# Tables, bounds and graphics of the smallest known sizes of complete caps in the spaces PG$(3, q)$ and PG$(4, q)$ *

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**Abstract**

In this paper we present and analyze computational results concerning small complete caps in the projective spaces PG$(N, q)$ of dimension $N = 3$ and $N = 4$ over the finite field of order $q$. The results have been obtained using randomized greedy algorithms and the algorithm with fixed order of points (FOP). The computations have been done in relatively wide regions of $q$ values; such wide regions are not considered in literature for $N = 3, 4$. The new complete caps are the smallest known. Basing on them, we obtained new upper bounds on $t_2(N, q)$, the minimum size of a complete cap in PG$(N, q)$, in particular,

$$t_2(N,q) < \sqrt{N+2} \cdot q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad q \in L_N, \quad N = 3, 4,$$

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$$t_2(N,q) < \left(\sqrt{N+1} + \frac{1.3}{\ln(2q)}\right) q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad q \in L_N, \quad N = 3,4,$$

where

$$L_3 := \{q \leq 4673,\ q \text{ prime}\} \cup \{5003, 6007, 7001, 8009\},$$
$$L_4 := \{q \leq 1361,\ q \text{ prime}\} \cup \{1409\}.$$

Our investigations and results allow to conjecture that these bounds hold for all $q$.

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# 1 Introduction

Let $\mathrm{PG}(N,q)$ be the $N$-dimensional projective space over the Galois field $\mathbb{F}_q$ of order $q$. A cap $\mathcal{K}$ in $\mathrm{PG}(N,q)$ is a set of points no three of which are collinear. A cap $\mathcal{K}$ is complete if it is not contained in a larger cap or, equivalently, if every point of $\mathrm{PG}(N,q) \setminus \mathcal{K}$ is collinear with two points of $\mathcal{K}$. Caps in $\mathrm{PG}(2,q)$ are also called arcs and they have been widely studied by many authors in the past decades. In particular, we refer to the surveys and the results in the works [1–6, 8, 9, 18, 19, 26–30] (see also the references therein) for the known constructions and bounds on the size of complete arcs in projective planes. If $N > 2$ only few constructions and bounds are known.

Caps and in particular arcs have been intensively studied for their connection with Coding Theory. A linear $q$-ary code with length $n$, dimension $k$, and minimum distance $d$ is denoted by $[n,k,d]_q$. If a parity-check matrix of a linear $q$-ary code is obtained by taking as columns the homogeneous coordinates of the points of a cap in $\mathrm{PG}(N,q)$, then the code has minimum distance 4 (with the exceptions of the complete 5-cap in $\mathrm{PG}(3,2)$ giving rise to the $[5,1,5]_2$ code and the complete 11-cap in $\mathrm{PG}(4,3)$ corresponding to the Golay $[11,6,5]_3$ code). In particular, complete caps of size $n$ in $\mathrm{PG}(N,q)$ correspond to non-extendable $[n, n-N-1, 4]_q$ codes. In the case $N = 2$ these codes are MDS, that is they attain the Singleton bound, whereas if $N = 3$ they are Almost MDS, since their Singleton defect is equal to 1.

Another important parameter concerning linear codes is the covering radius. The covering radius of an $[n,k,d]_q$ code $\mathcal{C}$ is the minimum integer $r = r(\mathcal{C})$ such that any vector of $\mathbb{F}_q^n$ has distance at most $r$ from $\mathcal{C}$. Complete caps correspond to quasi-perfect linear codes, that is codes with $r(\mathcal{C}) = \left\lfloor \frac{d-1}{2} \right\rfloor + 1$, since they have minimum distance 4 and covering radius 2; see also [13–17]. The covering density $\mu(\mathcal{C})$, introduced in [17], is one of the parameters characterizing the covering quality of an $[n,k,d]$-code $\mathcal{C}$ and it is defined by

$$\mu(\mathcal{C}) = \frac{1}{q^{n-k}} \sum_{i=0}^{r(\mathcal{C})} (q-1)^i \binom{n}{i}.$$

Note also that caps are connected with quantum codes; see e.g. [23, 34].

In general, a central problem concerning caps is to determine the spectrum of the possible sizes of complete caps in a given space; see [26, 27] and the references therein. Of particular interest for applications to Coding Theory is the lower part of the spectrum; in fact, small complete caps in projective Galois spaces correspond to quasi-perfect linear codes with minimum distance 4 and small density; see for example [16, 19].

Let $t_2(N, q)$ be *the minimum size of a complete cap in* $\mathrm{PG}(N, q)$.

The exact values of $t_2(N, q)$ are known only for small $q$. For instance, $t_2(3, q)$ is known only for $q \leq 7$; see [19, Tab. 3].

Whereas the trivial lower bound for $t_2(N, q)$ is $\sqrt{2}q^{(N-1)/2}$, general constructions of complete caps whose size is close to this lower bound are only known for $q$ even; see [19, 20, 24, 25, 31]. According to the survey paper [27], the smallest known complete caps in $\mathrm{PG}(3, q)$, with $q$ arbitrary large, have size approximately $q^{3/2}/2$ and were presented by Pellegrino in 1998 [32]. However, Pellegrino's completeness proof appears to present a major gap, and counterexamples can be found; see [10, Sect. 2]. Recently, using a modification of the approach of [29], the probabilistic upper bound $cq^{\frac{N-1}{2}} \log^{300} q$, with $c$ constant, for the value $t_2(N, q)$ has been obtained; see [11, 12]. Computer assisted results on small complete caps in $\mathrm{PG}(N, q)$ and $\mathrm{AG}(N, q)$ are given in [10, 19, 21, 22, 30, 33]. Here and further, $\mathrm{AG}(N, q)$ is the $N$-dimensional affine space over the field $\mathbb{F}_q$.

In this paper we obtain by computer searches results concerning *upper bounds* on the functions $t_2(3, q)$ and $t_2(4, q)$. These searches requested a huge amount of memory and execution time. In particular, we constructed small complete caps in $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$ using two different approaches[1]: the algorithm with fixed order of points (FOP), for $q \in L_3$ in $\mathrm{PG}(3, q)$ and $q \in L_4$ in $\mathrm{PG}(4, q)$, and randomized greedy algorithms, for $q \in G_3$ in $\mathrm{PG}(3, q)$ and $q \in G_4$ in $\mathrm{PG}(4, q)$, where

$$L_3 := \{q \leq 4673,\ q \text{ prime}\} \cup \{5003, 6007, 7001, 8009\}, \tag{1.1}$$

$$G_3 := \{q \leq 3701,\ q \text{ prime}\} \cup \{3803, 3907, 4001, 4289\}, \tag{1.2}$$

$$L_4 := \{q \leq 1361,\ q \text{ prime}\} \cup \{1409\}, \tag{1.3}$$

$$G_4 := \{q \leq 463,\ q \text{ prime}\}. \tag{1.4}$$

Note that such relatively wide regions of $q$ values are not considered in literature for $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$.

[1] In this work, calculations were performed using computational resources of Multipurpose Computing Complex of National Research Centre "Kurchatov Institute", http://computing.kiae.ru

Using the data obtained by the computer searches we present different functions which approximate the values $t_2(3,q)$ and $t_2(4,q)$, as done in [1–6, 8, 9] for the minimum size of complete arcs in projective planes $\mathrm{PG}(2,q)$. The main estimates obtained in this paper are given in the following theorem, see also Sections 5 and 6.

**Theorem 1.1.** *Let $t_2(N,q)$ be the minimum size of a complete cap in the projective space $\mathrm{PG}(N,q)$. Let $L_3$ and $L_4$ be the sets of values of $q$ given by relations (1.1) and (1.3), respectively. The following upper bounds on $t_2(N,q)$ hold.*

***A.*** *Upper bounds with the constant multiplier $\sqrt{N+2}$:*

$$t_2(N,q) < \sqrt{N+2} \cdot q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad q \in L_N, \quad N = 3,4. \tag{1.5}$$

***B.*** *Upper bounds with a decreasing multiplier $\beta_N(q)$:*

$$t_2(N,q) < \beta_N(q) q^{\frac{N-1}{2}} \sqrt{\ln q}, \quad \beta_N(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)}, \quad q \in L_N, \quad N = 3,4. \tag{1.6}$$

Our investigations and results (see figures and observations in Sections 5 and 6) allow to conjecture that the estimates of Theorem 1.1, especially the bound with constant multiplier $\sqrt{N+2}$, hold for every prime power $q$.

**Conjecture 1.2.** *In $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$, the upper bounds (1.5), (1.6) hold for all $q$.*

**Remark 1.3.** In the works [2–4], the sizes of small complete arcs in $\mathrm{PG}(2,q)$ are given for all power prime $q \leq 301813$. In this work, we obtained complete arcs in $\mathrm{PG}(2,q)$ for $301813 < q \leq 321007$, $q$ power prime. The results of [2–4] and of this work give the following upper bounds for $\mathrm{PG}(2,q)$:

$$t_2(2,q) < 1.05\sqrt{3q \ln q} < \sqrt{2+2} \cdot q^{\frac{2-1}{2}} \sqrt{\ln q}, \quad q \leq 321007.$$

So, the upper bounds (1.5) hold also for $N = 2$ in a wide region of $q$ values.

As far as this is known to the authors, complete caps obtained in this work are the smallest known in literature for $\mathrm{PG}(3,q)$ with $q \in \{61, 67, 71, 73, 79, 83\}$, $97 \leq q \in L_3$, and $\mathrm{PG}(4,q)$ with $17 \leq q \in L_4$. In particular, the results of this work improve ones of the papers [19, 22, 33].

The paper is organized as follows. In Section 2, we describe the main features of the algorithms used in our searches. In Section 3, some types of upper bounds on $t_2(N,q)$ are discussed. In Section 4, we shortly give the content of tables collecting sizes of small complete caps obtained with the help of the algorithms of Section 2 (the tables are placed in Appendix). In Sections 5 and 6, we analyze the results presented in the tables and illustrated the analysis by graphics. In Section 7, we do some conclusions from the present work.

Some results of this paper were briefly presented in [7].

# 2 Algorithms for small caps in $\mathrm{PG}(N, q)$

In this section we describe two different algorithms used to construct small complete caps in $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$. First of all note that the number of points of $\mathrm{PG}(N, q)$ is of order $q^N$ and for instance, if $q \simeq 5\cdot 10^3$ then $|\mathrm{PG}(3, q)| \simeq 1.2\cdot 10^{11}$: this represents a strong constraint for any algorithm which investigates subsets of points in projective spaces.

## 2.1 Algorithm with fixed order of points (FOP)

This algorithm is a particular type of random algorithm. Some variants of the algorithm FOP for $\mathrm{PG}(2, q)$ and $\mathrm{PG}(3, q)$ are given in [1, 2, 4, 7, 8]. In this work we describe the algorithm FOP for the arbitrary space $\mathrm{PG}(N, q)$.

Firstly we fix a particular order on the points of $\mathrm{PG}(N, q)$. The algorithm builds a complete cap step by step adding a new point at each step, until a complete cap is obtained.

Let $K^{(i-1)}$ be the cap obtained at the $(i-1)$-th step. Among the points not lying on bisecants of $K^{(i-1)}$, the first point in the fixed order is added to $K^{(i-1)}$ to obtain $K^{(i)}$.

Suppose that the points of $\mathrm{PG}(N, q)$ are ordered as $A_1, A_2, \ldots, A_{\frac{q^{N+1}-1}{q-1}}$. Consider the empty set as root of the search and let $K^{(j)}$ be the partial solution obtained in the $j$-th step, as extension of the root. We put

$$K^{(0)} = \emptyset,\ K^{(1)} = \{A_1\},\ K^{(2)} = \{A_1, A_2\},\ m(1) = 2,\ K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},$$
$$m(j) = \min\left\{i \in \left[m(j-1)+1, \frac{q^{N+1}-1}{q-1}\right] \,\Big|\, \nexists\, P, Q \in K^{(j)} :\ A_i, P, Q \text{ are collinear}\right\},$$

i.e. $m(j)$ is the minimum subscript $i$ such that the corresponding point $A_i$ does not lie on a bisecant of $K^{(j)}$. The process ends when a complete cap is obtained, that is no other points can be added.

We decided to choose a particular order on the points of $\mathrm{PG}(N, q)$. For seek of simplicity, we considered only $q$ prime. Let the elements of the field $\mathbb{F}_q = \{0, 1, \ldots, q-1\}$ be treated as integers modulo $q$. Let the points $A_i$ of $\mathrm{PG}(N, q)$ be represented in homogenous coordinates so that

$$A_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_N^{(i)}),\ x_j^{(i)} \in \mathbb{F}_q,$$

where the leftmost non-zero element is 1. The points of $\mathrm{PG}(N, q)$ are sorted according to the lexicographic order on the $(N+1)$-tuples of their coordinates. This order is called a *lexicographical order of points*. We call *lexicap* a cap obtained by the algorithm FOP with the lexicographical order of points. We denote by $t_2^L(N, q)$ the size of a complete lexicap in $\mathrm{PG}(N, q)$. It is important that for such a lexicographical order for prime $q$, *the size* $t_2^L(N, q)$ *of a complete lexicap and its set of points depend on* $N$ *and* $q$ *only.*

From a geometrical point of view the lexicographical order of points is a random order. Clearly, different orders on the points of $\mathrm{PG}(N,q)$ can determine different size of the complete cap obtained by the algorithm. Due to our experiences in similar types of search (see [1,2,5,6,8,9]) we can conjecture that the choice of the order determines only a small perturbation on the size of the complete caps obtained. For instance, in [8, Fig. 11], sizes of complete arcs in $\mathrm{PG}(2,q)$ obtained by the algorithm FOP with the lexicographical and the so-called Singer orders of points are compared. The percentage difference between the sizes is approximately in the interval [-4%, +4%] for $q \geq 1000$.

Connections of the algorithm FOP with algorithms of Coding Theory are noted in [2, Remark 3.1] and [8, Remark 2.1].

## 2.2 Randomized greedy algorithms

A different approach can be used to obtain small complete caps in $\mathrm{PG}(N,q)$. In general, small complete caps in $\mathrm{PG}(N,q)$ obtained using randomized greedy algorithms have size smaller than those obtained with programs of type FOP as described in the previous subsection; see [1–6, 8, 9, 18, 21].

The main difference between the two types of algorithm is that at every step a randomized greedy algorithm maximizes an objective function $f$ and only some steps are executed in a random manner. The number of these steps, their ordinal numbers, and some other parameters of the algorithm have been taken intuitively. Also, if the same maximum of $f$ can be obtained in distinct ways, one way is chosen randomly.

We start constructing a complete cap by using a starting point set $S_0$. In the $i$-th step one point is added to the set $S_{i-1}$ and we obtain a point set $S_i$. As the value of the objective function $f$ we consider the number of covered points in $\mathrm{PG}(N,q)$, that is, points that lie on bisecants of $S_i$.

On every random $i$-th step we take $d_{q,i}$ randomly chosen points of $\mathrm{PG}(N,q)$ not covered by $S_{i-1}$ and compute the objective function $f$ adding each of these $d_{q,i}$ points to $S_{i-1}$. The point providing the maximum of $f$ is included into $S_i$. On every non-random $j$-th step we consider all points not covered by $S_{j-1}$ and add to $S_{j-1}$ the point providing the maximum of $f$.

As $S_0$ we can use a subset of points of an arc obtained in previous stages of the search.

A generator of random numbers is used for random choices. To obtain caps with distinct sizes, starting conditions of the generator are changed for the same set $S_0$. In this way the algorithm works in a convenient limited region of the search space to obtain examples improving the size of the cap from which the fixed points have been taken.

In order to obtain arcs with new sizes, sufficiently many attempts should be made with randomized greedy algorithms. "Predicted" sizes could be useful for understanding if a good result has been obtained. If the result is not close to the predicted size, the attempts are continued.

We obtain small complete caps in PG$(N,q)$ in two stages.

At the 1-st stage, we take the frame as $S_0$ and create a starting complete cap $K_0$ using in the beginning of the process $\delta_q$ random steps with distinct $d_{q,i}$. All the subsequent steps are non-random.

At the 2-nd stage we execute $n_q$ attempts to get a complete cap. For every attempt, the starting conditions of the random generator are different from the previous ones, whereas the set $S_0$ is the same. Two or three among the first five steps of every attempt are random, the rest of them are non-random.

The values $d_{q,i}$, $\delta_q$, and $n_q$ are given intuitively depending on $q$ and (for $d_{q,i}$) on $|S_{i-1}|$ and on the stage of the process. Of course, CPU performance affects the algorithm parameters choice.

Cap sizes obtained by the randomized greedy algorithms depend on many factors, but in general the results are better than the ones obtained by the algorithm FOP. Unfortunately, this approach requires a huge amount of execution time and therefore this type of search has been executed only for a relatively small region of values of $q$.

# 3 General types of bounds for the value $t_2(N,q)$

Let $t_2(N,q)$ be the size of the smallest complete cap in PG$(N,q)$. In this section we propose different types of bounds for these values, generalizing the approach proposed for estimates on $t_2(2,q)$ in [2,6,8]. Also, let $t_2^G(N,q)$ denote the smallest size of a complete cap in PG$(N,q)$ obtained using greedy algorithms. Finally, remind that $t_2^L(N,q)$ is the size of the complete lexicap in PG$(N,q)$ obtained by the algorithm FOP with the lexicographical order of points.

Let $\beta_N(q)$, $\beta_N^G(q)$, and $\beta_N^L(q)$ be some functions of $q$ defined as follows:

$$\beta_N(q)=\frac{t_2(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}},\quad \beta_N^G(q)=\frac{t_2^G(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}},\quad \beta_N^L(q)=\frac{t_2^L(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}}. \tag{3.1}$$

From (3.1) we obtain

$$\begin{aligned}&t_2(N,q)=\beta_N(q)q^{\frac{N-1}{2}}\sqrt{\ln q},\quad t_2^G(N,q)=\beta_N^G(q)q^{\frac{N-1}{2}}\sqrt{\ln q},\\&t_2^L(N,q)=\beta_N^L(q)q^{\frac{N-1}{2}}\sqrt{\ln q}.\end{aligned} \tag{3.2}$$

Clearly $t_2(N,q)\le\min\{t_2^G(N,q),t_2^L(N,q)\}$ and in general, due to the main features of the two algorithms, $t_2^G(N,q)\le t_2^L(N,q)$ always holds. This implies

$$\beta_N(q)\le\beta_N^G(q)\le\beta_N^L(q),\quad \beta_N(q)\le\min\{\beta_N^G(q),\beta_N^L(q)\}. \tag{3.3}$$

We consider **two types of upper bounds** on $t_2(N,q)$.

A. *Upper bounds with the constant multiplier* $\sqrt{N+2}$. For this type, we consider upper bounds on $\beta_N(q)$ equal to a value dependent on $N$ but independent of $q$.

B. *Upper bounds with a decreasing multiplier* $\beta_N(q)$. For this type, we find upper bounds on $\beta_N(q)$ as a decreasing function of $q$ denoted by $\beta_N^{up}(q)$. This function looks like

$$\beta_N^{up}(q) = a + \frac{b}{\ln(cq)},$$

where $a$ is a value dependent on $N$ but independent of $q$, whereas $b, c$ are constants independent of $N$ and $q$.

# 4 The content of tables

Results of our computer searches are collected in tables given in Appendix.

In Table 1, for $q \in L_3$, we collected the sizes $t_2^L(3,q)$ ($t_2^L$ for short) of complete lexicaps in $\mathrm{PG}(3,q)$ obtained using the algorithm FOP with the lexicographical order of points, see Section 2.1.

In Table 2, for $q \in G_3$, the sizes $t_2^G(3,q)$ ($t_2^G$ for short) of complete caps in $\mathrm{PG}(3,q)$ obtained using randomized greedy algorithms, see Section 2.2, are given.

Note that for $q \in \{61, 67, 71, 73, 79, 83, 97\}$ sizes $t_2^G(3,q)$ in Table 2 improve the ones from [19, Table 7]. Also, the values of $t_2^G(3,q)$ given in Table 2 are smaller than the sizes of complete caps in $\mathrm{AG}(3,q)$ obtained in [33, Section 3]. The improvements are written in Table 2 in bold font.

In Table 3 we collected the sizes $t_2^L(4,q)$ ($t_2^L$ for short) of complete lexicaps in $\mathrm{PG}(4,q)$, $q \in L_4$, obtained by the algorithm FOP with lexicographical order of points, see Section 2.1.

In Table 4 we give the sizes $t_2^G(4,q)$ ($t_2^G$ for short) of complete caps in $\mathrm{PG}(4,q)$, $q \in G_4$, obtained by the randomized greedy algorithms, see Section 2.2.

Note that the size $t_2^G(4,17)$ in Table 4 improves the one from [19, Table 8]. Also, the values of $t_2^G(4,q)$ given in Table 4 are smaller than the sizes of complete caps in $\mathrm{AG}(4,q)$ obtained in [22, Theorem 1.1]. The improvements are written in Table 4 in bold font.

# 5 Small complete caps in $\mathrm{PG}(3,q)$

The values $t_2^L(3,q)$ written in Table 1 are shown in Figure 1 by the 2-nd solid black curve. In turn, the values $t_2^L(4,q)$, given in Table 3, are shown by the 2-nd solid black curve in Figure 2.

The values $t_2^G(3,q)$ from Table 2 are shown in Figure 1 by the bottom dashed blue curve. In the scale of Figure 1 the curves $t_2^L(3,q)$ and $t_2^G(3,q)$ are very close to each other.

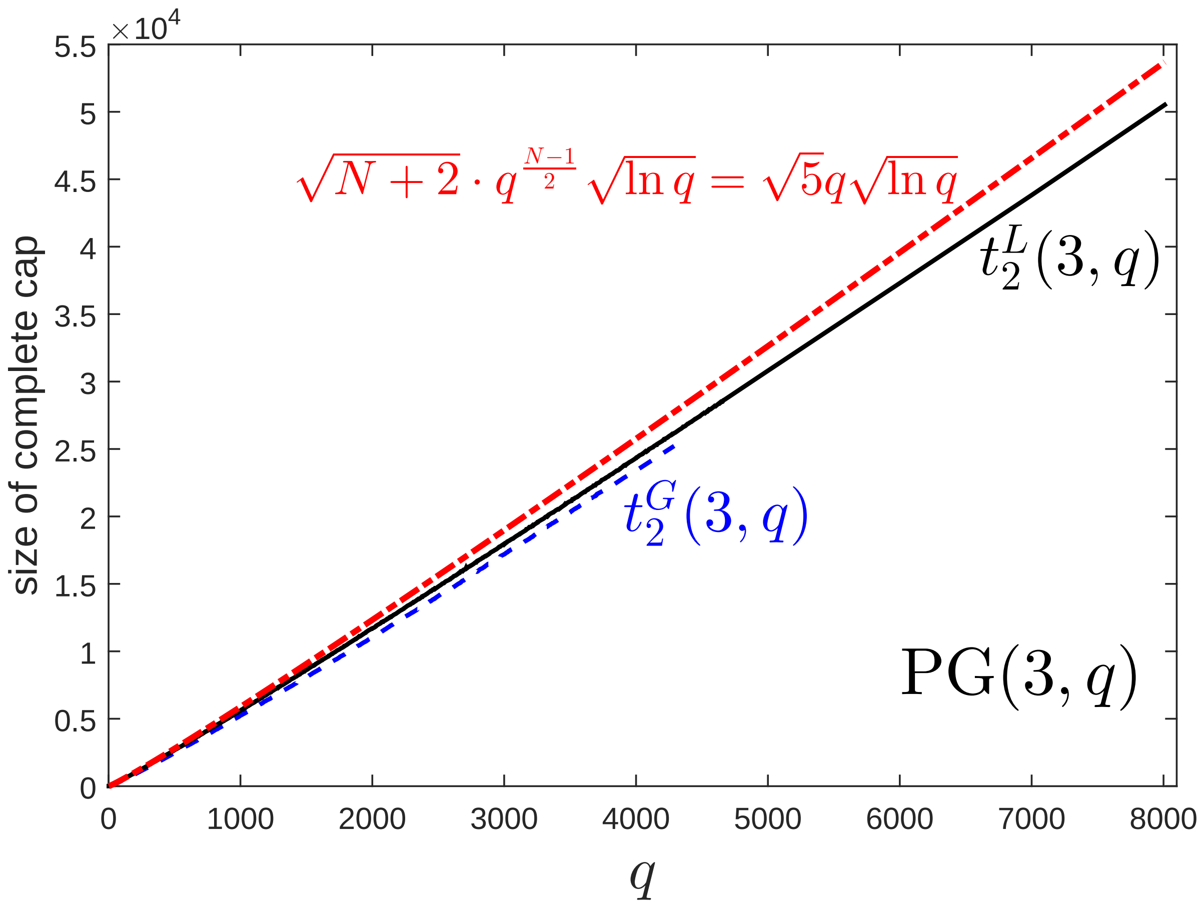


Figure 1: $\mathrm{PG}(3, q)$. **Upper bound** $t_2(3, q) < \sqrt{3+2} \cdot q^{\frac{3-1}{2}} \sqrt{\ln q} = \sqrt{5} q \sqrt{\ln q}$ (*top dashed-dotted red curve*) **vs sizes** $t_2^L(3, q)$ of complete lexicaps, $q \in L_3$ (*the 2-nd solid black curve*) and **sizes** $t_2^G(3, q)$ of complete caps obtained by greedy algorithms, $q \in G_3$ (*bottom dashed blue curve*).

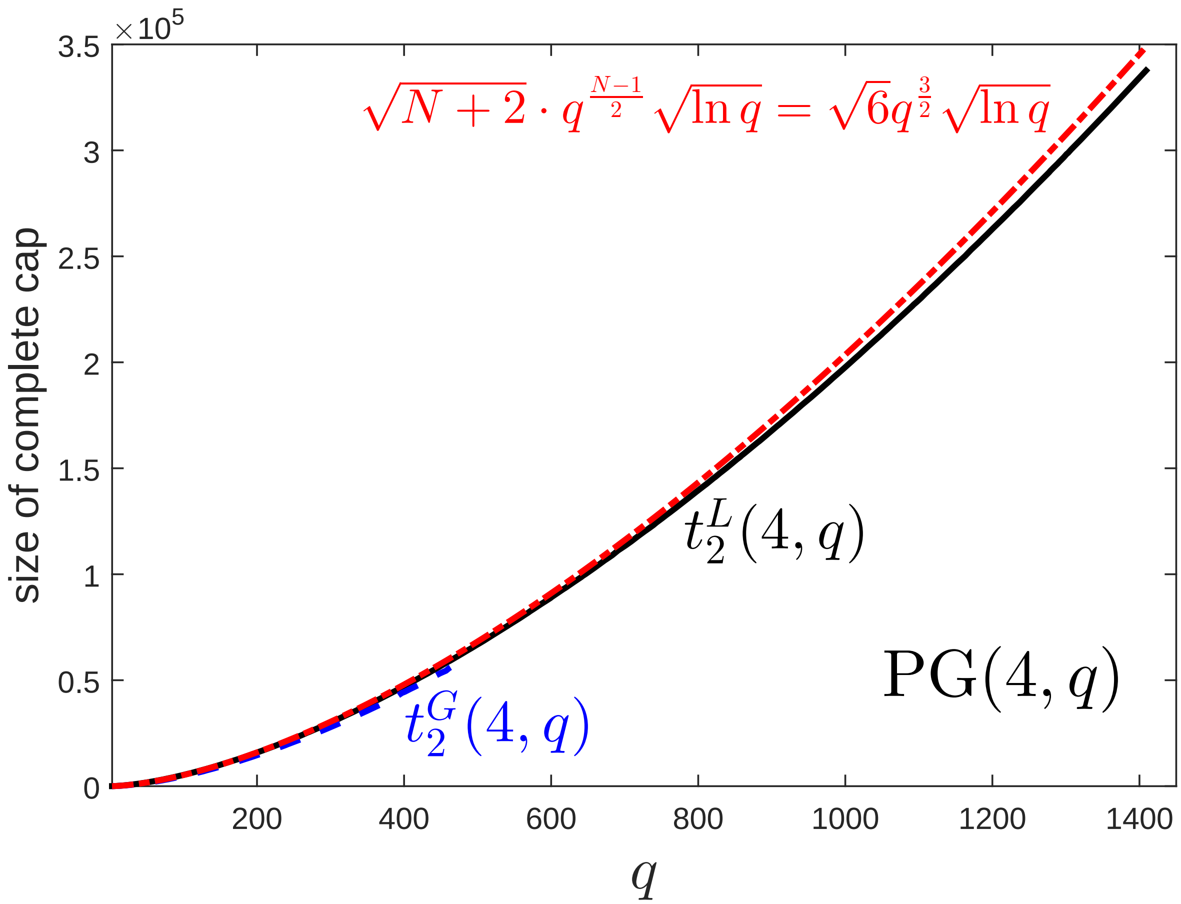


Figure 2: $\mathrm{PG}(4,q)$. **Upper bound** $t_2(4,q) < \sqrt{4+2} \cdot q^{\frac{4-1}{2}} \sqrt{\ln q} = \sqrt{6} q^{\frac{3}{2}} \sqrt{\ln q}$ (*top dashed-dotted red curve*) **vs sizes** $t_2^L(4,q)$ of complete lexicaps, $q \in L_4$ (*the 2-nd solid black curve*) and **sizes** $t_2^G(4,q)$ of complete caps obtained by greedy algorithms, $q \in G_4$ (*bottom dashed blue curve*).

Note that for all $q \in G_3$, we have $t_2^G(3,q) < t_2^L(3,q)$. So, as already pointed out above, the use of greedy algorithms provides better results, that is the size of the complete caps obtained is smaller. However, randomized greedy algorithms require in general more execution time than algorithm FOP, since they require more investigations at each step, trying to maximize a particular objective function as illustrated in Section 2.2. For this reason we have been able to obtain the data for a smaller region of values of $q$ than by the FOP algorithm.

Figure 3a shows the percentage difference between $t_2^L(3,q)$ and $t_2^G(3,q)$.

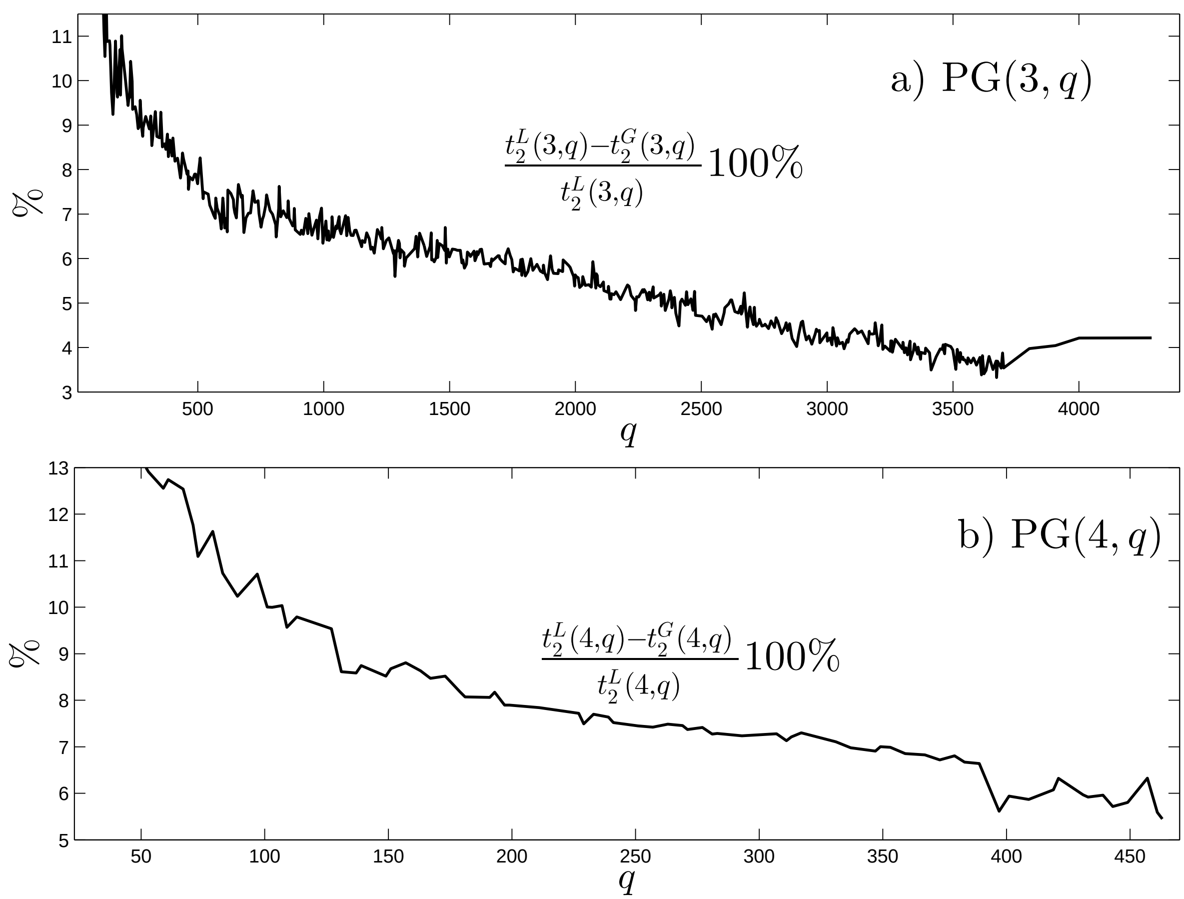


Figure 3: **Percentage difference between $t_2^L(N,q)$ and $t_2^G(N,q)$.** a) $N=3$, PG$(3,q)$; b) $N=4$, PG$(4,q)$

**Observation 5.1.** *From Tables* 1 *and* 2 *and Figure* 3a*, one sees that the percentage difference between $t_2^L(3,q)$ and $t_2^G(3,q)$ given by*

$$\frac{t_2^L(3,q)-t_2^G(3,q)}{t_2^L(3,q)}100\%$$

*is relatively small and it tends to decrease when $q$ grows. In particular, in the region $q \in [503 \dots 3701]$ this difference decreases approximately from 7% to 4%.*

Figure 4a shows the values $\beta_3^L(q)$ and $\beta_3^G(q)$ obtained by (3.1) from the sizes collected in Tables 1 and 2. Also, in this figure, upper bounds $\beta_3^{up}(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)} = \sqrt{3+1} + \frac{1.3}{\ln(2q)} = 2 + \frac{1.3}{\ln(2q)}$ and the line-bound $y = \sqrt{N+2} = \sqrt{5}$ are presented in red color.

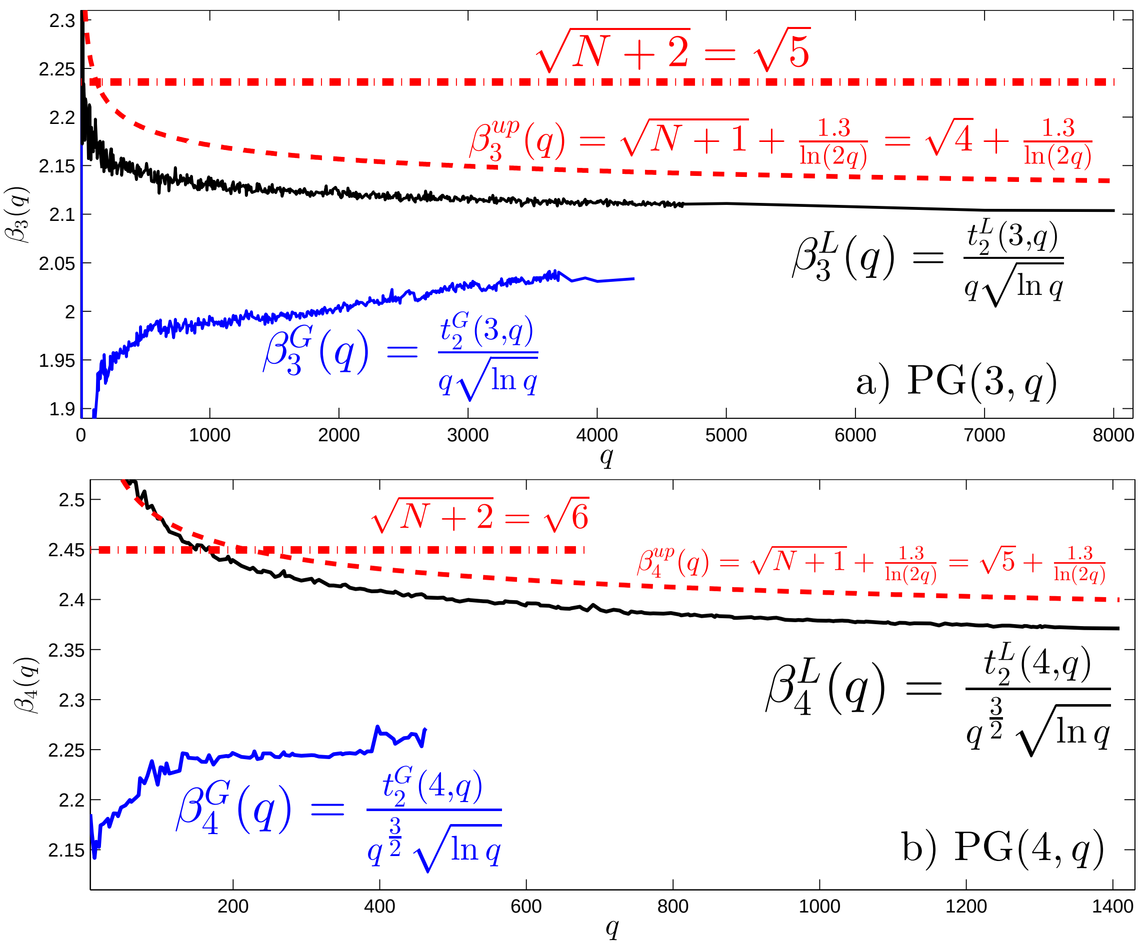


Figure 4: **Upper bounds** $\beta_N(q) = \frac{t_2(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}} < \sqrt{N+2}$ (*dashed-dotted red line* $y = \sqrt{N+2}$) and $\beta_N(q) < \beta_N^{up}(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)}$ (*top dashed red curve*) **vs values** of $\beta_N^L(q)$, $q \in L_N$ (*the 2-nd solid black curve*) and $\beta_N^G(q)$, $q \in G_N$ (*bottom solid blue curve*). a) $N = 3$, PG$(3, q)$; b) $N = 4$, PG$(4, q)$

By Tables 1, 2 and Figure 4a, it holds that, see (3.3),

$$\beta_3(q) \le \min\{\beta_3^G(q), \beta_3^L(q)\} < \sqrt{N+2} = \sqrt{5}, \quad q \in L_3; \tag{5.1}$$

$$\beta_3(q) \le \min\{\beta_3^G(q), \beta_3^L(q)\} < \beta_3^{up}(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)} = \sqrt{3+1} + \frac{1.3}{\ln(2q)} = 2 + \frac{1.3}{\ln(2q)}, \quad q \in L_3. \tag{5.2}$$

This implies upper bounds for $\mathrm{PG}(3,q)$ in Theorem 1.1.

The upper bound (1.5) for $N = 3$, based on (5.1), is shown by the dashed-dotted red curve in Figure 1. This bound is presented also by the dashed-dotted red line $y = \sqrt{N+2} = \sqrt{5}$ in Figure 4a. The bound (1.6) for $N = 3$, based on (5.2), is given by the dashed red curve in Figure 4a.

Figure 5a shows the percentage differences between $\sqrt{5}$ and $\beta_3^L(q)$ and $\sqrt{5}q\sqrt{\ln q}$ and $t_2^L(3,q)$. (These percentage differences are equal to each other.)

**Observation 5.2.** *From Table* 1 *and Figure* 4a *one sees that the curve* $\beta_3^L(q)$ *has a decreasing trend. Therefore the difference* $\sqrt{5} - \beta_3^L(q)$*, and the corresponding percent differences*

$$\frac{\sqrt{5} - \beta_3^L(q)}{\sqrt{5}} 100\% = \frac{\sqrt{5}q\sqrt{\ln q} - t_2^L(3,q)}{\sqrt{5}q\sqrt{\ln q}} 100\%,$$

*tend to increase when* $q$ *grows, see Figure* 5a*. This raises confidence in the correctness of the bound* (1.5) *for* $N = 3$.

Concerning the execution time, the search for the small complete cap in $\mathrm{PG}(3, 7001)$ lasted 2 months with a processor AMD Opteron(TM) Processor 6212, 2.6 Ghz, and used 85GB of memory.

# 6 Small complete caps in $\mathrm{PG}(4,q)$

Figures 2 and 4b show the values $t_2^L(4,q)$, $t_2^G(4,q)$ collected in Tables 3, 4 and the corresponding values $\beta_4^L(q)$, $\beta_4^G(q)$ obtained by (3.1). Also, in these figures, upper bounds are presented. Note that in the scale of Figure 2 the curves $\sqrt{6}q^{\frac{3}{2}}\sqrt{\ln q}$, $t_2^L(4,q)$, and $t_2^G(4,q)$ are very closed to each other.

Figure 3b shows the percentage difference between $t_2^L(4,q)$ and $t_2^G(4,q)$.

**Observation 6.1.** *Even if for all* $q \in G_4$ *the inequality* $t_2^G(4,q) < t_2^L(4,q)$ *holds, see Figure* 2*, the difference in percentage between these two values given by*

$$\frac{t_2^L(4,q) - t_2^G(4,q)}{t_2^L(4,q)} 100\%$$

*is relatively small and it tends to decrease when* $q$ *grows, see Figure* 3b*. In particular, in the region* $q \in [101 \dots 443]$ *this difference decreases approximately from 10% to 5%.*

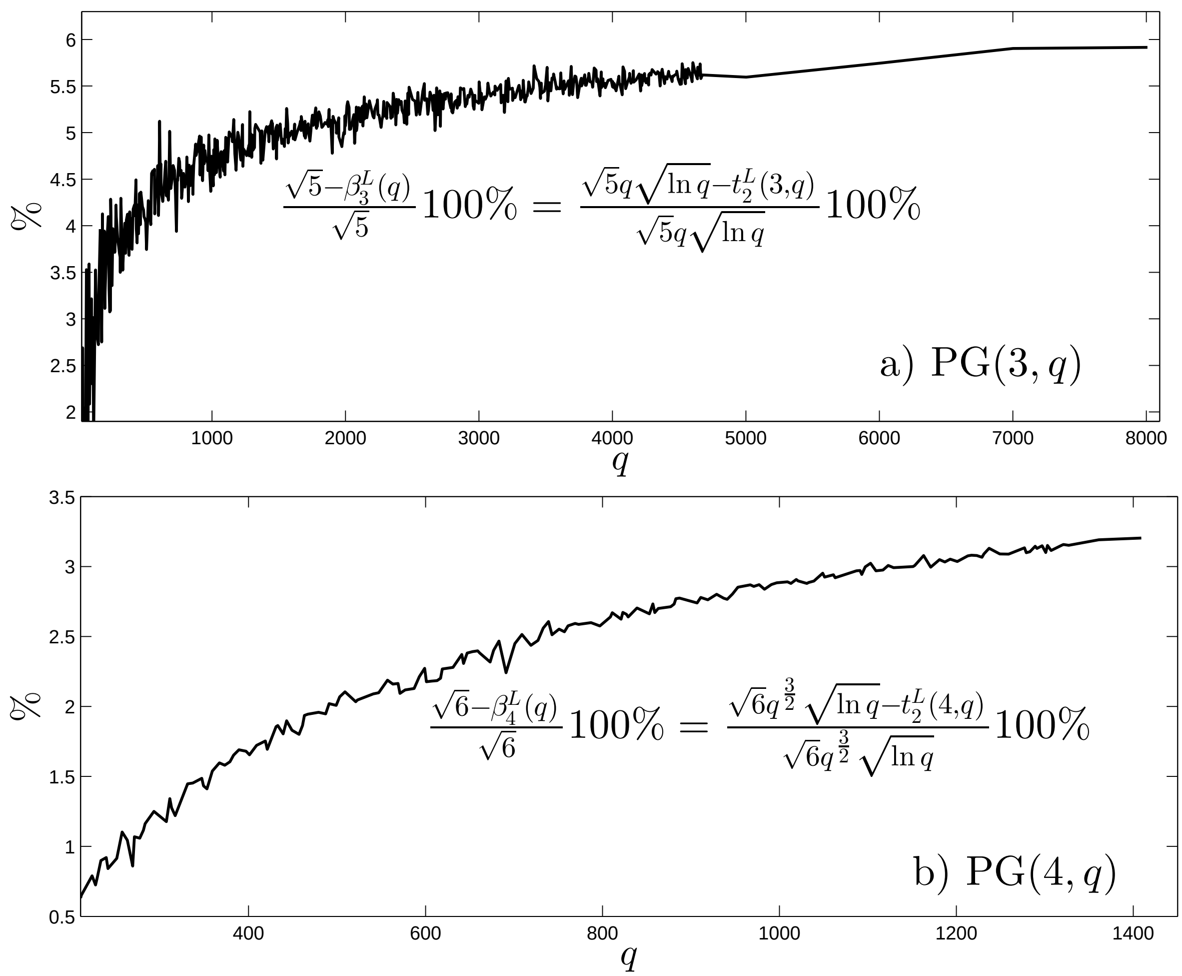


Figure 5: **Percentage difference between $\sqrt{N+2}\cdot q^{\frac{N-1}{2}}\sqrt{\ln q}$ and $t_2^L(N,q)$.** a) $N=3$, PG$(3,q)$; b) $N=4$, PG$(4,q)$

Figure 4b shows the values $\beta_4^L(q)$ and $\beta_4^G(q)$ obtained by (3.1) from the sizes collected in Tables 3 and 4. Also, in this figure, upper bounds $\beta_4^{up}(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)} = \sqrt{4+1} + \frac{1.3}{\ln(2q)} = \sqrt{5} + \frac{1.3}{\ln(2q)}$ and the line-bound $y = \sqrt{N+2} = \sqrt{6}$ are presented in red color.

From Tables 3, 4 and Figure 4b, we have, see (3.3),

$$\beta_4(q) \leq \min\{\beta_4^G(q), \beta_4^L(q)\} < \sqrt{N+2} = \sqrt{6}, \quad q \in L_4; \tag{6.1}$$

$$\beta_4(q) \leq \min\{\beta_4^G(q), \beta_4^L(q)\} < \beta_4^{up}(q) = \sqrt{N+1} + \frac{1.3}{\ln(2q)} = \sqrt{4+1} + \frac{1.3}{\ln(2q)} = \sqrt{5} + \frac{1.1}{\ln q}, \quad q \in L_4. \tag{6.2}$$

This implies bounds for $\mathrm{PG}(4,q)$ in Theorem 1.1.

The upper bounds (1.5) for $N = 4$, based on (6.1), are shown by the dashed-dotted red curves in Figure 2. This bound is presented also by the dashed-dotted red line $y = \sqrt{N+2} = \sqrt{6}$ in Figure 4b. The bound (1.6) for $N = 4$, based on (6.2), is given by the dashed red curve in Figure 4b.

Figure 5b shows the percentage differences between $\sqrt{6}$ and $\beta_4^L(q)$ and $\sqrt{6}q\sqrt{\ln q}$ and $t_2^L(4,q)$. (These percentage differences are equal to each other.)

**Observation 6.2.** *From Table* 3 *and Figure* 4b *one sees that the curve* $\beta_4^L(q)$ *have a decreasing trend. Therefore the difference* $\sqrt{6} - \beta_4^L(q)$ *and the corresponding percent differences*

$$\frac{\sqrt{6} - \beta_4^L(q)}{\sqrt{6}} 100\% = \frac{\sqrt{6}q^{\frac{3}{2}}\sqrt{\ln q} - t_2^L(3,q)}{\sqrt{6}q^{\frac{3}{2}}\sqrt{\ln q}} 100\%$$

*tend to increase when* $q$ *grows, see Figure* 5b. *This raises confidence in the correctness of the bound* (1.5) *for* $N = 4$.

# 7 Conclusion

In this paper we presented and analyze computational results concerning small complete caps in $\mathrm{PG}(3,q)$, $q \leq 4673$, $q$ prime, and $q = 5003, 6007, 7001$, $8009$, and $\mathrm{PG}(4,q)$, $q \leq 1361$, $q$ prime, and $q = 1409$.

The results have been obtained using randomized greedy algorithms and the algorithm with fixed order of points (FOP). Tables 1–4 and Figures 1–5 show that the sizes $t_2^G(N,q)$ of complete caps obtained by greedy algorithms are smaller than sizes $t_2^L(N,q)$ of complete caps formed by the algorithm FOP with the lexicographical order of points. This allows, in particular, to increase the regions of $q$ values where the proposed upper bounds hold, see Figures 1, 4 and relations (3.3), (5.1), (5.2), (6.1), (6.2).

In the other side, the percent difference between $t_2^L(N,q)$ and $t_2^G(N,q)$ is relatively small and it decreases when $q$ grows, see Observations 5.1 and 6.1. Execution time of greedy algorithms is essentially greater than for the algorithm FOP. The sizes $t_2^G(N,q)$ depend not only on $q$ and $N$ but also on parameters $d_{q,i}$, $\delta_q$, $n_q$ of greedy algorithms, see Section 2.2. These parameters are not always chosen optimal due to restrictions of the computer time.

At the same time, the sizes $t_2^L(N,q)$ depend on $q$ and $N$ only. Therefore the behavior of the curves $\beta_N^L(q)$ obtained from $t_2^L(N,q)$ allows to understand the order of value and effectively estimate the smallest sizes $t_2(N,q)$ of complete caps for $N=3,4$ in the considered regions of $q$, see Figures 4, 5.

Moreover, the decreasing trend of the curves $\beta_N^L(q)$, see Figures 4, 5, allow us to conjecture that the upper bounds on $t_2(3,q)$ and $t_2(4,q)$ we obtained, especially the bounds (1.5) with constant multiplier $\sqrt{N+2}$, hold for any $q$ prime power.

As far as this is known to the authors, new complete caps obtained in this work are the smallest known in literature.

# 8 Appendix. Tables of sizes of the small complete caps in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$

In Table 1, for $q \in L_3$, we collected the sizes $t_2^L(3,q)$ ($t_2^L$ for short) of complete lexicaps in $\mathrm{PG}(3,q)$ obtained using the algorithm FOP with the lexicographical order of points, see Section 2.1.

In Table 2, for $q \in G_3$, the sizes $t_2^G(3,q)$ ($t_2^G$ for short) of complete caps in $\mathrm{PG}(3,q)$ obtained using randomized greedy algorithms, see Section 2.2, are given.

In Table 3 we collected the sizes $t_2^L(4,q)$ ($t_2^L$ for short) of complete lexicaps in $\mathrm{PG}(4,q)$, $q \in L_4$, obtained by the algorithm FOP with lexicographical order of points, see Section 2.1.

In Table 4 we give the sizes $t_2^G(4,q)$ ($t_2^G$ for short) of complete caps in $\mathrm{PG}(4,q)$, $q \in G_4$, obtained by the randomized greedy algorithms, see Section 2.2.

**Table 1.** Sizes $t_2^L(3,q) = t_2^L$ of complete lexicaps in PG$(3,q)$, $q \in L_3$

| $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 8 | 3 | 8 | 5 | 16 | 7 | 23 | 11 | 37 | 13 | 49 | 17 | 69 |
| 19 | 71 | 23 | 91 | 29 | 118 | 31 | 125 | 37 | 156 | 41 | 175 | 43 | 183 |
| 47 | 202 | 53 | 232 | 59 | 257 | 61 | 273 | 67 | 304 | 71 | 324 | 73 | 328 |
| 79 | 356 | 83 | 382 | 89 | 410 | 97 | 449 | 101 | 474 | 103 | 481 | 107 | 502 |
| 109 | 512 | 113 | 540 | 127 | 603 | 131 | 626 | 137 | 660 | 139 | 671 | 149 | 725 |
| 151 | 732 | 157 | 761 | 163 | 790 | 167 | 814 | 173 | 854 | 179 | 874 | 181 | 893 |
| 191 | 944 | 193 | 951 | 197 | 981 | 199 | 990 | 211 | 1050 | 223 | 1112 | 227 | 1138 |
| 229 | 1154 | 233 | 1179 | 239 | 1212 | 241 | 1208 | 251 | 1275 | 257 | 1300 | 263 | 1334 |
| 269 | 1368 | 271 | 1381 | 277 | 1412 | 281 | 1429 | 283 | 1442 | 293 | 1501 | 307 | 1582 |
| 311 | 1605 | 313 | 1619 | 317 | 1628 | 331 | 1720 | 337 | 1750 | 347 | 1801 | 349 | 1813 |
| 353 | 1841 | 359 | 1868 | 367 | 1912 | 373 | 1952 | 379 | 1989 | 383 | 2002 | 389 | 2043 |
| 397 | 2083 | 401 | 2113 | 409 | 2149 | 419 | 2205 | 421 | 2219 | 431 | 2267 | 433 | 2291 |
| 439 | 2329 | 443 | 2342 | 449 | 2384 | 457 | 2419 | 461 | 2447 | 463 | 2462 | 467 | 2476 |
| 479 | 2549 | 487 | 2596 | 491 | 2621 | 499 | 2668 | 503 | 2692 | 509 | 2735 | 521 | 2791 |
| 523 | 2801 | 541 | 2913 | 547 | 2931 | 557 | 2988 | 563 | 3024 | 569 | 3057 | 571 | 3084 |
| 577 | 3112 | 587 | 3170 | 593 | 3195 | 599 | 3248 | 601 | 3254 | 607 | 3260 | 613 | 3317 |
| 617 | 3334 | 619 | 3356 | 631 | 3430 | 641 | 3482 | 643 | 3493 | 647 | 3512 | 653 | 3543 |
| 659 | 3592 | 661 | 3601 | 673 | 3676 | 677 | 3693 | 683 | 3706 | 691 | 3777 | 701 | 3832 |
| 709 | 3873 | 719 | 3934 | 727 | 3992 | 733 | 4044 | 739 | 4056 | 743 | 4080 | 751 | 4117 |
| 757 | 4154 | 761 | 4184 | 769 | 4229 | 773 | 4266 | 787 | 4337 | 797 | 4403 | 809 | 4468 |
| 811 | 4471 | 821 | 4544 | 823 | 4565 | 827 | 4578 | 829 | 4582 | 839 | 4652 | 853 | 4725 |
| 857 | 4764 | 859 | 4769 | 863 | 4784 | 877 | 4856 | 881 | 4886 | 883 | 4897 | 887 | 4920 |
| 907 | 5030 | 911 | 5076 | 919 | 5102 | 929 | 5187 | 937 | 5214 | 941 | 5249 | 947 | 5277 |
| 953 | 5318 | 967 | 5404 | 971 | 5433 | 977 | 5447 | 983 | 5507 | 991 | 5566 | 997 | 5580 |
| 1009 | 5671 | 1013 | 5672 | 1019 | 5724 | 1021 | 5713 | 1031 | 5779 | 1033 | 5807 | 1039 | 5822 |
| 1049 | 5900 | 1051 | 5908 | 1061 | 5979 | 1063 | 5976 | 1069 | 6010 | 1087 | 6132 | 1091 | 6142 |
| 1093 | 6158 | 1097 | 6195 | 1103 | 6205 | 1109 | 6242 | 1117 | 6307 | 1123 | 6332 | 1129 | 6358 |
| 1151 | 6495 | 1153 | 6510 | 1163 | 6554 | 1171 | 6628 | 1181 | 6701 | 1187 | 6716 | 1193 | 6745 |
| 1201 | 6797 | 1213 | 6886 | 1217 | 6912 | 1223 | 6935 | 1229 | 6957 | 1231 | 7002 | 1237 | 7025 |
| 1249 | 7100 | 1259 | 7164 | 1277 | 7265 | 1279 | 7286 | 1283 | 7274 | 1289 | 7343 | 1291 | 7341 |
| 1297 | 7397 | 1301 | 7411 | 1303 | 7415 | 1307 | 7452 | 1319 | 7523 | 1321 | 7519 | 1327 | 7552 |

**Table 1. Continue 1.** Sizes $t_2^L(3,q) = t_2^L$ of complete lexicaps in PG$(3,q)$, $q \in L_3$

| $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1361 | 7766 | 1367 | 7830 | 1373 | 7838 | 1381 | 7913 | 1399 | 7997 | 1409 | 8054 | 1423 | 8147 |
| 1427 | 8178 | 1429 | 8176 | 1433 | 8207 | 1439 | 8236 | 1447 | 8281 | 1451 | 8318 | 1453 | 8320 |
| 1459 | 8381 | 1471 | 8441 | 1481 | 8495 | 1483 | 8538 | 1487 | 8530 | 1489 | 8557 | 1493 | 8579 |
| 1499 | 8613 | 1511 | 8676 | 1523 | 8769 | 1531 | 8814 | 1543 | 8895 | 1549 | 8914 | 1553 | 8947 |
| 1559 | 8955 | 1567 | 9020 | 1571 | 9064 | 1579 | 9103 | 1583 | 9131 | 1597 | 9212 | 1601 | 9240 |
| 1607 | 9276 | 1609 | 9294 | 1613 | 9315 | 1619 | 9340 | 1621 | 9369 | 1627 | 9403 | 1637 | 9460 |
| 1657 | 9575 | 1663 | 9608 | 1667 | 9643 | 1669 | 9640 | 1693 | 9810 | 1697 | 9818 | 1699 | 9849 |
| 1709 | 9902 | 1721 | 9969 | 1723 | 9990 | 1733 | 10063 | 1741 | 10101 | 1747 | 10127 | 1753 | 10157 |
| 1759 | 10206 | 1777 | 10297 | 1783 | 10358 | 1787 | 10368 | 1789 | 10393 | 1801 | 10450 | 1811 | 10517 |
| 1823 | 10605 | 1831 | 10653 | 1847 | 10748 | 1861 | 10837 | 1867 | 10869 | 1871 | 10902 | 1873 | 10906 |
| 1877 | 10918 | 1879 | 10943 | 1889 | 11006 | 1901 | 11122 | 1907 | 11113 | 1913 | 11154 | 1931 | 11272 |
| 1933 | 11296 | 1949 | 11376 | 1951 | 11413 | 1973 | 11563 | 1979 | 11591 | 1987 | 11632 | 1993 | 11671 |
| 1997 | 11672 | 1999 | 11698 | 2003 | 11718 | 2011 | 11746 | 2017 | 11778 | 2027 | 11848 | 2029 | 11890 |
| 2039 | 11952 | 2053 | 12027 | 2063 | 12082 | 2069 | 12159 | 2081 | 12184 | 2083 | 12229 | 2087 | 12247 |
| 2089 | 12250 | 2099 | 12297 | 2111 | 12392 | 2113 | 12387 | 2129 | 12482 | 2131 | 12523 | 2137 | 12536 |
| 2141 | 12544 | 2143 | 12588 | 2153 | 12629 | 2161 | 12699 | 2179 | 12787 | 2203 | 12957 | 2207 | 12985 |
| 2213 | 13025 | 2221 | 13062 | 2237 | 13153 | 2239 | 13161 | 2243 | 13208 | 2251 | 13259 | 2267 | 13378 |
| 2269 | 13388 | 2273 | 13405 | 2281 | 13448 | 2287 | 13464 | 2293 | 13524 | 2297 | 13539 | 2309 | 13628 |
| 2311 | 13616 | 2333 | 13760 | 2339 | 13811 | 2341 | 13829 | 2347 | 13849 | 2351 | 13890 | 2357 | 13890 |
| 2371 | 13997 | 2377 | 14066 | 2381 | 14073 | 2383 | 14063 | 2389 | 14123 | 2393 | 14142 | 2399 | 14164 |
| 2411 | 14247 | 2417 | 14280 | 2423 | 14332 | 2437 | 14427 | 2441 | 14466 | 2447 | 14468 | 2459 | 14582 |
| 2467 | 14581 | 2473 | 14654 | 2477 | 14634 | 2503 | 14818 | 2521 | 14934 | 2531 | 15021 | 2539 | 15045 |
| 2543 | 15058 | 2549 | 15121 | 2551 | 15140 | 2557 | 15184 | 2579 | 15283 | 2591 | 15345 | 2593 | 15387 |
| 2609 | 15485 | 2617 | 15563 | 2621 | 15582 | 2633 | 15625 | 2647 | 15723 | 2657 | 15809 | 2659 | 15805 |
| 2663 | 15820 | 2671 | 15934 | 2677 | 15945 | 2683 | 15933 | 2687 | 15985 | 2689 | 15992 | 2693 | 16045 |
| 2699 | 16062 | 2707 | 16090 | 2711 | 16180 | 2713 | 16139 | 2719 | 16172 | 2729 | 16255 | 2731 | 16275 |
| 2741 | 16310 | 2749 | 16371 | 2753 | 16385 | 2767 | 16473 | 2777 | 16552 | 2789 | 16641 | 2791 | 16627 |
| 2797 | 16699 | 2801 | 16713 | 2803 | 16739 | 2819 | 16820 | 2833 | 16905 | 2837 | 16946 | 2843 | 16962 |
| 2851 | 17029 | 2857 | 17062 | 2861 | 17059 | 2879 | 17176 | 2887 | 17225 | 2897 | 17324 | 2903 | 17357 |
| 2909 | 17395 | 2917 | 17429 | 2927 | 17496 | 2939 | 17556 | 2953 | 17657 | 2957 | 17704 | 2963 | 17749 |

**Table 1. Continue 2.** Sizes $t_2^L(3,q) = t_2^L$ of complete lexicaps in PG$(3,q)$, $q \in L_3$

| $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2969 | 17759 | 2971 | 17782 | 2999 | 17966 | 3001 | 17935 | 3011 | 18024 | 3019 | 18073 | 3023 | 18132 |
| 3037 | 18200 | 3041 | 18220 | 3049 | 18276 | 3061 | 18344 | 3067 | 18357 | 3079 | 18454 | 3083 | 18480 |
| 3089 | 18524 | 3109 | 18643 | 3119 | 18713 | 3121 | 18707 | 3137 | 18818 | 3163 | 18971 | 3167 | 19036 |
| 3169 | 19034 | 3181 | 19109 | 3187 | 19155 | 3191 | 19210 | 3203 | 19218 | 3209 | 19274 | 3217 | 19353 |
| 3221 | 19327 | 3229 | 19402 | 3251 | 19540 | 3253 | 19580 | 3257 | 19586 | 3259 | 19613 | 3271 | 19689 |
| 3299 | 19841 | 3301 | 19853 | 3307 | 19903 | 3313 | 19962 | 3319 | 19981 | 3323 | 20011 | 3329 | 20059 |
| 3331 | 20033 | 3343 | 20159 | 3347 | 20185 | 3359 | 20246 | 3361 | 20261 | 3371 | 20305 | 3373 | 20305 |
| 3389 | 20432 | 3391 | 20447 | 3407 | 20537 | 3413 | 20523 | 3433 | 20681 | 3449 | 20790 | 3457 | 20855 |
| 3461 | 20907 | 3463 | 20885 | 3467 | 20909 | 3469 | 20946 | 3491 | 21082 | 3499 | 21118 | 3511 | 21153 |
| 3517 | 21272 | 3527 | 21292 | 3529 | 21323 | 3533 | 21345 | 3539 | 21389 | 3541 | 21394 | 3547 | 21409 |
| 3557 | 21476 | 3559 | 21476 | 3571 | 21571 | 3581 | 21654 | 3583 | 21661 | 3593 | 21730 | 3607 | 21820 |
| 3613 | 21819 | 3617 | 21872 | 3623 | 21911 | 3631 | 21987 | 3637 | 21993 | 3643 | 22053 | 3659 | 22143 |
| 3671 | 22211 | 3673 | 22228 | 3677 | 22286 | 3691 | 22303 | 3697 | 22397 | 3701 | 22439 | 3709 | 22463 |
| 3719 | 22507 | 3727 | 22560 | 3733 | 22613 | 3739 | 22691 | 3761 | 22789 | 3767 | 22821 | 3769 | 22854 |
| 3779 | 22909 | 3793 | 22979 | 3797 | 23032 | 3803 | 23071 | 3821 | 23164 | 3823 | 23168 | 3833 | 23255 |
| 3847 | 23346 | 3851 | 23385 | 3853 | 23396 | 3863 | 23411 | 3877 | 23507 | 3881 | 23563 | 3889 | 23607 |
| 3907 | 23769 | 3911 | 23768 | 3917 | 23769 | 3919 | 23824 | 3923 | 23832 | 3929 | 23865 | 3931 | 23897 |
| 3943 | 23967 | 3947 | 24005 | 3967 | 24114 | 3989 | 24253 | 4001 | 24351 | 4003 | 24349 | 4007 | 24391 |
| 4013 | 24419 | 4019 | 24472 | 4021 | 24481 | 4027 | 24530 | 4049 | 24631 | 4051 | 24665 | 4057 | 24705 |
| 4073 | 24838 | 4079 | 24798 | 4091 | 24888 | 4093 | 24919 | 4099 | 24989 | 4111 | 25051 | 4127 | 25136 |
| 4129 | 25139 | 4133 | 25184 | 4139 | 25232 | 4153 | 25311 | 4157 | 25312 | 4159 | 25358 | 4177 | 25468 |
| 4201 | 25606 | 4211 | 25720 | 4217 | 25712 | 4219 | 25756 | 4229 | 25802 | 4231 | 25807 | 4241 | 25863 |
| 4243 | 25896 | 4253 | 25968 | 4259 | 25994 | 4261 | 26014 | 4271 | 26091 | 4273 | 26073 | 4283 | 26133 |
| 4289 | 26178 | 4297 | 26213 | 4327 | 26421 | 4337 | 26482 | 4339 | 26491 | 4349 | 26560 | 4357 | 26651 |
| 4363 | 26656 | 4373 | 26704 | 4391 | 26864 | 4397 | 26882 | 4409 | 26967 | 4421 | 27021 | 4423 | 27043 |
| 4441 | 27173 | 4447 | 27221 | 4451 | 27230 | 4457 | 27270 | 4463 | 27306 | 4481 | 27417 | 4483 | 27422 |
| 4493 | 27512 | 4507 | 27619 | 4513 | 27641 | 4517 | 27683 | 4519 | 27678 | 4523 | 27658 | 4547 | 27877 |
| 4549 | 27845 | 4561 | 27949 | 4567 | 27958 | 4583 | 28080 | 4591 | 28144 | 4597 | 28195 | 4603 | 28173 |
| 4621 | 28322 | 4637 | 28446 | 4639 | 28421 | 4643 | 28454 | 4649 | 28487 | 4651 | 28525 | 4657 | 28527 |
| 4663 | 28614 | 4673 | 28667 | 5003 | 30823 | 6007 | 37344 | 7001 | 43831 | 8009 | 50515 | | |

**Table 2.** Sizes $t_2^G(3,q) = t_2^G$ of complete caps[2] in $\mathrm{PG}(3,q)$ obtained using randomized greedy algorithms, $q \in G_3$

| $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 5 | 3 | 8 | 5 | 12 | 7 | 17 | 11 | 30 | 13 | 36 | 17 | 51 |
| 19 | 58 | 23 | 72 | 29 | 96 | 31 | 104 | 37 | 128 | 41 | 145 | 43 | 153 |
| 47 | 169 | 53 | 195 | 59 | 220 | **61** | **229** | **67** | **255** | **71** | **273** | **73** | **282** |
| **79** | **309** | **83** | **327** | 89 | 355 | **97** | **392** | **101** | **412** | **103** | **422** | **107** | **437** |
| **109** | **447** | **113** | **466** | **127** | **536** | **131** | **560** | **137** | **583** | **139** | **598** | **149** | **646** |
| **151** | **653** | **157** | **687** | **163** | **717** | **167** | **734** | **173** | **761** | **179** | **789** | **181** | **807** |
| **191** | **843** | **193** | **859** | **197** | **873** | **199** | **884** | **211** | **944** | **223** | **1007** | **227** | **1028** |
| **229** | **1043** | **233** | **1056** | **239** | **1091** | **241** | **1095** | **251** | **1155** | **257** | **1180** | **263** | **1215** |
| **269** | **1245** | **271** | **1249** | **277** | **1286** | **281** | **1304** | **283** | **1313** | **293** | **1363** | **307** | **1441** |
| **311** | **1459** | **313** | **1470** | **317** | **1489** | **331** | **1560** | **337** | **1597** | **347** | **1644** | **349** | **1655** |
| 353 | 1670 | 359 | 1709 | 367 | 1748 | 373 | 1787 | 379 | 1814 | 383 | 1836 | 389 | 1866 |
| 397 | 1910 | 401 | 1929 | 409 | 1973 | 419 | 2023 | 421 | 2037 | 431 | 2090 | 433 | 2110 |
| 439 | 2134 | 443 | 2149 | 449 | 2191 | 457 | 2228 | 461 | 2252 | 463 | 2276 | 467 | 2282 |
| 479 | 2351 | 487 | 2391 | 491 | 2414 | 499 | 2463 | 503 | 2478 | 509 | 2509 | 521 | 2586 |
| 523 | 2591 | 541 | 2696 | 547 | 2720 | 557 | 2777 | 563 | 2813 | 569 | 2846 | 571 | 2857 |
| 577 | 2890 | 587 | 2949 | 593 | 2982 | 599 | 3009 | 601 | 3018 | 607 | 3042 | 613 | 3088 |
| 617 | 3114 | 619 | 3103 | 631 | 3174 | 641 | 3227 | 643 | 3243 | 647 | 3265 | 653 | 3298 |
| 659 | 3322 | 661 | 3325 | 673 | 3405 | 677 | 3419 | 683 | 3462 | 691 | 3516 | 701 | 3563 |
| 709 | 3601 | 719 | 3638 | 727 | 3702 | 733 | 3750 | 739 | 3760 | 743 | 3794 | 751 | 3841 |
| 757 | 3868 | 761 | 3893 | 769 | 3925 | 773 | 3949 | 787 | 4030 | 797 | 4095 | 809 | 4166 |
| 811 | 4181 | 821 | 4220 | 823 | 4217 | 827 | 4248 | 829 | 4264 | 839 | 4323 | 853 | 4398 |
| 857 | 4431 | 859 | 4421 | 863 | 4454 | 877 | 4529 | 881 | 4555 | 883 | 4548 | 887 | 4593 |
| 907 | 4701 | 911 | 4739 | 919 | 4768 | 929 | 4829 | 937 | 4872 | 941 | 4881 | 947 | 4921 |
| 953 | 4971 | 967 | 5037 | 971 | 5060 | 977 | 5096 | 983 | 5125 | 991 | 5169 | 997 | 5226 |
| 1009 | 5283 | 1013 | 5308 | 1019 | 5345 | 1021 | 5347 | 1031 | 5402 | 1033 | 5404 | 1039 | 5445 |
| 1049 | 5503 | 1051 | 5514 | 1061 | 5564 | 1063 | 5578 | 1069 | 5616 | 1087 | 5705 | 1091 | 5742 |
| 1093 | 5745 | 1097 | 5766 | 1103 | 5794 | 1109 | 5835 | 1117 | 5896 | 1123 | 5912 | 1129 | 5936 |
| 1151 | 6088 | 1153 | 6093 | 1163 | 6137 | 1171 | 6192 | 1181 | 6269 | 1187 | 6299 | 1193 | 6323 |
| 1201 | 6381 | 1213 | 6428 | 1217 | 6455 | 1223 | 6482 | 1229 | 6509 | 1231 | 6533 | 1237 | 6588 |
| 1249 | 6645 | 1259 | 6701 | 1277 | 6821 | 1279 | 6836 | 1283 | 6867 | 1289 | 6889 | 1291 | 6889 |
| 1297 | 6923 | 1301 | 6941 | 1303 | 6969 | 1307 | 6990 | 1319 | 7062 | 1321 | 7081 | 1327 | 7098 |
| 1361 | 7283 | 1367 | 7321 | 1373 | 7365 | 1381 | 7393 | 1399 | 7494 | 1409 | 7567 | 1423 | 7636 |
| 1427 | 7640 | 1429 | 7688 | 1433 | 7716 | 1439 | 7748 | 1447 | 7782 | 1451 | 7785 | 1453 | 7819 |
| 1459 | 7850 | 1471 | 7910 | 1481 | 7970 | 1483 | 7966 | 1487 | 8027 | 1489 | 8036 | 1493 | 8046 |
| 1499 | 8093 | 1511 | 8137 | 1523 | 8225 | 1531 | 8269 | 1543 | 8345 | 1549 | 8388 | 1553 | 8413 |
| 1559 | 8437 | 1567 | 8490 | 1571 | 8507 | 1579 | 8548 | 1583 | 8579 | 1597 | 8644 | 1601 | 8690 |

[2]The sizes improving the ones from [19, Table 7] and [33, Section 3] are written in bold font

**Table 2. Continue.** Sizes $t_2^G(3,q) = t_2^G$ of complete caps in $\mathrm{PG}(3,q)$ obtained using randomized greedy algorithms, $q \in G_3$

| $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1607 | 8715 | 1609 | 8722 | 1613 | 8748 | 1619 | 8764 | 1621 | 8788 | 1627 | 8819 | 1637 | 8904 |
| 1657 | 9011 | 1663 | 9049 | 1667 | 9065 | 1669 | 9067 | 1693 | 9215 | 1697 | 9223 | 1699 | 9258 |
| 1709 | 9314 | 1721 | 9383 | 1723 | 9384 | 1733 | 9437 | 1741 | 9488 | 1747 | 9522 | 1753 | 9578 |
| 1759 | 9616 | 1777 | 9708 | 1783 | 9739 | 1787 | 9785 | 1789 | 9771 | 1801 | 9864 | 1811 | 9917 |
| 1823 | 9969 | 1831 | 10042 | 1847 | 10117 | 1861 | 10218 | 1867 | 10250 | 1871 | 10283 | 1873 | 10268 |
| 1877 | 10298 | 1879 | 10324 | 1889 | 10398 | 1901 | 10448 | 1907 | 10473 | 1913 | 10522 | 1931 | 10634 |
| 1933 | 10647 | 1949 | 10727 | 1951 | 10732 | 1973 | 10887 | 1979 | 10919 | 1987 | 10971 | 1993 | 11018 |
| 1997 | 11044 | 1999 | 11039 | 2003 | 11062 | 2011 | 11092 | 2017 | 11148 | 2027 | 11208 | 2029 | 11225 |
| 2039 | 11307 | 2053 | 11376 | 2063 | 11435 | 2069 | 11438 | 2081 | 11534 | 2083 | 11537 | 2087 | 11559 |
| 2089 | 11587 | 2099 | 11637 | 2111 | 11714 | 2113 | 11733 | 2129 | 11828 | 2131 | 11850 | 2137 | 11898 |
| 2141 | 11906 | 2143 | 11932 | 2153 | 11974 | 2161 | 12032 | 2179 | 12138 | 2203 | 12262 | 2207 | 12283 |
| 2213 | 12325 | 2221 | 12387 | 2237 | 12488 | 2239 | 12525 | 2243 | 12529 | 2251 | 12578 | 2267 | 12669 |
| 2269 | 12686 | 2273 | 12694 | 2281 | 12746 | 2287 | 12781 | 2293 | 12815 | 2297 | 12854 | 2309 | 12897 |
| 2311 | 12920 | 2333 | 13047 | 2339 | 13089 | 2341 | 13147 | 2347 | 13145 | 2351 | 13168 | 2357 | 13219 |
| 2371 | 13285 | 2377 | 13324 | 2381 | 13348 | 2383 | 13375 | 2389 | 13403 | 2393 | 13417 | 2399 | 13488 |
| 2411 | 13608 | 2417 | 13563 | 2423 | 13603 | 2437 | 13713 | 2441 | 13706 | 2447 | 13750 | 2459 | 13839 |
| 2467 | 13876 | 2473 | 13883 | 2477 | 13943 | 2503 | 14121 | 2521 | 14250 | 2531 | 14315 | 2539 | 14367 |
| 2543 | 14394 | 2549 | 14404 | 2551 | 14431 | 2557 | 14462 | 2579 | 14593 | 2591 | 14611 | 2593 | 14634 |
| 2609 | 14714 | 2617 | 14774 | 2621 | 14792 | 2633 | 14874 | 2647 | 14972 | 2657 | 15030 | 2659 | 15054 |
| 2663 | 15059 | 2671 | 15101 | 2677 | 15180 | 2683 | 15223 | 2687 | 15235 | 2689 | 15228 | 2693 | 15257 |
| 2699 | 15309 | 2707 | 15364 | 2711 | 15387 | 2713 | 15394 | 2719 | 15448 | 2729 | 15514 | 2731 | 15521 |
| 2741 | 15579 | 2749 | 15635 | 2753 | 15644 | 2767 | 15744 | 2777 | 15791 | 2789 | 15897 | 2791 | 15887 |
| 2797 | 15919 | 2801 | 15942 | 2803 | 15968 | 2819 | 16095 | 2833 | 16150 | 2837 | 16174 | 2843 | 16219 |
| 2851 | 16257 | 2857 | 16326 | 2861 | 16342 | 2879 | 16485 | 2887 | 16484 | 2897 | 16533 | 2903 | 16560 |
| 2909 | 16636 | 2917 | 16702 | 2927 | 16751 | 2939 | 16841 | 2953 | 16896 | 2957 | 16923 | 2963 | 16974 |
| 2969 | 16981 | 2971 | 17052 | 2999 | 17188 | 3001 | 17192 | 3011 | 17264 | 3019 | 17339 | 3023 | 17326 |
| 3037 | 17466 | 3041 | 17451 | 3049 | 17530 | 3061 | 17589 | 3067 | 17628 | 3079 | 17684 | 3083 | 17719 |
| 3089 | 17727 | 3109 | 17819 | 3119 | 17897 | 3121 | 17900 | 3137 | 17995 | 3163 | 18196 | 3167 | 18205 |
| 3169 | 18214 | 3181 | 18287 | 3187 | 18329 | 3191 | 18335 | 3203 | 18442 | 3209 | 18474 | 3217 | 18481 |
| 3221 | 18562 | 3229 | 18620 | 3251 | 18775 | 3253 | 18768 | 3257 | 18774 | 3259 | 18849 | 3271 | 18865 |
| 3299 | 19054 | 3301 | 19036 | 3307 | 19130 | 3313 | 19165 | 3319 | 19215 | 3323 | 19218 | 3329 | 19228 |
| 3331 | 19287 | 3343 | 19367 | 3347 | 19361 | 3359 | 19466 | 3361 | 19449 | 3371 | 19547 | 3373 | 19515 |
| 3389 | 19643 | 3391 | 19632 | 3407 | 19740 | 3413 | 19806 | 3433 | 19897 | 3449 | 19965 | 3457 | 20030 |
| 3461 | 20058 | 3463 | 20062 | 3467 | 20102 | 3469 | 20093 | 3491 | 20234 | 3499 | 20285 | 3511 | 20411 |
| 3517 | 20430 | 3527 | 20511 | 3529 | 20494 | 3533 | 20548 | 3539 | 20580 | 3541 | 20597 | 3547 | 20612 |
| 3557 | 20697 | 3559 | 20667 | 3571 | 20782 | 3581 | 20839 | 3583 | 20850 | 3593 | 20947 | 3607 | 20996 |
| 3613 | 21081 | 3617 | 21037 | 3623 | 21164 | 3631 | 21220 | 3637 | 21193 | 3643 | 21215 | 3659 | 21367 |
| 3671 | 21390 | 3673 | 21489 | 3677 | 21463 | 3691 | 21514 | 3697 | 21529 | 3701 | 21645 | 3803 | 22178 |
| 3907 | 22859 | 4001 | 23401 | 4289 | 25225 | | | | | | | | |

**Table 3.** Sizes $t_2^L(4,q) = t_2^L$ of complete lexicaps in PG$(4,q)$, $q \in L_4$

| $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ | $q$ | $t_2^L$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 16 | 3 | 16 | 5 | 44 | 7 | 74 | 11 | 157 | 13 | 203 |
| 17 | 316 | 19 | 378 | 23 | 509 | 29 | 745 | 31 | 833 | 37 | 1095 |
| 41 | 1296 | 43 | 1396 | 47 | 1602 | 53 | 1937 | 59 | 2302 | 61 | 2433 |
| 67 | 2831 | 71 | 3086 | 73 | 3228 | 79 | 3681 | 83 | 3960 | 89 | 4436 |
| 97 | 5069 | 101 | 5409 | 103 | 5581 | 107 | 5920 | 109 | 6095 | 113 | 6445 |
| 127 | 7761 | 131 | 8138 | 137 | 8737 | 139 | 8943 | 149 | 9967 | 151 | 10201 |
| 157 | 10855 | 163 | 11503 | 167 | 11972 | 173 | 12620 | 179 | 13312 | 181 | 13544 |
| 191 | 14763 | 193 | 15026 | 197 | 15489 | 199 | 15755 | 211 | 17255 | 223 | 18818 |
| 227 | 19371 | 229 | 19633 | 233 | 20157 | 239 | 20985 | 241 | 21282 | 251 | 22687 |
| 257 | 23511 | 263 | 24404 | 269 | 25342 | 271 | 25588 | 277 | 26497 | 281 | 27092 |
| 283 | 27386 | 293 | 28913 | 307 | 31160 | 311 | 31754 | 313 | 32100 | 317 | 32772 |
| 331 | 35017 | 337 | 36027 | 347 | 37724 | 349 | 38090 | 353 | 38793 | 359 | 39792 |
| 367 | 41182 | 373 | 42261 | 379 | 43332 | 383 | 44038 | 389 | 45118 | 397 | 46601 |
| 401 | 47359 | 409 | 48830 | 419 | 50717 | 421 | 51132 | 431 | 52980 | 433 | 53365 |
| 439 | 54573 | 443 | 55309 | 449 | 56538 | 457 | 58157 | 461 | 58926 | 463 | 59287 |
| 467 | 60094 | 479 | 62545 | 487 | 64212 | 491 | 64999 | 499 | 66689 | 503 | 67494 |
| 509 | 68745 | 521 | 71375 | 523 | 71800 | 541 | 75708 | 547 | 77032 | 557 | 79195 |
| 563 | 80569 | 569 | 81925 | 571 | 82440 | 577 | 83791 | 587 | 86086 | 593 | 87403 |
| 599 | 88749 | 601 | 89304 | 607 | 90711 | 613 | 92127 | 617 | 93061 | 619 | 93474 |
| 631 | 96338 | 641 | 98664 | 643 | 99215 | 647 | 100114 | 653 | 101572 | 659 | 103041 |
| 661 | 103550 | 673 | 106602 | 677 | 107510 | 683 | 108942 | 691 | 111219 | 701 | 113524 |
| 709 | 115495 | 719 | 118167 | 727 | 120203 | 733 | 121660 | 739 | 123174 | 743 | 124346 |
| 751 | 126409 | 757 | 128030 | 761 | 129041 | 769 | 131161 | 773 | 132247 | 787 | 136021 |
| 797 | 138787 | 809 | 141999 | 811 | 142507 | 821 | 145352 | 823 | 145837 | 827 | 146975 |
| 829 | 147565 | 839 | 150278 | 853 | 154310 | 857 | 155338 | 859 | 156008 | 863 | 157105 |
| 877 | 161115 | 881 | 162240 | 883 | 162756 | 887 | 163911 | 907 | 169825 | 911 | 170936 |
| 919 | 173333 | 929 | 176239 | 937 | 178683 | 941 | 179900 | 947 | 181636 | 953 | 183357 |
| 967 | 187579 | 971 | 188822 | 977 | 190636 | 983 | 192545 | 991 | 194948 | 997 | 196781 |
| 1009 | 200504 | 1013 | 201779 | 1019 | 203602 | 1021 | 204253 | 1031 | 207445 | 1033 | 208062 |
| 1039 | 209945 | 1049 | 213006 | 1051 | 213705 | 1061 | 216874 | 1063 | 217563 | 1087 | 225218 |
| 1091 | 226516 | 1093 | 227235 | 1097 | 228418 | 1103 | 230322 | 1109 | 232422 | 1117 | 235049 |
| 1123 | 236956 | 1129 | 238987 | 1151 | 246325 | 1153 | 246977 | 1163 | 250167 | 1171 | 253092 |
| 1181 | 256352 | 1187 | 258447 | 1193 | 260448 | 1201 | 263240 | 1213 | 267271 | 1217 | 268644 |
| 1223 | 270732 | 1229 | 272856 | 1231 | 273487 | 1237 | 275468 | 1249 | 279794 | 1259 | 283322 |
| 1277 | 289573 | 1279 | 290388 | 1283 | 291795 | 1289 | 293821 | 1291 | 294584 | 1297 | 296679 |
| 1301 | 298264 | 1303 | 298829 | 1307 | 300381 | 1319 | 304605 | 1321 | 305314 | 1361 | 319836 |
| 1409 | 337667 | | | | | | | | | | |

**Table 4.** Sizes $t_2^G(4,q) = t_2^G$ of complete caps[3] in $\mathrm{PG}(4,q)$, $q \in G_4$, obtained by greedy algorithms

| $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ | $q$ | $t_2^G$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 9 | 3 | 11 | 5 | 31 | 7 | 56 | 11 | 121 | 13 | 162 | **17** | **254** |
| 19 | 309 | 23 | 425 | **29** | **625** | 31 | 695 | 37 | 935 | 41 | 1106 | 43 | 1194 |
| 47 | 1386 | **53** | **1687** | 59 | 2013 | 61 | 2123 | **67** | **2476** | 71 | 2723 | **73** | **2870** |
| 79 | 3253 | **83** | **3535** | **89** | **3982** | **97** | **4526** | **101** | **4868** | **103** | **5023** | **107** | **5326** |
| **109** | **5512** | **113** | **5814** | **127** | **7021** | 131 | 7437 | 137 | 7987 | 139 | 8161 | 149 | 9118 |
| 151 | 9316 | 157 | 9899 | 163 | 10510 | 167 | 10958 | 173 | 11545 | 179 | 12223 | 181 | 12451 |
| 191 | 13573 | 193 | 13798 | 197 | 14266 | 199 | 14511 | 211 | 15902 | 223 | 17360 | 227 | 17876 |
| 229 | 18162 | 233 | 18605 | 239 | 19382 | 241 | 19682 | 251 | 20997 | 257 | 21766 | 263 | 22577 |
| 269 | 23453 | 271 | 23702 | 277 | 24532 | 281 | 25122 | 283 | 25391 | 293 | 26821 | 307 | 28892 |
| 311 | 29490 | 313 | 29785 | 317 | 30380 | 331 | 32529 | 337 | 33513 | 347 | 35118 | 349 | 35423 |
| 353 | 36082 | 359 | 37065 | 367 | 38371 | 373 | 39422 | 379 | 40383 | 383 | 41100 | 389 | 42123 |
| 397 | 43984 | 401 | 44547 | 409 | 45964 | 419 | 47636 | 421 | 47899 | 431 | 49819 | 433 | 50206 |
| 439 | 51320 | 443 | 52148 | 449 | 53257 | 457 | 54478 | 461 | 55632 | 463 | 56057 | | |

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[3] The sizes improving the ones from [19, Table 8] and [22, Theorem 1.1] are written in bold font

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