[1]

Abstract

We provide a framework for modal logic of model-theoretic constructions and calculate the modal logic of submodels, which is S4.2.1 or S4 for models in any signature with finitely many functional symbols, at least one binary functional symbol, and with or without constant symbols, respectively.

modal logic, logic of submodels, expressible relation, large frame

Introduction

Given a class $\mathcal{C}$ of models in a fixed signature and a binary relation $\mathcal{R}$ between models in $\mathcal{C}$, it is natural to use modal language to describe properties of this (possibly large) frame. To this aim, one can interpret propositional modal formulas as sentences of a given language L in such a way that $\Diamond\varphi$ is true at a model $\mathfrak{A}$ iff φ is true in some model $\mathfrak{B}$ related to $\mathfrak{A}$ via $\mathcal{R}$. There are a few examples of such an approach dealing with models of powerful theories like PA ([7], [5]) or ZF ([3], [4], [1]) allowing to put the interpretation inside the theory.

However, if we consider arbitrary models, we get an obstacle since $\Diamond\psi$, which should be considered as a sentence of L, may appear to be not expressible in L. Once this problem is overcome, the notions of truth and validity can be defined like in the standard Kripke semantics. We notice that in this case, the set of all valid modal formulas, the *modal theory of* $(\mathcal{C}, \mathcal{R})$, turns out to be a logic of a (large) general frame, so we obtain a normal modal logic. This provides a natural approach to modal logics of model-theoretic relations.

Then, we apply this approach to the case of the *submodel* relation, where $\mathfrak{A}\,\mathcal{R}\,\mathfrak{B}$ means that $\mathfrak{B}$ is a submodel of $\mathfrak{A}$. It can be seen that the first-order language is not powerful enough to express $\mathcal{R}$. However, in the second-order language, it is possible to express $\mathcal{R}$ on the class of all models $\mathcal{C}$ in any signature Ω whose set of functional symbols is finite (and the set of predicate symbols is arbitrary). If Ω contains at least one binary functional symbol, the modal theory of $(\mathcal{C}, \mathcal{R})$ is either S4 or S4.2.1 $=$ S4 $+ \Diamond\Box p \leftrightarrow \Box\Diamond p$, depends on whether Ω is without or with constant symbols.

1 Modal theories of translations

Fix a signature Ω and a language L based on Ω. We do not fix a specific language in advance, which may be first- or higher-order, finitary or not,

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although our subsequent results on logic of submodels use the second-order finitary language. Let L_s denote the set of all sentences of L.

A *translation* is a function $t : L_s \to L_s$. A *(propositional) valuation in* L is a function $\theta : \mathrm{PV} \to L_s$, where $\mathrm{PV} = \{p_0, p_1, \ldots\}$ is an infinite set of propositional variables. MF denotes the set of modal formulas. For a valuation θ in L and a translation t, define the map $\theta_t : \mathrm{MF} \to L_s$ by induction on the length of a modal formula: $\theta_t(p) = \theta(p)$ for $p \in \mathrm{PV}$, θ_t commutes with Boolean connectives, and $\theta_t(\Diamond\alpha) = t(\theta_t(\alpha))$.

For a translation t and a set of sentences $T \subseteq L_s$, the *modal theory of* (T, t) is the set

$$\mathrm{MTh}^{L,\Omega}(T, t) = \{\alpha \in \mathrm{MF} : \theta_t(\alpha) \in T \text{ for every valuation } \theta \text{ in } L\}.$$

Let $\mathcal{C}$ be a class of Ω-structures and t a translation. A modal formula α is *t-valid in* $\mathcal{C}$ iff for every valuation θ we have $\mathcal{C} \vDash \theta_t(\alpha)$, i.e. $\mathfrak{A} \vDash \theta_t(\alpha)$ for every $\mathfrak{A} \in \mathcal{C}$; in symbols: $(\mathcal{C}, t) \vDash \alpha$. The *modal theory of* $(\mathcal{C}, t)$ is the set of all modal formulas t-valid in $\mathcal{C}$; in symbols: $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, t)$. Note that $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, t) = \mathrm{MTh}^{L,\Omega}(T, t)$ for $T = \{\varphi \in L_s : \mathfrak{A} \vDash \varphi \text{ for every } \mathfrak{A} \in \mathcal{C}\}$.

Let L be the language of arithmetic and let $t(\varphi)$ express consistency of a sentence φ. By well-known Solovay's results [6], if T is PA then $\mathrm{MTh}^{L,\Omega}(T, t)$ is the Gödel–Löb logic GL, and if T is TA (the set of all sentences true in the standard model of arithmetic) then $\mathrm{MTh}^{L,\Omega}(T, t)$ is the (quasi-normal but not normal) Solovay logic S.

Let L be the language of set theory and let $t(\varphi)$ express that the sentence φ holds in a generic extension. By [4], if T is ZFC then $\mathrm{MTh}^{L,\Omega}(T, t)$ is S4.2.

2 Modal theories of relations

In provability logic and logic of forcing, modalities are expressible in the language of arithmetic and set theory, respectively. We want to have this feature for model-theoretic constructions under our studying.

Let $\mathcal{C}$ be a class of Ω-structures. A translation t in a language L *expresses* a binary relation $\mathcal{R}$ on $\mathcal{C}$ iff for every $\varphi \in L_s$ and $\mathfrak{A} \in \mathcal{C}$,

$$\mathfrak{A} \vDash t(\varphi) \;\Leftrightarrow\; (\exists\, \mathfrak{B} \in \mathcal{C})\, \mathfrak{A}\, \mathcal{R}\, \mathfrak{B} \;\&\; \mathfrak{B} \vDash \varphi.$$

$\mathcal{R}$ is *expressible* in L iff some translation expresses it. Some examples will be given below.

Let $\mathcal{C}$ be a class of Ω-structures, $\mathcal{R}$ a binary relation on $\mathcal{C}$. If translations t and s in L both express $\mathcal{R}$, then $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, t) = \mathrm{MTh}^{L,\Omega}(\mathcal{C}, s)$.

This fact allows to replace translations with relations between models. Let $\mathcal{C}$ be a non-empty class of Ω-structures, $\mathcal{R}$ an expressible binary relation on $\mathcal{C}$. We define $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \mathcal{R})$ as $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, t)$ where t expresses $\mathcal{R}$, and call it the *modal theory* of $(\mathcal{C}, \mathcal{R})$.

[Soundness theorem for normal logics] Suppose that $\mathcal{C}$ is a non-empty class of Ω-structures and $\mathcal{R}$ an expressible in L relation on $\mathcal{C}$. Then $\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R})$ is a normal modal logic.

3 General frames of models

Fix a nonempty set $\mathcal{C}$ of Ω-structures and a language L. Put $V_\varphi = \{\mathfrak{A} \in \mathcal{C} : \mathfrak{A} \vDash \varphi\}$ for any sentence $\varphi \in L_s$, and $\mathcal{V} = \{V_\varphi : \varphi \in L_s\}$.

If $\mathcal{R}$ is expressible on $\mathcal{C}$ then $(\mathcal{C},\mathcal{R},\mathcal{V})$ is a general frame and

$$\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R}) = \mathrm{MLog}(\mathcal{C},\mathcal{R},\mathcal{V}).$$

Ω-structures $\mathfrak{A}$ and $\mathfrak{B}$ are *L-equivalent* iff $\mathfrak{A} \vDash \varphi \Leftrightarrow \mathfrak{B} \vDash \varphi$ for all sentences $\varphi \in L_s$; in symbols: $\mathfrak{A} \equiv_L \mathfrak{B}$. For an $\mathcal{R}$ expressible in L on $\mathcal{C}$, by identifying structures with the same theories in L, we obtain the so-called *refinement* of the frame $(\mathcal{C},\mathcal{R},\mathcal{V})$.

Namely, let $\mathcal{C}_L = \mathcal{C}/{\equiv_L}$. For an $\mathfrak{A}$ in $\mathcal{C}$ let $[\mathfrak{A}]$ denote its $\equiv_L$-class in $\mathcal{C}$. Given $\mathcal{R}$ on $\mathcal{C}$, define $\mathcal{R}_L$ on $\mathcal{C}_L$: for $\equiv_L$-classes $[\mathfrak{A}],[\mathfrak{B}] \in W_L$ put

$$[\mathfrak{A}]\mathcal{R}_L[\mathfrak{B}] \;\Leftrightarrow\; (\exists\, \mathfrak{A}' \equiv_L \mathfrak{A})(\exists\, \mathfrak{B}' \equiv_L \mathfrak{B})\; \mathfrak{A}' \,\mathcal{R}\, \mathfrak{B}'.$$

If t expresses $\mathcal{R}$ in $\mathcal{C}$, put

$$[\mathfrak{A}]\mathcal{R}_L^{\mathrm{ref}}[\mathfrak{B}] \;\Leftrightarrow\; (\forall\, V \in \mathcal{V})\; \mathfrak{B} \in V \Rightarrow \mathfrak{A} \in \mathcal{R}^{-1}(V),$$

thus, in other words,

$$[\mathfrak{A}]\mathcal{R}_L^{\mathrm{ref}}[\mathfrak{B}] \;\Leftrightarrow\; (\forall\, \varphi \in L_s)\; \mathfrak{B} \vDash \varphi \Rightarrow \mathfrak{A} \vDash t(\varphi).$$

Finally, put $\overline{V}_\varphi = \{[\mathfrak{A}] \in \mathcal{C}_L : \mathfrak{A} \vDash \varphi\}$ for $\varphi \in L_s$, and $\mathcal{V}_L = \{\overline{V}_\varphi : \varphi \in L_s\}$.

If $\mathcal{R}$ is expressible on $\mathcal{C}$, then $(\mathcal{C}_L,\mathcal{R}_L,\mathcal{V}_L)$ and $(\mathcal{C}_L,\mathcal{R}_L^{\mathrm{ref}},\mathcal{V}_L)$ are also general frames and their modal algebras are isomorphic to the modal algebra of the frame $(\mathcal{C},R,\mathcal{V})$ (see e.g. [2, Proposition 8.45]); in particular, $(\mathcal{C}_L,\mathcal{R}_L^{\mathrm{ref}},\mathcal{V}_L)$ is the refinement of $(\mathcal{C},\mathcal{R},\mathcal{V})$.

Let $\mathcal{R}$ be expressible in L on $\mathcal{C}$. Then

$$\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R}) = \mathrm{MLog}(\mathcal{C}_L,\mathcal{R}_L,\mathcal{V}_L) = \mathrm{MLog}(\mathcal{C}_L,\mathcal{R}_L^{\mathrm{ref}},\mathcal{V}_L).$$

If moreover $\mathcal{C}_L$ is finite then $\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R})$ is $\mathrm{MLog}(\mathcal{C}_L,\mathcal{R}_L)$, the modal logic of the Kripke frame $(\mathcal{C}_L,\mathcal{R}_L)$.

Let $\mathcal{R}$ be an expressible binary relation on $\mathcal{C}$. Is it true that

$$\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R}) = \mathrm{MLog}(\mathcal{C}_L,\mathcal{R}_L)?$$

Is it true, at least, that $\mathrm{MTh}^{L,\Omega}(\mathcal{C},\mathcal{R})$ is Kripke complete?

4 Logics of submodels

Let $\mathfrak{A} \sqsupseteq \mathfrak{B}$ mean "$\mathfrak{A}$ contains $\mathfrak{B}$ as a submodel". We are going to calculate the modal logic of this relation. As the first step, we note:

The relation $\sqsupseteq$ is not expressible in the first-order language and is expressible in any second-order language with finitely many functional (including constant) symbols.

We illustrate the non-expressibility as follows:

Let Ω contain a binary predicate symbol $\leq$, and let L be the first-order language.

1. Let $(\mathbb{Z}, \leq)$ be the integers with their usual ordering and let $X \cdot Y$ denote the anti-lexicographic product of X and Y. If φ is the first-order sentence characterizing dense linear orders without end-points, then $\Diamond\varphi$ means the non-scatteredness of linear orders. However, we cannot distinguish scattered orders by first-order means since e.g. $(\mathbb{Z} \cdot \mathbb{Q}, \leq)$ and $(\mathbb{Z}, \leq)$ are elementary equivalent.

2. Suppose that t expresses $\sqsupseteq$. Let ψ say that $\leq$ is a partial order and let φ say that there is a $\leq$-minimal element. Then $\psi \wedge \neg\, t(\neg\, \varphi)$ is a first-order sentence saying that $\leq$ is well-founded, a contradiction.

A second-order formula φ is *atomic* iff it is of form $P(t, \ldots)$ for a predicate constant P, or $X(t, \ldots)$ for a predicate variable X, and (second-order) terms $t, \ldots$. Let φ be any second-order formula and $\chi(x)$ a second-order formula with one first-order parameter. The *relativization* φ^χ of φ to χ is defined by recursion:

(i) φ^χ is φ for atomic φ,

(ii) $(\neg\, \varphi)^\chi$ is $\neg\, \varphi^\chi$, and $(\varphi \wedge \psi)^\chi$ is $\varphi^\chi \wedge \psi^\chi$,

(iii) $(\exists x\, \varphi)^\chi$ is $\exists x\, \chi(x) \wedge \varphi^\chi$,

(iv) $(\exists X\, \varphi)^\chi$ is $\exists X \left(\forall v_0 \ldots \forall v_{n-1}\, X(v_0, \ldots, v_{n-1}) \to \bigwedge_{i<n} \chi(v_i)\right) \wedge \varphi^\chi$.

The relativization of other connectives and quantifiers easily follows. Note also that $(\varphi^\psi)^\chi \leftrightarrow \varphi^{\psi \wedge \chi}$, in particular, $(\varphi^\chi)^\chi \leftrightarrow \varphi^\chi$.

Let now Ω have only finitely many functional (including constant) symbols (and arbitrarily many predicate symbols). Then the relation $\sqsupseteq$ can be expressed by the existential second-order formula

$$\exists X \left(X \text{ is a submodel} \wedge \varphi^X\right)$$

where X is a one-parameter second-order variable and "X is a submodel" is the formula

$$(\exists x\, X(x)) \wedge \bigwedge \{\psi_F(X) : F \in \Omega \text{ is a functional symbol}\}$$

where, in turn, $\psi_F(X)$ is the formula

$$\forall x_0 \dots \forall x_{n-1} \Big(\bigwedge_{i<n} X(x_i) \to X(F(x_0, \dots, x_{n-1})\Big),$$

meaning that X is closed under F, for each functional symbol F in Ω.

Henceforth we assume that L is the second-order language and Ω has only finitely many functional symbols, so $\sqsupseteq$ turns out be expressible in L.

Let $\mathcal{C}$ be a class of Ω-structures. Then $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \sqsupseteq)$ is a normal modal logic including S4. If Ω contains a constant symbol then $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \sqsupseteq)$ includes S4.2.1.

For an Ω-structure $\mathfrak{A}$, let $Sub(\mathfrak{A})$ denote the set of all its substructures.

Let $\mathcal{C}$ be a class of Ω-structures closed under substructures. Then

$$\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \sqsupseteq) = \bigcap_{\mathfrak{A} \in \mathcal{C}} \mathrm{MTh}^{L,\Omega}(Sub(\mathfrak{A}), \sqsupseteq).$$

Let Q_n be the lexicographical product of $(n^{<n}, \subseteq)$ (an n-ramified tree of height n) and $(n, n \times n)$ (a cluster of size n), and Q'_n the ordered sum of Q_n and a reflexive singleton. The following fact is well-known: $\mathrm{S4} = \mathrm{MLog}\{Q_n : n \in \omega\}$, $\mathrm{S4.2.1} = \mathrm{MLog}\{Q'_n : n \in \omega\}$.

For any positive $n < \omega$, there exists models $\mathfrak{A}_n$ and $\mathfrak{A}'_n$ such that $(Sub(\mathfrak{A}_n)/{\equiv_L}, \sqsupseteq_L)$ is isomorphic to Q_n and $(Sub(\mathfrak{A}'_n)/{\equiv_L}, \sqsupseteq_L)$ to Q'_n.

Now our main result follows: Let Ω contain a binary functional symbol and let $\mathcal{C}$ be the class of all Ω-structures. Then $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \sqsupseteq) = \mathrm{S4}$ whenever Ω has no constant symbols, and $\mathrm{MTh}^{L,\Omega}(\mathcal{C}, \sqsupseteq) = \mathrm{S4.2.1}$ otherwise.

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