane, $\mu \ge 2$

For $\mu \ge 2$, we construct a $(1,\mu)$-saturating set $\mathcal{S}_\mu$ in Π_q by joining a $(\mu-1)$-saturating set $\mathcal{S}_{\mu-1}$ and a "usual" saturating set *disjoint* from $\mathcal{S}_{\mu-1}$.

Let $w > 0$ be a fixed integer; we consider a random $(w+1)$-point subset $\mathcal{H}_{w+1}$ of Π_q disjoint from $\mathcal{S}_{\mu-1}$. Let k denote the size of $\mathcal{S}_{\mu-1}$. Then the total number of such subsets is $\binom{q^2+q+1-k}{w+1}$. Clearly, if $\mathcal{H}_{w+1}$ is a saturating set then $\mathcal{S}_{\mu-1} \cup \mathcal{H}_{w+1}$ is a $(1,\mu)$-saturating set.

We argue as in Section II. For a fixed point A of Π_q we estimate

$$\lambda := \mathrm{Prob}(A \text{ not covered by } \mathcal{H}_{w+1})$$

as the ratio of the number of $(w+1)$-point subsets not covering A and disjoint from $\mathcal{S}_{\mu-1}$ over the total number of subsets of size $(w+1)$ disjoint from $\mathcal{S}_{\mu-1}$. Similarly to (9), we have

$$\lambda < \frac{q^{w+1}\binom{q+1}{w+1}}{\binom{q^2+q+1-k}{w+1}}. \tag{15}$$

In fact, the number of $(w+1)$-point subsets not covering A and disjoint from $\mathcal{S}_{\mu-1}$ is smaller than the numerator of (15).

By straightforward calculations similar to Section II,

$$\lambda < \prod_{i=0}^{w}\left(1 - \frac{i(q-1)-k}{q^2+q+1}\right).$$

We denote

$$\Psi(k,w,q) = -\frac{w^2}{2q+2} + \frac{k(w+1)}{2q(q+1)}.$$

Now, under the condition (11), see also (10), we obtain

$$\lambda < e^{-\sum_{i=0}^{w}\frac{i(q-1)-k}{q^2+q+1}} < e^{-\frac{(w^2+w)(q-1)}{2(q^2+q+1)}+\frac{k(w+1)}{2q(q+1)}} < e^{\Psi(k,w,q)}.$$

This implies that

$$\mathrm{Prob}\left(\mathcal{H}_{w+1} \text{ is not saturating}\right) \le (q^2+q+1)\lambda < (q+1)^2 e^{\Psi(k,w,q)},$$

$$\mathrm{Prob}\left(\mathcal{S}_{\mu-1} \cup \mathcal{H}_{w+1} \text{ is } (1,\mu)\text{-saturating}\right) > 1-(q+1)^2 e^{\Psi(k,w,q)}. \tag{16}$$

Throughout this section, M and δ are as in (6).

We represent w and k in the following form:

$$w = \left\lceil d\sqrt{(2q+2)\ln((q+1)^2)} \right\rceil,\ d > 1 \text{ independent of } q;$$
$$k \le 2D\sqrt{(q+1)\ln(q+1)} + 2,\ D \ge 1 \text{ independent of } q. \tag{17}$$

Then the following holds:

$$w+1 \le 2(d+\delta)\sqrt{(q+1)\ln(q+1)}; \tag{18}$$
$$k \le 2(D+\delta)\sqrt{(q+1)\ln(q+1)}; \tag{19}$$
$$(q+1)^2 e^{\Psi(k,w,q)} < (q+1)^{\frac{2}{q}(D+\delta)(d+\delta)-2(d^2-1)}.$$

Let

$$d = 1 + \frac{D+\delta}{q}. \tag{20}$$

Then

$$d-1 = \frac{D+\delta}{q} > \frac{2(D+\delta)(d+\delta)}{2q(d+1)};$$
$$\frac{2}{q}(D+\delta)(d+\delta) - 2(d^2-1) < 0;$$
$$(q+1)^2 e^{\Psi(k,w,q)} < 1.$$

The last inequality means that the probability in (16) is positive. As $\#(\mathcal{S}_{\mu-1}\cup\mathcal{H}_{w+1}) = k+w+1$, taking into account (6), (18)–(20), we have proved the following lemma.

Lemma 8. *Let Π_q be a projective plane of order q. Let $\mu \ge 2$. Assume that for some $D \ge 1$ in Π_q there exists a $(1,\mu-1)$-saturating set with size $k \le 2D\sqrt{(q+1)\ln(q+1)}+2$. Then in Π_q there exists a $(1,\mu)$-saturating set with size*

$$v \le 2\left(D+1+\frac{D+\delta}{q}+\delta\right)\sqrt{(q+1)\ln(q+1)}+2. \tag{21}$$

Corollary 9. *Let $\mu \ge 2$.*

1) *In Π_q there is a $(1,\mu)$-saturating set with size*
$$k \le 2D_\mu\sqrt{(q+1)\ln(q+1)}+2,$$
where $D_1 = 1,\ D_i = D_{i-1}+1+\frac{D_{i-1}+\delta}{q}+\delta,\ i \ge 2.$

2) *In Π_q there is a $(1,\mu)$-saturating set with size*
$$k \le 2(\mu+1)\sqrt{(q+1)\ln(q+1)}+2, \quad \mu \le \sqrt{q}.$$

3) *In Π_q there is a $(1,\mu)$-saturating set with size*
$$k \le 2(2\mu-1)\sqrt{(q+1)\ln(q+1)}+2 \text{ for } \mu \le M.$$

Proof.

1) For $\mu = 1$ we use Theorem 2 and put $D_1 = 1$. Then we iteratively apply (21).
2) Clearly, $D_i = i + A_i, i \ge 2$, where $A_i := \frac{1}{q}\sum_{j=1}^{i-1} D_j + \left(\frac{1}{q}+1\right)(i-1)\delta$. By induction, it can be shown that for $\mu \le \sqrt{q}$, we have $A_i \le 1$. So, $D_i \le i+1$, $i = 2,3,\dots\mu$.
3) By the proof of the point 2), $D_2 \le 3$ holds. Assume that $D_i \le 2i-1$, $i = 2,3,\dots,h$, with $h \le \mu-1 \le M-1$. Then $D_{h+1} = 2h+1$.

□

IV. Improved upper bounds on the smallest size of a $(1,\mu)$-saturating set in a projective plane, $\mu \le 4$

Let $w > 0$ be a fixed integer. We consider a random $(w+1)$-point subset $\mathcal{K}_{w+1}$ of Π_q. The total number of such subsets is $\binom{q^2+q+1}{w+1}$. As above, let A be a fixed point of Π_q.

We say that $\mathcal{K}_{w+1}$ covers A *exactly* i times if the number of secants of $\mathcal{K}_{w+1}$ through A is *exactly* i, counted with multiplicity. Denote by T_i the number of $(w+1)$-subsets covering A exactly i times, $i = 0, 1, 2, \ldots$, where $i = 0$ means that A is not covered by $\mathcal{K}_{w+1}$. Similarly to the numerator of (9) we have

$$T_0 = q^{w+1}\binom{q+1}{w+1}. \tag{22}$$

According to Definition 5, we say that a fixed point A of Π_q is *μ-covered* by $\mathcal{K}_{w+1}$ if the number of secants of $\mathcal{K}_{w+1}$ through A is at least μ, counted with multiplicity. We estimate

$$\pi_\mu := \mathrm{Prob}(A \text{ not } \mu\text{-covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w+1)$-point subsets that do not μ-cover A over the total number of $(w+1)$-subsets. So,

$$\pi_\mu = \frac{\sum_{i=0}^{\mu-1} T_i}{\binom{q^2+q+1}{w+1}} = R_{w,q}\sum_{i=0}^{\mu-1}\frac{T_i}{T_0} \tag{23}$$

where

$$R_{w,q} = \frac{T_0}{\binom{q^2+q+1}{w+1}}. \tag{24}$$

The set $\mathcal{K}_{w+1}$ is not $(1,\mu)$-saturating if at least one point $A \in \Pi_q$ is not μ-covered by $\mathcal{K}_{w+1}$. As in (12), (13), we have

$$\mathrm{Prob}\,(\mathcal{K}_{w+1} \text{ is not } (1,\mu)\text{-saturating}) \le (q^2+q+1)\pi_\mu < (q+1)^2\pi_\mu.$$

So, the probability that all the points of Π_q are μ-covered is

$$\mathrm{Prob}\,(\mathcal{K}_{w+1} \text{ is } (1,\mu)\text{-saturating}) \ge 1 - (q^2+q+1)\pi_\mu > 1-(q+1)^2\pi_\mu. \tag{25}$$

Throughout this section, we represent w in the form (17). Also, from now, we assume

$$w < \frac{q+1}{2}. \tag{26}$$

Theorem 10. *For the smallest size $s_\mu(2,q)$ of a $(1,\mu)$-saturating set in a projective plane of order q the following upper bounds hold:*

$$s_2(2,q) \le 2.4\sqrt{(q+1)\ln(q+1)} + 2, \quad q \ge 97;$$
$$s_3(2,q) \le 2.6\sqrt{(q+1)\ln(q+1)} + 2, \quad q \ge 181;$$
$$s_4(2,q) \le 2.8\sqrt{(q+1)\ln(q+1)} + 2, \quad q \ge 125.$$

Proof. Let

$$\beta = d\sqrt{(2q+2)\ln((q+1)^2)}. \tag{27}$$

From (9), (10), (22), (24), (26), (27), we establish some inequalities that will be useful in the proof below.

$$q+1-2w > 0,\ 2(q+f-w) > q+1 \text{ if } f \ge 1,$$
$$\beta \le w < \beta+1,\ R_{w,q} < e^{-\frac{w^2}{2q+2}} \le e^{-\frac{\beta^2}{2q+2}},\ w^2 \pm w < 2\beta^2,$$
$$(w-2)(w-1)w(w+1) < 2\beta^4,\ (w-1)w(w+1) < 3\beta^3,$$
$$(w-4)(w-3)(w-2)(w-1)w(w+1) < \beta^6.$$

A set $\mathcal{K}_{w+1}$ covers A exactly once if one line through A contains two points of $\mathcal{K}_{w+1}$, whereas each of the remaining q lines contains at most one point of $\mathcal{K}_{w+1}$. So,

$$T_1 = (q+1)\binom{q}{2}\cdot q^{w-1}\binom{q}{w-1}. \tag{28}$$

A set $\mathcal{K}_{w+1}$ covers A exactly twice if some two lines through A contains two points of $\mathcal{K}_{w+1}$, whereas each of the remaining $q-1$ lines contains at most one point of $\mathcal{K}_{w+1}$. So,

$$T_2 = \binom{q+1}{2}\binom{q}{2}^2\cdot q^{w-3}\binom{q-1}{w-3}. \tag{29}$$

Finally, a set $\mathcal{K}_{w+1}$ covers A exactly 3 times in the following two cases: - one line through A contains three points of $\mathcal{K}_{w+1}$, whereas each of the remaining q lines contains at most one point of $\mathcal{K}_{w+1}$; - three lines through A contain two points of $\mathcal{K}_{w+1}$, whereas each of the remaining $q-2$ lines contains at most one point of $\mathcal{K}_{w+1}$. Therefore,

$$T_3 = (q+1)\binom{q}{3}\cdot q^{w-2}\binom{q}{w-2} + \binom{q+1}{3}\binom{q}{2}^3\cdot q^{w-5}\binom{q-2}{w-5}. \tag{30}$$

Let $\mu = 2$. Taking into account (17), (22), (23), (27), (28), we can show that

$$\pi_2 = R_{w,q}\left(1+\frac{T_1}{T_0}\right) = R_{w,q}\left(1+\frac{w(w+1)(q-1)}{2q(q+1-w)}\right) < \frac{2q+2+2\beta^2}{q+1}e^{-\frac{\beta^2}{2q+2}} = \frac{2+8d^2\ln(q+1)}{(q+1)^{2d^2}}.$$

By a computer aided computation, we have

$$(q+1)^2\pi_2 = \frac{2+8d^2\ln(q+1)}{(q+1)^{2d^2-2}} < 1 \text{ if } d = 1.2,\ q \ge 97.$$

In this case the probability in (25) is positive. So, taking into account (17), the upper bound for $\mu = 2$ is proved.

Let $\mu = 3$. Taking into account (17), (22), (23), (27) – (29), it can be shown that

$$\pi_3 = R_{w,q}\left(1+\frac{w(w+1)(q-1)}{2q(q+1-w)} + \frac{(w-2)(w-1)w}{8q^2(q+2-w)}\times\frac{(w+1)(q-1)^2}{(q+1-w)}\right) < e^{-\frac{\beta^2}{2q+2}}\left(1+\frac{2\beta^2}{q+1}+\frac{2\beta^4}{2(q+1)^2}\right) = \frac{1+8d^2\ln(q+1)+16d^4\ln^2(q+1)}{(q+1)^{2d^2}}.$$

Now by a computer aided computation, we can obtain

$$(q+1)^2\pi_3 < 1 \text{ if } d = 1.3,\ q \ge 181.$$

In this case the probability in (25) is positive. So, taking into account (17), the upper bound for $\mu = 3$ is proved.

The case $\mu = 4$ can be proved similarly to $\mu = 2, 3$. $\square$

V. Upper bounds on the smallest size of a saturating set in the projective space $PG(N, q)$

A point set $S \subset PG(N, q)$ is *saturating* if any point of $PG(N, q) \setminus S$ is collinear with two points in S. Results on saturating sets in $PG(N, q)$ can be found in [6]–[9], [14], [20].

Let $[n, n-r]_q R$ be a linear q-ary code of length n, codimension r, and covering radius R. The homogeneous coordinates of the points of a saturating set with size n in $PG(r-1, q)$, form a parity check matrix of an $[n, n-r]_q 2$ code; see [6]–[9], [14]. Let $s(N, q)$ be the smallest size of a saturating set in $PG(N, q)$, $N \geq 3$. In terms of covering codes, we recall the equality $s(N, q) = \ell(2, N+1, q)$.

Proposition 11. *For the smallest size $s(N, q)$ of a saturating set in the projective space $PG(N, q)$ and for the length function $\ell(2, N+1, q)$, the following upper bound holds:*

$$s(N,q) = \ell(2, N+1, q) \leq \qquad (31)$$
$$\left(2\sqrt{(q+1)\ln(q+1)} + 2\right) q^{\frac{N-2}{2}} + 2q^{\frac{N-4}{2}} \sim 2q^{\frac{N-1}{2}}\sqrt{\ln q},$$

where $N = 2t-2 \geq 6$, $t = 4, 6$ and $t \geq 8$, $N \neq 8, 12$, $q \geq 79$.

Proof. By Theorem 2, there is a saturating set with size $n_q = 2\sqrt{(q+1)\ln(q+1)}+2$ in $PG(2, q)$. From the corresponding $[n_q, n_q - 3]_q 2$ code, by using the construction of [8, Ex. 6], see also [9, Th. 4.4], one can obtain an $[n, n-r]_q 2$ code with $r = 2t-1 \geq 7$, $r \neq 9, 13$, $n = n_q q^{t-2} + 2q^{t-3}$, under condition $q+1 \geq 2n_q$ that holds for $q \geq 79$. $\square$

Surveys of the known $[n, n-r]_q 2$ codes and saturating sets in $PG(N, q)$ can be found in [8], [9], [14]. In many cases bounds (31) are better than the known ones.

A point set $S \subset PG(N, q)$ is $(1, \mu)$*-saturating* if for every point Q of $PG(N, q) \setminus S$ the number of secants of S through Q is at least μ, counted with multiplicity. The multiplicity of a secant ℓ is computed as $\binom{\#(\ell \cap S)}{2}$.

Let $[n, n-r]_q(R, \mu)$ be a linear q-ary (R, μ)-MCF code of length n, codimension r, and covering radius R. The points of a $(1, \mu)$-saturating set with size n in $PG(r-1, q)$ form a parity check matrix of an $[n, n-r]_q(2, \mu)$ code; see [1], [14]. Let $s_\mu(N, q)$ be the smallest size of a $(1, \mu)$-saturating set in $PG(N, q)$, $N \geq 3$.

Proposition 12. *For the smallest size $s_\mu(N, q)$ of a $(1, \mu)$-saturating set in the projective space $PG(N, q)$, $N \geq 4$ even, and for the μ-length function, it holds that:*

$$s_\mu(N,q) = \ell_\mu(2, N+1, q) \leq$$
$$q^{\frac{N-2}{2}} n_{q,\mu} + \max(3, \mu)\frac{q^{\frac{N-2}{2}} - 1}{q-1} \sim 2\Omega_\mu q^{\frac{N-1}{2}}\sqrt{\ln q},$$

where $n_{q,\mu} = 2\Omega_\mu\sqrt{(q+1)\ln(q+1)} + 2$, Ω_μ is as in (5), $q^{\frac{N-2}{2}} + 1 - \mu \geq n_{q,\mu}$.

Proof. By Theorem 6, there is a $(1, \mu)$-saturating set with size $n_{q,\mu}$ in $PG(2, q)$. We directly apply [1, Cor. 6.5] to the corresponding MCF code and use the one-to-one correspondence between $(1, \mu)$-saturating sets and $(2, \mu)$-MCF codes. $\square$

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