Local tabularity without transitivity

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Abstract

According to the classical result by Segerberg and Maksimova, a modal logic containing **K4** is locally tabular iff it is of finite height. The notion of finite height can also be defined for logics, in which the master modality is expressible ('pretransitive' logics). We observe that every locally tabular logic is a pretransitive logic of finite height. Then we prove some semantic criteria of local tabularly. By applying them we extend the Segerberg – Maksimova theorem to a certain larger family of pretransitive logics.

Keywords: local tabularity, pretransitive logic, logic of finite height

1 Introduction

Recall that a propositional logic is called tabular (or finite) if it is characterized by a finite logical matrix (by a finite Kripke frame in the case of normal modal or intermediate logics). A weaker property is local tabularity (or local finiteness): a logic is locally tabular if each of its finite-variable fragments is tabular. Both these properties have been studied since the 1920s. They also appeared in universal algebra as properties of varieties [14], and they were investigated for different kinds of algebras: lattices, groups, rings, Lie algebras, etc. The most famous example is the Burnside problem on local tabularity of varieties of groups with the identity $x^n = 1$, cf. [1].

In this paper we are interested in modal logics, so let us mention the corresponding results in this area. Tabularity for modal and intuitionistic logics has been studied quite well, cf. [6], chapter 12. As for local tabularity, our knowledge is still very incomplete. The first well-known examples of logics with this

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property are the modal logic **S5** and the intermediate logic **LC**. Beginning from the 1970s, the studies of local tabularity became more systematic. First of all, a nice theorem was found by Krister Segerberg [15] and Larisa Maksimova [13], characterizing locally tabular extensions of **K4** as logics of finite height (in another terminology, logics of finite depth, or logics of finite slices).

Around the same time an analogue to Segerbergs theorem was established for intermediate logics: Alexander Kuznetsov [12], and Yuichi Komori [9] proved local tabularity of intermediate logics of finite height. However, unlike the modal case, for intermediate logics the converse theorem does not hold, and **LC** is the simplest counterexample. Characterization of locally tabular intermediate logics remains a challenging open problem.

Research of local tabularity for intuitionistic modal logics and the corresponding class of monadic Heyting algebras was started by Guram Bezhanishvili in [2] and continued in his joint work with Revaz Grigolia [4]. These papers considered extensions of the well-known logic **MIPC**.

Another interesting result on local tabularity was obtained by Nick Bezhanishvili for bimodal logics [5]: he proved that the logic $\mathbf{S5}^2$ is pre-locally tabular. G. Bezhanishvili [3] considered some other bimodal logics, e.g. extensions of the fusions $\mathbf{Grz} * \mathbf{S5}$ and $\mathbf{GL} * \mathbf{S5}$ with (half)-commutation axioms. In these cases local tabularity can be reduced to local tabularity of intuitionistic modal logics studied in [2], [4].

For non-transitive normal extensions of **K** the first family of locally tabular logics was probably the logics $\mathbf{K} + \Box^n\bot$ considered in [7] (also, cf. [17], where local tabularity is extended to $(\mathbf{K} + \Box^n\bot) \times \mathbf{S5}$).

Characterization of locally tabular modal logics above **K** is an open problem. This paper makes a step towards its solution. The notion of finite height can also be defined for logics, in which the master modality is expressible ('pretransitive' logics). We observe that every locally tabular logic is a pretransitive logic of finite height (Theorem 3.7). Then we prove some semantic criteria of local tabularity (Theorems 4.3 and 5.2). By applying them we extend the Segerberg – Maksimova theorem to a certain family of pretransitive logics (Theorem 6.2).

2 Preliminaries

By a *logic* we mean a normal propositional monomodal logic, cf. [6]. ML denotes the set of all modal formulas; they are built from a countable set $\mathrm{PL} = \{p_1, p_2, \ldots\}$ of proposition letters, the classical connectives $\to$, $\bot$, and the modal connectives $\Diamond$, $\Box$.

An *n-formula* is a formula in proposition letters $p_1, p_2, \ldots, p_n$. $\mathrm{ML}\lceil n$ denotes the set of all n-formulas. For a set $\Gamma \subseteq \mathrm{ML}$, put $\Gamma\lceil n = \Gamma \cap (\mathrm{ML}\lceil n)$.

For a logic L, formulas φ, ψ are called L-*equivalent* if $(\varphi \leftrightarrow \psi) \in \mathrm{L}$.

L is called *locally tabular* if for any finite n there exist finitely many n-formulas up to L-equivalence. Equivalently, a logic is locally tabular if the variety of L-algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

By *a frame* we mean a Kripke frame (W, R), $W \neq \varnothing, R \subseteq W \times W$. For a frame $\mathrm{F} = (W, R)$, $\varnothing \neq V \subseteq W$, put $\mathrm{F}{\restriction}V = (V, R{\restriction}V)$, where $R{\restriction}V = R \cap (V \times V)$; $\mathrm{F}{\restriction}V$ is called *a generated subframe* of F if $R(V) \subseteq V$.

A *model* (W, R, θ) over a frame (W, R) is a tuple (W, R, θ), where $\theta : \mathrm{PL} \to 2^W$ is a *valuation.* An *n-weak model* is (W, R, θ), where $\theta : \mathrm{PL}\lceil n \to 2^W$ is an *n-valuation*; in this case we can only evaluate n-formulas.

$\mathrm{M_L} = (W_\mathrm{L}, R_\mathrm{L}, \theta_\mathrm{L})$ denotes the canonical model of L. $\mathrm{M}_{\mathrm{L}\lceil n} = (W_{\mathrm{L}\lceil n}, R_{\mathrm{L}\lceil n}, \theta_{\mathrm{L}\lceil n})$ denotes the canonical model of $\mathrm{L}\lceil n$ [17]. $\mathrm{M}_{\mathrm{L}\lceil n}$ is also called a *weak canonical model of* L.

Note that if $\mathrm{M}_{\mathrm{L}\lceil n}$ is finite, its points bijectively correspond to atoms in the Lindenbaum algebra of $\mathrm{L}\lceil n$.

Proposition 2.1 *A modal logic is locally tabular iff all its weak canonical models are finite.*

Proposition 2.2 *Every extension of a locally tabular logic is locally tabular.*

As usual, *a partition* $\mathcal{A}$ of a non-empty set W is a set of non-empty pairwise disjoint sets such that $W = \cup\mathcal{A}$. The corresponding equivalence relation is denoted by $\sim_\mathcal{A}$, so $\mathcal{A} = W/{\sim_\mathcal{A}}$. By a partition of a Kripke model or a frame we mean a partition of its set of worlds.

For a partition $\mathcal{A}$ of W and nonempty $V \subseteq W$, put

$$\mathcal{A}{\restriction}V = \{V \cap X \mid X \in \mathcal{A} \;\&\; V \cap X \neq \varnothing\}.$$

Let $\mathrm{M} = (W, R, \theta)$ be a model, Γ a set of formulas. Put

$$x \equiv_\Gamma y \text{ iff } \forall \varphi \in \Gamma(\mathrm{M}, x \vDash \varphi \iff \mathrm{M}, y \vDash \varphi).$$

A partition $\mathcal{A}$ of M *respects* Γ if $\sim_\mathcal{A} \subseteq \equiv_\Gamma$; $\mathcal{A}$ is *induced by* Γ if $\sim_\mathcal{A} = \equiv_\Gamma$.

If M is n-weak, the partition of M induced by $\mathrm{ML}\lceil n$ is called *canonical* and denoted by $\mathcal{A}(\mathrm{M})$. For $V \in \mathcal{A}(\mathrm{M})$ put

$$\mathfrak{t}_n(V) = \{\varphi \in \mathrm{ML}\lceil n \mid \mathrm{M}, x \vDash \varphi \text{ for some (for all) } x \in V\}.$$

Proposition 2.3 *If* M *is* n*-weak and* $\mathrm{M} \vDash \mathrm{L}\lceil n$*, then* $\mathfrak{t}_n$ *is an injection* $\mathcal{A}(\mathrm{M}) \to W_{\mathrm{L}\lceil n}$.

Proof. Note that membership in $\mathfrak{t}_n(V)$ respects all Boolean connectives. Since $\mathrm{M} \vDash \mathrm{L}\lceil n$, we have $\mathfrak{t}_n(V) \in W_{\mathrm{L}\lceil n}$ for any $V \in \mathcal{A}(\mathrm{M})$. By definition, if $V_1 \neq V_2$ for some $V_1, V_2 \in \mathcal{A}(\mathrm{M})$, then $\mathfrak{t}_n(V_1) \neq \mathfrak{t}_n(V_2)$. □

Proposition 2.4 *If* $\mathcal{A}(\mathrm{M})$ *is finite, then every* $U \in \mathcal{A}(\mathrm{M})$ *is definable in* M*, i.e.,* $\mathrm{M}, x \vDash \varphi_U \iff x \in U$ *for some* n*-formula* φ_U.

Proof. For any distinct $U, V \in \mathcal{A}(\mathrm{M})$, there is an n-formula φ_{UV} such that $\varphi_{UV} \in \mathfrak{t}_n(U)$ and $\varphi_{UV} \notin \mathfrak{t}_n(V)$; then put

$$\varphi_U = \bigwedge_{V \neq U} \varphi_{UV}.$$

□

Proposition 2.5 *Suppose* M *is n-weak,* $\mathcal{A}(\mathrm{M})$ *is finite and for any n-formula* φ*,*

$$\mathrm{M} \vDash \varphi \iff \varphi \in \mathrm{L}\lceil n.$$

Then $\mathsf{t}_n : \mathcal{A}(\mathrm{M}) \to W_{\mathrm{L}\lceil n}$ *is a bijection.*

Proof. Since $\mathrm{M} \vDash \mathrm{L}\lceil n$, t_n is an injection (Proposition 2.3). To check the surjectivity, suppose $\mathcal{A}(\mathrm{M}) = \{V_0, \ldots, V_l\}$, $x_i = \mathsf{t}_n(V_i)$ and there exists $x \in W_{\mathrm{L}\lceil n}$ such that $x \neq x_i$ for each $i \leq l$. For each $i \leq l$ choose a formula φ_i such that $\varphi_i \in x$ and $\varphi_i \notin x_i$; put $\varphi = \wedge_{i \leq l} \varphi_i$. Then $\varphi \in x$, so $\neg\varphi \notin \mathrm{L}$, thus for some w in M we have $\mathrm{M}, w \vDash \varphi$. On the other hand, for each $i \leq l$ we have $\mathrm{M}, w \vDash \varphi_i$ and $\varphi_i \notin \mathsf{t}_n(V_i)$. Hence $w \notin V_i$ for each $i \leq l$, which is a contradiction. □

3 All locally tabular logics are pretransitive logics of finite height

In this section we formulate a necessary condition for local tabularity.

For a binary relation R on W, R^* denotes its transitive reflexive closure: $R^* = \bigcup\limits_{i \geq 0} R^i$, where $R^0 = Id(W) = \{(x,x) \mid x \in W\}$, $R^{i+1} = R^i \circ R$. $\sim_R$ denotes the equivalence relation $R^* \cap R^{*-1}$. A *cluster* in a frame (W, R) is an equivalence class modulo $\sim_R$; for a cluster C, the frame $(W,R)\restriction C$ is also called a cluster.

For clusters C, D, put $C \leq_R D$ iff xR^*y for some $x \in C, y \in D$. The frame $(W/{\sim_R}, \leq_R)$ is a poset; it is called the *skeleton of* F.

A poset F is of *finite height* $\leq n$ (in symbols, $ht(\mathrm{F}) \leq n$) if every of its chains contains at most n elements. F is *of height* n (in symbols, $ht(\mathrm{F}) = n$) if $ht(\mathrm{F}) \leq n$ and $ht(\mathrm{F}) \not\leq n-1$.

More generally, the *height of a frame* F is the height of its skeleton; it is also denoted by $ht(\mathrm{F})$. A class $\mathcal{F}$ of frames is of *(uniformly) finite height* if there exists $h \in \mathbb{N}$ such that for every $\mathrm{F} \in \mathcal{F}$ we have $ht(\mathrm{F}) \leq h$.

For any transitive F, $\mathrm{F} \vDash B_h \iff ht(\mathrm{F}) \leq h$, where

$$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

(see e.g. [6, Proposition 3.44]). [3]

Theorem 3.1 *[15,13] A logic* $\mathrm{L} \supseteq \mathbf{K4}$ *is locally tabular iff it contains* B_h *for some* $h \geq 0$*.*

Let $R^{\leq m} = \bigcup\limits_{0 \leq i \leq m} R^i$. A binary relation R is called *m-transitive* if $R^{\leq m} = R^*$, or equivalently, $R^{m+1} \subseteq R^{\leq m}$. R is called *pretransitive* if it is m-transitive for some $m \geq 0$.

Put

$$\Diamond^0\varphi = \varphi, \ \Diamond^{i+1}\varphi = \Diamond\Diamond^i\varphi, \ \Diamond^{\leq m}\varphi = \bigvee_{i=0}^{m} \Diamond^i\varphi, \ \Box^{\leq m}\varphi = \neg\Diamond^{\leq m}\neg\varphi.$$

[3] Is another terminology, a poset of height n is called a poset of depth $(n-1)$.

Proposition 3.2 *R is m-transitive iff* $(W,R) \vDash \Diamond^{m+1}p \to \Diamond^{\leq m}p$.

$\varphi^{[m]}$ denotes the formula obtained from φ by replacing $\Diamond$ with $\Diamond^{\leq m}$ and $\Box$ with $\Box^{\leq m}$. In particular,

$$B_1^{[m]} = p_1 \to \Box^{\leq m}\Diamond^{\leq m}p_1, \quad B_{i+1}^{[m]} = p_{i+1} \to \Box^{\leq m}(\Diamond^{\leq m}p_{i+1} \vee B_i^{[m]}).$$

Since in the case of m-transitive relation $\Diamond^{\leq m}$ corresponds to the master modality, we have

Proposition 3.3 *[11] For an m-transitive frame* F,

$$\mathrm{F} \vDash B_h^{[m]} \iff ht(\mathrm{F}) \leq h.$$

A logic L is called *m-transitive* if $\mathrm{L} \vdash \Diamond^{m+1}p \to \Diamond^{\leq m}p$. In this case, it is easy to see that the formula $\Diamond^{\leq m}\varphi$ is equivalent to every $\Diamond^{\leq n}\varphi$ for $n \geq m$. L is *pretransitive* if it is m-transitive for some $m \geq 0$. Note that every m-transitive logic is n-transitive for $n > m$.

Definition 3.4 An m-transitive logic L is of *finite height* $\leq h$ if $\mathrm{L} \vdash B_h^{[m]}$.

If L is pretransitive, there exists the least m such that L is m-transitive; then put $\mathrm{L}[h] = \mathrm{L} + B_h^{[m]}$. Note that $\mathrm{L}[h] \vdash B_h^{[n]}$ for $n > m$.

Theorem 3.5 *[11] Let* L *be a pretransitive logic. Then*

(i) $\mathrm{L}[1] \supseteq \mathrm{L}[2] \supseteq \mathrm{L}[3] \supseteq \ldots \supseteq \mathrm{L}$.

(ii) *If* L *is consistent, then* L[1] *(and, consequently, every* L[h]*) is consistent.*

(iii) *If* L *is canonical, then every* L[h] *is canonical.*

Theorem 3.6 *For an m-transitive logic* L*, the set* $\{\varphi \mid \mathrm{L} \vdash \varphi^{[m]}\}$ *is a logic containing* **S4**.

This is an easy consequence of Lemma 1.3.45 from [8].

Theorem 3.7 *Every locally tabular logic is a pretransitive logic of finite height.*

Proof. If L is locally tabular, $W_{\mathrm{L}\lceil 1}$ is finite. It follows that $R_{\mathrm{L}\lceil 1}^{m+1} \subseteq R_{\mathrm{L}\lceil 1}^{\leq m}$ for some m. So $\Diamond^{m+1}p \to \Diamond^{\leq m}p \in \mathrm{L}\lceil 1 \subset \mathrm{L}$, hence, L is m-transitive.

Then ${}^{[m]}\mathrm{L} = \{\varphi \mid \varphi^{[m]} \in \mathrm{L}\}$ is a logic containing **S4** (Theorem 3.6). It follows that ${}^{[m]}\mathrm{L}$ is locally tabular. So by Theorem 3.1 ${}^{[m]}\mathrm{L} \vdash B_h$ for some $h > 0$; hence $\mathrm{L} \vdash B_h^{[m]}$. □

4 Ripe frames

Now let us show that local tabularity of a Kripke complete logic is equivalent to a quite simple semantic condition on its frames.

Definition 4.1 Let $\mathrm{F} = (W,R)$ be a frame. A partition $\mathcal{A}$ of W *is R-tuned* if for any $U, V \in \mathcal{A}$

$$\exists u \in U\ \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uRv. \tag{1}$$

A frame F is called *f-partitionable* for a function $f : \mathbb{N} \to \mathbb{N}$ if for any $n \in \mathbb{N}$, for any finite partition $\mathcal{A}$ of W with $|\mathcal{A}| \leq n$ there is an R-tuned refinement $\mathcal{B}$ such that $|\mathcal{B}| \leq f(n)$.

A class of frames $\mathcal{F}$ is called *ripe* if there exists $f : \mathbb{N} \to \mathbb{N}$ such that every $\mathrm{F} \in \mathcal{F}$ is f-partitionable.

Remark 4.2 If F is *f-partitionable*, then it is f'-partitionable for some monotonic f': put $f'(n) = \max\{f(n_0) \mid n_0 \leq n\}$. On the other hand, if f is monotonic, then F is f-partitionable iff for every finite partition $\mathcal{A}$ of W there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

Theorem 4.3 *For a class $\mathcal{F}$ of Kripke frames, $Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is ripe.*

This theorem can be proved by algebraic methods from [14],[3]. Our proof is based on the subsequent simple propositions and (more technical) Lemma 4.8 giving an upper bound for f.

Proposition 4.4 *[16] Let $\mathrm{F} = (W, R)$ be a frame, $\mathcal{A}$ an R-tuned partition of* F*. Then there is a p-morphism from* F *onto $(\mathcal{A}, R_{\mathcal{A}})$, where*

$$UR_{\mathcal{A}}V \iff \exists u \in U \ \exists v \in V \ uRv.$$

Proof. The required p-morphism is $x \mapsto V$ for $x \in V \in \mathcal{A}$. □

Proposition 4.5 *Let $\mathrm{M} = (W, R, \theta)$ be an n-weak model, $\mathcal{A}$ an R-tuned partition of* M*. If $\mathcal{A}$ respects $\{p_1, \ldots, p_n\}$ then $\mathcal{A}$ refines $\mathcal{A}(\mathrm{M})$.*

Proof. By induction on the length of an n-formula φ, let us show that $x \sim_{\mathcal{A}} y$ implies $\mathrm{M}, x \vDash \varphi \iff \mathrm{M}, y \vDash \varphi$. The base and the Boolean cases are trivial. Suppose $\mathrm{M}, x \vDash \Diamond\varphi$, so $\mathrm{M}, z \vDash \varphi$ and xRz for some z. Since $x \sim_{\mathcal{A}} y$, we have yRz' and $z' \sim_{\mathcal{A}} z$ for some z'. By the induction hypothesis, $\mathrm{M}, z' \vDash \varphi$, so $\mathrm{M}, y \vDash \Diamond\varphi$. □

Proposition 4.6 *Let $\mathrm{M} = (W, R, \theta)$ be an n-weak model. If $\mathcal{A}(\mathrm{M})$ is finite, then $\mathcal{A}(\mathrm{M})$ is R-tuned.*

Proof. Suppose $U, V \in \mathcal{A}(\mathrm{M})$, $u, u' \in U$, $v \in V$, uRv. By Proposition 2.4 we have $\mathrm{M}, v' \vDash \varphi_V \iff v' \in V$. Then $\mathrm{M}, u \vDash \Diamond\varphi_V$, so $\mathrm{M}, u' \vDash \Diamond\varphi_V$, thus $u'Rv'$ for some $v' \in V$. □

Proposition 4.7 *Suppose $f : \mathbb{N} \to \mathbb{N}$,* M *is an n-weak model over an f-partitionable frame. Then $\mathcal{A}(\mathrm{M})$ is finite and $|\mathcal{A}(\mathrm{M})| \leq f(2^n)$.*

Proof. Let $\mathcal{A}$ be the partition of M induced by $\{p_1, \ldots, p_n\}$. Clearly, $|\mathcal{A}| \leq 2^n$. There exists a refinement $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is R-tuned (where R is the accessability relation in M) and $|\mathcal{B}| \leq f(2^n)$. By Proposition 4.5, $\mathcal{B}$ refines $\mathcal{A}(\mathrm{M})$. □

Lemma 4.8 *Suppose $f : \mathbb{N} \to \mathbb{N}$ is monotonic, $(\mathrm{F}_i)_{i \in I}$ is a non-empty family of f-partitionable frames. Then the disjoint sum $\sum\limits_{i \in I} \mathrm{F}_i$ is g-partitionable for the function $g(n) = 2^{f(n)^2} \cdot n^{f(n)} \cdot f(n)^2$.*

Proof. Let $\mathrm{F}_i = (W_i, R_i)$, $i \in I$. Let $\sum\limits_{i\in I} \mathrm{F}_i = (W, R)$, i.e.,

$$W = \{(w,i) \mid i \in I,\ w \in W_i\},\ (w,i)R(v,j) \iff i = j \& wR_iv.$$

Fix $n \in \mathbb{N}$. Suppose $\mathcal{A}$ is a partition of W, $|\mathcal{A}| = n$, $\mathcal{A} = \{A_1, \ldots, A_n\}$.

For $V \subseteq W$, $i \in I$, put $pr_i(V) = \{w \mid (w,i) \in V\}$. For $i \in I$, put $\mathcal{A}_i = \{pr_i(A) \mid A \in \mathcal{A},\ pr_i(A) \neq \varnothing\}$; so $\mathcal{A}_i$ is a partition of W_i and $|\mathcal{A}_i| \leq n$. For every $i \in I$ there exists an R_i-tuned partition $\mathcal{B}_i$ of W_i such that $\mathcal{B}_i$ refines $\mathcal{A}_i$, and $|\mathcal{B}_i| \leq f(n)$.

Consider the signature $\Omega_n = (S^{(2)}, P_1^{(1)}, \ldots, P_n^{(1)}, c)$, where $S^{(2)}$ is a binary relation symbol, $P_1^{(1)}, \ldots, P_n^{(1)}$ are unary relation symbols, c is a constant.

For every $(w,i) \in W$ we define an Ω_n-structure $\mathrm{S}(w,i)$ as follows.

For $i \in I$, let S_i be the binary relation on $\mathcal{B}_i$ such that

$$US_iV \iff \exists u \in U\ \exists v \in V\ uR_iv.$$

For $i \in I$, $1 \leq l \leq n$ let

$$P_{il} = \{B \in \mathcal{B}_i \mid B \subseteq pr_i(A_l)\}.$$

For $w \in W_i$, $[w]_i$ denotes the element of $\mathcal{B}_i$ containing w. For $(w,i) \in W$ put

$$\mathrm{S}(w,i) = (\mathcal{B}_i; S_i, P_{i1}, \ldots, P_{in}, [w]_i).$$

For $(w,i), (u,j) \in W$ put $(w,i) \sim (v,j)$ iff the structures $\mathrm{S}(w,i)$ and $\mathrm{S}(v,j)$ are isomorphic. Clearly, $\sim$ is an equivalence on W.

We claim that $W/{\sim}$ is the required partition of F.

Note that for any $(w,i) \in W$, for any l,

$$(w,i) \in A_l \iff [w]_i \in P_{il}. \tag{2}$$

Indeed, $(w,i) \in A_l$ iff $w \in pr_i(A_l)$ iff $[w]_i \subseteq pr_i(A_l)$ (since $\mathcal{B}_i$ is a refinement of $\mathcal{A}_i$) iff $[w]_i \in P_{il}$.

Let us show that $W/{\sim}$ is a refinement of $\mathcal{A}$. Suppose $(w,i) \sim (v,j)$ and $(w,i) \in A_l$ for some l. We have to check that $(w,i) \sim_{\mathcal{A}} (v,j)$, i.e., $(v,j) \in A_l$. Since $(w,i) \in A_l$, by (2) we have $[w]_i \in P_{il}$. By the definition of $\sim$, there exists an isomorphism $h : \mathrm{S}(w,i) \to \mathrm{S}(v,j)$; then $h([w]_i) \in P_{jl}$ and $h([w]_i) = [v]_j$. It follows that $[v]_j \in P_{jl}$. By (2), $(v,j) \in A_l$.

Let us show that $W/{\sim}$ is R-tuned. To this end, suppose $(w,i) \sim (v,j)$ and $(w,i)R(w',i')$. So $i' = i$ and wR_iw'. Then we have $[w]_iS_i[w']_i$. Now, there exists an isomorphism $h : \mathrm{S}(w,i) \to \mathrm{S}(v,j)$. Then $h([w]_i)S_jh([w']_i)$ and $h([w]_i) = [v]_j$. So $[v]_jS_jh([w']_i)$. Since $\mathcal{B}_j$ is R_j-tuned, we have vR_jv' for some $v' \in h([w']_i)$. Hence $(v,j)R(v',j)$. h is also an isomorphism between $\mathrm{S}(w',i)$ and $\mathrm{S}(v',j)$, since $[v']_j = h([w']_i)$. Thus $(w',i) \sim (v',j)$.

Finally, let us check that $|W/{\sim}| \leq 2^{f(n)^2} \cdot n^{f(n)} \cdot f(n)^2$. Indeed, up to isomorphism, every $\mathrm{S}(w,i)$ is a structure with the carrier $\{1, \ldots, k\}$ for some $k > 0$.

In this case there are 2^{k^2} binary relations. An interpretation of $P_1^{(1)}, \ldots, P_n^{(1)}$ is a sequence of pairwise disjoint sets covering the carrier, so it corresponds to a function $\{1, \ldots, k\} \to \{1, \ldots, n\}$; thus there are n^k possible interpretations of these predicates. There are also k interpretations of the constant. Since $|\mathcal{B}_i| \leq f(n)$ for every i, we have

$$|W/{\sim}| \leq \sum_{k=1}^{f(n)} \left(2^{k^2} \cdot n^k \cdot k\right) \leq f(n) \cdot 2^{f(n)^2} \cdot n^{f(n)} \cdot f(n).$$

□

Proof of Theorem 4.3 Suppose $\mathrm{L} = \mathit{Log}(\mathcal{F})$ is locally tabular. Put $f(n) = |W_{\mathrm{L}\lceil n}|$. Suppose $\mathrm{F} = (W, R) \in \mathcal{F}$, $\mathcal{A} = \{V_1, \ldots, V_n\}$ is a finite partition of F. Put $\theta(p_i) = V_i$, $1 \leq i \leq n$ and consider the model $\mathrm{M} = (\mathrm{F}, \theta)$. The partition $\mathcal{A}(\mathrm{M})$ refines $\mathcal{A}$. Since $\mathrm{M} \vDash \mathrm{L}\lceil n$, by Proposition 2.3 we have $|\mathcal{A}(\mathrm{M})| \leq f(n)$. By Proposition 4.6, $\mathcal{A}(\mathrm{M})$ is R-tuned.

Suppose there is $f : \mathbb{N} \to \mathbb{N}$ such that every frame from $\mathcal{F}$ is f-partitionable. Since $\mathrm{L} = \mathit{Log}(\mathcal{F})$, then there exists a model $\mathrm{M} = (\mathrm{G}, \eta)$ such that G is a disjoint sum of some frames from $\mathcal{F}$, and $\mathrm{M} \vDash \varphi$ iff $\varphi \in \mathrm{L}$. By Lemma 4.8, G is g-partitionable for $g(n) = 2^{f(n)^2} \cdot n^{f(n)} \cdot f(n)^2$. By Propositions 2.5 and 4.7, $|W_{\mathrm{L}\lceil n}| \leq g(2^n)$ for every n. □

Since every locally tabular logic is Kripke complete, in view of Theorem 4.3 we readily obtain

Corollary 4.9 *The following conditions are equivalent:*

- *a logic* L *is locally tabular;*
- L *is the logic of a ripe class of frames;*
- L *is Kripke complete and the class of all its frames is ripe.*

The following fact from [11] gives many examples of ripe classes: if $\mathcal{F}$ is of uniformly finite height, and there exists a common finite upper bound for cardinalities of clusters in frames from $\mathcal{F}$, then $\mathcal{F}$ is ripe. By Theorem 4.3, logics of these classes are locally tabular. In the next section we will prove a stronger result: instead of boundedness of clusters, we only need them to be ripe.

5 Ripe cluster property

Definition 5.1 For a class $\mathcal{F}$ of frames let $cl\mathcal{F}$ be the class of all clusters occurring in frames from $\mathcal{F}$.

$\mathcal{F}$ has the *ripe cluster property* if $cl\mathcal{F}$ is ripe. A logic L has the *ripe cluster property* if the class of all its frames has.

Theorem 5.2 *A logic $\mathit{Log}(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the ripe cluster property.*

Before proving this theorem let us discuss some examples.

Example 5.3 **S4** has the ripe cluster property.

Indeed, every cluster in a preorder is a frame with the universal relation $(C, C \times C)$. Every partition of C is $(C \times C)$-tuned. So every cluster in a preorder is f-partitionable, where f is the identity function on $\mathbb{N}$.

Example 5.4 $\mathbf{wK4} = \mathbf{K} + \Diamond\Diamond p \to \Diamond p \vee p$ has the ripe cluster property.

To show this, consider a cluster (C, R) in a **wK4**-frame. Then $R \cup R^0 = C \times C$. Every partition $\mathcal{A}$ of C is R-tuned, due to the following proposition.

Proposition 5.5 *If $R \cup R^0 = W \times W$, then every partition $\mathcal{A}$ of W is R-tuned.*

Proof. Let $x, y \in U$, $z \in V$, xRz for some $U, V \in \mathcal{A}$. Suppose $z \neq y$; then yRz. Suppose $z = y$; in this case $U = V$, so $x \in V$; since R is symmetric, we have yRx. □

Example 5.6 $\mathbf{K} + \Diamond\Diamond\Diamond p \to \Diamond p$ has the ripe cluster property.

Clusters in frames validating $\Diamond\Diamond\Diamond p \to \Diamond p$ are frames (C, R) such that R^* is universal and $R^3 \subseteq R$. Let us show that these clusters are f-partitionable for $f(n) = 2n$. Suppose $\mathcal{A}$ is a finite partition of C. Put $x \sim y$ iff there exists $l \geq 0$ such that $xR^{2l}y$. It is not difficult to see that $\sim$ is an equivalence relation on C and $|C/{\sim}| \leq 2$ (more details will be given in Lemma 6.4). Let $\mathcal{B} = C/({\sim} \cap {\sim_{\mathcal{A}}})$. Then $|\mathcal{B}| \leq 2|\mathcal{A}|$. To show that $\mathcal{B}$ is R-tuned, suppose $x, y \in U$, $z \in V$, xRz for some $U, V \in \mathcal{B}$. Then $yR^{2l}x$ for some $l \geq 0$, so $yR^{2l+1}z$. Since $R^3 \subseteq R$, we have $R^{2l+1} \subseteq R$, thus yRz.

Proposition 5.7 *If $\mathcal{B}$ is R-tuned, then $\mathcal{B}$ is R^l-tuned for any $l \geq 0$.*

Proof. By an easy indiction on l. □

Lemma 5.8 *Suppose* $\mathrm{F} = (W, R)$ *is a frame,* $\{X_1, X_2\}$ *is a partition of* W*,* $\mathrm{F}{\restriction}X_1$ *is a generated subframe of* F*,* $\mathrm{F}{\restriction}X_1$ *is* f_1*-partitionable,* $\mathrm{F}{\restriction}X_2$ *is* f_2*-partitionable for some monotonic* $f_1, f_2 : \mathbb{N} \to \mathbb{N}$*. Then* F *is* g*-partitionable for*

$$g(n) = f_2\left(n \cdot 2^{f_1(n)}\right) + f_1(n).$$

Proof. Let $\mathcal{A}$ be a partition of W, $|\mathcal{A}| = n \in \mathbb{N}$. Put $\mathcal{A}_i = \mathcal{A}{\restriction}X_i$, $i = 1, 2$.

There exists a partition $\mathcal{B}_1$ of X_1 such that $\mathcal{B}_1$ is $(R{\restriction}X_1)$-tuned, $\mathcal{B}_1$ is a refinement of $\mathcal{A}_1$, and $|\mathcal{B}_1| \leq f_1(n)$. Let ${\sim_1} = {\sim_{\mathcal{B}_1}}$.

For $x \in X_2$ put $\mathcal{V}(x) = \{B \in \mathcal{B}_1 \mid \exists y \in B\ xRy\}$. For $x, y \in X_2$ put $x \equiv y$ iff x and y belong to the same element of $\mathcal{A}_2$ and $\mathcal{V}(x) = \mathcal{V}(y)$. Clearly, $\equiv$ is an equivalence on X_2. Since $\mathcal{V}(x) \subseteq \mathcal{B}_1$ for every $x \in X_2$, we have

$$|X_2/{\equiv}| \leq |\mathcal{A}_2| \cdot 2^{|\mathcal{B}_1|} \leq n \cdot 2^{f_1(n)}.$$

Then there exists a partition $\mathcal{B}_2$ of X_2 such that $\mathcal{B}_2$ is $(R{\restriction}X_2)$-tuned, $\mathcal{B}_2$ is a refinement of $X_2/{\equiv}$, and

$$|\mathcal{B}_2| \leq f_2\left(n \cdot 2^{f_1(n)}\right).$$

Let ${\sim_2} = {\sim_{\mathcal{B}_2}}$.

Let $\sim$ be the union of $\sim_1$ and $\sim_2$; clearly, $\sim$ is an equivalence on W.

We claim that $\mathcal{C} = W/{\sim}$ is the required partition of W. By the construction, $\mathcal{C}$ refines $\mathcal{A}$ and

$$|\mathcal{C}| \leq f_2\left(n \cdot 2^{f_1(n)}\right) + f_1(n).$$

Let us check that $\mathcal{C}$ is R-tuned. Suppose $x \sim y$ and xRx' for some $x, y, x' \in W$.

First assume that $x \in X_1$. Then $x \sim_1 y$. Since $R \cap (X_1 \times X_2) = \varnothing$, we have $x' \in X_1$. Since $\mathcal{B}_1$ is $(R{\restriction}X_1)$-tuned, for some $y' \in X_1$ we have yRy' and $x' \sim_1 y'$; the latter implies $x' \sim y'$.

Assume now that $x \in X_2$ and $x' \in X_1$. Let B be the element of $\mathcal{B}_1$ containing x'. Then $B \in \mathcal{V}(x)$. Since $x \equiv y$, $\mathcal{V}(x) = \mathcal{V}(y)$. So $B \in \mathcal{V}(y)$. It follows that for some y' we have yRy' and $y' \in B$; the latter implies $x' \sim y'$.

Finally, assume that $x, x' \in X_2$. Since $x \sim_2 y$ and $\mathcal{B}_2$ is $(R_2{\restriction}X_2)$-tuned, for some y' we have yRy' and $x' \sim_2 y'$; the latter implies $x' \sim y'$. □

Lemma 5.9 *Suppose $f : \mathbb{N} \to \mathbb{N}$, (W, R) is an f-partitionable frame, $V \subset W$, $V \neq \varnothing$. Then $(V, R{\restriction}V)$ is g-partitionable for $g(n) = f(n+1) - 1$.*

Proof. Let $\mathcal{A}$ be a partition of V, $|\mathcal{A}| \leq n \in \mathbb{N}$. Put $\mathcal{A}' = \mathcal{A} \cup \{W \setminus V\}$. So $\mathcal{A}'$ is a partition of W and $|\mathcal{A}'| \leq n + 1$. There exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}'$ such that $|\mathcal{B}| \leq f(n+1)$. Then $\mathcal{C} = \{B \in \mathcal{B} \mid B \subseteq V\}$ is a partition of V, $\mathcal{C}$ refines $\mathcal{A}$ and is $(R{\restriction}V)$-tuned. There exists $B_0 \in \mathcal{B}$ such that $B_0 \subseteq W \setminus V$. Then $B_0 \notin \mathcal{C}$, so $|\mathcal{C}| < |\mathcal{B}|$. □

Lemma 5.10 *For $h \in \mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}$, let $\mathcal{F}_{h,f}$ be the class*

$$\{\mathrm{F} \mid ht(\mathrm{F}) \leq h \textit{ and every cluster in } \mathrm{F} \textit{ is } f\textit{-partitionable}\}$$

Then $\mathcal{F}_{h,f}$ is ripe.

Proof. By induction on h, let us show that there exists a monotonic $g_h : \mathbb{N} \to \mathbb{N}$ such that every $\mathrm{F} \in \mathcal{F}_{h,f}$ is g_h-partitionable.

If $\mathrm{F} \in \mathcal{F}_{1,f}$, then $ht(\mathrm{F}) = 1$, so F is a disjoint sum of clusters. So F is g_1-partitionable for $g_1(n) = 2^{f(n)^2} \cdot n^{f(n)} \cdot f(n)^2$ by Lemma 4.8 (without any loss of generality, we may assume that f is monotonic).

For the induction step, suppose $\mathrm{F} = (W, R) \in \mathcal{F}_{h+1,f}$.

If $ht(\mathrm{F}) \leq h$, then F is g_h-partitionable by the induction hypothesis.

Suppose $ht(\mathrm{F}) = h + 1$. Let X_1 be the union of all maximal clusters in F, $X_2 = W \setminus X_1$. Then $\mathrm{F}{\restriction}X_1$ is a disjoint sum of some clusters in F; this frame is g_1-partitionable by Lemma 4.8. The height of the frame $\mathrm{F}{\restriction}X_2$ is h, so this frame is g_h-partitionable by the induction hypothesis. Since $\mathrm{F}{\restriction}X_1$ is a generated subframe of F, by Lemma 5.8, F is g-partitionable for $g(n) = g_h\left(n \cdot 2^{g_1(n)}\right) + g_1(n)$.

Since $g_h(n) \leq g(n)$, if follows that F is g-partitionable. □

Lemma 5.11 *A class $\mathcal{F}$ of Kripke frames is ripe iff $\mathcal{F}$ is of uniformly finite height and has the ripe cluster property.*

Proof. Suppose $\mathcal{F}$ is ripe. Then its logic is locally tabular by Theorem 4.3. It follows that all frames in $\mathcal{F}$ are of height $\leq h$ for some $h \in \mathbb{N}$ (Theorem 3.7). Since $\mathcal{F}$ is ripe, the class $cl\mathcal{F}$ is also ripe by Lemma 5.9.

The right-to-left direction: for some f, h, the class $\mathcal{F}$ is a subclass of the class $\mathcal{F}_{h,f}$ described in Lemma 5.10; thus $\mathcal{F}$ is a subclass of a ripe class. □

Proof of Theorem 5.2 Follows from Lemma 5.11 and Theorem 4.3. □

Corollary 5.12 *A logic* L *is locally tabular iff* L *is a Kripke complete pretransitive logic of finite height with the ripe cluster property.*

Theorem 5.13 *Suppose* L_0 *is a canonical pretransitive logic with the ripe cluster property. Then for any logic* $\mathrm{L} \supseteq \mathrm{L}_0$*:*

$$\mathrm{L} \textit{ is locally tabular iff } \mathrm{L} \textit{ is of finite height.}$$

Proof. If an extension L of L_0 is locally tabular, then L is of finite height by Theorem 3.7.

To show that every extension of L_0 of finite height is locally tabular, it is sufficient to show that $\mathrm{L}_0[h]$ is locally tabular for every $h > 0$ (Proposition 2.2). By Theorem 3.5, $\mathrm{L}_0[h]$ is canonical, so it is Kripke complete. By Theorem 5.2, $\mathrm{L}_0[h]$ is locally tabular. □

6 Criterion for logics containing $\Diamond^{m+1}p \to \Diamond p \vee p$

Proposition 6.1 *Let* $m \geq 1$.

(i) $\mathbf{K} + \Diamond^{m+1}p \to \Diamond p \vee p$ *is* m*-transitive.*

(ii) $\Diamond^{m+1}p \to \Diamond p \vee p$ *is a Sahlqvist formula, the logic* $\mathbf{K} + \Diamond^{m+1}p \to \Diamond p \vee p$ *is canonical and Kripke complete, and*

$$(W, R) \vDash \Diamond^{m+1}p \to \Diamond p \vee p \textit{ iff } R^{m+1} \subseteq R \cup R^0.$$

(iii) (W, R) *is a cluster in a frame validating* $\Diamond^{m+1}p \to \Diamond p \vee p$ *iff* $R^{\leq m} = W \times W$ *and* $R^{m+1} \subseteq R \cup R^0$.

Theorem 6.2 *Let* $m \geq 1$, $\Diamond^{m+1}p \to \Diamond p \vee p \in \mathrm{L}$. *Then* L *is locally tabular iff* $\mathrm{L} \vdash B_h^{[m]}$ *for some* $h > 0$.

We will prove this theorem later on.

Consider a frame (W, R). For $d > 0$ put $x \preceq_d y$ iff $wR^{ld}y$ for some $l \geq 0$. Clearly, $\preceq_d$ is a preorder.

Proposition 6.3 *Let* $R^* = W \times W$. *Then for every* $k > 0$ *there exists* d *such that* $0 < d \leq k$, $\preceq_d$ *is an equivalence on* W, $|W/{\preceq_d}| \leq d$, *and* $\preceq_d \subseteq \preceq_k$.

For the proof cf. [11]. The idea of is that d is a common divisor of k and the lengths of all cycles in (W, R).

Lemma 6.4 *Let* $m \geq 1$, $R^* = W \times W$, $R^{m+1} \subseteq R$. *Then* (W, R) *is* f*-partitionable for* $f(n) = mn$.

Proof. Consider a finite partition $\mathcal{A}$ of W. Let $\preceq_d$ be the equivalence described in Proposition 6.3 for $k = m$. Put $\mathcal{B} = W/(\preceq_d \cap \sim_{\mathcal{A}})$. Since $|W/\preceq_d| \leq d \leq m$, we have $|\mathcal{B}| \leq m|\mathcal{A}|$.

Suppose $x \sim_{\mathcal{B}} y$, xRz. We have $y \preceq_d x$, and then $y \preceq_m x$ (Proposition 6.3), so for some l we have $yR^{ml}x$, and thus $yR^{ml+1}z$. Since $R^{m+1} \subseteq R$, we have $R^{ml+1} \subseteq R$ (by an easy induction on l), thus yRz. It follows that $\mathcal{B}$ is R-tuned. □

Lemma 6.5 *Let $m \geq 1$, $R^* = W \times W$, and suppose $R^{m+1} \subseteq R \cup R^0$ and $R^{m+1} \not\subseteq R$. Then:*

- *there exists a cyclic path $x_0Rx_1R\ldots Rx_mRx_0$;*
- *if $W \neq \{x_0, \ldots, x_m\}$, then $R \cup R^0 = W \times W$.*

Proof. There exist x_0, y_0 such that $x_0R^{m+1}y_0$ and not x_0Ry_0. Since $R^{m+1} \subseteq R \cup R^0$, we have $x_0 = y_0$. So $x_0R^{m+1}x_0$, i.e., there exists a cyclic path $x_0Rx_1R\ldots Rx_mRx_0$. Put $U = \{x_0, x_1, \ldots, x_m\}$.

If $m = 1$, then (W, R) is a **wK4**-cluster, so $R \cup R^0 = C \times C$ (see Example 5.4).

Suppose $m \geq 2$.

Let $\overrightarrow{x_i}$ and $\overleftarrow{x_i}$ be the successor and the predecessor of x_i in our cycle, i.e.: $\overrightarrow{x_i} = x_{i+1}$ for $i < m$, $\overrightarrow{x_m} = x_0$; $\overleftarrow{x_i} = x_{i-1}$ for $i > 0$, $\overleftarrow{x_0} = x_m$.

Suppose $z \notin U$. Then for $0 \leq i \leq m$

- zRx_i implies $zR\overleftarrow{x_i}$;
- x_iRz implies $\overrightarrow{x_i}Rz$.

Indeed, $x_iR^m\overleftarrow{x_i}$, so zRx_i implies $zR^{m+1}\overleftarrow{x_i}$; since $z \neq \overleftarrow{x_i}$ we have $zR\overleftarrow{x_i}$. Similarly, $\overrightarrow{x_i}R^mx_i$, so x_iRz implies $\overrightarrow{x_i}R^{m+1}z$; since $\overrightarrow{x_i} \neq z$ we have $\overrightarrow{x_i}Rz$. It follows that

(1) $\exists x \in U\ zRx \iff \forall x \in U\ zRx$,

(2) $\exists x \in U\ xRz \iff \forall x \in U\ xRz$.

Let us check that

(3) $\forall z \notin U\ \forall x \in U\ (zRx\ \&\ xRz)$.

Indeed, since $R^{\leq m}$ is universal on W and $z \notin U$, for some l, $0 < l \leq m$, we have $zR^lx_0R^{m+1-l}x_{m+1-l}$, so $zR^{m+1}x_{m+1-l}$, and since $z \neq x_{m+1-l}$ we obtain zRx_{m+1-l}. By (1), it follows that $\forall x \in U\ zRx$.

Similarly, we have x_mR^lz for some l, $0 < l \leq m$, so $x_{l-1}R^{m+1}z$, and since $x_{l-1} \neq z$ we obtain $x_{l-1}Rz$. By (2), it follows that $\forall x \in U\ xRz$.

Now let us show that

(4) $\forall y \forall y'(y \neq y' \Rightarrow yR^{m+1}y')$.

Consider the following cases.

(a) $y, y' \notin U$. By (3) we have $yRx_0R^{m-1}x_{m-1}Ry'$, so $yR^{m+1}y'$.

(b) $y \notin U$, $y' \in U$. Then by (3) $yR\overrightarrow{y'}R^my'$, and again $yR^{m+1}y'$.

(c) $y \in U$, $y' \notin U$. Then by (3) $yR^m\overleftarrow{y'}Ry'$, and thus $yR^{m+1}y'$.

(d) $y, y' \in U$. By (3), for $z \notin U$ we have $yR^{m-1}\overleftarrow{y}RzRy'$, so $yR^{m+1}y'$ again.

If $y \neq y'$ and $yR^{m+1}y'$, then we have yRy'. Thus $R \cup R^0 = W \times W$. □

Lemma 6.6 $\mathbf{K} + \Diamond^{m+1}p \to \Diamond p \vee p$ *has the ripe cluster property for every* $m \geq 1$.

Proof. Let(C, R) be a cluster in a frame validating $\Diamond^{m+1}p \to \Diamond p \vee p$, $\mathcal{A}$ a finite partition of C. We have $R^{m+1} \subseteq R \cup R^0$.

If $R^{m+1} \subseteq R$, then by Lemma 6.4 there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq m|\mathcal{A}|$.

Otherwise, by Lemma 6.5, we have two cases: $|C| \leq m+1$, or $R \cup R^0 = C \times C$. In the first case, put $\mathcal{B} = \{\{x\} \mid x \in C\}$; trivially, $\mathcal{B}$ is R-tuned and $|\mathcal{B}| \leq m+1$. In the second case, $\mathcal{A}$ is R-tuned by Proposition 5.5.

It follows that every cluster in any frame validating $\Diamond^{m+1}p \to \Diamond p \vee p$ is f-partitionable for $f(n) = \max\{mn, m+1\}$. □

Proof of Theorem 6.2 If L is locally tabular, then $B_h^{[m]} \in \mathrm{L}$ for some $h > 0$ by Theorem 3.7.

Suppose $B_h^{[m]} \in \mathrm{L}$ for some $h > 0$. L has the ripe cluster property by Lemma 6.6. Since L is of finite height, by Theorem 5.13, L is locally tabular.□

7 Properties of partitionable frames

Theorem 3.7 implies that every f-partitionable frame is pretransitive. The next two propositions give more details.

Proposition 7.1 *If* F *is f-partitionable then* F *is* $(f(2)-1)$*-transitive.*

Proof. Let $\mathrm{F} = (W, R)$. Put $m = f(2) - 1$. Suppose that $xR^{m+1}y$, so there is a path $x = x_0Rx_1 \dots Rx_{m+1} = y$.

If $x_i = x_j$ for some $i < j$, then $xR^{m+1-(j-i)}y$, so $xR^{\leq m}y$.

Suppose all the x_i are different. Consider the two-element partition $\{\{x_{m+1}\}, W \setminus \{x_{m+1}\}\}$. There exists its refinement $\mathcal{B}$ such that $\mathcal{B}$ is R-tuned and $|\mathcal{B}| \leq f(2)$. Since $\{x_{m+1}\} \in \mathcal{B}$, $\mathcal{B}$ splits $W \setminus \{x_{m+1}\}$ into $|\mathcal{B}| - 1$ parts. Since $|\mathcal{B}| - 1 \leq f(2) - 1 = m$, there are at least two $\sim_{\mathcal{B}}$-equivalent points among $x_0, \dots, x_m$. Suppose $x_i \sim_{\mathcal{B}} x_j$ for $i < j$.

By Proposition 5.7, $\mathcal{B}$ is R^{m+1-j}-tuned, and $x_i \sim_{\mathcal{B}} x_j$, so there exists y such that $x_iR^{m+1-j}y$ and $y \sim_{\mathcal{B}} x_{m+1}$. But the element of $\mathcal{B}$ containing x_{m+1} is the singleton $\{x_{m+1}\}$, thus $y = x_{m+1}$, and therefore $xR^{\leq m}y$. □

Proposition 7.2 *If all clusters in* F *are f-partitionable and* $ht(\mathrm{F}) = h \in \mathbb{N}$*, then* F *is m-transitive for* $m = h \cdot f(2) - 1$.

Proof. Put $m_0 = f(2) - 1$. By Proposition 7.1, all clusters in F are m_0-transitive. Suppose xR^*y. Let C and D be the clusters containing x and y, respectively. In the skeleton of F, consider a maximal chain $C = C_1 \leq_R \cdots \leq_R C_l = D$ from C to D. Then for each $i < l$ we have y_iRx_{i+1} for some $y_i \in C_i$ and $x_{i+1} \in C_{i+1}$ (indeed, there are no clusters between C_i and C_{i+1}). Since C_i is m_0-transitive, we have $x_iR^{\leq m_0}y_i$; we also have y_iRx_{i+1}. Then $xR^{\leq l \cdot m_0 + l - 1}y$.

To finish the proof, note that $l \leq h$, so $l \cdot m_0 + l - 1 \leq h \cdot f(2) - 1$. □

For $m \in \mathbb{N}$, consider the following first-order property P_m:

$$\forall x_0, \ldots, x_{m+1} \left(x_0 R x_1 \ldots R x_{m+1} \to \bigvee_{i<j} x_i = x_j \vee \bigvee_{i+1<j} x_i R x_j \right).$$

Note that this property implies m-transitivity. The next theorem gives another necessary condition for local tabularity.

Theorem 7.3 *If $\mathcal{F}$ is a ripe class, then $\mathcal{F}$ satisfies P_m for some m.*

Thus theorem follows from

Proposition 7.4 *If (W, R) is f-partitionable, then $\mathrm{P}_{f(3)-1}$ holds in (W, R).*

Proof. Put $m = f(3) - 2$. Let $X = \{x_0, \ldots, x_{m+1}\}$ and suppose $x_0 R x_1 \ldots R x_{m+1}$.

First, suppose $X = W$. Then $\mathrm{P}_{f(3)-1}$ holds in (W, R), since $|W| \leq f(3)$ in this case.

Next, suppose $X \neq W$. By Lemma 5.9, $(X, R{\restriction}X)$ is g-partitionable for $g(n) = f(n+1) - 1$. By Proposition 7.1, $(X, R{\restriction}X)$ is $(g(2)-1)$-transitive and $g(2) - 1 = f(3) - 2 = m$. Thus for some $l \leq m$ we have $x_0 R^l x_{m+1}$. Thus there is a path $x_{i_0} R x_{i_1} R \ldots R x_{i_l}$, $i_0 = 0$, $i_l = m+1$.

Now suppose that $|X| = m+2$ and $i_{k+1} \leq i_k + 1$ for each $k \leq m$. In this case $i_k \leq k$ for each $k \leq l$ (by trivial induction on k), so $i_l \neq m+1$, which is a contradiction.

It follows that $|X| < m+2$ or $i_{k+1} > i_k + 1$ for some $k \leq l$.

Thus we have $\mathrm{P}_{f(3)-2}$ in (W, R).

Therefore, $\mathrm{P}_{f(3)-1}$ holds in (W, R) anyway. □

8 Conclusion

Let us summarize our main results.

- The necessary syntactic condition for local tabularity: pretransitivity and finite height.
- Two semantic criteria of local tabularity: (i) a logic is locally tabular iff it is complete with respect to a ripe class of frames; (ii) a logic is locally tabular iff it is complete with respect to a class of frames of uniformly finite height with the ripe cluster property.
- A syntactic criterion of local tabularity above $\mathbf{K} + \Diamond^{m+1} p \to \Diamond p \vee p$.

A syntactic criterion of local tabularity above $\mathbf{K}$ remains an open problem. The necessary syntactic condition from Theorem 3.7 is not sufficient: for example, the logic $\mathbf{K}_3^2 = \mathbf{K} + \Diamond\Diamond\Diamond p \to \Diamond\Diamond p$ is 2-transitive, but even $\mathbf{K}_3^2[1]$ has Kripke incomplete extensions [10], so it is not locally tabular. However, there is a hope for a criterion. In fact, every ripe class has the property P_m for some m (Theorem 7.3). So one may ask if the following holds:

Problem 8.1 *Suppose that $\mathcal{F}$ is a class of clusters satisfying P_m for some m. Is $\mathcal{F}$ ripe?*

In view of Theorem 7.3, the positive solution of the above problem would provide us with a syntactic criterion of local tabularity over **K**.

Another important field is local tabularity of polymodal logics. The notion of a ripe class and the ripe cluster property can be reformulated in a straightforward way, and the analogues of Theorems 3.7, 4.3 and 5.2 can be proved for logics with finitely many modalities.

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