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Tanaka structures (non holonomic G-structures) and Cartan connections

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ABSTRACT

Let $\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_l$ ($k > 0, l \geq 0$) be a finite dimensional graded Lie algebra, with a Euclidean metric $\langle \cdot, \cdot \rangle$ adapted to the gradation. The metric $\langle \cdot, \cdot \rangle$ is called admissible if the codifferentials $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ ($k \geq 0$) are Q-invariant ($\mathrm{Lie}(Q) = \mathfrak{h}_0 \oplus \mathfrak{h}_+$). We find necessary and sufficient conditions for a Euclidean metric, adapted to the gradation, to be admissible, and we develop a theory of normal Cartan connections, when these conditions are satisfied. We show how the treatment from Cap and Slovak (2009), about normal Cartan connections of semisimple type, fits into our theory. We also consider in detail the case when $\mathfrak{h} := t^*(\mathfrak{g})$ is the cotangent Lie algebra of a non-positively graded Lie algebra $\mathfrak{g}$.

1. Introduction

The theory of G-structures, which is a coordinate free version of the *répére mobile* (moving frame) method by E. Cartan, provides a powerful tool for the investigation of different geometric structures on an n-dimensional manifold M. If a geometric structure, say, a tensor field A, is infinitesimally homogeneous, i.e. takes the same constant value A_0 at any point $p \in M$ with respect to an appropriate frame $f_p = (e_1, \dots, e_n)$, then the set of such adapted frames forms a G-structure $\pi : P \to M$. Applying the prolongation procedure to this G-structure, one can construct an absolute parallelism (and, in best cases, a Cartan connection), which can be used to find invariants for the given geometric structure. More precisely, recall that the first prolongation of the G-structure $\pi : P \to M$ is a $G^{(1)}$-structure $\pi^{(1)} : P^{(1)} \to P$, with commutative structure group $G^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2(V^*)$ (where $\mathfrak{g} = \mathrm{Lie}(G)$). The bundle $\pi^{(1)} : P^{(1)} \to P$ is not unique: it depends on the choice of a subspace $D \subset V \otimes S^2(V^*)$, complementary to $\partial(\mathfrak{g} \otimes V)$, where $\partial : \mathfrak{g} \otimes V \to V \otimes S^2(V^*)$ is the skew-symmetrization. If the group G is simple, then (in most cases) $G^{(1)} = \{e\}$ and $\pi^{(1)} : P^{(1)} \to P$ is an $\{e\}$-structure, or an absolute parallelism, on the bundle of frames P. If moreover D is G-invariant, then the $\{e\}$-structure $\pi^{(1)} : P^{(1)} \to P$ is identified with a Cartan connection of type $V \oplus \mathfrak{g}$, i.e. a G-equivariant map $\kappa : TP \to V \oplus \mathfrak{g}$ such that $\kappa_p : T_pP \to V \oplus \mathfrak{g}$ (for any $p \in P$) is an isomorphism, which is an extension of the vertical parallelism $i_p : T_p^{\mathrm{vert}}P \to \mathfrak{g}$. In the more general case when $\pi : P \to M$ is a G-structure of finite type, i.e. the k-th prolongation $G^{(k)} = \mathfrak{g} \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*)$, for some $k > 0$, is trivial, we get an absolute parallelism on the k-th prolongation $\pi^{(k)} : P^{(k)} \to P^{(k-1)}$ of the G-structure $\pi : P \to M$ (which, in general, is not a Cartan connection).

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The approach described above does not work if the G-structure has infinite type, i.e. the full prolongation $\mathfrak{g}^{(\infty)}$ is infinite dimensional (e.g. for symplectic or contact structures). To overcome this difficulty, N. Tanaka developed in [1,2] a generalization of the theory of G-structures to "non-holonomic G-structures" (called Tanaka structures in [3] and infinitesimal flag structures in [4]), which are roughly speaking principal subbundles of frames on a non-holonomic distribution. Tanaka defined the prolongation of non-positively graded Lie algebras and under some assumptions, he associated to a non-holonomic G-structure a canonical Cartan connection. There are several expositions of different versions and generalizations of Tanaka theory [3–8]. The aim of this paper is to give a self contained exposition of the Tanaka theory and to investigate relations between Tanaka structures and Cartan connections. We generalize the results by A. Čap and J. Slovak [4], where the special case of parabolic geometry is studied in detail.

The paper is structured as follows. In Section 2, intended to fix notation, we recall basic definitions about graded Lie algebras, Tanaka structures and Cartan connections. Our approach follows closely [3,4].

In Section 3 we study the relation between Cartan connections and Tanaka structures. We show that a regular Cartan connection $\kappa \in \Omega^1(P, \mathfrak{h})$ on a principal Q-bundle $\pi : P \to M$ induces a Tanaka structure $(\mathcal{D}, \pi_G : P_G \to M)$ and conversely, that a Tanaka structure is always induced by a regular Cartan connection (see Propositions 2 and 5). In the semisimple case (i.e. $\mathfrak{h}$ semisimple with a $|k|$-gradation and $\mathrm{Lie}(Q) = \mathfrak{q} = \mathfrak{h}_0 \oplus \mathfrak{h}_+$), this was done in [4]. We remark that the arguments from [4] used to prove these statements require no semisimplicity assumptions. We simplify and adapt them to our more general setting.

In Section 4 we generalize the theory of **normal** Cartan connections, defined in [4] for Cartan connections of semisimple type, to the non-semisimple case. Let $\pi : P \to M$ be a principal Q-bundle and $\kappa : TP \to \mathfrak{h}$ a Cartan connection, where $\mathfrak{h}$ is a graded semisimple Lie algebra with non-negative subalgebra $\mathfrak{q} = \mathrm{Lie}(Q) = \mathfrak{h}_0 \oplus \mathfrak{h}_+$. Using the (non-degenerate) Killing form B and an appropriate Cartan involution θ of $\mathfrak{h}$, one can define a Euclidean metric $B_\theta = -B(\theta\cdot, \cdot)$, adapted to the gradation, see (4.7). It turns out that the codifferentials $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ (defined as the metric adjoints of the Lie algebra differentials $\partial : C^k(\mathfrak{h}_-, \mathfrak{h}) \to C^{k+1}(\mathfrak{h}_-, \mathfrak{h})$, with respect to the metric induced by B_θ on $\{C^k(\mathfrak{h}_-, \mathfrak{h}),\ k \geq 0\}$) are Q-invariant (with Q-action induced by the adjoint action). This invariancy is the crucial property of the theory of normal Cartan connections (of semisimple type), which are defined as Cartan connections with coclosed curvature form, see [4]. Using the semisimple case as motivation, in Section 4.1 we consider $\mathfrak{h}$ a graded (not necessarily semisimple) Lie algebra with a fixed Euclidean metric $\langle\cdot,\cdot\rangle$, adapted to the gradation. We call the metric $\langle\cdot,\cdot\rangle$ admissible if the codifferentials $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ $(k \geq 0)$ are Q-invariant (see Definition 4). We characterize admissible metrics (see Proposition 6). As a consequence, we reobtain in our setting that the standard metric B_θ on a graded semisimple Lie algebra is admissible and we show that admissible metrics exist also on (graded) non-semisimple Lie algebras (see Corollary 1 and Remark 1). We develop a theory of normal Cartan connections of type $\mathfrak{h}$, where $\mathfrak{h}$ is a graded Lie algebra with an admissible metric, and we show that various facts from the theory of normal Cartan connections of semisimple type [4] are preserved in this more general setting. In particular, any Tanaka structure which is induced by a regular Cartan connection of type $\mathfrak{h}$ is induced also by a normal Cartan connection of type $\mathfrak{h}$ and the cohomology group $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h})$ is the only obstruction for the uniqueness (up to bundle automorphisms) of a normal Cartan connection of type $\mathfrak{h}$ inducing a given Tanaka structure (see Theorems 1 and 2). Our proofs for these two theorems are based on the arguments from [4]. We simplify these arguments and show that no semisimplicity assumptions are required. While for most graded semisimple Lie algebras $\mathfrak{h}$, the cohomology group $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h})$ is trivial, our computations from the next section show that this is not true in general.

In Section 5, which is entirely algebraic, we consider in detail the cotangent Lie algebra $\mathfrak{h} = t^*(\mathfrak{g})$ of a non-positively graded Lie algebra $\mathfrak{g}$. We show that $\mathfrak{h}$ inherits a gradation from the gradation of $\mathfrak{g}$, with negative part $\mathfrak{g}_-$. In Section 5.1 we compute the cohomology group $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h})$. In general, it is non-trivial. As an illustration of our computations, in Section 5.2 we describe $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h})$, in the simplest case when $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ has a non-positive gradation of depth one. Cohomology groups for cotangent Lie algebras are of independent interest and appear in the literature in various settings (see e.g. [9], for the cohomology of the cotangent bundle of Heisenberg group). In Section 5.3 we return to the topic of admissible metrics. We assume that $\mathfrak{g}$ has a Euclidean metric $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, adapted to the gradation, and we define a metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$, which coincides with the given metric $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ and $\mathfrak{g}^*$ (identified with $\mathfrak{g}$ using $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$) and such that $\mathfrak{g}$ is orthogonal to $\mathfrak{g}^*$ with respect to $\langle\cdot,\cdot\rangle$. Our motivation to consider such a metric on $\mathfrak{h}$ comes from its formal similarity with the standard metric B_θ on semisimple Lie algebras. More precisely, both $\langle\cdot,\cdot\rangle$ and B_θ are of the form $B(\theta(\cdot), \cdot)$, where $\theta : \mathfrak{h} \to \mathfrak{h}$ is linear, bijective, and $B \in S^2(\mathfrak{h}^*)$ is a bi-invariant non-degenerate symmetric form: when $\mathfrak{h} = t^*(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ is the cotangent Lie algebra, B is the standard metric of neutral signature of $\mathfrak{h}$ and $\theta : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$, $\theta(\mathfrak{g}) = \mathfrak{g}^*$, $\theta(\mathfrak{g}^*) = \mathfrak{g}$ is induced by the Riemannian duality defined by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$; when $\mathfrak{h}$ is semisimple, B the Killing form of $\mathfrak{h}$ and $\theta : \mathfrak{h} \to \mathfrak{h}$ is (minus) a Cartan involution. Our main result in this section shows that, despite this formal analogy, there are strong obstructions for $\langle\cdot,\cdot\rangle$ to be admissible (see Theorem 4). The problem to find admissible metrics on cotangent Lie algebras remains open and will be studied in the future.

At the end of this introduction, we mention that the general question of existence of a canonical Cartan connection associated to a Tanaka structure was considered, in large generality, also in the paper [10] by T. Morimoto, in the framework of his remarkable theory of filtered manifolds. There it was associated, to any Tanaka structure of type $(\mathfrak{m}, G)$, for which the so called (C)-condition is satisfied (see Definition 3.10.1 of [10]), a canonical Cartan connection θ (see Theorem 3.10.1 of [10]). The principal bundle P on which θ is defined was constructed by an elaborated, induction procedure. The Lie algebra of the structure group K of P is the non-negative part of the prolongation $(\mathfrak{m} \oplus \mathfrak{g})^\infty$ of $\mathfrak{m} \oplus \mathfrak{g}$ (when this prolongation is finite dimensional). The canonical Cartan connection θ takes values in $(\mathfrak{m} \oplus \mathfrak{g})^\infty$. Now, as remarked in the proof of Proposition

3.10.1 of [10], if $(\mathfrak{m} \oplus \mathfrak{g})^{\infty}$ admits a Euclidean metric $\langle \cdot, \cdot \rangle$, adapted to the gradation, and such that the codifferentials $\partial^* : C^{k+1}(\mathfrak{m}, (\mathfrak{m} + \mathfrak{g})^{\infty}) \to C^k(\mathfrak{m}, (\mathfrak{m} \oplus \mathfrak{g})^{\infty})$ $(k \geq 0)$ are K-invariant (i.e., in our language, $\langle \cdot, \cdot \rangle$ is admissible), then the condition (C) is satisfied. Note that Morimoto gives a general construction of a canonical Cartan connection associated to a Tanaka structure, but he does not consider normal Cartan connections, i.e. connections with coclosed curvature. While Morimoto's theory covers a large class of situations, it is also much more technical compared to our simple, direct approach.

2. Preliminary material

In this section, intended to fix notation, we recall the basic facts we need about Tanaka structures and Cartan connections (see e.g. [3,4]).

2.1. Tanaka structures

2.1.1. Graded Lie algebras

Let $\mathfrak{h}$ be a finite dimensional Lie algebra. A decomposition $\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_{\ell}$ (where $k > 0$ and $\ell \geq 0$) of $\mathfrak{h}$ is called a **gradation of depth** k if $[\mathfrak{h}_i, \mathfrak{h}_j] \subset \mathfrak{h}_{i+j}$, for any i, j (with the convention that $\mathfrak{h}_i = 0$ when $i < -k$ and $i > l$). We will always assume that such a gradation is **fundamental**, i.e. the subalgebra $\mathfrak{h}_- := \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_{-1}$ is generated by $\mathfrak{h}_{-1}$, the subspaces $\mathfrak{h}_0$, $\mathfrak{h}_{-k}$ and $\mathfrak{h}_{\ell}$ are non-trivial and the adjoint action of $\mathfrak{h}_0$ on $\mathfrak{h}_{-1}$ is exact. Let $\mathfrak{h}^i := \oplus_{j \geq i} \mathfrak{h}_j$ be the associated filtration.

The space $C^i(\mathfrak{h}_-, \mathfrak{h}) = \Lambda^i(\mathfrak{h}_-^*) \otimes \mathfrak{h}$ of $\mathfrak{h}$-valued i-forms on $\mathfrak{h}_-$ inherits a gradation $C^i(\mathfrak{h}_-, \mathfrak{h}) = \oplus_j C^i_j(\mathfrak{h}_-, \mathfrak{h})$, where

$$C^i_j(\mathfrak{h}_-, \mathfrak{h}) := \sum_{s_1, \ldots, s_i} (\mathfrak{h}_{s_1} \wedge \cdots \wedge \mathfrak{h}_{s_i})^* \otimes \mathfrak{h}_{s_1 + \cdots + s_i + j}$$

is the space of i-forms of homogeneous degree j. We shall use the notation

$$\mathrm{gr}_j : C^i(\mathfrak{h}_-, \mathfrak{h}) \to C^i_j(\mathfrak{h}_-, \mathfrak{h}), \qquad \phi \to \mathrm{gr}_j(\phi) = \phi_j$$

for the natural projection and $C^i(\mathfrak{h}_-, \mathfrak{h})^m = \oplus_{j \geq m} C^i_j(\mathfrak{h}_-, \mathfrak{h})$ for the filtration associated to the gradation. The Lie algebra differential $\partial : C^k(\mathfrak{h}_-, \mathfrak{h}) \to C^{k+1}(\mathfrak{h}_-, \mathfrak{h})$ defined by

$$\begin{aligned}(\partial\phi)(X_0, \ldots, X_k) := &\sum_{0 \leq i \leq k} (-1)^i [X_i, \phi(X_0, \ldots, \widehat{X_i}, \ldots, X_k)] \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \phi\left([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k\right)\end{aligned} \tag{2.1}$$

(for $X_i \in \mathfrak{h}_-$, $0 \leq i \leq k$) is homogeneous of degree zero, i.e. it preserves gradations (the hat means that the argument is omitted).

2.2. Definition of Tanaka structure

Let $\mathcal{D}$ be a distribution on a manifold M. It determines a filtration

$$\cdots \mathcal{D}_{-d-1}(p) = \mathcal{D}_{-d}(p) \supset \mathcal{D}_{-d+1}(p) \supset \cdots \supset \mathcal{D}_{-1}(p) = \mathcal{D}(p) \tag{2.2}$$

of any tangent space T_pM, where each subspace $\mathcal{D}_{-j}(p)$ is spanned by $\mathcal{D}(p)$ and the values at p of commutators of vector fields $X_1, \ldots, X_s$ $(2 \leq s \leq j)$ in $\mathcal{D}$. The commutator of vector fields induces the structure of a graded Lie algebra on each graded vector space

$$\mathfrak{m}(p) = \mathrm{gr}_{\mathcal{D}}(T_pM) = \mathfrak{m}_{-d}(p) \oplus \cdots \oplus \mathfrak{m}_{-1}(p)$$

associated to the filtered space T_pM, where $\mathfrak{m}_{-1}(p) = \mathcal{D}_{-1}(p)$ and $\mathfrak{m}_{-i}(p) = \mathcal{D}_{-i}(p)/\mathcal{D}_{-i+1}(p)$, for any $2 \leq i \leq d$.

A distribution $\mathcal{D}$ is called **regular of type** $\mathfrak{m}$ (**and depth** k) if all Lie algebras $\mathfrak{m}(p)$, $p \in M$, are isomorphic to a given negatively graded fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}_{-k} \oplus \cdots \oplus \mathfrak{m}_{-1}$. If $\mathcal{D}$ is regular of type $\mathfrak{m}$, an isomorphism $\xi : \mathfrak{m}_{-1} \to \mathcal{D}(p)$ is called an **adapted frame** if it can be extended to an isomorphism $\hat{\xi} : \mathfrak{m} \to \mathrm{gr}_{\mathcal{D}}(T_pM)$ of graded Lie algebras. We remark that if such an extension $\hat{\xi}$ exists, then it is unique. The group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$ of (grading preserving) automorphisms of the Lie algebra $\mathfrak{m}$ acts simply transitively on the set $\mathbb{L}(M, \mathcal{D})$ of all adapted frames. The natural projection $\pi_0 : \mathbb{L}(M, \mathcal{D}) \to M$ is a principal $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$-bundle, called the **bundle of** $\mathcal{D}$**-frames**.

Definition 1. Let G be a Lie subgroup of $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{m})$. A **Tanaka structure of type** $(\mathfrak{m}, G)$ on M is a regular distribution $\mathcal{D}$ of type $\mathfrak{m}$ together with a principal G-subbundle $\pi_G : P_G \subset \mathbb{L}(M, \mathcal{D}) \to M$ of the bundle of $\mathcal{D}$-frames.

Let $\mathfrak{g} := \mathrm{Lie}(G)$. We denote by $\mathfrak{m}(\mathfrak{g}) = \mathfrak{m} \oplus \mathfrak{g}$ the non-positively graded Lie algebra associated to $(\mathfrak{m}, G)$ and by $\mathfrak{m}(\mathfrak{g})^{\infty}$ its Tanaka prolongation, defined as the maximal graded Lie algebra with non-positive part isomorphic to $\mathfrak{m}(\mathfrak{g})$ (see [2]).

2.3. Cartan connections

2.3.1. Definition of Cartan connections

Let H be a Lie group and Q a subgroup of H. We denote by $\mathfrak{h}$ and $\mathfrak{q}$ the Lie algebras of H and Q respectively. Let $\pi : P \to M$ be a principal Q-bundle. For $a \in \mathfrak{q}$, we denote by $X^a(p) = p \cdot \exp(a)$ the fundamental vector field on P generated by a.

Definition 2. A 1-form $\kappa \in \Omega^1(P, \mathfrak{h})$ is called a **Cartan connection of type** $\mathfrak{h}$ if:

(i) $\kappa(X^a) = a$, for any $a \in \mathfrak{q}$;
(ii) $\kappa|_{T_uP} : T_uP \to \mathfrak{h}$ is an isomorphism, for any $u \in P$;
(iii) $(r^g)^*(\kappa) = \mathrm{Ad}(g^{-1}) \circ \kappa$, for any $g \in Q$.

The following simple lemma will be useful for our purposes.

Lemma 1. *The map*

$$P \times_Q (\mathfrak{h}/\mathfrak{q}) \to TM, \qquad [u, \hat{a}] \to \pi_*(X^a_u), \qquad (u, a) \in P \times \mathfrak{h} \tag{2.3}$$

is an isomorphism. Above $a \in \mathfrak{h}$ is any representative of $\hat{a} \in \mathfrak{h}/\mathfrak{q}$ and $X^a := \kappa^{-1}(a) \in \mathfrak{X}(P)$ (is called a ***constant vector field****).*

The **curvature** of a Cartan connection κ is a 2-form $\Omega \in \Omega^2(P, \mathfrak{h})$ defined by:

$$\Omega(X, Y) = d\kappa(X, Y) + [\kappa(X), \kappa(Y)], \quad \forall X, Y \in TP.$$

From definitions,

$$\Omega(X^a, X^b) = -\kappa([X^a, X^b]) + [a, b], \quad \forall a, b \in \mathfrak{h}, \tag{2.4}$$

i.e. Ω measures the failure of $\mathfrak{h} \ni a \to X^a \in \mathfrak{X}(P)$ to be a Lie algebra homomorphism. The curvature $\Omega \in \Omega^2(P, \mathfrak{h})$ is invariant, i.e. $(r^g)^*(\Omega) = \mathrm{Ad}(g^{-1}) \circ \Omega$, for any $g \in Q$, and horizontal, i.e. $\Omega(X^a, \cdot) = 0$, for any $a \in \mathfrak{q}$. In particular, Ω is a section of $\Lambda^2(M) \otimes \mathcal{A}(M)$, where $\mathcal{A}(M) = P \times_Q \mathfrak{h}$ is the **adjoint tractor bundle**. The **curvature function** is defined by

$$K : P \to \Lambda^2(\mathfrak{h}/\mathfrak{q})^* \otimes \mathfrak{h}, \qquad K(u)(\hat{a}, \hat{b}) = \Omega(X^a_u, X^b_u), \quad u \in P,\ \hat{a}, \hat{b} \in \mathfrak{h}/\mathfrak{q}.$$

2.3.2. Cartan connections of graded type

Let $\kappa \in \Omega^1(P, \mathfrak{h})$ be a Cartan connection on a principal Q-bundle $\pi : P \to M$. It is called of **graded type** if a fundamental gradation

$$\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_l = \mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+ \tag{2.5}$$

is given, such that $\mathfrak{q} = \mathrm{Lie}(Q) = \mathfrak{h}_0 \oplus \mathfrak{h}_+$. The gradation of $\mathfrak{h}$ induces a gradation of TP:

$$TP = (TP)_{-k} \oplus \cdots \oplus (TP)_\ell, \ (TP)_i := \kappa^{-1}(\mathfrak{h}_i). \tag{2.6}$$

The gradations of TP and $\mathfrak{h}$ allow decompositions of the curvature Ω and the curvature function K into homogeneous components

$$\Omega = \sum_{i,j,m} \Omega_{ij,m}, \qquad K = \sum_{i,j,m} K_{ij,m}$$

where $\Omega_{ij,m}$ (respectively $K_{ij,m}$) are identified with sections of $(TP)^*_i \wedge (TP)^*_j \otimes \mathfrak{h}_m$ (respectively, functions : $P \to (\mathfrak{h}_i)^* \otimes (\mathfrak{h}_j)^* \otimes \mathfrak{h}_m$) and have homogeneous degree $d := m - i - j$.

Definition 3. A Cartan connection κ is called **regular** if it is of graded type and all the homogeneous components of its curvature function K have positive degree.

3. Cartan connections and Tanaka structures

3.1. From Cartan connections to Tanaka structures

Let $\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_l$ be a graded Lie algebra and $\kappa \in \Omega^1(P, \mathfrak{h})$ a Cartan connection of graded type (not necessarily regular) on a principal Q-bundle $\pi : P \to M$. Assume that the Lie group Q is decomposed into a semidirect product $Q = G \cdot N := H_0 \cdot H^+$, according to the semidirect decomposition $\mathfrak{q} = \mathfrak{h}_0 \oplus \mathfrak{h}_+$. We define a Q-invariant flag of distributions

$$T^iP := \kappa^{-1}(\mathfrak{h}^i) = \sum_{j \geq i} (TP)_j, \quad -k \leq i \leq l \tag{3.1}$$

of TP associated to the gradation (2.6). Like in [4], the quotient $P_G := P/N$, with the natural projection $\pi_G : P_G \to M$ is a principal G-bundle. The natural projections $\tilde{\pi} : P \to P_G$ and $\pi_G : P_G \to M$ map the flag of distributions on P to a G-invariant flag of distributions

$$TP_G = T^{-k}P_G \supset T^{-k+1}P_G \supset \cdots \supset T^0P_G = T^{\mathrm{vert}}(\pi_G)$$

on P_G and to a flag of distributions

$$TM = T^{-k}M \supset T^{-k+1}M \cdots \supset T^{-1}M \tag{3.2}$$

on M. We define the graded tangent bundle

$$\mathrm{gr}(TM) = (TM)_{-k} \oplus (TM)_{-k+1} \oplus \cdots \oplus (TM)_{-1}, \qquad \mathrm{gr}_{-i}(TM) = (TM)_{-i} := T^{-i}M/T^{-i+1}M$$

and we denote by $q_j : T^jM \to (TM)_j$ the natural projections. From Lemma 1, $T^iM \cong P \times_Q (\mathfrak{h}^i/\mathfrak{q})$, for any i. It follows that

$$\mathrm{gr}_i(TM) \cong P \times_Q (\mathfrak{h}^i/\mathfrak{h}^{i+1}) = P_G \times_G \mathfrak{h}_i, \quad \forall i < 0, \qquad \mathrm{gr}(TM) \cong P_G \times_G \mathfrak{h}_-. \tag{3.3}$$

Since G acts on $\mathfrak{h}_-$ as a group of automorphisms, the Lie bracket of $\mathfrak{h}_-$ induces a Lie bracket $[\cdot,\cdot]_{\mathfrak{h}}$ on $\mathrm{gr}(TM)$. The following proposition can be proved easily [4].

Proposition 1. *The Cartan connection κ is regular if and only if the following two conditions hold:*

(i) *The flag of distributions T^iM defines a filtration $\Gamma(T^iM)$ of the Lie algebra $\mathfrak{X}(M)$ of vector fields on M (i.e. $[\Gamma(T^iM), \Gamma(T^jM)] \subset \Gamma(T^{i+j}M)$, for any i, j);*

(ii) *The bracket $[\cdot,\cdot]$ on $\mathrm{gr}(TM)$, induced by the Lie bracket of vector fields, coincides with $[\cdot,\cdot]_{\mathfrak{h}}$.*

The next proposition is our main result from this section. It shows that any regular Cartan connection defines a Tanaka structure.

Proposition 2. *Let $\kappa \in \Omega^1(P, \mathfrak{h})$ be a regular Cartan connection.*

(i) *The flag of distributions $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ defined by κ is the derived flag of the distribution $\mathcal{D} = T^{-1}M$, which is regular of type $\mathfrak{h}_-$.*

(ii) *The principal G-bundle $\pi_G : P_G \to M$ is a reduction to the structure group G of the $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{h}_-)$-bundle of adapted frames $\pi_0 : \mathbb{L}(M, \mathcal{D}) \to M$.*

Equivalently, $(\mathcal{D}, \pi_G : P_G \to M)$ is a Tanaka structure.

Proof. The first claim follows from the assumption that $\mathfrak{h}_{-1}$ generates $\mathfrak{h}_-$ and from $[\cdot,\cdot]_{\mathfrak{h}} = [\cdot,\cdot]$ (because κ is regular). Since $\mathrm{gr}_{\mathcal{D}}(TM) \cong P_G \times_G \mathfrak{h}_-$ (see (3.3)), the distribution $\mathcal{D}$ is regular of type $\mathfrak{h}_-$. For the second claim, we notice that any $p_0 \in P_G$ determines an isomorphism

$$\hat{p}_0 : \mathfrak{h}_- \to \mathrm{gr}_{\mathcal{D}}(T_{\pi_G(p_0)}M), \qquad a \to \hat{p}_0(a), \tag{3.4}$$

where $\hat{p}_0(a) = [p_0, a]$ under the identification $\mathrm{gr}_{\mathcal{D}}(TM) = P_G \times_G \mathfrak{h}_-$. The map

$$i : P_G \to \mathbb{L}(M, \mathcal{D}), \qquad p_0 \to i(p_0) := \hat{p}_0 \in \mathbb{L}_{\pi_G(p_0)}(M, \mathcal{D})$$

is injective and covers the natural inclusion $G \subset \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{h}_-)$, i.e. is a reduction of $\pi_0 : \mathbb{L}(M, \mathcal{D}) \to M$ to G. □

3.2. The space of Cartan connections inducing a given Tanaka structure

We now study the relation between the regular Cartan connections which take values in a given Lie algebra and induce the same Tanaka structure. Propositions 3 and 4 below will be our main computational tool in the theory of normal Cartan connections. Their proofs follow like in the semisimple case treated in [4]. To keep the text short, we chose not to reproduce them here.

Let $\kappa \in \Omega^1(P, \mathfrak{h})$ be a regular Cartan connection of type $\mathfrak{h}$ on a principal Q-bundle $\pi : P \to M$. Any other regular Cartan connection $\tilde{\kappa} \in \Omega^1(P, \mathfrak{h})$ is given by $\tilde{\kappa} = \kappa + \Phi$, where $\Phi : TP \to \mathfrak{h}$ vanishes on $T^{\mathrm{vert}}P = \kappa^{-1}(\mathfrak{q})$ and will be considered as a function on the quotient $TP/T^{\mathrm{vert}}P$. Define

$$\phi : P \to C^1(\mathfrak{h}_-, \mathfrak{h}), \qquad \phi := \Phi \circ \hat{\kappa}^{-1} = (\tilde{\kappa} - \kappa) \circ \hat{\kappa}^{-1}. \tag{3.5}$$

Above $\hat{\kappa}_p^{-1} : \mathfrak{h}_- \to T_pP/T_p^{\mathrm{vert}}P$ (for any $p \in P$) is the inverse of the isomorphism $T_pP/T_p^{\mathrm{vert}}P \cong \mathfrak{h}_-$ induced by $\kappa_p : T_pP \to \mathfrak{h}$. The function ϕ is Q-invariant. The following holds:

Proposition 3. (i) *The (regular) Cartan connections κ and $\tilde{\kappa}$ define the same Tanaka structure if and only if the function ϕ from (3.5) takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^1$.*

(ii) *Suppose that ϕ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^m$ ($m \geq 1$) and let $K, \tilde{K} : P \to C^2(\mathfrak{h}_-, \mathfrak{h})$ the curvature functions of κ and $\tilde{\kappa}$. Then $\tilde{K} - K$ takes values in $C^2(\mathfrak{h}_-, \mathfrak{h})^m$ and*

$$\tilde{K}_m = K_m + \partial(\phi_m). \tag{3.6}$$

For the following proposition, see the proof of Proposition 3.1.14 from [4, p. 270].

Proposition 4. *Let $\tilde{\kappa}, \kappa \in \Omega^1(P, \mathfrak{h})$ be two regular Cartan connections, which induce the same Tanaka structure, and $\psi : P \to \mathfrak{h}^l$ a Q-invariant function ($l \geq 1$). Then $\psi^{\mathrm{iso}}(p) := p \cdot \exp(\psi(p))$, $p \in P$, is a principal bundle automorphism and $(\psi^{\mathrm{iso}})^*(\tilde{\kappa})$ is a regular Cartan connection. Let $\tilde{\phi} := ((\psi^{\mathrm{iso}})^*(\tilde{k}) - \kappa) \circ \hat{\kappa}^{-1}$ and $\phi := (\tilde{\kappa} - \kappa) \circ \hat{\kappa}^{-1}$ the functions associated to the pairs $(\kappa, (\psi^{\mathrm{iso}})^*(\tilde{k}))$ and $(\kappa, \tilde{\kappa})$, as in Proposition 3. Then $\tilde{\phi} - \phi + \partial(\psi) : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^{l+1}$. In particular, $\tilde{\phi} : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^1$ and $(\psi^{\mathrm{iso}})^*(\tilde{\kappa})$ induces the same Tanaka structure as κ (or $\tilde{\kappa}$).*

3.3. Construction of a Cartan connection

In this section we associate to a Tanaka structure $\pi_G : P_G \subset \mathbb{L}(M, \mathcal{D}) \to M$ of type $(\mathfrak{m}, G)$ a regular Cartan connection, which induces the given Tanaka structure. We assume that the Tanaka structure has finite type, i.e. the Tanaka prolongation $\mathfrak{m}(\mathfrak{g})^\infty = \mathfrak{m} \oplus \mathfrak{g} \oplus \mathfrak{g}_+$ of the non-positively graded Lie algebra $\mathfrak{m}(\mathfrak{g}) = \mathfrak{m} \oplus \mathfrak{g}$ is finite dimensional. Let $\mathfrak{n} := \mathfrak{g}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l$ be the positive graded part and $\mathfrak{q} := \mathfrak{g} \oplus \mathfrak{n}$ the non-negative part of $\mathfrak{m}(\mathfrak{g})^\infty$. We denote by Q the Lie group with Lie algebra $\mathfrak{q}$ and we assume that it is the semidirect product $Q = G \cdot N$ of closed subgroups G and N, generated by $\mathfrak{g}$ and $\mathfrak{n}$ respectively.

We will construct a regular Cartan connection of type $\mathfrak{m}(\mathfrak{g})^\infty$ on the principal Q-bundle

$$P := P_G \times_G Q = P_G \times_G (G \cdot N) = P_G \times N.$$

Above G acts on Q by the left action. The (right) Q-action which makes $P = P_G \times_G Q$ a principal Q-bundle is given by

$$q[p_0, q'] = [p_0, q'q], \quad q, q' \in Q,\ p_0 \in P_G$$

or in terms of the representation $P = P_G \times N$ by

$$(g \cdot n)(p_0, n') = [p_0, n'gn] = (p_0 g, (g^{-1} n' g)n), \quad g \in G,\ n, n' \in N,\ p_0 \in P_G.$$

Proposition 5. *Let $(\mathcal{D}, \pi_G : P_G \to M)$ be a Tanaka structure. There is a regular Cartan connection of type $\mathfrak{m}(\mathfrak{g})^\infty$ on the principal Q-bundle $\pi : P_G \times N \to M$ which induces $(\mathcal{D}, \pi_G)$.*

Proof. In a first stage, we construct a Cartan connection of type $\mathfrak{m}(\mathfrak{g})^\infty_{\leq 0} = \mathfrak{m} \oplus \mathfrak{g}$ on $\pi_G : P_G \to M$. For this, we chose a principal connection $\omega \in \Omega^1(P_G, \mathfrak{g})$, with horizontal bundle $H \subset TP_G$, and a gradation $TM = (TM)_{-k} \oplus \cdots \oplus (TM)_{-1}$, consistent with the filtration (2.2) determined by $\mathcal{D}$. (In particular, $(TM)_{-1} = \mathcal{D}$.) The gradation defines a bundle isomorphism $i : TM \to \mathrm{gr}_{\mathcal{D}}(TM)$. From the definition of Tanaka structures, $\pi_G : P_G \to M$ is a G-subbundle of the principal bundle of $\mathcal{D}$-frames. In particular, any point $p_0 \in P_G$ defines an isomorphism $\hat{p}_0 : \mathfrak{m} \to \mathrm{gr}_{\mathcal{D}}(T_{\pi_G(p_0)} M)$. By combining the maps π_G, i and $\hat{p}_0$, we obtain an isomorphism

$$(\kappa_G)^{\mathrm{hor}}_{p_0} : H_{p_0} \to \mathfrak{m}, \qquad (\kappa_G)^{\mathrm{hor}}_{p_0} := (\hat{p}_0)^{-1} \circ i \circ (\pi_G)_*.$$

Extending it to $T_{p_0}(P_G)$ using the vertical parallelism $T^{\mathrm{vert}}_{p_0}(P_G) \cong \mathfrak{g}$, we obtain an isomorphism $(\kappa_G)_{p_0} : T_{p_0} P_G \to \mathfrak{m} \oplus \mathfrak{g}$. It is easy to check that $\kappa_G : TP_G \to \mathfrak{m} \oplus \mathfrak{g}$ constructed in this way is a Cartan connection on $\pi_G : P_G \to M$, of type $\mathfrak{m}(\mathfrak{g})^\infty_{\leq 0} = \mathfrak{m} \oplus \mathfrak{g}$.

In a second stage, we extend the Cartan connection $\kappa_G \in \Omega^1(P_G, \mathfrak{m}(\mathfrak{g})^\infty_{\leq 0})$ to a Cartan connection $\kappa \in \Omega^1(P, \mathfrak{m}(\mathfrak{g})^\infty)$ on the $Q = G \cdot N$-principal bundle $\pi : P = P_G \times N \to M$. Namely, we define $\kappa = \kappa_G \oplus \mu_N$, where $\mu_N \in \Omega^1(N, \mathfrak{n})$ is the left-invariant Maurer Cartan form on N. It is easy to check that κ is a Cartan connection which induces the given Tanaka structure. $\square$

4. Normal Cartan connections

4.1. Admissible metrics on graded Lie algebras

Let $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+$ be a graded Lie algebra, H a connected Lie group, with Lie algebra $\mathfrak{h}$, $G = H_0$ and Q connected subgroups with Lie algebras $\mathfrak{h}_0$ and $\mathfrak{q} = \mathfrak{h}_0 \oplus \mathfrak{h}_+$ respectively. We fix a Euclidean metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$, **adapted to the gradation**, i.e. $\langle \mathfrak{h}_i, \mathfrak{h}_j \rangle = 0$ for $i \neq j$. It induces, in a natural way, a Euclidean metric, also denoted by $\langle \cdot, \cdot \rangle$, on the spaces $C^k(\mathfrak{h}_-, \mathfrak{h})$, ($k \geq 0$). We denote by $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ the codifferential, or the metric adjoint of the Lie algebra differential $\partial : C^k(\mathfrak{h}_-, \mathfrak{h}) \to C^{k+1}(\mathfrak{h}_-, \mathfrak{h})$ defined by (2.1). Since ∂ preserves the homogeneous degree and $\langle \cdot, \cdot \rangle$ is adapted to the gradation, ∂^* preserves homogeneous degree, too.

We consider $\mathfrak{h}_- \cong \mathfrak{h}/\mathfrak{q}$ and its dual $\mathfrak{h}_-^*$ as Q-modules, with Q-action induced by the adjoint action of Q on $\mathfrak{h}$. We use the same notation Ad^-_g for the action of $g \in Q$ on $\mathfrak{h}_-$ and its dual $\mathfrak{h}_-^*$. We denote by $\mathrm{Ad}^*_g : \mathfrak{h} \to \mathfrak{h}$ and $(\mathrm{Ad}^-_g)^* : \mathfrak{h}_- \to \mathfrak{h}_-$ the metric adjoints of $\mathrm{Ad}_g : \mathfrak{h} \to \mathfrak{h}$ and $\mathrm{Ad}^-_g : \mathfrak{h}_- \to \mathfrak{h}_-$ respectively. Similarly, for any $X \in \mathfrak{h}$, we denote by $(\mathrm{ad}_X)^* : \mathfrak{h} \to \mathfrak{h}$ the metric adjoint of $\mathrm{ad}_X : \mathfrak{h} \to \mathfrak{h}$. Let $\mathfrak{h}_- \ni X \to X^\sharp = \langle X, \cdot\rangle \in \mathfrak{h}_-^*$ and its inverse $\mathfrak{h}_-^* \ni \gamma \to \gamma^\flat \in \mathfrak{h}_-$.

For any $k \geq 0$, $C^k(\mathfrak{h}_-, \mathfrak{h})$ is a Q-module, with Q-action given by:

$$(g\cdot\phi)(X_1,\dots,X_k) = \mathrm{Ad}_g\left(\phi(\mathrm{Ad}^-_{g^{-1}}(X_1),\dots,\mathrm{Ad}^-_{g^{-1}}(X_k))\right), \tag{4.1}$$

for any $g \in Q$, $\phi \in C^k(\mathfrak{h}_-, \mathfrak{h})$ and $X_i \in \mathfrak{h}_-$.

Definition 4. The metric $\langle\cdot,\cdot\rangle$ is called **admissible** if the codifferentials $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ are Q-invariant, for any $k \geq 0$.

The following proposition characterizes admissible metrics.

Proposition 6. (i) *The codifferential $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ has the following expression: for any $\alpha_i \in \mathfrak{h}_-^*$ and $V \in \mathfrak{h}$,*

$$\begin{aligned}\partial^*((\alpha_0\wedge\cdots\wedge\alpha_k)\otimes V) &= \sum_{i=0}^{k}(-1)^i(\alpha_0\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\alpha_k)\otimes(\mathrm{ad}_{\alpha_i^\flat})^*(V)\\ &\quad+\sum_{i<j}(-1)^{i+j}\left([\alpha_i^\flat,\alpha_j^\flat]^\sharp\wedge\alpha_0\wedge\cdots\wedge\widehat{\alpha_i}\wedge\cdots\wedge\widehat{\alpha_j}\wedge\cdots\wedge\alpha_k\right)\otimes V.\end{aligned} \tag{4.2}$$

(ii) *The metric $\langle\cdot,\cdot\rangle$ is admissible if and only if*

$$\mathrm{Ad}^*_g([Z,W]) = [(\mathrm{Ad}^-_g)^*(Z), (\mathrm{Ad}_g)^*(W)], \quad \forall g\in Q,\ \forall Z\in\mathfrak{h}_-,\ \forall W\in\mathfrak{h} \tag{4.3}$$

and

$$(\mathrm{Ad}^-_g)^*([Z,W]) = [(\mathrm{Ad}^-_g)^*(Z), (\mathrm{Ad}^-_g)^*(W)], \quad \forall g\in Q,\ \forall Z,\ W\in\mathfrak{h}_-. \tag{4.4}$$

Proof. Claim (i) follows from direct computation. For claim (ii), we notice, from (4.1) and (4.2), that for any $\alpha := (\alpha_0\wedge\cdots\wedge\alpha_k)\otimes V \in C^{k+1}(\mathfrak{h}_-, \mathfrak{h})$ and $g \in Q$,

$$\begin{aligned}\partial^*(g\cdot\alpha) &= \sum_i(-1)^i\left((\mathrm{Ad}^-_g)(\alpha_0)\wedge\cdots\wedge\widehat{(\mathrm{Ad}^-_g)(\alpha_i)}\wedge\cdots\wedge(\mathrm{Ad}^-_g)(\alpha_k)\right)\otimes(\mathrm{ad}_{(\mathrm{Ad}_g(\alpha_i))^\flat})^*\mathrm{Ad}_g(V)\\ &\quad+\sum_{i<j}(-1)^{i+j}\left([(\mathrm{Ad}_g(\alpha_i))^\flat,(\mathrm{Ad}_g(\alpha_j))^\flat]^\sharp\wedge\mathrm{Ad}^*_g(\alpha_0)\cdots\widehat{\mathrm{Ad}_g(\alpha_i)}\cdots\widehat{\mathrm{Ad}_g(\alpha_j)}\cdots\wedge\mathrm{Ad}_g(\alpha_k)\right)\otimes\mathrm{Ad}_g(V).\end{aligned}$$

Similarly,

$$\begin{aligned}g\cdot(\partial^*\alpha) &= \sum_i(-1)^i\left(\mathrm{Ad}_g(\alpha_0)\wedge\cdots\wedge\widehat{\mathrm{Ad}_g(\alpha_i)}\wedge\cdots\wedge\mathrm{Ad}_g(\alpha_k)\right)\otimes\mathrm{Ad}_g(\mathrm{ad}_{\alpha_i^\flat})^*(V)\\ &\quad+\sum_{i<j}(-1)^{i+j}(\mathrm{Ad}_g([\alpha_i^\flat,\alpha_j^\flat]^\sharp)\wedge\mathrm{Ad}_g(\alpha_0)\cdots\widehat{\mathrm{Ad}_g(\alpha_i)}\cdots\widehat{\mathrm{Ad}_g(\alpha_j)}\cdots\wedge\mathrm{Ad}_g(\alpha_k))\otimes\mathrm{Ad}_g(V).\end{aligned}$$

In particular, $\partial^*(g\cdot\alpha) = g\cdot\partial^*(\alpha)$ for any $\alpha\in C^{k+1}(\mathfrak{h}_-, \mathfrak{h})$ (and $k\geq 0$), if and only if for any $\alpha, \beta \in (\mathfrak{h}_-)^*$ and $g \in Q$,

$$(\mathrm{ad}_{(\mathrm{Ad}_g(\alpha))^\flat})^*\mathrm{Ad}_g(V) = \mathrm{Ad}_g(\mathrm{ad}_{\alpha^\flat})^*(V), \qquad [(\mathrm{Ad}_g(\alpha))^\flat,(\mathrm{Ad}_g(\beta))^\flat]^\sharp = \mathrm{Ad}_g([\alpha^\flat,\beta^\flat]^\sharp). \tag{4.5}$$

On the other hand, it is easy to check that

$$(\mathrm{Ad}^-_g)^*\left((\mathrm{Ad}_g(\alpha))^\flat\right) = \alpha^\flat, \quad \forall g\in Q,\ \forall\alpha\in\mathfrak{h}_-^*. \tag{4.6}$$

Using (4.6), it is easy to see that relations (4.5) are equivalent to (4.3) and (4.4). For example, to prove (4.3) we consider the first relation (4.5) and we take its inner product with $W \in \mathfrak{h}$. This gives $(\mathrm{Ad}_g)^*[(\mathrm{Ad}_g(\alpha))^\flat, W] = [\alpha^\flat, (\mathrm{Ad}_g)^*(W)]$. Combining this relation with (4.6), we obtain (4.3). In a similar way we obtain (4.4). □

As an application of Proposition 6, we now give examples of admissible metrics. First, we consider the situation treated in [4], namely $\mathfrak{h} = \mathfrak{h}_{-l}\oplus\cdots\oplus\mathfrak{h}_l$ a semisimple Lie algebra with an $|l|$-gradation and the standard metric B_θ, defined in the following way. Let B be the (non-degenerate) Killing form of $\mathfrak{h}$ and $\theta : \mathfrak{h}\to\mathfrak{h}$ a Cartan involution, with $\theta(\mathfrak{h}_i) = \mathfrak{h}_{-i}$, for any i (see [4, p. 342]). Then B_θ is defined by

$$B_\theta(X,Y) := -B(X,\theta(Y)), \quad \forall X,Y\in\mathfrak{h}. \tag{4.7}$$

It is positive definite, and adapted to the gradation.

Corollary 1. *The metric $\langle\cdot,\cdot\rangle = B_\theta$ is admissible.*

Proof. A straightforward computation which uses $\theta(\mathfrak{h}_-) = \mathfrak{h}_+$, $B(\theta(X), \theta(Y)) = B(X, Y)$ and $B(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)) = B(X, Y)$, for any $X, Y \in \mathfrak{h}$ and $g \in Q$, implies that $\mathrm{Ad}_g^* = \theta \mathrm{Ad}_{g^{-1}} \theta$ and $(\mathrm{Ad}_g^-)^* = (\theta \mathrm{Ad}_{g^{-1}} \theta)|_{\mathfrak{h}_-}$. Since θ and $\mathrm{Ad}_{g^{-1}}$ preserve Lie brackets, relations (4.3) and (4.4) hold. □

Remark 1. Admissible metrics exist also on non-semisimple (graded) Lie algebras. For example, let $\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0$ be a (non-semisimple) non-positively graded Lie algebra of depth one. Suppose there are given Ad_{H_0}-invariant Euclidean metrics $\langle \cdot, \cdot \rangle_{-1}$ and $\langle \cdot, \cdot \rangle_0$ on $\mathfrak{h}_{-1}$ and $\mathfrak{h}_0$ respectively. They induce a metric $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$, with respect to which $\mathfrak{h}_{-1}$ and $\mathfrak{h}_0$ are orthogonal, and an easy computation shows that $\mathrm{Ad}_g^* = \mathrm{Ad}_{g^{-1}} : \mathfrak{h} \to \mathfrak{h}$ and $(\mathrm{Ad}_g^-)^* = \mathrm{Ad}_{g^{-1}} : \mathfrak{h}_{-1} \to \mathfrak{h}_{-1}$, for any $g \in H_0$. Therefore, conditions (4.3) and (4.4) are satisfied and $\langle \cdot, \cdot \rangle$ is admissible. We also mention that Proposition 3.1.10 of [10] provides other examples of non-semisimple Lie algebras which support admissible metrics (see also Remark 3.10.4 from the same reference).

4.2. Normal Cartan connections

In this section we define and study normal Cartan connections of type $\mathfrak{h}$, where $\mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_+$ is a graded Lie algebra, with a fixed admissible metric $\langle \cdot, \cdot \rangle$. We assume that the Lie group Q with Lie algebra $\mathfrak{q} = \mathfrak{h}_0 \oplus \mathfrak{h}_+$ is the semidirect product $G \cdot N$ of the Lie groups G and N, with Lie algebras $\mathfrak{h}_0$ and $\mathfrak{n} = \mathfrak{h}_+$ respectively. Since $\langle \cdot, \cdot \rangle$ is positive definite, the induced metric on the complex $\{C^k(\mathfrak{h}_-, \mathfrak{h}),\ k \geq 0\}$ is also positive definite. The adjoint action of G induces an action on each space of this complex. With respect to this action, there is a G-invariant Hodge decomposition

$$C^k(\mathfrak{h}_-, \mathfrak{h}) = \mathrm{Ker}\left(\Delta|_{C^k(\mathfrak{h}_-,\mathfrak{h})}\right) \oplus \partial^*(C^{k+1}(\mathfrak{h}_-, \mathfrak{h})) \oplus \partial(C^{k-1}(\mathfrak{h}_-, \mathfrak{h})), \tag{4.8}$$

where $\partial : C^{k-1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ denotes as usual the Lie algebra differential, ∂^* its (metric) adjoint and $\Delta = \partial\partial^* + \partial^*\partial$ the Laplacian. Since $\langle \cdot, \cdot \rangle$ is admissible, the codifferentials $\partial^* : C^{k+1}(\mathfrak{h}_-, \mathfrak{h}) \to C^k(\mathfrak{h}_-, \mathfrak{h})$ are Q-invariant (not only G-invariant).

Definition 5. A regular Cartan connection $\kappa \in \Omega^1(P, \mathfrak{h})$ on a principal Q-bundle $\pi : P \to M$ is called **normal** if its curvature function $K : P \to C^2(\mathfrak{h}_-, \mathfrak{h})$ satisfies $\partial^*(K) = 0$.

Remark 2. From Q-invariance, the codifferentials induce bundle maps $\partial^* : \Lambda^{k+1}(M) \otimes \mathcal{A}(M) \to \Lambda^k(M) \otimes \mathcal{A}(M)$ $(k \geq 0)$ between forms on M with values in the adjoint bundle $\mathcal{A}(M) = P \times_Q \mathfrak{h}$. The Cartan connection is normal if its curvature $\Omega \in \Omega^2(M, \mathcal{A}(M))$ satisfies $\partial^*(\Omega) = 0$.

The following simple lemma will play an important role in our treatment of normal Cartan connections.

Lemma 2. *Let $\kappa \in \Omega^1(P, \mathfrak{h})$ be a regular Cartan connection on a principal Q-bundle $\pi : P \to M$ and $(\mathcal{D}, \pi_G : P_G \to M)$ the induced Tanaka structure.*

(i) *Let $f : P \to C^k(\mathfrak{h}_-, \mathfrak{h})^l$ be Q-invariant. Then its component of degree l is constant on the orbits of N and it descends to a (G-invariant) function $f_l : P_G \to C^k_l(\mathfrak{h}_-, \mathfrak{h})$.*
(ii) *Let $f : P_G \to C^k_l(\mathfrak{h}_-, \mathfrak{h})$ be a G-invariant function. Then there is a Q-invariant function $f^Q : P \to C^k(\mathfrak{h}_-, \mathfrak{h})^l$ such that $(f^Q)_l = f$.*
(iii) *Let $f : P \to C^k(\mathfrak{h}_-, \mathfrak{h})$ be Q-invariant, such that $\partial^*(f) : P \to C^{k-1}(\mathfrak{h}_-, \mathfrak{h})^l$, for $l \geq 0$. Then there is $\tilde{f} : P \to C^k(\mathfrak{h}_-, \mathfrak{h})^l$, Q-invariant, such that $\partial^*(f) = \partial^*(\tilde{f})$.*

Proof. Claim (i) follows from the Q-invariance $f(u \cdot g) = g^{-1} \cdot f(u)$ (for any $u \in P$ and $g \in Q = G \cdot N$) and from the fact that N acts trivially on $C^k_l(\mathfrak{h}_-, \mathfrak{h}) = C^k(\mathfrak{h}_-, \mathfrak{h})^l / C^k(\mathfrak{h}_-, \mathfrak{h})^{l+1}$. For claim (ii), consider the following general situation. Assume that V is a Q-module (in our case $V = C^k(\mathfrak{h}_-, \mathfrak{h})$), endowed with a Q-invariant filtration $\{V^i\}$ (in our case $V^i = C^k(\mathfrak{h}_-, \mathfrak{h})^i$) and that N acts trivially on the associated graded vector space $\mathrm{gr}(V) = \oplus_i V^i/V^{i+1}$ (in our case, $V^i/V^{i+1} = C^k_i(\mathfrak{h}_-, \mathfrak{h})$). The vector bundle $P \times_Q V$ inherits a filtration $(P \times_Q V)^i := P \times_Q V^i$ and $\mathrm{gr}_i(P \times_Q V) = (P \times_Q V^i/V^{i+1})$ is isomorphic to $P_G \times_G V^i/V^{i+1}$. Claim (ii) follows by noticing that sections of $\mathrm{gr}_i(P \times_G V)$ can be lifted to sections of $P \times_Q V^i$. Claim (iii) follows from the fact that ∂^* is Q-invariant and filtration preserving. □

Our main results from this section are the following two theorems, about existence and uniqueness of normal Cartan connections which induce a given Tanaka structure.

Theorem 1. *Let $(\mathcal{D}, \pi_G : P_G \to M)$ be a Tanaka structure, which is induced by a regular Cartan connection $\kappa \in \Omega^1(P, \mathfrak{h})$. We fix an admissible metric on $\mathfrak{h}$. Then the given Tanaka structure is induced also by a normal Cartan connection $\kappa^n \in \Omega^1(P, \mathfrak{h})$.*

Proof. Let $K^{(0)} = \sum_{i \geq 1} K_i^{(0)}$ be the curvature function of the regular Cartan connection $\kappa^{(0)} := \kappa$, where $K_i^{(0)} : P \to C^2_i(\mathfrak{h}_-, \mathfrak{h})$, $i \geq 1$. From Lemma 2(i), $K_1^{(0)} : P_G \to C^2_1(\mathfrak{h}_-, \mathfrak{h})$ is G-invariant. Using (4.8) for $C^2_1(\mathfrak{h}_-, \mathfrak{h})$, we can decompose

$K_1^{(0)} = (K_1^{(0)})^{\mathrm{harm}} + \partial^*(\eta_1^{(0)}) + \partial(\phi_1^{(0)})$, where $\eta_1^{(0)} : P_G \to C_1^3(\mathfrak{h}_-, \mathfrak{h})$ and $\phi_1^{(0)} : P_G \to C_1^1(\mathfrak{h}_-, \mathfrak{h})$ are G-invariant. From Lemma 2(ii), there is $(\phi_1^{(0)})^Q : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^1$ Q-invariant, with the homogeneous degree one component equal to $\phi_1^{(0)}$. Define a Cartan connection $\kappa^{(1)} := \kappa^{(0)} - (\phi_1^{(0)})^Q \circ \kappa^{(0)}$ and let $K^{(1)}$ its curvature function. We apply Proposition 3 to the pair $(\kappa^{(0)}, \kappa^{(1)})$. Since $(\phi_1^{(0)})^Q$ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^1$, we deduce that $\kappa^{(1)}$ induces the given Tanaka structure, the difference $K^{(1)} - K^{(0)}$ takes values in $C^2(\mathfrak{h}_-, \mathfrak{h})^1$ and

$$K_1^{(1)} = K_1^{(0)} - \partial(\phi_1^{(0)}) = (K_1^{(0)})^{\mathrm{harm}} + \partial^*(\eta_1^{(0)})$$

is coclosed. To summarize: $\kappa^{(1)}$ is a regular Cartan connection, which induces the given Tanaka structure, and whose curvature function $K^{(1)}$ has the property that its homogeneous degree one component $K_1^{(1)}$ is coclosed (or $\partial^*(K^{(1)}) : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^2$).

Using Lemma 2(iii), let $\widetilde{K^{(1)}} : P \to C^2(\mathfrak{h}_-, \mathfrak{h})^2$ Q-invariant, such that $\partial^*(\widetilde{K^{(1)}}) = \partial^*(K^{(1)})$. As before, $(\widetilde{K^{(1)}})_2 : P_G \to C_2^2(\mathfrak{h}_-, \mathfrak{h})$ is G-invariant and from (4.8) for $C_2^2(\mathfrak{h}_-, \mathfrak{h})$, we can decompose $(\widetilde{K^{(1)}})_2 = (\widetilde{K^{(1)}})_2^{\mathrm{harm}} + \partial^*(\eta_2^{(1)}) + \partial(\phi_2^{(1)})$, where $\eta_2^{(1)} : P_G \to C_2^3(\mathfrak{h}_-, \mathfrak{h})$ and $\phi_2^{(1)} : P_G \to C_2^1(\mathfrak{h}_-, \mathfrak{h})$ are G-invariant. Let $(\phi_2^{(1)})^Q : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^2$ Q-invariant, with the homogeneous degree two component equal to $\phi_2^{(1)}$. Define a new Cartan connection $\kappa^{(2)} := \kappa^{(1)} - (\phi_2^{(1)})^Q \circ \kappa^{(1)}$ and let $K^{(2)}$ its curvature function. Again from Proposition 3, applied to $(\kappa^{(1)}, \kappa^{(2)})$, $K^{(2)} - K^{(1)}$ takes values in $C^2(\mathfrak{h}_-, \mathfrak{h})^2$ and $K_2^{(2)} = K_2^{(1)} - \partial(\phi_2^{(1)})$. Thus $K_1^{(2)} = K_1^{(1)}$ is coclosed (because $K_1^{(1)}$ is coclosed) and

$$\partial^*(K_2^{(2)}) = \partial^*(K_2^{(1)}) - \partial^*\partial(\phi_2^{(1)}) = \partial^*(\widetilde{K^{(1)}})_2 - \partial^*\partial(\phi_2^{(1)}) = \partial^*\partial(\phi_2^{(1)}) - \partial^*\partial(\phi_2^{(1)}) = 0,$$

where in the second equality we used $\partial^*(K^{(1)}) = \partial^*(\widetilde{K^{(1)}})$ and in the third equality we used the Hodge decomposition of $(\widetilde{K^{(1)}})_2$. To summarize: $\kappa^{(2)}$ is a regular Cartan connection, which induces the given Tanaka structure, and the homogeneous degree one and two components of its curvature function are coclosed. Repeating inductively the argument we obtain a normal Cartan connection $\kappa^n \in \Omega^1(P, \mathfrak{h})$, as required. □

Theorem 2. *Let $\mathfrak{h}$ be a graded Lie algebra, with $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h}) = 0$. Fix an admissible metric on $\mathfrak{h}$. For any two normal Cartan connections $\kappa, \tilde{\kappa} \in \Omega^1(P, \mathfrak{h})$, which induce the same Tanaka structure, there is a bundle automorphism $\psi : P \to P$ such that $\tilde{\kappa} = \psi^*(\kappa)$.*

Proof. Let $K, \tilde{K}$, the curvature functions of κ, $\tilde{\kappa}$ respectively, and $\phi^{(0)} := (\tilde{\kappa} - \kappa) \circ \hat{\kappa}^{-1} : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^1$, which is Q-invariant. Then $\phi_1^{(0)} : P_G \to C_1^1(\mathfrak{h}_-, \mathfrak{h})$ is G-invariant and $\tilde{K}_1 - K_1 = \partial(\phi_1^{(0)})$ (see Proposition 3). Since κ and $\tilde{\kappa}$ are normal, $\tilde{K}_1, K_1$ and therefore also $\partial(\phi_1^{(0)})$ are coclosed. The Hodge decomposition of $C_1^2(\mathfrak{h}_-, \mathfrak{h})$ implies $\partial(\phi_1^{(0)}) = 0$. Since $H_1^1(\mathfrak{h}_-, \mathfrak{h}) = 0$, there is $\mu_1^{(0)} : P_G \to \mathfrak{h}_1$, G-invariant, such that $\phi_1^{(0)} = \partial(\mu_1^{(0)})$. Using Lemma 2(ii), let $(\mu_1^{(0)})^Q : P \to \mathfrak{h}^1$, Q-invariant, with the homogeneous degree one component equal to $\mu_1^{(0)}$, and define $\psi_0 : P \to P$ by $\psi_0(p) := p \cdot \exp((\mu_1^{(0)})^Q(p))$. Then $\psi_0^*(\tilde{\kappa})$ is a normal Cartan connection, which induces the given Tanaka structure. We claim that $\phi^{(1)} := (\psi_0^*(\tilde{\kappa}) - \kappa) \circ \hat{\kappa}^{-1}$ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^2$. To prove this claim, we notice, from Proposition 4, that $\phi^{(1)} - \phi^{(0)} + \partial((\mu_1^{(0)})^Q)$ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^2$. In particular, $\phi_1^{(1)} = \phi_1^{(0)} - \partial(\mu_1^{(0)}) = 0$. It follows that $\phi^{(1)} : P \to C^1(\mathfrak{h}_-, \mathfrak{h})^2$, as claimed.

To summarize: $\psi_0^*(\tilde{\kappa})$ is a normal Cartan connection, which induces the given Tanaka structure, and the function $\phi^{(1)} = (\psi_0^*(\tilde{\kappa}) - \kappa) \circ \hat{\kappa}^{-1}$ associated to the pair $(\kappa, \psi_0^*(\tilde{\kappa}))$ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^2$. An argument as before, with $\tilde{\kappa}$ replaced by $\psi_0^*(\tilde{\kappa})$ (and κ the same), which uses $H_2^1(\mathfrak{h}_-, \mathfrak{h}) = 0$, shows that the homogeneous degree two component $\phi_2^{(1)}$ of $\phi^{(1)}$ is of the form $\phi_2^{(1)} = \partial(\mu_2^{(1)})$, for a G-invariant function $\mu_2^{(1)} : P_G \to \mathfrak{h}_2$. Let $(\mu_2^{(1)})^Q : P \to \mathfrak{h}^2$, Q-invariant, with the homogeneous degree two component equal to $\mu_2^{(1)}$, and define $\psi_1 : P \to P$, by $\psi_1(p) := p \cdot \exp((\mu_2^{(1)})^Q(p))$. As before, one shows that the function $\phi^{(2)} := (\psi_1^*\psi_0^*(\tilde{\kappa}) - \kappa) \circ \hat{\kappa}^{-1}$ associated to the pair $(\kappa, \psi_1^*\psi_0^*(\tilde{\kappa}))$ takes values in $C^1(\mathfrak{h}_-, \mathfrak{h})^3$. An induction argument concludes the proof. □

5. The cotangent Lie algebra

Let $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ be a non-positively graded fundamental Lie algebra. We shall consider $\mathfrak{g}^*$ with the induced non-negative gradation $(\mathfrak{g}^*)_i := (\mathfrak{g}_{-i})^*$. For $X \in \mathfrak{g}$, we denote by L_X its coadjoint action on $\mathfrak{g}^*$, given by $L_X(\alpha)(Y) := -\alpha([X, Y])$, for any $\alpha \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$. The cotangent Lie algebra $\mathfrak{h} = t^*(\mathfrak{g})$, defined as the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Lie bracket

$$[X + \xi, Y + \eta] = [X, Y] + L_X(\eta) - L_Y(\xi),$$

inherits a natural (fundamental) gradation $\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_0 \oplus \cdots \oplus \mathfrak{h}_k$, where

$$\mathfrak{h}_i = \begin{cases} \mathfrak{g}_i, & i < 0 \\ \mathfrak{g}_0 \oplus (\mathfrak{g}_0)^*, & i = 0 \\ (\mathfrak{g}_{-i})^*, & i > 0. \end{cases}$$

In the following section we compute the cohomology group $H^1_{\geq 1}(\mathfrak{h}_-, \mathfrak{h}) = \oplus_{l \geq 1} H^1_l(\mathfrak{h}_-, \mathfrak{h})$.

5.1. The cohomology group $H^1_{\geq 1}(\mathfrak{g}_-, t^*(\mathfrak{g}))$

We begin by fixing notation. We consider $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{g}^*$ as $\mathfrak{h}_-$-modules, via the adjoint representation in the cotangent Lie algebra $\mathfrak{h}$, and we denote by ∂_V the Lie algebra differential in the complex $\{C^k(\mathfrak{h}_-, V),\ k \geq 0\}$, where $V = \mathfrak{h}, \mathfrak{g}$ or $\mathfrak{g}^*$. We denote by $C^k_l(\mathfrak{h}_-, V) \subset C^k(\mathfrak{h}_-, V)$ the subspace of forms of homogeneous degree l. For $\alpha \in C^1(\mathfrak{h}_-, \mathfrak{h})$, we denote by $\alpha_{\mathfrak{g}} \in C^1(\mathfrak{h}_-, \mathfrak{g})$ and $\alpha_{\mathfrak{g}^*} \in C^1(\mathfrak{h}_-, \mathfrak{g}^*)$ the composition of α with the natural projections from $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}^*$ onto its factors $\mathfrak{g}$ and $\mathfrak{g}^*$. The isomorphism

$$C^1(\mathfrak{h}_-, \mathfrak{h}) \ni \alpha \to (\alpha_{\mathfrak{g}}, \alpha_{\mathfrak{g}^*}) \in C^1(\mathfrak{h}_-, \mathfrak{g}) \oplus C^1(\mathfrak{h}_-, \mathfrak{g}^*)$$

is compatible with homogeneous degree and

$$\partial_{\mathfrak{h}}(\alpha) = \partial_{\mathfrak{g}}(\alpha_{\mathfrak{g}}) + \partial_{\mathfrak{g}^*}(\alpha_{\mathfrak{g}^*}), \quad \forall \alpha \in C^1(\mathfrak{h}_-, \mathfrak{h}). \tag{5.1}$$

Lemma 3. *For any $l \geq 1$,*

$$H^1_l(\mathfrak{h}_-, \mathfrak{h}) = \mathrm{Ker}\left(\partial_{\mathfrak{g}} : C^1_l(\mathfrak{g}_-, \mathfrak{g}) \to C^2_l(\mathfrak{g}_-, \mathfrak{g})\right) \oplus H^1_l(\mathfrak{g}_-, \mathfrak{g}^*). \tag{5.2}$$

Proof. For $\beta \in C^0_l(\mathfrak{h}_-, \mathfrak{h}) = (\mathfrak{g}_{-l})^*$, $\partial_{\mathfrak{h}}(\beta) \in C^1_l(\mathfrak{h}_-, \mathfrak{h})$ is given by $\partial_{\mathfrak{h}}(\beta)(X) = L_X(\beta)$, for any $X \in \mathfrak{h}_-$. In particular, $\partial_{\mathfrak{h}}(\beta)$ takes values in $\mathfrak{g}^*$ and belongs to $C^1_l(\mathfrak{g}_-, \mathfrak{g}^*)$. Our claim follows from (5.1). □

Therefore, in order to compute $H^1_l(\mathfrak{h}_-, \mathfrak{h})$, we need to determine both terms in the right hand side of (5.2). This is done in the next two propositions.

Proposition 7. (i) *For any $l \geq 2$, $\mathrm{Ker}\left(\partial_{\mathfrak{g}} : C^1_l(\mathfrak{g}_-, \mathfrak{g}) \to C^2_l(\mathfrak{g}_-, \mathfrak{g})\right) = \{0\}$;*
(ii) *The restriction map $C^1_1(\mathfrak{g}_-, \mathfrak{g}) \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ defines an injection*

$$\mathrm{Ker}\left(\partial_{\mathfrak{g}} : C^1_1(\mathfrak{g}_-, \mathfrak{g}) \to C^2_1(\mathfrak{g}_-, \mathfrak{g})\right) \to \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$$

whose image $S(\mathfrak{g}) \subset \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ is the vector space of all linear functions $f : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ with the property: for any $X_1, \ldots, X_s, X'_1, \ldots, X'_s \in \mathfrak{g}_{-1}$, such that

$$\mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_{s-1}}(X_s) = \mathrm{ad}_{X'_1} \cdots \mathrm{ad}_{X'_{s-1}}(X'_s),$$

the following equality holds:

$$\begin{aligned} &\mathrm{ad}_{f(X_1)}\mathrm{ad}_{X_2} \cdots \mathrm{ad}_{X_{s-1}}(X_s) + \cdots + \mathrm{ad}_{X_1} \cdots \mathrm{ad}_{f(X_{s-1})}(X_s) + \mathrm{ad}_{X_1} \cdots \mathrm{ad}_{X_{s-1}} f(X_s) \\ &\quad = \mathrm{ad}_{f(X'_1)}\mathrm{ad}_{X'_2} \cdots \mathrm{ad}_{X'_{s-1}}(X'_s) + \cdots + \mathrm{ad}_{X'_1} \cdots \mathrm{ad}_{f(X'_{s-1})}(X'_s) + \mathrm{ad}_{X'_1} \cdots \mathrm{ad}_{X'_{s-1}} f(X'_s). \end{aligned} \tag{5.3}$$

Proof. Let $\beta \in C^1_l(\mathfrak{g}_-, \mathfrak{g})$ with $\partial_{\mathfrak{g}}(\beta) = 0$. Then

$$\beta([X, Y]) = [X, \beta(Y)] - [Y, \beta(X)], \quad \forall X, Y \in \mathfrak{g}_-. \tag{5.4}$$

Suppose first that $l \geq 2$. Since β is of homogeneous degree l, $\beta|_{\mathfrak{g}_i} = 0$, for any $i \geq -l+1$. In particular, $\beta|_{\mathfrak{g}_{-1}} = 0$. Since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$, relation (5.4) implies $\beta = 0$, as required.

It remains to consider the case $l = 1$. Then $\beta(\mathfrak{g}_i) \subset \mathfrak{g}_{i+1}$, for any $i \leq -1$, and from (5.4) β is determined by its restriction $\beta|_{\mathfrak{g}_{-1}} : \mathfrak{g}_{-1} \to \mathfrak{g}_0$. It is straightforward to check that a linear map $f : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ can be extended to a map $\beta : \mathfrak{g}_- \to \mathfrak{g}$ of homogeneous degree one, with $\partial_{\mathfrak{g}}(\beta) = 0$, if and only if it satisfies (5.3). Our second claim follows. □

Next, we need to compute $H^1_l(\mathfrak{h}_-, \mathfrak{g}^*)$ for $l \geq 1$. Any $\gamma \in C^1_l(\mathfrak{g}_-, \mathfrak{g}^*)$ such that $\partial_{\mathfrak{g}^*}(\gamma) = 0$, i.e.

$$\gamma([X, Y]) = \gamma(X) \circ \mathrm{ad}_Y - \gamma(Y) \circ \mathrm{ad}_X, \quad \forall X, Y \in \mathfrak{g}_-, \tag{5.5}$$

is determined by its restriction $\gamma|_{\mathfrak{g}_{-1}} : \mathfrak{g}_{-1} \to (\mathfrak{g}_{1-l})^*$. With this preliminary remark, we state:

Proposition 8. *Let $l \geq 1$. The restriction map $C^1_l(\mathfrak{g}_-, \mathfrak{g}^*) \to \mathrm{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}_{1-l})^*)$ induces an isomorphism*

$$H^1_l(\mathfrak{g}_-, \mathfrak{g}^*) = Z_l(\mathfrak{g})/B_l(\mathfrak{g}).$$

Above $Z_1(\mathfrak{g}) = \mathrm{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}_0)^*)$ and for $l \geq 2$, $Z_l(\mathfrak{g}) \subset \mathrm{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}_{1-l})^*)$ is the vector space of all maps $\alpha : \mathfrak{g}_{-1} \to (\mathfrak{g}_{1-l})^*$ which satisfy: for any $X_1, \ldots, X_s, X'_1, \ldots, X'_s \in \mathfrak{g}_{-1}$ $(2 \leq s \leq l)$ with

$$\mathrm{ad}_{X_1}\mathrm{ad}_{X_2} \cdots \mathrm{ad}_{X_{s-1}}(X_s) = \mathrm{ad}_{X'_1}\mathrm{ad}_{X'_2} \cdots \mathrm{ad}_{X'_{s-1}}(X'_s),$$

the following relation holds:

$$\sum_{i=1}^{s}(-1)^{i+1}\alpha(X_i)\mathrm{ad}_{\mathrm{ad}_{X_{i+1}}\mathrm{ad}_{X_{i+2}}\cdots\mathrm{ad}_{X_{s-1}}(X_s)}\mathrm{ad}_{X_{i-1}}\cdots\mathrm{ad}_{X_1}|_{\mathfrak{g}_{s-l}}$$
$$= \sum_{i=1}^{s}(-1)^{i+1}\alpha(X_i')\mathrm{ad}_{\mathrm{ad}_{X_{i+1}'}\mathrm{ad}_{X_{i+2}'}\cdots\mathrm{ad}_{X_{s-1}'}(X_s')}\mathrm{ad}_{X_{i-1}'}\cdots\mathrm{ad}_{X_1'}|_{\mathfrak{g}_{s-l}}. \quad (5.6)$$

Moreover, for any $l \geq 1$, $B_l(\mathfrak{g}) = \{\hat{\beta} : \mathfrak{g}_{-1} \to (\mathfrak{g}_{1-l})^*,\ \beta \in (\mathfrak{g}_{-l})^*\}$, *where*

$$\hat{\beta}(X)(Z) := -\beta([X, Z]), \quad \forall X \in \mathfrak{g}_{-1},\ Z \in \mathfrak{g}_{1-l}. \quad (5.7)$$

Proof. Let $l = 1$. Any $\gamma \in C^1_1(\mathfrak{g}_-, \mathfrak{g}^*)$ is trivial on $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-2} \subset \mathfrak{g}_-$ and is determined by its restriction $\gamma|_{\mathfrak{g}_{-1}} : \mathfrak{g}_{-1} \to (\mathfrak{g}_0)^*$. Conversely, the trivial extension of any map $\alpha : \mathfrak{g}_{-1} \to (\mathfrak{g}_0)^*$ to $\mathfrak{g}_-$ belongs to $C^1_1(\mathfrak{g}_-, \mathfrak{g}^*)$ and is $\partial_{\mathfrak{g}^*}$-closed.

Let $l \geq 2$ and $\gamma \in C^1_l(\mathfrak{h}_-, \mathfrak{g}^*)$ such that $\partial_{\mathfrak{g}^*}(\gamma) = 0$. Relation (5.5) and an induction argument shows that the restriction $\alpha := \gamma|_{\mathfrak{g}_{-1}} : \mathfrak{g}_{-1} \to (\mathfrak{g}_{1-l})^*$ satisfies (5.6). Conversely, it may be shown that any linear map $\alpha : \mathfrak{g}_{-1} \to (\mathfrak{g}_{1-l})^*$ which satisfies (5.6) can be extended (uniquely) to an homogeneous degree l linear map $\gamma : \mathfrak{g}_- \to \mathfrak{g}^*$, which satisfies $\partial_{\mathfrak{g}^*}(\gamma) = 0$.

So far, we proved that for any $l \geq 1$, the restriction $C^1_l(\mathfrak{g}_-, \mathfrak{g}^*) \to \mathrm{Hom}(\mathfrak{g}_{-1}, (\mathfrak{g}_{1-l})^*)$ maps $\mathrm{Ker}(\partial_{\mathfrak{g}^*})$ isomorphically onto $Z_l(\mathfrak{g})$. To conclude the proof, we need to show that it also maps $\mathrm{Im}(\partial_{\mathfrak{g}^*})$ onto $B_l(\mathfrak{g})$. For this, we first notice that for any $\beta \in C^0_l(\mathfrak{g}_-, \mathfrak{g}^*) = (\mathfrak{g}_{-l})^*$, $\hat{\beta}$ defined in (5.7) is equal to $\partial_{\mathfrak{g}^*}(\beta)|_{\mathfrak{g}_{-1}}$. Consider now $\gamma \in C^1_l(\mathfrak{g}_-, \mathfrak{g}^*)$ with $\partial_{\mathfrak{g}^*}(\gamma) = 0$. We need to show that $\gamma|_{\mathfrak{g}_{-1}} = \hat{\beta}$ implies $\gamma = \partial_{\mathfrak{g}^*}(\beta)$. This follows by an induction argument. More precisely, assume that $\gamma|_{\mathfrak{g}_{-1}} = \hat{\beta}$. Then, for any $X, Y \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_{2-l}$,

$$\begin{aligned}\gamma([X, Y])(Z) &= \gamma(X)([Y, Z]) - \gamma(Y)([X, Z]) = -\beta([X, [Y, Z]]) + \beta([Y, [X, Z]]) \\ &= -\beta([[X, Y], Z]) = \partial_{\mathfrak{g}^*}(\beta)([X, Y])(Z),\end{aligned}$$

i.e. $\gamma|_{\mathfrak{g}_{-2}} = \partial_{\mathfrak{g}^*}(\beta)|_{\mathfrak{g}_{-2}}$. By induction, we obtain $\gamma = \partial_{\mathfrak{g}^*}(\beta)$, as required. □

As a consequence of our above considerations we obtain:

Theorem 3. *Let* $l \geq 1$ *and* $B_l(\mathfrak{g}), Z_l(\mathfrak{g})$ *and* $S(\mathfrak{g})$, *as in Propositions 7 and 8. Then*

$$H^1_1(\mathfrak{h}_-, \mathfrak{h}) = S(\mathfrak{g}) \oplus Z_1(\mathfrak{g})/B_1(\mathfrak{g}),\ H^1_l(\mathfrak{h}_-, \mathfrak{h}) = Z_l(\mathfrak{g})/B_l(\mathfrak{g}), \quad l \geq 2.$$

5.2. Example: $|1|$-graded cotangent Lie algebra $t^*(\mathfrak{g})$

As an illustration of the above computations, we consider now the simple case when $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 = V \oplus \mathfrak{g}_0$ has a non-positive gradation of depth one defined by a representation $\mu : \mathfrak{g}_0 \otimes V \to V$ of a Lie algebra $\mathfrak{g}_0$ on a vector space $V = \mathfrak{g}_{-1}$. Then $\mathfrak{h} = t^*(\mathfrak{g})$ has a $|1|$-gradation

$$t^*(\mathfrak{g}) = \mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_1 = \mathfrak{g}_{-1} \oplus (\mathfrak{g}_0 \oplus (\mathfrak{g}_0)^*) \oplus \mathfrak{g}^*_{-1} = V \oplus (\mathfrak{g}_0 \oplus (\mathfrak{g}_0)^*) \oplus V^*.$$

We denote by $\mu^* : V^* \to (\mathfrak{g}_0 \otimes V)^*$ the dual map of μ and by $\mathfrak{g}_0^{[1]} := \mathfrak{g}_0 \otimes V^* \cap V \otimes \Lambda^2 V^*$ the first skew-prolongation of $\mathfrak{g}_0$, considered as a linear Lie algebra. (For descriptions of skew-prolongations and various applications, see e.g. [11].) We denote by $(\Lambda^2 V^*)^{\mathfrak{g}_0}$ the space of $\mathfrak{g}_0$-invariant skew-symmetric forms and by $(S^2 V^*)_{\mathfrak{g}_0}$ the space of symmetric bilinear forms with respect to which all endomorphisms from $\mathfrak{g}_0$ are symmetric. From our previous computations we obtain:

Proposition 9. *The cohomology groups* $H^1_l(V, \mathfrak{h})$ *are trivial for* $l \geq 3$ *and*

$$H^1_1(V, \mathfrak{h}) = \mathfrak{g}_0^{[1]} \oplus (V \otimes \mathfrak{g}_0)^*/\mu^*(V^*), \qquad H^1_2(V, h) = (\Lambda^2 V^*)^{\mathfrak{g}_0} \oplus (S^2 V^*)_{\mathfrak{g}_0}.$$

For most linear Lie algebras $\mathfrak{g}_0$, $(\Lambda^2 V^*)^{\mathfrak{g}_0}$, $(S^2 V^*)_{\mathfrak{g}_0}$ and $\mathfrak{g}_0^{[1]}$ are trivial. However, $\mu^*(V^*) \neq V \otimes (\mathfrak{g}_0)^*$, if $\dim \mathfrak{g}_0 > 1$. Thus, if $\dim \mathfrak{g}_0 > 1$, then $H^1_1(V, \mathfrak{h}) \neq 0$.

5.3. A class of metrics on $t^*(\mathfrak{g})$

In this section we are interested in admissible metrics on the cotangent Lie algebra $\mathfrak{h} = t^*(\mathfrak{g})$ of a graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0$. Recall, from the beginning of Section 5, that $\mathfrak{h}$ inherits a gradation $\mathfrak{h} = \mathfrak{h}_{-k} \oplus \cdots \oplus \mathfrak{h}_k$.

Let $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be a Euclidean metric on $\mathfrak{g}$, adapted to the gradation. It induces a metric $\langle\cdot,\cdot\rangle_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$. Define a metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$, which on $\mathfrak{g}$ and $\mathfrak{g}^*$ coincides with $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ and $\langle\cdot,\cdot\rangle_{\mathfrak{g}^*}$ respectively and such that $\mathfrak{g}$ is orthogonal to $\mathfrak{g}^*$. The metric $\langle\cdot,\cdot\rangle$ is adapted to the gradation of $\mathfrak{h}$. Let

$$B(X+\xi, Y+\eta) = \xi(Y) + \eta(X), \quad X+\xi, Y+\eta \in \mathfrak{h} = \mathfrak{g}\oplus\mathfrak{g}^*$$

be the standard Ad_H-invariant metric of neutral signature of $\mathfrak{h}$. It easy to check that $\langle\cdot,\cdot\rangle$ is related to B by

$$\langle X+\xi, Y+\eta\rangle = B(X+\xi, \theta(Y+\eta)), \quad X+\xi, Y+\eta\in\mathfrak{h}. \tag{5.8}$$

Above $\theta : \mathfrak{h}\to\mathfrak{h}$, with $\theta(X) = X^{\sharp}$, $\theta(\alpha) = \alpha^{\flat}$ $(X\in\mathfrak{g}, \alpha\in\mathfrak{g}^*)$ is the involution defined by the Riemannian duality induced by $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. In particular, θ is B-orthogonal and $\theta(\mathfrak{h}_i) = \mathfrak{h}_{-i}$, for any i.

Our main result in this section is the following.

Theorem 4. *Suppose that $\langle\cdot,\cdot\rangle$ is admissible. Then $\mathfrak{g}_0$ is abelian and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is not Ad_{G_0}-invariant.*

Proof. We divide our argument in several steps.

Step 1: We claim that

$$\langle \xi\circ \mathrm{ad}_{\theta(\beta)}, \alpha\circ\mathrm{ad}_Y\rangle_{\mathfrak{g}^*} = \langle \xi\circ \mathrm{ad}_{\theta(\alpha)}, \beta\circ\mathrm{ad}_Y\rangle_{\mathfrak{g}^*}, \quad \forall Y\in\mathfrak{g}_-,\ \alpha,\beta,\xi\in\mathfrak{g}^*. \tag{5.9}$$

The above relation is proved in the following way. Using (5.8) and the properties of $\langle\cdot,\cdot\rangle$, B and θ, we obtain (like in Corollary 1) $\mathrm{Ad}_g^* = \theta\mathrm{Ad}_{g^{-1}}\theta$ and $(\mathrm{Ad}_g^-)^* = (\theta\mathrm{Ad}_{g^{-1}}\theta)|_{\mathfrak{h}^-}$, for any $g\in Q$. Because $\langle\cdot,\cdot\rangle$ is admissible, relations (4.3) and (4.4) hold and they reduce to the single condition:

$$(\theta\mathrm{Ad}_g\theta)[Y, Z+\alpha] = [(\theta\mathrm{Ad}_g\theta)(Y), (\theta\mathrm{Ad}_g\theta)(Z+\alpha)], \quad \forall g\in Q,\ Y\in\mathfrak{g}_-,\ Z+\alpha\in\mathfrak{h},$$

which is equivalent to

$$\theta([X+\xi, \theta([Y, Z+\alpha])]) = [\theta([X+\xi,\theta(Y)]), Z+\alpha] + [Y, \theta([X+\xi, \theta(Z+\alpha)])], \tag{5.10}$$

for any $X+\xi\in\mathfrak{h}_0\oplus\mathfrak{h}_+ = \mathrm{Lie}(Q)$, $Y\in\mathfrak{g}_-$ and $Z+\alpha\in\mathfrak{h}$. Letting $X=0$ in the above relation we get

$$\theta([\xi, \theta([Y,\alpha])]) = [Y, \theta([\xi,\theta(\alpha)])]. \tag{5.11}$$

Contracting with $B(\cdot,\beta)$, with $\beta\in\mathfrak{g}^*$, we obtain

$$B(\theta([\xi, \theta L_Y(\alpha)]), \beta) = B([Y, \theta([\xi,\theta(\alpha)])], \beta), \quad \forall\beta\in\mathfrak{g}^*.$$

Using (5.8), that B is Ad_H-invariant and θ is B-orthogonal, we obtain (5.9), as required. Relation (5.11) with $Y\in\mathfrak{g}_{-1}$ and $\alpha,\xi\in(\mathfrak{g}_0)^*$ implies $[Y, \theta L_{\theta(\alpha)}(\xi)] = 0$, i.e. $L_{\theta(\alpha)}(\xi) = 0$ (because the adjoint action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is exact), i.e. $\mathfrak{g}_0$ is abelian.

Step 2: Assume, from now on, by absurd, that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is Ad_{G_0}-invariant. We claim that for any $0<j\leq k$ and $Z,V\in\mathfrak{g}_{-j}$,

$$\sum_i [Z, X_i]\otimes[V, X_i] = \sum_i [V, X_i]\otimes[Z, X_i], \tag{5.12}$$

where $\{X_i\}$ is an orthonormal basis of $\mathfrak{g}_0$. To prove the claim, let $\alpha := V^{\sharp}$, $\beta := Z^{\sharp}$, with $Z,\ V\in\mathfrak{g}_{-j}$, $\xi\in(\mathfrak{g}_{-j})^*$ and $Y\in\mathfrak{g}_{-j}$. Then $L_{\theta(\alpha)}(\xi), L_Y(\beta)\in(\mathfrak{g}_0)^*$ and

$$\begin{aligned}\langle \xi\circ\mathrm{ad}_{\theta(\alpha)}, \beta\circ\mathrm{ad}_Y\rangle_{\mathfrak{g}^*} &= \langle\xi\circ\mathrm{ad}_V, \beta\circ\mathrm{ad}_Y\rangle_{\mathfrak{g}^*} = \sum_i \xi([V,X_i])\langle Z, [Y, X_i]\rangle_{\mathfrak{g}}\\ &= \sum_i \xi([V,X_i])\langle [X_i, Z], Y\rangle_{\mathfrak{g}},\end{aligned}$$

where in the last equality we used the Ad_{G_0}-invariance of $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$. From (5.9) the above expression is symmetric in Z and V. Relation (5.12) holds.

Step 3: Using relation (5.12) we now arrive at a contradiction as follows. The (exact) action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ is by skew-symmetric endomorphisms (because $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is Ad_{G_0}-invariant) and it decomposes into 1- and 2-dimensional irreducible representations. The 1-dimensional representations are trivial. The 2-dimensional ones are of the form (by choosing an orthonormal base in $\mathfrak{g}_{-1}$)

$$\mathfrak{g}_0\ni X\to\begin{pmatrix}0 & -\lambda(X)\\ \lambda(X) & 0\end{pmatrix}$$

where $\lambda(X)\in\mathbb{R}$. Therefore, we can find $Z,\ V\in\mathfrak{g}_{-1}$ (linearly independent), such that

$$[X,Z] = \lambda(X)V, \qquad [X,V] = -\lambda(X)Z, \quad \forall X\in\mathfrak{g}_0$$

and $\lambda\in(\mathfrak{g}_0)^*$ is non-trivial. We obtain $\sum_i [Z,X_i]\otimes[V,X_i] = -\sum_i \lambda(X_i)^2 Z\otimes V$, which is non-symmetric in V, Z. This contradicts relation (5.12). Our claim follows. □

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