HOMOGENEOUS 2^{nd} ORDER PDES ON THE ADJOINT CONTACT MANIFOLD OF A SIMPLE COMPLEX LIE GROUP

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Abstract. There is a natural method to obtain 2^{nd} order PDEs with prescribed simple group of symmetries G. However, one observes that the so–obtained PDEs are, in many cases, of high degree. In this paper we consider the problem of existence of lower degree PDEs with the same group of symmetries.

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Introduction

'Inverse problems' in Mathematics are usually harder than their direct counterparts. For instance, describing the group of symmetries of a given PDEs is rather algorithmic, whereas the problem of *realising* a PDE with a *prescribed* group of symmetries poses nontrivial challenges. In spite of the extensive literature concerning specific instances (see, e.g., [11, 13, 14, 14, 8, 16, 3] and references therein), a unified treatment of the general problem appears to be unpractical. Even the very particular case of classifying homogeneous PDEs on homogeneous contact manifolds, dating back the paper [1] by one of the present authors (DVA), is simply too large to be dealt with in a single work.

In the present paper we further restrict ourselves to a particularly well–behaved setting, narrow enough to obtain a reasonably complete result, but general enough to avoid trivialities. Before explaining what our setting is going to be, and before stating the main results, we recall a few generalities about homogeneous PDEs.

Invariant 2^{nd} order PDEs on a homogeneous contact manifold. Traditionally, a scalar n–dimensional 2^{nd} order PDE $\mathcal{E}$ is understood as a hypersurface $\mathcal{E}\subset J^2(n,1):=J^2(\mathbb{R}^n,\mathbb{R})$ in the space of 2^{nd} order jets of $\mathbb{R}$–valued functions on $\mathbb{R}^n$. For many purposes—most notably, the study of the characteristics of $\mathcal{E}$—one needs also the underlying 1^{st} order jet space $J^1(n,1)$, which is a contact $(2n+1)$–dimensional manifold. All the interesting interactions between the equation $\mathcal{E}$ and its characteristics are mediated by the bundle structure $J^2\to J^1$, which is an affine one, modelled over $S^2\mathbb{R}^{n\,*}$. Such a framework is extremely convenient for local analysis, but it has no topological content: its obvious 'global' analogous can be obtained as follows.

Replace $J^1(n,1)$ with an arbitrary $(2n+1)$–dimensional contact manifold $(M,\mathcal{C})$, where $\mathcal{C}$ is a completely non–integrable one–codimensional distribution defined on M, and replace $J^2(n,1)$ with the corresponding Lagrangian Grassmannian bundle:

$$M^{(1)}:=\coprod_{p\in M}\mathrm{LGr}(\mathcal{C}_p)\longrightarrow M\,. \tag{1}$$

In this context, a (scalar) 2^{nd} order PDE on M is defined as a codimension–one sub–bundle $\mathcal{E}\subset M^{(1)}$; a *solution* of $\mathcal{E}$ is an n–dimensional Lagrangian submanifold $L\subset M$ such that $TL\subset\mathcal{E}$. If now $(M,\mathcal{C})$ is a G–homogeneous contact manifold, for some group G, one can focus on *G–invariant PDEs*, which we conveniently collect into the set

$$\mathsf{Inv}(M,G):=\{\mathcal{E}\mid\mathcal{E}\subset M^{(1)}\text{ is a }2^{\text{nd}}\text{ order PDE on }M\text{ such that }G\cdot\mathcal{E}=\mathcal{E}\}\,. \tag{2}$$

The problem of studying G–invariant 2^{nd} order PDEs is the problem of studying the sets (2). But one can hope to obtain a reasonably complete picture about $\mathsf{Inv}(M,G)$ only by imposing some restrictions both on the group G and on the contact manifold $(M,\mathcal{C})$.

Date: June 9, 2016.
1991 *Mathematics Subject Classification.* 30, 80.

Setup and assumptions. The main results of this paper are formulated in the context of complex contact manifolds and their (complex) Lagrangian Grassmannian bundles.[1] More precisely, we shall assume that G is a complex, connected and simply–connected, simple Lie group, and that $(M, \mathcal{C})$ is not *any* G–homogeneous contact manifold, but the so–called *adjoint* manifold (associated with G), i.e., the one infinitesimally given by the negative part of the *contact grading*

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \,, \tag{3}$$

where $\mathfrak{g}$ is the (complex) simple Lie algebra of G. According to the Killing–Cartan classification, we label the group G by the type of the corresponding Dynkin diagram, herewith denoted by $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}$. As we shall see, the case of G of type C can be safely ruled out, as it leads to trivial results.

Recall that, up to conjugacy, the above grading is uniquely defined (see, e.g., [5], Section 3.2.4) by the fact that the Lie bracket equips $\mathfrak{g}_{-1}$ with a symplectic form ω (up to a trivialisation of the one–dimensional subspace $\mathfrak{g}_{-2}$). So, even in our particular setting, it makes sense to associate with the adjoint variety M the Lagrangian Grassmannian bundle $M^{(1)}$, defined the same way as in (1). The typical fibre of this bundle is the Lagrangian Grassmannian

$$\mathrm{LGr}(\mathfrak{g}_{-1}) \equiv \mathrm{LGr}(\omega) \equiv \mathrm{LGr}(n, 2n) \,, \tag{4}$$

where $n = \frac{1}{2}\dim \mathfrak{g}_{-1}$. Just observe that now $\mathrm{LGr}(\mathfrak{g}_{-1})$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$, since it sits into the so–called *Plücker embedding space* $\mathbb{P}\Lambda^n_0\mathfrak{g}_{-1}$. The symbol $\Lambda^n_0\mathfrak{g}_{-1}$ stands for the 'trace–free part' of the space of n–vectors on $\mathfrak{g}_{-1}$, with respect to the obvious pairing between n–vectors (with values in $\Lambda^{2n}\mathfrak{g}_{-1} \equiv \mathbb{C}$).

Another way to introduce the Plücker embedding space, more convenient for our purposes, is by fixing a Cartan subalgebra in $\mathsf{Sp}(\omega) \equiv \mathsf{Sp}_n$ (which is of type C_n) and a set of fundamental characters $\lambda_1, \ldots, \lambda_n$, following the Bourbaki labelling (see Remark 3 later on). Then, it can be proved that

$$V_{\lambda_n} \simeq \Lambda^n_0\mathfrak{g}_{-1} \,, \tag{5}$$

and that $\mathrm{LGr}(\mathfrak{g}_{-1})$ is the orbit of the weight vector v_{λ_n} corresponding to λ_n. In particular, $\mathbb{P}V_{\lambda_n}$ is the smallest projective space $\mathrm{LGr}(\mathfrak{g}_{-1})$ can be embedded into. As usual, we denote by $\mathcal{O}(1)$ the canonical (tautological) line bundle on $\mathrm{LGr}(\mathfrak{g}_{-1})$ induced from the surrounding projective space $\mathbb{P}V_{\lambda_n}$, and by $\mathcal{O}(d)$ its d^{th} tensor powers. By definition of the Plücker embedding, the fibre of $\mathcal{O}(1)$ over a Lagrangian plane $L \in \mathrm{LGr}(\mathfrak{g}_{-1})$ is the line spanned by the 'volume' of L, i.e., the product $e_1 \wedge \cdots \wedge e_n$, with $e_1, \ldots, e_n$ an arbitrary basis of L.

Now we are in position of defining scalar 2^{nd} order PDEs over the adjoint contact manifold M associated with a complex, connected and simply–connected, simple Lie group G. We are aware that the term 'scalar' may be misleading, since in our setting scalars are complex numbers.

Definition 1. *A* scalar 2^{nd} order PDE *on the adjoint manifold M is a codimension–one sub–bundle $\mathcal{E}$ of $M^{(1)}$, such that all the fibres of $\mathcal{E}$ are closed (analytic) hypersurfaces (possibly singular) in the fibres of $M^{(1)}$. The set* $\mathsf{Inv}(M, G)$ *of G–*invariant PDEs *is defined by the same formula* (2)*, according to the present definition of a PDE.*

Recall that, by the definition of the adjoint manifold M, the contact plane $\mathcal{C}_p$ at a point $p \in M$ identifies with $\mathfrak{g}_{-1}$ (whereas $\mathfrak{g}_{-2}$ is spanned by the Reeb vector field). In contrast with the traditional definition of a PDE $\mathcal{E}$, where each fibre $\mathcal{E}_p \subset \mathrm{LGr}(\mathcal{C}_p)$ is a smooth hypersurface (possibly open), in the present setup $\mathcal{E}_p$ is always closed, and it can be presented as the union of projective hypersurfaces (i.e., the zeroes of homogeneous polynomials, instead of smooth functions). Even at risk of sound pedantic, we also warn the reader that the *order*, or *differential degree*, of our equations is fixed to 2, whereas their *degree*, or *algebraic degree*, which is a novelty with respect to the traditional setting for PDEs, is just the degree of a generic fibre $\mathcal{E}_p$, understood as an algebraic variety. The latter notion is central in our main results.

The main results. Before announcing the main results, we need to introduce a remarkable subgroup G_0^{ss} of $\mathsf{Sp}(\omega)$, which represents the 'residue action' of $\mathsf{Sp}(\omega)$ on the contact plane $\mathfrak{g}_{-1}$. A convenient way to introduce G_0^{ss} is via the adjoint map $\mathrm{ad} : \mathfrak{g}_0 \to \mathrm{End}\,\mathfrak{g}_{-1}$ (see grading (3)). Indeed, this map allows to unambiguously define the connected closed subgroup $G_0^{\text{ss}} \subset \mathsf{Sp}(\omega)$, by requiring its Lie algebra to coincide with the image of $[\mathfrak{g}_0, \mathfrak{g}_0]$.

Definition 2. *The* ring of invariants *for G_0^{ss} on $\mathrm{LGr}(\omega)$ is*

$$R := \bigoplus_{d \geq 0} R_d = \bigoplus_{d \geq 0} \Gamma(\mathrm{LGr}(\omega), \mathcal{O}(d))^{G_0^{ss}} \,. \tag{6}$$

In general, the adjoint map $\mathrm{ad} : \mathfrak{g}_0 \to \mathrm{End}\,\mathfrak{g}_{-1}$ is injective, so that we may consider $\mathfrak{g}_0$ itself as the Lie algebra of a connected subgroup $G_0 \subset \mathsf{CSp}(\omega)$ of the group preserving the conformal class $[\omega]$ of symplectic forms. We may then decompose

$$G_0 = G_0^{\text{ss}} \times Z(G_0)$$

where $Z(G_0)$ the centre of G_0. By semisimplicity, the factor G_0^{ss} is always contained in $\mathsf{Sp}(\omega)$. On the other hand, the centre behaves as follows:

- for $\mathfrak{g}$ not of type A we have $Z(G_0) \simeq \mathbb{C}^\times$ and $Z(G_0) \cap \mathsf{Sp}(\omega) \simeq \{1\}$,
- for $\mathfrak{g}$ of type A we have $Z(G_0) \simeq \mathbb{C}^\times \times \mathbb{C}^\times$ and $Z(G_0) \cap \mathsf{Sp}(\omega) \simeq \mathbb{C}^\times$.

In what follows we shall use $\mathbb{C}^\times$ to denote precisely $Z(G_0) \cap \mathsf{Sp}(\omega)$ in type A, unless explicitly stated otherwise.

[1] Only in the concluding Section 4 we go back to the real smooth context.

Definition 3. *For $\mathfrak{g}$ of type A we shall say that a nonzero element $r \in R$ of degree d has* weight $m \in \mathbb{Z}$ *if and only if $\lambda \cdot r = \lambda^m r$ for $\lambda \in \mathbb{C}^\times$.*

In particular, we conclude the following.

Lemma 1. *Let $r \in R_d$ be a nonzero element. Then:*

(1) *for $\mathfrak{g}$ not of type A, the element $r \in R_d$ is a* relative *G_0-invariant.*
(2) *for $\mathfrak{g}$ of type A, the element $r \in R_d$ is a* relative *G_0-invariant if and only if it has a weight.*

Theorem 1 (Main)**.** *For $\mathfrak{g}$ of given type, the following table provides a $d > 0$ such that $R_d \neq 0$:*

type	A	B	D	E	F	G
d	*1*	*4*	*2*	*2?*	*4?*	*3*

where the numbers for types E and F are only conjectural. Furthermore:

- *in type A the ring R is generated by a pair of elements of degree 1 and opposite, nonzero weights,*
- *in type G_2 the ring R is generated by a single element of degree 3.*

The relationship between Theorem 1 and the general question of realising certain groups as the groups of symmetries of a PDE is clarified by the following proposition.

Proposition 1. *Assume G is a connected, simply connected complex simple Lie group not of type C, with Lie algebra $\mathfrak{g}$, and that M is its adjoint variety equipped with the canonical G–invariant contact structure. Then:*

(1) *The elements of $\mathsf{Inv}(M, G)$ are in one–to–one correspondence with projective classes of reduced homogeneous elements of R of positive degree and—in type A—definite weight.*
(2) *Let $\mathcal{E} \in \mathsf{Inv}(M, G)$. Assume that either $\mathfrak{g}$ is not of type A or $\mathcal{E}$ is not irreducible. Then the local infinitesimal symmetry algebra of $\mathcal{E}$ at each point of M is isomorphic to $\mathfrak{g}$.*

Another way to state part (1) of the above Proposition is that degree d elements of $\mathsf{Inv}(M, G)$ are in one–to–one correspondence with:

- $\mathbb{P}R_d$ for $\mathfrak{g}$ not of type A,
- fixed points of the $\mathbb{C}^\times$-action on $\mathbb{P}R_d$ for $\mathfrak{g}$ of type A.

The table displayed in the statement of Theorem 1 may now be read as follows: for any group G of the above–mentioned type, we are able to exhibit an element on $\mathsf{Inv}(M, G)$, i.e., in view of Proposition 1, a 2^{nd} order G–invariant PDE $\mathcal{E}$ on the corresponding adjoint contact manifold M, whose degree is 1,4,2,2,4,3, depending on G being of type $\mathsf{A}, \mathsf{B}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}$, respectively. Moreover, modulo the assumptions in item (2) of Proposition 1, not only is $\mathcal{E}$ a G–invariant PDE, but it actually allows us to recover the Lie algebra $\mathfrak{g}$ from local data (the local infinitesimal symmetry algebra of $\mathcal{E}$ at $m \in M$ means the Lie algebra of germs at m of infinitesimal contactormorphisms whose lifts to $M^{(1)}$ are tangent to $\mathcal{E}$). This way, Proposition 1 allows us to derive from Theorem 1 our second main result in the form of the following:

Corollary 1. *Assume G is a connected, simply connected complex simple Lie group not of type C, and let M be its adjoint variety equipped with the canonical G-invariant contact structure.*

- *In type A the set $\mathsf{Inv}(M, G)$ consists of three elements: a pair of G–invariant PDEs of degree 1, and their union, itself a G–invariant PDE of degree 2. The latter has $\mathfrak{g}$ as its local infinitesimal symmetry algebra.*
- *In type G_2 the set $\mathsf{Inv}(M, G)$ consists of a single G–invariant PDE of degree 3.*
- *In types B and, conjecturally, F, the set $\mathsf{Inv}(M, G)$ contains a G–invariant PDE of degree 4.*
- *In types D and, conjecturally, E, the set $\mathsf{Inv}(M, G)$ contains a G–invariant PDE of degree 2.*

Furthermore, the local infinitesimal symmetry algebra of each of the elements described above, except for those of degree 1, is precisely $\mathfrak{g}$.

Background, motivations and perspectives. The study of the ring of invariants (6) can be carried out by various means, one of which deserves a special attention for its naturality. In spite of its evident simplicity, it was apparently overlooked so far, though being independently observed by D. The in his recent preprint [17].

The idea is to produce invariant functions on the Lagrangian Grassmannian $\mathrm{LGr}(\mathcal{C}) = \mathrm{LGr}(n, 2n)$ out of $(n-1)$–dimensional submanifolds in $\mathbb{P}^{2n-1} = \mathbb{P}\mathcal{C}$, by means of a 'Lagrangian' variant of the well–known Chow form, introduced by Wei-Liang Chow in 1937 [6]. Indeed, the projectiveised contact plane $\mathbb{P}\mathcal{C} = \mathbb{P}\mathfrak{g}_{-1}$ of the adjoint contact manifold M associated with G contains (essentially) a unique $(n-1)$–dimensional submanifold, known as the sub–adjoint variety associated with G (found, e.g., in the works of J. Landsberg and J. Buczyǹski [12, 4]), if G is not of type C.

Hence, there is a natural way to associate with G a nontrivial element of the ring (6), the 'Lagrangian Chow form', and hence an element of $\mathsf{Inv}(M, G)$, i.e., in view of Proposition 1, a 2^{nd} order G–invariant PDE on the corresponding adjoint contact manifold M. This method, which we shall review below, is very direct and very nice, but its main drawback is that of producing PDEs of possibly high algebraic degree.

However, one should note that the degree of $\mathcal{E}_G$, coinciding with the degree of the variety X, can be very high. It would thus be practical to find lower–degree PDEs with the same symmetries.

The main Theorem 1 makes use of less evident tricks than the 'Lagrangian Chow form' in order to produce invariant PDEs of a reasonably low degree. We underline that in [17] the problem of the uniqueness of the 'exceptionally simple PDEs" found there (i.e., those obtained as the Lagrangian Chow forms of the subadjoint varietites) is not addressed.

Here we show that for certain types (namely A and G, see Sections 2.2 and 2.3) the Lagrangian Chow form is in fact unique, whereas in other cases there are other invariant PDEs of significantly lower degree.

Finally, it is also worth stressing that, in spite of our decision to work with semisimple complex Lie groups, analogous structures to the ones exploited here are, to some degree, present in the more general case. Indeed, as proved in [1], contact homogeneous manifolds of a general Lie group G correspond to co–adjoint orbits of G, with a contact structure induced from the Kirillov–Kostant–Souriau symplectic structure (in our case, we are dealing the projectivisation of the highest–weight orbit). Furthermore, as an immediate extension of our setting, one may consider different real forms of the complex situation presented here: this is briefly discussed in the last Section 4

The reader will notice that our treatment is, to a certain extent, projective–geometric. However, instead of thinking in terms of projective geometry in a projective contact space, it is possible to work more generally with not necessarily closed, conical C^∞–submanifolds of the underlying symplectic vector spaces. Applying our construction to this setting is much more subtle, and it would require a separate study.

Arrangement. The preliminary Section 1 contains all material necessary to state and prove the main results. We review there the construction of the adjoint contact manifold, and the subadjoint variety associated to a simple complex Lie group. We recall the standard construction of the Chow form. We introduce the Lagrangian Grassmannian, define its Plücker embedding space, and study their basic properties.

In Section 2 we prove the main Theorem 1, which establishes the existence of low–degree invariants, and we review D. The's result on the Lagrangian Chow forms of the subadjoint varieties [17]. In Section 3 we clarify the link between the ring of invariants and the set of invariant PDEs, by proving Proposition 1, which links those invariants to 2^{nd} order nonlinear PDEs on the adjoint contact manifold.

Finally, in Section 4 we discuss some real forms.

1. Preliminary

1.1. **The adjoint variety associated with a complex simple Lie group.** We recall how to introduce the contact grading (3) on the Lie algebra $\mathfrak{g}$ corresponding to the connected, simply connected complex simple Lie group G. To this end, we need to choose a Cartan subalgebra $\mathfrak{h}$ and a decomposition $\Phi = \Phi^+ \cup \Phi^-$ of the corresponding root system Φ into positive and negative roots. Then, there is a one–to–one correspondence between isomorphism classes of finite–dimensional irreducible representations of G (irreducible G-modules) and dominant integral weights $\lambda \in \mathfrak{h}^*$. We denote by V_λ the irreducible G-module associated with a dominant weight $\lambda \in \mathfrak{h}^*$, and by v_λ the corresponding weight vector.

Lemma 2. *The orbit $G[v_\lambda] \subset \mathbb{P}V_\lambda$ of $[v_\lambda] \in \mathbb{P}V_\lambda$ is the only closed G–orbit.*

Proof. Since the stabiliser P of $[v_\lambda]$ is a parabolic subgroup, the orbit $G[v_\lambda]$ is a compact simply connected submanifold of $\mathbb{P}V_\lambda$. Conversely, assume that the orbit $G[v]$ is compact. Then, by Borel theorem, the action of Borel subgroup $B \subset G$ has a fixed point $[v'] \in G[v]$. The stability subgroup $P := G_{[v']}$ contains B. This shows that v' is a highest weight vector and $G[v] = G[v']$ is the orbit of a highest weight vector. □

In virtue of the adjoint representation, we can specialise Lemma 2 to the particular case when the G–module is the Lie algebra $\mathfrak{g}$ itself. To this end, denote by $\alpha \in \Phi^+$ (resp., by E_α) the maximal root (resp., root vector) and by $H_\alpha := [E_\alpha, E_{-\alpha}]$ the associated element of $\mathfrak{h}$. Let $\theta \in \mathfrak{g}^*$ be the 1–form on $\mathfrak{g}$ dual to $E_{-\alpha}$. We also denote by $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_+$ the non–negative part of the contact grading (3) and by P (resp., G_0, G_+) the unique connected subgroup of G corresponding to $\mathfrak{p}$ (resp., $\mathfrak{g}_0$, $\mathfrak{g}_+$). Recall that (3) is precisely the eingenspace decomposition for the endomorphism ad_{H_α}.

Corollary 2. *The following results hold.*

1. *The orbit $M = \mathrm{Ad}_G[E_\alpha] \subset \mathbb{P}\mathfrak{g}$ is the only closed orbit in the projective space $\mathbb{P}\mathfrak{g}$.*
2. *The stability subalgebra is $\mathfrak{p}$, and hence $M = G/P$.*
3. *The isotropy representation of $\mathfrak{p}$ (resp., of $P = G_0 \cdot G_+$) in the tangent space $T_oM = \mathfrak{g}_-$ has kernel $\mathfrak{g}_+$ (resp., G_+) and reduces to the adjoint action $\mathrm{ad}_{\mathfrak{g}_0}$ (resp., Ad_{G_0}).*
4. *The 2–form $\omega := d\theta$ is non–degenerate on $\mathfrak{g}_{-1}$, and*
$$\omega(X, Y) = -\theta([X, Y]), \quad X, Y \in \mathfrak{g}_{-1}.$$
The conformal class $[\theta]$ is invariant with respect to the isotropy representation and defines an invariant line bundle of contact forms on M. This bundle possesses no invariant sections.
5. *If $\mathfrak{g}$ is not of type A, then the Lie algebra $\mathfrak{g}_0$ has a direct sum decomposition $\mathfrak{g}_0 = \langle H_\alpha \rangle \oplus \mathfrak{g}_0^{ss}$ where $\mathfrak{g}_0^{ss}$ is the semisimple part of $\mathfrak{g}_0$ and the isotropy action of the subalgebra $\mathfrak{g}_0^{ss}$ preserves the form ω.*
6. *M has a canonical invariant complex contact structure $\mathcal{C} \subset TM$ whose value $\mathcal{C}_o$ at the origin $o = eP$ is $\mathfrak{g}_{-1}$.*

Proof. Well–known, see, e.g., [5]. □

Definition 4. *The* adjoint variety *M associated with the connected, simply connected complex simple Lie group G is the contact manifold $(M, \mathcal{C})$, where M is the orbit G/P and $\mathcal{C}$ is the canonical G–invariant contact structure on M.*

TABLE 1. Table of adjoint varieties for types different than C

type	$\mathcal{C}_o$	G_0^{ss}	X_o	$\deg X_o$
A_{n+1}	$\mathbb{C}^n \oplus \mathbb{C}^{n*}$	SL_n	$\mathbb{P}^{n-1} \cup \mathbb{P}^{n-1}$	2
$\mathsf{B}_\bullet$, $\mathsf{D}_\bullet$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathsf{SL}_2 \times \mathsf{SO}_n$	$\mathbb{P}^1 \times Q$	$2(n-1)$
G_2	$S^3\mathbb{C}^2$	SL_2	Veronese	3
F_4	$\Lambda_0^3\mathbb{C}^6$	Sp_3	$\mathrm{LGr}(3,6)$	16
E_6	$\Lambda^3\mathbb{C}^6$	SL_6	$\mathrm{Gr}(3,6)$	42
E_7	spinor	Spin_{12}	$\mathrm{NGr}(6,12)$	large
E_8	$\mathrm{FTS}(\mathcal{A})$	E_7	Freudenthal	very large

The Lie group G_0^{ss}, whose Lie algebra appears in Corollary 2 (5), is precisely the Lie group involved in our definition of the ring of invariants (Definition 2). It can be proved that G_0^{ss} is a maximal connected subgroup (except in type A, where it is the intersection of two maximal parabolic subgroups, see Lemma 7 later on). But the true importance of the group G_0 is that the isotropy representation of P factors through it. In particular, P–invariant varieties in $\mathbb{P}\mathfrak{g}_{-1} = \mathbb{P}\mathcal{C}_o$ are the same as the G_0–invariant ones, or even – except for type A – G_0^{ss}-invariant ones (in type A one additionally need invariance under $\mathbb{C}^\times$).

1.2. **The sub–adjoint variety.** We show now that the action of G_0^{ss} on $\mathbb{P}\mathcal{C}_o$ displays an interesting property: it possesses an (essentially) unique closed orbit, whose dimension is $n-1$ (if G is not of type C).

Let G be a connected simply–connected simple complex Lie group, and consider the G_0^{ss}–module $\mathcal{C}_o$ (recall Corollary 2 (6)). If G is not of type A, then $\mathcal{C}_o$ is irreducible. If G is of type A, then $\mathcal{C}_o$ consists of two mutually dual $\mathbb{C}^\times$-invariant irreducible components, say $\mathcal{C}_o^+$ and $\mathcal{C}_o^-$, which are not equivalent one to the other (see first row of Table 1). Then, by Lemma 2, $\mathbb{P}\mathcal{C}_o$ contains either a unique irreducible closed orbit (if G is of type different than A), or the union of two irreducible closed orbits (if G is of type A).

Definition 5. *The unique closed G_0^{ss}–orbit $X_o \subset \mathbb{P}\mathcal{C}_o$ (resp., the union X_o of the closed G_0–orbits $X_o^\pm \subset \mathbb{P}\mathcal{C}_o^\pm$) is called the* subadjoint variety *at $o \in M$, if G is not of type A (resp., is of type A).*

Proposition 2. *The sub–adjoint variety X_o is $(n-1)$–dimensional and irreducible, except in the case A, when it consists of two $(n-1)$–dimensional irreducible components, and in the case C, when it is the whole of $\mathbb{P}\mathcal{C}_o$ (see Table 1. Furthermore, its stabiliser in $\mathsf{Sp}(\mathcal{C}_o) \equiv \mathsf{Sp}(\omega)$ is precisely G_0.*

Proof. See, e.g., Table A.1 in [4]. □

Remark 1. Interestingly, the sub–adjoint variety X is the first member of an entire flag

$$V_1 \subset V_2 \subset V_3 \subset \mathbb{P}\mathcal{C}_o \equiv \mathbb{P}\mathfrak{g}_{-1} \tag{7}$$

of so–called Freudenthal sub–varieties, where $V_i = \mathbb{P}\{v \in \mathfrak{g}_{-1} \mid \mathrm{ad}_v^{i+1}\,\mathfrak{g}_2 = 0\}$ [10].

1.3. **The Chow transform.** Recall that the 'projective Grassmannian'

$$\mathbb{G}(k, 2n-1) := \mathrm{Gr}(k+1, 2n) \tag{8}$$

is the space of k–dimensional projective subspaces of $\mathbb{P}^{2n-1}$, which is in an obvious one–to–one correspondence with the space of $(k+1)$–dimensional projective subspaces of $\mathbb{C}^{2n}$. For instance, $\mathbb{G}(0, 2n-1)$ is the set of points of $\mathbb{P}^{2n-1}$, i.e, $\mathbb{P}^{2n-1}$ itself.

We are only interested in the case $k = n-1$, for which we have that $\dim \mathbb{G}(n-1, 2n-1) = n^2$.

Recall that $\mathbb{G}(n-1, 2n-1)$ is a projective variety in $\mathbb{P}\Lambda^n\mathbb{C}^{2n}$, via the Plücker embedding. In particular, this allows us to speak about *lines* (or $\mathbb{P}^1$'s) in $\mathbb{G}(n-1, 2n-1)$. As it turns out, such lines admit a nice description in terms of pairs of $(n-2)$– and n–dimensional projective subspaces of $\mathbb{P}^{2n-1}$.

Lemma 3. *The generic line $\mathbb{P}^1 \subset \mathbb{G}(n-1, 2n-1)$ has the form $\mathbb{P}^1 = \ell(\mathbb{P}^{n-2}, \mathbb{P}^n)$, where*

$$\ell(\mathbb{P}^{n-2}, \mathbb{P}^n) := \{\mathbb{P}^{n-1} \in \mathbb{G}(n-1, 2n-1) \mid \mathbb{P}^{n-2} \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n\}\,. \tag{9}$$

We shall need the so–obtained lines in order to determine how the degree behave under the Chow transform.

The second feature of $\mathbb{G}(n-1, 2n)$, which will be useful to make the main point clear, is the corresponding incidence relation

$$\begin{tikzcd} & \mathbb{F}(0, n-1, 2n-1) \arrow[dl, "\widetilde{p}"'] \arrow[dr, "\widetilde{q}"] & \\ \mathbb{P}^{2n-1} \equiv \mathbb{G}(0, 2n-1) & & \mathbb{G}(n-1, 2n-1)\,, \end{tikzcd} \tag{10}$$

where

$$\mathbb{F}(0, n-1, 2n-1) := \{(P, \mathbb{P}^{n-1}) \in \mathbb{P}^{2n-1} \times \mathbb{G}(n-1, 2n-1) \mid P \in \mathbb{P}^{n-1}\}\,. \tag{11}$$

The key property of diagram (10) is that the fibres of both projections are Grassmannian themselves, and the two projections are mutually transversal.

Lemma 4. *For any $P \in \mathbb{P}^{2n-1}$ and $\mathbb{P}^{n-1} \in \mathbb{G}(n-1, 2n-1)$, it is true that*

(1) *the fibre $\widetilde{p}^{-1}(P)$ identifies with $\mathbb{G}(n-2, 2n-2)$;*
(2) *the fibre $\widetilde{q}^{-1}(\mathbb{P}^{n-1})$ identifies with $\mathbb{P}^{n-1}$ itself;*
(3) *the two fibres above intersect transversally.*

Definition 6. *Let $X \subset \mathbb{P}^{2n-1}$ be an $(n-1)$-dimensional subvariety. The variety*

$$\widetilde{X} := \widetilde{q}(\widetilde{p}^{-1}(X)) \subset \mathbb{G}(n-1, 2n-1) \tag{12}$$

is called the Chow transform[2] *of X.*

Our next result is to show that the Chow transform produces hypersurfaces in $\mathbb{G}(n-1, 2n-1)$ out of $(n-1)$–folds in $\mathbb{P}^{2n-1}$, which is exactly what is needed here.

Proposition 3. *If $X \subset \mathbb{P}^{2n-1}$ is an $(n-1)$–dimensional irreducible subvariety, then $\widetilde{X}$ is an irreducible hypersurface of degree equal to the degree of $\widetilde{X}$.*

Proof. The preimage $\widetilde{p}^{-1}(X)$ is an irreducible subvariety of dimension

$$\dim \widetilde{p}^{-1}(X) = n - 1 + n(n-1) = n^2 - 1,$$

so that $\widetilde{X}$ is an irreducible closed subvariety of codimension $1 + e$, where e is the dimension of the general fibre of $\widetilde{p}^{-1}(X) \to \widetilde{X}$. If $e > 0$, it follows that every $(n-1)$-plane in $\mathbb{P}^{2n-1}$ intersecting X intersects it in a positive-dimensional subset; choosing a general n-plane $\Lambda \subset \mathbb{P}^{2n-1}$ so that $\dim(\Lambda \cap X) = 0$, we then find that every $(n-1)$-plane $\Lambda' \subset \Lambda$ intersecting $\Lambda \cap X$ intersects it in a positive-dimensional subset – a contradiction. Hence $e = 0$ and $\widetilde{X}$ is an irreducible hypersurface in $\mathbb{G}(n-1, 2n-1)$. In order to compute its dimension, let us once again choose a general n-plane $\Lambda \subset \mathbb{P}^{2n-1}$ so that $\Lambda \cap X$ consists of d points. Furthermore, choose a general $(n-2)$-plane $\Lambda' \subset \Lambda$ so that none of these points is contained in Λ', and the $(n-1)$-planes containing Λ' and contained in Λ form a general line $\mathbb{P}^1_{\Lambda',\Lambda} \subset \mathbb{G}(n-1, 2n-1)$. Since there is precisely one such $(n-1)$-plane through each of the d points of $\Lambda \cap X$, we have a bijection

$$\mathbb{P}^1_{\Lambda',\Lambda} \cap \widetilde{X} \simeq \Lambda \cap X$$

whence the degree of $\widetilde{X}$ in $\mathbb{G}(n-1, 2n-1)$ equals that of X in $\mathbb{P}^{2n-1}$. □

1.4. **The Lagrangian Grassmannian** $\mathrm{LGr}(n, 2n)$ **and its Plücker embedding.** We clarify now the definition and the main properties of the (complex) Lagrangian Grassmannian (4), and its Plücker embedding space (5). Since all the linear symplectic spaces are isomorphic each other, throughout this section we denote an arbitrary $2n$–dimensional symplectic space, whose symplectic form ω is defined up to a projective factor, by the same symbol $\mathcal{C}$ elsewhere used for the contact distribution. To keep things simple, we may take $\mathcal{C} = \mathbb{C}^n \oplus \mathbb{C}^{n*}$, with ω given by the standard pairing and we can forget, throughout this section, all the relationships with the contact grading (3) of $\mathfrak{g}$.

Remark 2. The condition

$$\omega|_L \equiv 0 \tag{13}$$

can be imposed on subspaces of any dimension—or even on flags of subspaces. For instance, we denote by $\mathbb{G}_\omega(k, 2n-1)$ the subset of $\mathbb{G}(k, 2n-1)$, consisting of solutions of the equation (13); similarly, $\mathbb{F}_\omega(l, k, 2n-1)$ consists of flags (L', L), such that L satisfies (13) (which is an hereditary condition).

In continuity with the original definition (4), we use symbol $\mathrm{LGr}(n, 2n)$ instead of $\mathrm{LGr}(\mathcal{C})$ to denote the subset of $\mathrm{Gr}(n, 2n) = \mathrm{Gr}(n, \mathcal{C})$ consisting of solutions of the equation (13).

Lemma 5. *Let $\mathrm{LGr}(n, 2n) = \mathrm{LGr}(C)$ be the complex Lagrangian Grassmannian. Then:*

(1) $\mathrm{LGr}(n, 2n)$ *is a complex non–singular manifold of dimension $\frac{n(n+1)}{2}$;*
(2) *the group Sp_n acts transitively on $\mathrm{LGr}(n, 2n)$, with stabiliser $P = \mathsf{GL}_n \ltimes S^2\mathbb{C}^n$;*
(3) $\mathrm{LGr}(n, 2n)$ *identifies with the Sp_n–orbit of the 'volume' $[e_1 \wedge \cdots \wedge e_n] \in \mathbb{P}\Lambda^n\mathcal{C}$ of any $L_0 = \langle e_1, \ldots, e_n\rangle \in$* $\mathrm{LGr}(n, 2n)$.

Proof. We fix a bi–Lagrangian decomposition $\mathcal{C} := L_0 \oplus L_0^*$ and using ω identify L_0^* as the dual space to L_0. Denote by pr_{L_0} the canonical projection of $\mathcal{C}$ onto L_0. Any n–dimensional subspace L such that $\mathrm{pr}_{L_0} L = L_0$ corresponds to a linear map $\gamma_L \in \mathrm{Hom}_\mathbb{C}(L_0, L_0^*) = L_0^* \otimes L_0^*$, whose graph is $L = \{v + \gamma_L(v) \mid v \in L_0\}$. The subspace L is Lagrangian if and only if γ_L is symmetric as a bilinear form. The open and dense domain $\mathrm{LGr}(\mathcal{C})_{L_0} := \{L \in \mathrm{LGr}(\mathcal{C}) \mid \mathrm{pr}_{L_0} L = L_0\}$ is thus identified with $S^2 L_0^*$, which gives a local chart in $\mathrm{LGr}(\mathcal{C})$. The coordinates of the symmetric bilinear form γ_L are the local coordinates of L.

We fix a basis $\{e_1, \ldots, e_n\}$ of L_0 together with the dual basis $\{e^1, \ldots, e^n\}$ of L_0^*. They form a symplectic basis e of $\mathcal{C}$. For any Lagrangian n–dimensional subspace $L \in \mathrm{LGr}(\mathcal{C})$ we can choose a symplectic basis $f = (f_1, \cdots, f_n, f^1, \cdots, f^n)$

[2]See the classical textbook [9], Chapter 3, Section 2. The *Chow form* mentioned there is precisely the homogeneous polynomial whose zero locus is the Chow transform mentioned here.

of $\mathcal{C}$, such that its first n vectors span L. Then the the linear transformation $\mathcal{A}$ which maps e to t transform L_0 onto L. It is a symplectic transformation $\mathcal{A} \in \mathsf{Sp}_n$, i.e., its matrix $\mathcal{A}$ has the form

$$\mathcal{A} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = 1 \right\} . \tag{14}$$

The new bi–Lagrangian decomposition $\mathcal{C} = \mathcal{A}L_0 + \mathcal{A}L_0^* = (AL + BL^*) + (CL + DL^*)$ gives a new chart $S^2L^* = S^2(\mathcal{A}L_0^*)$ in a neighborhood of $\mathcal{A}L_0$. The transition function between these two charts is given by the fractional linear transformation $L\hat{\mathcal{A}} : \gamma \mapsto \gamma' = (C + D\gamma)(A + B\gamma)^{-1}$. Obviously, $L = L_0$ if and only if $C = 0$, This shows that the stability subgroup of L_0 is the upper block triangular (parabolic) subgroup P. This proves (1), (2).
Note that the symplectic Lie algebra is given by

$$\mathfrak{sp}_n = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid C = C^t, B = B^t \right\} , \tag{15}$$

Now we consider the 'volume' map

$$\begin{array}{rcl} \mathrm{vol} : \mathrm{LGr}(n, 2n) & \longrightarrow & \mathbb{P}\Lambda^nC \\ L = \langle e_1, \dots, e_n \rangle & \longmapsto & [e_1 \wedge \cdots \wedge e_n] =: \mathrm{vol}(L) . \end{array} \tag{16}$$

Obviously it is injective and Sp_n–equivariant. Since Sp_n acts transitively on $\mathrm{LGr}(n, 2n)$, the image $\mathrm{vol}(\mathrm{LGr}(n, 2n))$ is the orbit $\mathsf{Sp}_n(\mathrm{vol}(L))$, and (3) is proved. □

Lemma 5 shows that $\mathrm{LGr}(n, 2n)$ is a non–singular projective variety in $\mathbb{P}\Lambda^n\mathcal{C}$, contained into $\mathrm{Gr}(n, 2n)$. In fact, it is cut out by a projective subspace (see (18) below). To see this, denote by ι_ω the following contraction:

$$\iota_\omega : \Lambda^\bullet\mathcal{C} \longrightarrow \Lambda^\bullet\mathcal{C} , \quad \eta \longrightarrow \omega \lrcorner \eta , \tag{17}$$

where ω is the symplectic form on $\mathcal{C}$. It is a Sp_n–equivariant derivation of degree -2 of the Grassman algebra $\Lambda^\bullet\mathcal{C}$.

Corollary 3. *The only irreducible component of the* Sp_n*–module* $\Lambda^n\mathcal{C}$ *whose projectivisation contains* $\mathrm{LGr}(n, 2n)$ *is* $\ker \iota_\omega|_{\Lambda^n\mathcal{C}}$.

Proof. By the definition (16), the volume of a Lagrangian n–plane is killed by the insertion (17). Hence, the image belongs to the projectivisation of the module $\Lambda^n_0\mathcal{C}$. One can check that the volume form of Lagrangian planes span all vector space $\Lambda^n_0\mathcal{C}$ which shows that this module is irreducible. □

In accordance with (5), we set $\Lambda^n_0\mathcal{C} := \ker \iota_\omega|_{\Lambda^n\mathcal{C}}$, and Corollary 3 immediately implies that

$$\mathrm{LGr}(n, 2n) = \mathrm{Gr}(n, 2n) \cap \mathbb{P}\Lambda^n_0C . \tag{18}$$

Remark 3. In Cartan's notation, Sp_n is a simple group of type C_n, and the highest weight of Λ^n_0C is precisely the n^{th} fundamental weight (Bourbaki labeling). In other words, the parabolic subgroup P is obtained by marking the long root (i.e., the rightmost vertex) of the Dynkin diagram of type C_n, viz.

$$\overset{0}{\circ} \frac{\quad}{} \overset{0}{\circ} \frac{\quad}{} \overset{0}{\circ} \frac{\quad}{} \cdots \frac{\quad}{} \overset{0}{\circ} \Longleftarrow \overset{1}{\bullet} . \tag{19}$$

2. The ring of invariants

This section contains the proof of the main Theorem 1, organised according to the type of G: type A is taken care of in Section 2.2, types B and D (in degree 4) in Section 2.7, type D (in degree 2) in Section 2.4, type E in Section 2.5, type F in Section 2.8, and type G in Section 2.3.

The 'higher–degree" invariants PDEs arising as the Lagrangian Chow form of the subadjoint variety (the 'exceptionally simple' PDEs exhamined in [17]) are discussed in Section 2.9. In Section 2.6 we discuss the general construction of a quartic invariant on $\mathfrak{g}_{-1}$. Section 2.1 serves only to refresh the setup and establish some preliminary results.

2.1. **Setup.** Recall that we assume G to be a complex, connected and simply–connected, simple Lie group of type different than C, with Lie algebra $\mathfrak{g}$, and that a Cartan subalgebra $\mathfrak{h}$ and a system of positive roots Φ^+ has been chosen. From these assumptions there follow many other things: the contact grading (3), the symplectic form $\omega \in \Lambda^2\mathfrak{g}^*_{-1}$ (defined up to a projective factor), the connected closed subgroup G_0^{ss} (see Corollary 2), and the Plücker embedding space $\mathbb{P}\Lambda^n_0\mathfrak{g}_{-1}$ of the Lagrangian Grassmannian $\mathrm{LGr}(\omega)$ (see Lemma 5 (3) and Corollary 3).

Recall also that $n = \frac{1}{2}\dim \mathfrak{g}_{-1}$, so that $\mathsf{Sp}(\omega)$ is a simple Lie group of type C_n.

Next lemma will clarify our definition of the rings of invariant (Definition 2). To this end, let $I \subset S^\bullet\Lambda^n_0\mathfrak{g}^*_{-1}$ be the homogeneous ideal of $\mathrm{LGr}(\omega)$. Write $I^{(d)} \subset S^d\Lambda^n_0\mathfrak{g}^*_{-1}$ for the degree d homogeneous component of I. Recall also the identification (5).

Lemma 6. *There are natural* $\mathsf{Sp}(\omega)$*–equivariant isomorphisms*

$$\Gamma(\mathrm{LGr}(\omega), \mathcal{O}(d)) \simeq S^d\Lambda^n_0\mathfrak{g}^*_{-1}/I^{(d)} \simeq V^*_{d\lambda_n} .$$

Proof. The first isomorphism follows by minimality and projective normality of the Plücker embedding (see Corollary 3). The second one follows from the identification (5), and by the definition of $\mathcal{O}(d)$ as the bundle of degree d forms on the tautological line bundle $\mathsf{Sp}(\omega)\cdot\langle v_\lambda\rangle$. □

Lemma 6 above allows to identify the space R_d (introduced in Definition 2) with the space of G_0^{ss}–invariants in $S^d\Lambda_0^n\mathfrak{g}_{-1}^*/I^{(d)}$.

Remark 4. Note that applying the same Definition 2 to $\mathfrak{g}$ of type C yields R consisting of constants only: that is because in type C the group G_0^{ss} coincides with the entire symplectic group $\mathsf{Sp}(\omega)$ acting transitively on $\mathrm{LGr}(\omega)$.

Now, propositions 4, 7, 6, 5 introduced in the following subsections will give a proof of our main Theorem 1 (except for the numbers marked with a question mark). We shall thus establish the existence of low-degree invariants in the ring R. It turns out that each of these actually suffices to reduce the symplectic group $\mathsf{Sp}(\omega)$ to G_0^{ss}:

Lemma 7. *Assume $\mathfrak{g}$ is not of type A. Then G_0^{ss} is a maximal connected subgroup of $\mathsf{Sp}(\omega)$.*

Corollary 4. *Assume $\mathfrak{g}$ is not of type A. Let $r\in R_d$ be a nonzero element with $d>0$. Then the identity component of its stabiliser in $\mathsf{Sp}(\omega)$ is precisely G_0^{ss}.*

Proof of Lemma 7. It is enough to decompose $S^2\mathfrak{g}_{-1}\simeq\mathfrak{sp}(\omega)$ into irreducible $\mathfrak{g}_0^{\mathrm{ss}}$-modules and observe that the complement of $\mathfrak{g}_0^{\mathrm{ss}}$ in $S^2\mathfrak{g}_{-1}$ is a single irreducible component. We refer to Table 1 for information on G_0 and its representation on $\mathfrak{g}_{-1}$. Types B,D:

$$S^2(\mathbb{C}^2\otimes\mathbb{C}^n)\simeq\Lambda^2\mathbb{C}^n\oplus S^2\mathbb{C}^2\oplus(S^2\mathbb{C}^2\otimes S_0^2\mathbb{C}^n)\simeq\mathfrak{so}_n\oplus\mathfrak{sl}_2\oplus(S^2\mathbb{C}^2\otimes S_0^2\mathbb{C}^n).$$

Type G_2:

$$S^2S^3\mathbb{C}^2\simeq S^2\mathbb{C}^2\oplus S^6\mathbb{C}^2\simeq\mathfrak{sl}_2\oplus S^6\mathbb{C}^2.$$

Type F_4:

$$S^2\Lambda_0^3\mathbb{C}^6\simeq S^2\mathbb{C}^6\oplus S_0^2\Lambda_0^3\mathbb{C}^6\simeq\mathfrak{sp}_3\oplus S_0^2\Lambda_0^3\mathbb{C}^6.$$

Type E_6:

$$S^2\Lambda^3\mathbb{C}^6\simeq\mathfrak{sl}_6\oplus S_0^2\Lambda^3\mathbb{C}^6.$$

Type E_7 (with $\mathcal{S}^+$ denoting one of the spinor representations of $\mathfrak{so}_{12}$):

$$S^2\mathcal{S}^+\simeq\Lambda^2\mathbb{C}^{12}\oplus S_0^2\mathcal{S}^+\simeq\mathfrak{so}_{12}\oplus S_0^2\mathcal{S}^+.$$

Type E_8 (with $\mathrm{FTS}(\mathcal{A})$ denoting the Freudenthal triple system over the Albert algebra):

$$S^2\mathrm{FTS}(\mathcal{A})\simeq\mathfrak{e}_7\oplus S_0^2\mathrm{FTS}(\mathcal{A}).$$

The last one is computed using `LiE`. □

2.2. **Results in type A.** For $\mathfrak{g}$ of type A, we have $G_0^{\mathrm{ss}}\simeq\mathsf{SL}_n$ and $\mathfrak{g}_{-1}\simeq\mathbb{C}^n\oplus\mathbb{C}^{n*}$ with a symplectic form induced by the canonical pairing:

$$\omega(x+\xi,y+\eta)=\xi(y)-\eta(x),\qquad x,y\in\mathbb{C}^n;\ \xi,\eta\in\mathbb{C}^{n*}.$$

A general Lagrangian subspace $L\subset\mathfrak{g}_{-1}$ may be viewed as the graph of a symmetric linear map $\mathbb{C}^n\to\mathbb{C}^{n*}$ or $\mathbb{C}^{n*}\to\mathbb{C}^n$, as explained in the proof of Lemma 5. Denoting by $U\subset\mathrm{LGr}(\omega)$ the locus of subspaces L projecting onto $\mathbb{C}^n$, we have that the complement $H=\mathrm{LGr}(\omega)\smallsetminus U$ is a hyperplane section. Dually, we define U' and its complement H'. Observe that U,U',H,H' are G_0^{ss}–invariant, and we may identify $U\simeq S^2\mathbb{C}^{n*}$, $U'\simeq S^2\mathbb{C}^n$ with the natural linear action of G_0^{ss}. In particular, $H'\cap U$ is the hypersurface in $S^2\mathbb{C}^n$ cut out by the determinant $\det: S^2\mathbb{C}^n\to(\det\mathbb{C}^n)^{-2}\simeq\mathbb{C}$.

Proposition 4. *For $\mathfrak{g}$ of type A the ring of invariants R is generated by a pair of elements of degree 1 cutting out H and H'. The subgroup $G_0^{ss}\times\mathbb{C}^\times\subset\mathsf{Sp}(\omega)$ is precisely the stabiliser of their product.*

Proof. Suppose $r\in R_d$ is a nonzero invariant of degree d and let D be the associated effective divisor on $\mathrm{LGr}(\omega)$. We may write $D=aH+b\overline{D}_U$ for some non-negative integers a,b where D_U is the restriction of D to $U\simeq S^2\mathbb{C}^{n*}$. Since D_U is necessarily G_0^{ss}-invariant, it will be enough to check that the ring of SL_n–invariants in $\mathbb{C}[S^2\mathbb{C}^{n*}]$ is generated by det. That is a classical result (see, e.g., [15]).

To check the second claim, we note that the stabilisers of H and H' in $\mathsf{Sp}(\omega)$ are a pair of opposite maximal parabolics $U\rtimes\mathsf{SL}_n$ and $\mathsf{SL}_n\ltimes U'$ intersecting precisely in $\mathsf{GL}_n\simeq G_0^{\mathrm{ss}}\times\mathbb{C}^\times$. □

2.3. **Results in type G_2.** For $\mathfrak{g}$ of type G_2, we have $G_0^{\mathrm{ss}}\simeq\mathsf{SL}_2$ and $\mathfrak{g}_{-1}\simeq S^3\mathbb{C}^2$ with a symplectic form defined by the natural SL_2-equivariant map

$$\Lambda^2S^3\mathbb{C}^2\to S^3\Lambda^2\mathbb{C}^2=(\det\mathbb{C}^2)^3\simeq\mathbb{C}.$$

The Plücker embedding space is identified with $S^4\mathbb{C}^2$ as a representation of SL_2, with $\mathrm{LGr}(\omega)\subset\mathbb{P}S^4\mathbb{C}^2$ being an SL_2–invariant quadric.

Proposition 5. *For $\mathfrak{g}$ of type G_2 the ring of invariants R is generated by a single element of degree 3.*

Proof. It is a classical result that the ring of SL_2-invariants in $\mathbb{C}[S^4\mathbb{C}^2]$ is generated by a pair of elements in degrees, respectively, 2 and 3 (see, e.g., [15]). Now, the ideal I of $\mathrm{LGr}(\omega)$ is generated by the quadric, whence the quotient $R\simeq\mathbb{C}[S^4\mathbb{C}^2]^{\mathsf{SL}_2}/I^{\mathsf{SL}_2}$ is generated by the cubic. □

2.4. **Results in type D.** For $\mathfrak{g}$ of type D, we have $G_0^{ss} \simeq \mathsf{SL}_2 \times \mathsf{SO}_n$ and $\mathfrak{g}_{-1} \simeq \mathbb{C}^2 \otimes \mathbb{C}^n$ with n even. The Plücker embedding space $\Lambda_0^n \mathfrak{g}_{-1}$ admits a G_0^{ss}–invariant inner product defined by

$$\langle \phi, \psi \rangle = \phi \wedge \psi$$

where we trivialise $\Lambda^{2n} \mathfrak{g}_{-1} \simeq \mathbb{C}$. We consider a G_0^{ss}–equivariant projection

$$\pi : \Lambda_0^n \mathfrak{g}_{-1} \to S^n \mathbb{C}^2$$

defined on decomposable elements as

$$\pi(\xi_1 \otimes v_1 \wedge \cdots \wedge \xi_n \otimes v_n) = \xi_1 \cdots \xi_n \cdot \mathrm{vol}(v_1, \ldots, v_n)$$

where $\mathrm{vol} \in \Lambda^n \mathbb{C}^{n*}$ is the volume form. Recall that $S^n \mathbb{C}^2$ admits an SL_2–invariant quadratic form $q \in S^2 S^n \mathbb{C}^{2*}$ defined by the map

$$S^2 S^n \mathbb{C}^2 \to S^n \Lambda^2 \mathbb{C}^2 = (\det \mathbb{C}^2)^n \simeq \mathbb{C}.$$

We may thus introduce a G_0^{ss}-invariant quadric $B \in S^2 \Lambda_0^n \mathfrak{g}_{-1}^*$ by

$$B(\phi) = q(\pi \phi)$$

for all $\phi \in \Lambda_0^n \mathfrak{g}_{-1}$.

Proposition 6. *For $\mathfrak{g}$ of type D, the residue class of B defines a non–zero element of degree 2 in the ring of invariants R.*

Proof. It suffices to show that $B(\phi) \neq 0$ for some $\phi \in \Lambda_0^n \mathfrak{g}_{-1}$ such that $[\phi] \in \mathrm{LGr}(\omega)$. Fix an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{C}^n$. Let $\eta \in S^n \mathbb{C}^2$ be such that $q(\eta) \neq 0$, and fix a factorisation $\eta = \xi_1 \cdots \xi_n$ with $\xi_i \in \mathbb{C}^2$, $1 \leq i \leq n$. Now, consider the linear subspace

$$L = \langle \xi_1 \otimes e_1, \ldots, \xi_n \otimes e_n \rangle \subset \mathfrak{g}_{-1}.$$

It is by construction Lagrangian, and furthermore its representing n-form

$$\phi = (\xi_1 \otimes e_1) \wedge \cdots \wedge (\xi_n \otimes e_n)$$

satisfies $B(\phi) = q(\eta) \neq 0$. □

2.5. **Results in types E_6, E_7, E_8.** For $\mathfrak{g}$ of type E, we always have $\dim \mathfrak{g}_{-1} = 2n$ for n even, so that $\Lambda_0^n \mathfrak{g}_{-1}$ admits an invariant inner product. Analogously to the situation in type D, we conjecture that one can obtain a quadric invariant from an orthogonal projection of $\Lambda_0^n \mathfrak{g}_{-1}$ onto a G_0^{ss}–invariant subspace.

Let us illustrate this approach in case of E_6. We have $G_0^{ss} \simeq \mathsf{SL}_6$ and $\mathfrak{g}_{-1} \simeq \Lambda^3 \mathbb{C}^6$ with $\dim \mathfrak{g}_{-1} = 2n = 20$. The symplectic form $\omega \in \Lambda^2 \Lambda^3 \mathbb{C}^{6*}$ is defined by $\wedge : \Lambda^3 \mathbb{C}^6 \times \Lambda^3 \mathbb{C}^6 \to \det \mathbb{C}^6 \simeq \mathbb{C}$, and we have

$$\Lambda^{10} \Lambda^3 \mathbb{C}^6 = \Lambda_0^{10} \Lambda^3 \mathbb{C} \oplus \omega \wedge \Lambda^8 \Lambda^3 \mathbb{C}$$

with a G_0^{ss}-invariant decomposition

$$\Lambda_0^{10} \Lambda^3 \mathbb{C}^6 = S_0^4 \Lambda^3 \mathbb{C}^6 \oplus (S^6 \mathbb{C}^6 \oplus S^6 \mathbb{C}^{6*}) \oplus (V_{2\varpi_2 + 4\varpi_5} \oplus V_{2\varpi_2 + 4\varpi_5}^*) \oplus V_{2\varpi_1 + 2\varpi_3 + 2\varpi_5}$$

where we used V_μ to denote the finite dimensional simple module with highest weight μ, a combination of the fundamental weights $\varpi_1, \ldots, \varpi_5$ of $\mathfrak{g}_0^{ss}$. Each of the modules in parentheses is self-dual by means of a unique (up to scale) G_0^{ss}-invariant inner product. The projection onto $S^6 \mathbb{C}^6$ may be symbolically written as

$$\pi = \epsilon^4 : \Lambda^{10} \Lambda^3 \mathbb{C}^6 \to S^6 \mathbb{C}^6$$

where $\epsilon \in \det \mathbb{C}^{6*}$ is a volume form. Introducing a dual form $\epsilon' \in \det \mathbb{C}^6$ and using ω to identify $\Lambda^3 \mathbb{C}^6 \simeq \Lambda^3 \mathbb{C}^{6*}$ we also have symbolically the other projection

$$\pi' = \epsilon'^4 \omega^{10} : \Lambda^{10} \Lambda^3 \mathbb{C}^6 \to S^6 \mathbb{C}^{6*}.$$

Finally thus, we have the pullback quadratic form

$$B(\phi) = \langle \pi \phi, \pi' \phi \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing $S^6 \mathbb{C}^6 \times S^6 \mathbb{C}^{6*} \to \mathbb{C}$. Certain computational evidence at our disposal leads us to conjecture that B does not vanish identically on $\mathrm{LGr}(\omega)$.

2.6. **Quartic invariants.** We recall the fundamental quartic invariant of the G_0^{ss} action on $\mathfrak{g}_{-1}$:

$$\Upsilon(x) = \mathrm{ad}_x^4 \in \mathfrak{g}_{-2}^2 \simeq \mathbb{C}$$

for $x \in \mathfrak{g}_{-1}$, i.e.

$$\Upsilon(x) = \langle z, [x, [x, [x, [x, z]]]] \rangle$$

with a fixed element $z \in \mathfrak{g}_2$ and $\langle \cdot, \cdot \rangle$ denoting the Killing form on $\mathfrak{g}$. It is straightforward to extend Υ to a quartic on the Plücker space $\Lambda_0^n \mathfrak{g}_{-1}$:

$$Q(\phi) = (\Lambda^n \Upsilon)(\phi \otimes \phi \otimes \phi \otimes \phi).$$

We wish to determine whether $Q \in S^4 V_{\lambda_n}^*$ vanishes modulo $I^{(4)}$.

2.7. **Results in type B (and D).** For $\mathfrak{g}$ of type B or D, we have again $G_0^{\mathrm{ss}} \simeq \mathsf{SL}_2 \times \mathsf{SO}_n$ and $\mathfrak{g}_{-1} \simeq \mathbb{C}^2 \otimes \mathbb{C}^n$ (if n is even, we already have the quadric, but we may still consider the quartic). Decomposing $S^4(\mathbb{C}^2 \otimes \mathbb{C}^n)$ into G_0^{ss}–irreducible subspaces, we find that the quartic Υ is necessarily given by the projection

$$S^4(\mathbb{C}^2 \otimes \mathbb{C}^n) \to S^2\Lambda^2\mathbb{C}^2 \otimes \ker\left[S^2\Lambda^2\mathbb{C}^n \xrightarrow{\wedge} \Lambda^4\mathbb{C}^n\right] \xrightarrow{\mathrm{id}\otimes\langle,\rangle} (\det\mathbb{C}^2)^2 \otimes \mathbb{C} \simeq \mathbb{C}$$

defined as

$$\Upsilon(\xi_1 \otimes e_1, \xi_2 \otimes e_2, \xi_3 \otimes e_3, \xi_4 \otimes e_4) = \epsilon(\xi_1, \xi_2)\epsilon(\xi_3, \xi_4)\left[\langle e_1, e_3\rangle\langle e_2, e_4\rangle - \langle e_2, e_3\rangle\langle e_1, e_4\rangle\right]$$

where $\epsilon \in \Lambda^2\mathbb{C}^{2*}$ is a volume form and $\langle\cdot,\cdot\rangle$ the SO_n-invariant inner product on $\mathbb{C}^n$.

Proposition 7. *For $\mathfrak{g}$ of type B or D, the residue class of Q defines a non–zero element of degree 4 in the ring of invariants R.*

Proof. Consider as before a Lagrangian subspace of the form

$$L = \langle \xi_1 \otimes e_1, \ldots, \xi_n \otimes e_n\rangle \subset \mathfrak{g}_{-1}$$

where $e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{C}^n$, while $\xi_1, \ldots, \xi_n \in \mathbb{C}^2$ is a general n–tuple. We compute $Q(\phi)$ for $\phi = \bigwedge_{i=1}^n (\xi_i \otimes e_i)$:

$$\begin{aligned} Q(\phi) = &\textstyle\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} && \mathrm{sgn}(\sigma^{(1)}\sigma^{(2)}\sigma^{(3)}) \\ &\textstyle\prod_{i=1}^n && \epsilon(\xi_i, \xi_{\sigma^{(1)}i})\epsilon(\xi_{\sigma^{(2)}i}, \xi_{\sigma^{(3)}i}) \\ & && [\langle e_i, e_{\sigma^{(2)}i}\rangle\langle e_{\sigma^{(1)}i}, e_{\sigma^{(3)}i}\rangle - \langle e_i, e_{\sigma^{(3)}i}\rangle\langle e_{\sigma^{(1)}i}, e_{\sigma^{(2)}i}\rangle] \\ = &\textstyle\sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} && \mathrm{sgn}(\sigma^{(1)}\sigma^{(2)}\sigma^{(3)}) \prod_{i=1}^n \left[\delta^i_{\sigma^{(2)}i}\delta^{\sigma^{(1)}i}_{\sigma^{(3)}i} - \delta^i_{\sigma^{(3)}i}\delta^{\sigma^{(1)}i}_{\sigma^{(2)}i}\right] \\ &\times\textstyle\prod_{i=1}^n && \epsilon(\xi_i, \xi_{\sigma^{(1)}i})\epsilon(\xi_{\sigma^{(2)}i}, \xi_{\sigma^{(3)}i}) \\ = & K(\xi_1 \cdots \xi_n) && \end{aligned}$$

Note that this way $Q(\phi)$ is computed as a SL_2–invariant quartic K on $S^n\mathbb{C}^2$. We only need to check that $K \neq 0$. We further rewrite:

$$K(\xi_1 \cdots \xi_n) = (-1)^n \sum_\sigma \left(\prod_i \epsilon(\xi_i, \xi_{\sigma i})^2\right) \times \left(\mathrm{sgn}(\sigma) \sum_{J \subset \{1,\ldots,n\}} C_{\sigma,J}\right)$$

where

$$C_{\sigma,J} = \mathrm{sgn}(\sigma^J)\,\mathrm{sgn}(\sigma^{J^c}), \quad \sigma^J(i) = \begin{cases} i & i \in J \\ \sigma(i) & i \notin J \end{cases}$$

and $\mathrm{sgn}(\sigma^J) = 0$ if σ^J is not a permutation. The sum over J may be restricted to σ-invariant sets, in which case $C_{\sigma,J} = \mathrm{sgn}(\sigma)$ and we obtain

$$K(\xi_1 \cdots \xi_n) = (-1)^n \sum_\sigma \left(\prod_i \epsilon(\xi_i, \xi_{\sigma i})^2\right) \times |\{J \subset \{1, \ldots, n\} \mid \sigma J = J\}|$$

Choosing $\xi_1, \ldots, \xi_n$ to define n distinct points in $\mathbb{P}^1$ with *real* coefficients, we find that $K(\xi_1, \ldots, \xi_n) > 0$. □

2.8. **Results in type F_4.** With the above definitions applied to G of type F_4, we conjecture that the quartic Q does not vanish identically on $\mathrm{LGr}(\omega)$.

2.9. **A distinguished invariant of geometric origin.** We have thus far described the low–degree elements in the ring of invariants R, defined earlier by (6). It turns out that there is also a 'canonical' higher-degree element of R, arising from the sub–adjoint variety $X_o \subset \mathbb{P}\mathfrak{g}_{-1}$ (see Definition 5) by means of the 'Lagrangian version' of the Chow transform (see Definition 6). These are precisely the 'exceptionally simple' PDEs introduced by D. The in [17].

2.9.1. *The Lagrangian Chow transform.* The Lagrangian Grassmannian inherits a rich structure from its parent Grassmannian $\mathrm{Gr}(n, 2n)$. We shall not cover here all the details, but just the particular submanifolds needed for our purposes.

Lemma 8. *If $\ell \in \mathbb{P}C$ is a line, then $\widetilde{\ell} := \{L \in \mathrm{LGr}(n, 2n) \mid L \supset l\}$ is a submanifold of $\mathrm{LGr}(n, 2n)$ isomorphic to $\mathrm{LGr}(n-1, 2(n-1))$.*

Proof. Just observe that there is a one–to–one correspondence between $\widetilde{\ell}$ and $\mathrm{LGr}\left(\frac{\ell^\perp}{\ell}\right)$. □

The sub–Grassmannians $\widetilde{\ell}$ are precisely the maximal smooth Schubert varieties.

We consider now a vector space $\mathcal{C}$ equipped with a symplectic form $\omega \in \Lambda^2\mathcal{C}^*$, thus working in the same setting as Section 1.4. We specialise the results obtained in Section 1.3 to this 'Lagrangian' context. From a choreographic standpoints, one just needs to decorate with a subscript 'ω' the double fibration (10), thus reading now

$$\begin{tikzcd}[column sep=small] & \mathbb{F}_\omega(0,n-1,2n-1) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^{2n-1} \equiv \mathbb{G}_\omega(0,2n-1) & & \mathbb{G}_\omega(n-1,2n-1) = \mathrm{LGr}(\mathcal{C})\,, \end{tikzcd} \tag{20}$$

in the sense that all the subspaces involved must be isotropic with respect to ω (see Remark 2). Comparing (20) with (10), one sees that the down–left endpoint is the same, since points are automatically ω–isotropic. Hence, Definition 6 admits a straightforward analogue in the Lagrangian context.

Definition 7. *The subset*

$$\mathcal{E}_X := q(p^{-1}(X)) \subset \mathbb{G}(n-1,2n-1) \tag{21}$$

is called the Lagrangian Chow transform *of X.*

In the spirit of this new definition, we extend the results contained in Proposition 3.

Proposition 8 (Lagrangian Chow transform)**.** *Let $X \subset \mathbb{P}C$ be an $(n-1)$–dimensional irreducible subvariety. Then the transform $\mathcal{E}_X \subset \mathrm{LGr}(C)$ is an irreducible hypersurface of the same degree as X.*

Proof. Since (compare Definition 6 and Definition 7)

$$\mathcal{E}_X = \widetilde{X} \cap \mathrm{LGr}(\mathcal{C})\,, \tag{22}$$

and $\widetilde{X}$ is an hypersurface in $\mathbb{G}(n-1,2n-1)$ by Proposition 3, either $\mathcal{E}_X$ is an hypersurface in $\mathrm{LGr}(\mathcal{C})$, or it has codimension zero. We wish to rule out the second possibility.

To this end, just observe (recall Lemma 8) that

$$\begin{aligned} \dim p^{-1}(X) &= \dim X + \dim \mathrm{LGr}(n-1,2n-2) = \\ &= n-1+n(n-1)/2 = (n-1)(n+2)/2 = n(n+1)/2-1 = \\ &= \dim \mathrm{LGr}(\mathcal{C}) - 1\,, \end{aligned}$$

and since $\dim p^{-1}(X) \geq \dim \mathcal{E}_X$, one must have $1 = \operatorname{codim} p^{-1}(X) \leq \operatorname{codim} \mathcal{E}_X$.

Finally, since $\mathrm{LGr}(\mathcal{C})$ is the intersection of $\mathbb{G}(n-1,2n-1)$ with a linear subspace of the Plücker embedding space, (recall formula (18)) it follows that the degree of $\mathcal{E}_X$ in $\mathrm{LGr}(\mathcal{C})$ equals that of $\widetilde{X}$ in $\mathbb{G}(n-1,2n-1)$. □

2.9.2. *The Lagrangian Chow form of the subadjoint variety.* Returning to the main context (see Section 2.1), we may now consider the Lagrangian Chow transform $\mathcal{E}_{X_o} \subset \mathrm{LGr}(\omega)$ of the sub-adjoint variety $X_o \subset \mathbb{P}\mathfrak{g}_{-1}$. It is an irreducible hypersurface, thus corresponding to an element $r_{X_o} \in R$, the *Lagrangian Chow form* of X_o. In view of Proposition 8, the degree of r_{X_o} is equal to the degree of X_o in $\mathbb{P}\mathfrak{g}_{-1}$, and thus may be rather high (see Table 1). Nevertheless, the invariant r_{X_o} is distinguished by its geometric definition, valid in all types (other than C).

3. Invariant PDEs

We shall now prove Proposition 1. Recall that its first claim is a one–to–one correspondence between the set $\mathsf{Inv}(M,G)$ of G–invariant PDEs and the set of reduced, positive–degree, homogeneous elements of the ring of invariants R.

Proof of part 1 of Proposition 1. Given $\mathcal{E} \in \mathsf{Inv}(M,G)$ we consider its fibre at $o \in M$, i.e. the G_0-invariant hypersurface $\mathcal{E}_o \subset \mathrm{LGr}(\omega)$. Then, letting d be the degree of $\mathcal{E}_o$ with respect to $\mathcal{O}(1)$, we have that $\mathcal{E}_o$ is the zero-locus of a reduced element $r \in R_d$, unique up to scale. Conversely, given such reduced $r \in R_d$, we let $\mathcal{E}_o \subset \mathrm{LGr}(\omega)$ be its zero-locus, a G_0-invariant degree d hypersurface. Then, we define $\mathcal{E} \in \mathsf{Inv}(M,G)$ as $\mathcal{E} = G \cdot \mathcal{E}_o$ so that the fibre $\mathcal{E}_m$ over $m \in M$ is $g \cdot E_o$ for any g such that $m = g \cdot o$. □

The second part of the Proposition states that for any G-invariant PDE $\mathcal{E}$, i.e. an element of $\mathsf{Inv}(M,G)$, the Lie algebra $\mathfrak{g}$ exhausts the local infinitesimal symmetries of $\mathcal{E}$. Recall that we are still in the context where $\mathfrak{g}$ is a complex simple Lie algebra not of type C.

Proof of part 2 of Proposition 1. Let $\mathcal{E} \in \mathsf{Inv}(M,G)$ satisfy the assumptions of part (2) of the Proposition. Denote by $\mathfrak{s}$ the Lie algebra of germs at $o \in M$ of infinitesimal contactomorphisms whose lifts to $M^{(1)}$ are tangent to $\mathcal{E}$. We equip $\mathfrak{s}$ with a natural decreasing filtration $\mathfrak{s}^\bullet$ induced by $\mathcal{C}$. Namely, letting $\mathrm{ev}_o : \mathfrak{s} \to T_oM$ be the evaluation map, we set $\mathfrak{s}^{-2} = \mathfrak{s}$, $\mathfrak{s}^{-1} = \mathrm{ev}_o^{-1}\mathcal{C}_o$, $\mathfrak{s}^0 = \ker \mathrm{ev}_o$; and then we further define

$$\mathfrak{s}^{i+1} = \{\xi \in \mathfrak{s}^i \mid [\xi, \mathfrak{s}^{-1}] \subset \mathfrak{s}^i\}$$

for $i \geq 0$.

The action of $\mathfrak{g}$ on M induces a natural inclusion $\iota : \mathfrak{g} \to \mathfrak{s}$ of filtered Lie algebras. By construction, the non-positive part of the associated graded Lie algebra is:

$$\mathrm{gr}_{-2}\,\mathfrak{s} \oplus \mathrm{gr}_{-1}\,\mathfrak{s} \oplus \mathrm{gr}_0\,\mathfrak{s} \simeq \mathfrak{g}_{\leq 0} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

where $\mathfrak{g}_\bullet$ is the grading on $\mathfrak{g}$ induced by the parabolic subalgebra $\mathfrak{p}$, and the isomorphism is induced by $\operatorname{gr}\iota : \operatorname{gr}\mathfrak{g} = \mathfrak{g} \to \operatorname{gr}\mathfrak{s}$. The only truly non-trivial statement here is the inclusion $\operatorname{gr}_0 \mathfrak{s} \subset \mathfrak{g}_0$. Since $\operatorname{gr}_0 \mathfrak{s}$ is contained in the infinitesimal stabiliser of the hypersurface $\mathcal{E}_o \subset \mathrm{LGr}(\omega)$, i.e. of the projective class of some positive degree homogeneous element $r \in R$ in the ring of invariants, it is enough to check that the stabiliser of r in $\mathsf{Sp}(\omega)$ is $\mathfrak{g}_0^{\mathrm{ss}}$. If $\mathfrak{g}$ is not of type A, this follows from Corollary 4; if $\mathfrak{g}$ is of type A, the additional hypothesis on $\mathcal{E}$ implies that r is the product of the two degree 1 generators of R, and we use Proposition 4.

Let us introduce the notion of a Tanaka prolongation of $\mathfrak{g}_{\leq 0}$ in terms of a universal property. Consider the category whose objects are graded Lie algebra monomorphisms $\kappa : \mathfrak{g}_{\leq 0} \to \mathfrak{a}_\bullet$ satisfying the conditions:

(1) $\mathfrak{a}_{\leq 0} = \kappa(\mathfrak{g}_{\leq 0})$,
(2) $\mathrm{ad} : \mathfrak{a}_{>0} \to \mathrm{Hom}(\mathfrak{a}_{-1}, \mathfrak{a})$ is injective.

The arrows from $(\kappa, \mathfrak{a}_\bullet)$ to $(\kappa', \mathfrak{a}'_\bullet)$ are graded Lie algebra homomorphisms $f : \mathfrak{a}_\bullet \to \mathfrak{a}'_\bullet$ such that $f \circ \kappa = \kappa'$. Now, a Tanaka prolongation of $\mathfrak{g}_{\leq 0}$ is a *terminal object* in this category; in particular, it is unique up to a canonical isomorphism. The universal property gives a canonical morphism from any object $(\iota, \mathfrak{a}_\bullet)$ of our category to the Tanaka prolongation; in fact, this morphism turns out to be injective by virtue of property (2) above. An explicit inductive construction of a Tanaka prolongation $\mathfrak{g}_{\leq 0} \to \tilde{\mathfrak{g}}$ is given by setting $\tilde{\mathfrak{g}}_i = \mathfrak{g}_i$ for $i \leq 0$, and then letting $\tilde{\mathfrak{g}}_i$ be the space of degree i derivations of $\mathfrak{g}_{<0}$ into the $\mathfrak{g}_{<0}$-module $\bigoplus_{j<i} \tilde{\mathfrak{g}}_j$ for $i > 0$.

Now, we have that both $\mathfrak{g}$ and $\operatorname{gr}\mathfrak{s}$ embed into the Tanaka prolongation of $\mathfrak{g}_{\leq 0}$, compatibly with respect to the inclusion $\mathfrak{g} \subset \operatorname{gr}\mathfrak{s}$. On the other hand, checking that the positive degree subspaces in $H^1(\mathfrak{g}_{<0}, \mathfrak{g})$ vanish, one proves by induction on degree that the Tanaka prolongation of $\mathfrak{g}_{\leq 0}$ coincides with $\mathfrak{g}$. It follows that $\operatorname{gr}\iota : \mathfrak{g} \to \operatorname{gr}\mathfrak{s}$ is an isomorphism, and so is thus $\iota : \mathfrak{g} \to \mathfrak{s}$. □

4. Real forms

4.1. **Preliminaries.** Recall that a real form of a complex simple Lie algebra $\mathfrak{g}$ are described as the fix point set $\mathfrak{g}^\sigma$ of a antilinear involution σ of $\mathfrak{g}$. Such real form is defined in terms of Satake diagrams. A real form $\mathfrak{g}^\sigma$ inherits a gradation $\mathfrak{g} = \sum \mathfrak{g}_i$ of the Lie algebra $\mathfrak{g}$ if and only is it preserves the grading element d such that $d^\sigma = d$, where $\mathrm{ad}_d|_{\mathfrak{g}_j} = j\mathrm{id}$.

The gradation is described by putting a cross on some simple roots of the Dynkin diagram of $\mathfrak{g}$. Let the crossed roots are $\alpha_{j_1}, \cdots, \alpha_{j_k}$. Then the degree of a root vector E_α is defined as $\deg E_\alpha = m := m_{j_1} + \cdots + m_{j_k}$ where $\alpha = \sum_{i=1}^n m_i \alpha_i$ is the decomposition of a root in terms of simple roots.

There is a simple criterion (see [7, 2]) when a real form $\mathfrak{g}^\sigma$ inherits the gradation in term of the Satake diagram of $\mathfrak{g}^\sigma$: the real form $\mathfrak{g}^\sigma$ inherits the gradation if and only if the crossed root s of associated Satake diagram are not black and two connected by an arrow white roots are both crossed or uncrossed. The classification of all real forms of complex simple Lie algebras which inherit the depth two gradation associated with adjoint variety are described by Chen.

Proposition 9. *A real form G^σ associated with an antilinear involution σ of the Lie algebra $\mathfrak{g}$ acts transitively on a compact contact manifold if (maybe, after a conjugation) σ preserves the grading element d, or equivalently, consistent with the gradation:*

$$\mathfrak{g}^\sigma = (\mathfrak{g}^{-2})^\sigma + (\mathfrak{g}^{-1})^\sigma + (\mathfrak{g}^{0})^\sigma + (\mathfrak{g}^{1})^\sigma + (\mathfrak{g}^{2})^\sigma.$$

The unique compact homogeneous contact G^σ manifold is the real form of N^μ that is $N^{\mu_\sigma} := P\mathrm{Ad}_G^\sigma E_\mu = G^\sigma / P^\sigma$ where P^σ is the parabolic subgroup generated by $\mathfrak{p}^\sigma = (\mathfrak{g}^0)^\sigma + (\mathfrak{g}^1)^\sigma + (\mathfrak{g}^2)^\sigma$.

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