SEGMENTS OF RECURSIVELY ENUMERABLE m-DEGREES

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Lachlan obtained in [3] a necessary and sufficient condition for and upper semilattice to be isomorphic to an initial segment of an upper semilattice of recursively enumerable (r.e.) m-degrees. In the present article we extend this result and show that Lachlan's condition is necessary in order for upper semilattices to be isomorphic to arbitrary segments of an upper semilattice of r.e. m-degrees. The sufficiency of the condition is proved in the following sense; for any incomplete r.e. m-degree a and any upper semilattice F, satisfying Lachlan's condition, there exists an r.e. m-degree u such that $a \leqslant u$; for any r.e. m-degree x, if $x \leqslant u$ then $x \leqslant a$ or $a \leqslant x$, and the segment formed by the set of m-degrees $\{x | a \leqslant x \leqslant u\}$ is isomorphic to F. The second of these results generalizes the results obtained in [1, 2] (in [2], a similar result was obtained for the case when F is a two-element lattice, while in [1] the result is generalized to the case when F is any finite distributive lattice). The proofs of the present article are based in an essential way on the proofs in [3], and an acquaintance with [3] is needed for understanding them. Proofs which are similar to those in [3] will be omitted, and the reader will be referred to [3]. We shall also mention many definitions given in [3] without dwelling on them in detail.

Let $(S, \leqslant)$ be a partially ordered set. We describe as a segment of it any partially ordered set $(S_{ab}, \leqslant_{ab})$, where $a, b \in S$, $a \leqslant b$, $S_{ab} = \{x \in S \mid a \leqslant x \leqslant b\}$, $\leqslant_{ab}$ is the restriction of $\leqslant$ onto S_{ab}. If a is the least element of $(S, \leqslant)$, we call $(S_{ab}, \leqslant_{ab})$ the initial segment.

A pair $(P, \preccurlyeq)$, where P is a set of natural numbers, and $\preccurlyeq$ is a transitive and reflexive relation in P, is called a preordered lattice. The preordered lattice $(P, \preccurlyeq)$ is finite if P is a finite set. A partially ordered set $(Q, \leqslant)$ is said to be a partially ordered set associated with the preordered lattice $(P, \preccurlyeq)$ if Q is the set of all classes, into which P is divided by the equivalence relation $x \equiv y \longleftrightarrow x \preccurlyeq y \,\&\, y \preccurlyeq x$, and $\leqslant$ is a partial ordering in Q, induced by the relation $\preccurlyeq$. If $a \in Q$ and $x \in a$, we say that x represents a. A sequence of preordered lattices $\{(P_i, \preccurlyeq_i)\}$ is said to be increasing if $P_i \subseteq P_{i+1}$ for all i and $x \preccurlyeq_i y \longrightarrow x \preccurlyeq_{i+1} y$ for all $x, y \in P_i$. We define the limit of such a sequence as the preordered lattice $(P_\omega, \preccurlyeq_\omega)$, where $P_\omega = \bigcup_{i \geqslant 0} P_i$ and $x \preccurlyeq_\omega y \longleftrightarrow \exists i\, (x \preccurlyeq_i y)$ for $x, y \in P_\omega$.

We denote by N the set of all natural numbers, by N^- the set of all odd

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numbers, by N^+ the set of all even numbers, and by $\emptyset$ the empty set. The set with the number i in the standard numbering of the class of all r.e. sets is denoted by W_i. We shall assume that a computable enumeration of the class of all partial recursive functions is given, where f_i is the function with number i in this numbering, and Γ_{f_i} is the graph of f_i. Let W_i^s be a finite set of elements, enumerated in W_i after S steps, let $\Gamma_{f_i}^s$ have the same meaning, and let f_{is} be the finite function with the graph $\Gamma_{f_i}^s$.

If $\mathcal{O}\!\mathcal{L}$ is the class of r.e. sets, then $(L(\mathcal{O}\!\mathcal{L}),\leqslant)$ will denote the upper semilattice of computable enumerations of this class. It is well known that the upper semilattice of r.e. m-degrees is isomorphic to the upper semilattices of computable enumerations of the class $\{\emptyset,\{0\}\}$. The following theorem extends the necessity of Lachlan's condition of [3] to segments of any semilattice of computable enumerations. The proof is a trivial extension of the proof of Theorem 3.1 of [3].

THEOREM 1. Let $\mathcal{O}\!\mathcal{L}$ be the class of r.e. sets, and let $a,b\in L(\mathcal{O}\!\mathcal{L})$ be such that $a\leqslant b$. Then there exists an increasing sequence of finite preordered lattices $\{(D_i,\leqslant_i)\}$ with limit $(D_\omega,\leqslant_\omega)$, such that the partially ordered set associated with $(D_i,\leqslant_\omega)$ is isomorphic to the segment $(L(\mathcal{O}\!\mathcal{L})_{ab},\leqslant_{ab})$ of the upper semilattice $(L(\mathcal{O}\!\mathcal{L}),\leqslant)$ and the following conditions hold:

1) $0,1\in D_i$, $0\leqslant_i 1$ for all i and $0\leqslant_i x\leqslant_i 1$ for all $x\in D_i$;

2) for any i, the partially ordered set, denoted by L_i, associated with $(D_i,\leqslant_i)$ is a distributive lattice;

3) for all i and all $x,y,z\in D_i$ if z is the union in L_i of the elements represented by x and y, then z is also the union in L_{i+1} of the elements represented by x and y;

4) $\{D_i\}$ is a strictly computable sequence of finite sets;

5) there exists a recursive relation R such that, for all i and all $x,y\in D_i$ we have $x\leqslant_i y \longleftrightarrow \forall u\exists v R(i,x,y,u,v)$;

6) there exist recursive functions n, u such that, for all x,y,i, if $x,y\in D_i$ then $n(i,x,y)$, $u(i,x,y)\in D_i$ and $n(i,x,y)$, $u(i,x,y)$ represent the intersection and union in L_i, respectively, of the elements represented by x and y.

Proof. Let $\beta\in b$. We shall require the operator Ψ defined as follows. Let W be an r.e. set such that $\{\beta(x)\mid x\in W\}=\mathcal{O}\!\mathcal{L}$, ω is a recursive function enumerating W. Then $\Psi(W)$ is the class of all numberings $\mathcal{O}\!\mathcal{L}$, equivalent to the numbering $\beta\omega$. For all remaining W, the value of $\Psi(W)$ is not defined. The value of $\Psi(W)$ is independent of the choice of the function ω. Let ω_1 be another recursive function, enumerating W. Then the function $\lambda x\mu y\,(\omega_1(y)=\omega(x))$ reduces $\beta\omega$ to $\beta\omega_1$. We can similarly show that $\beta\omega_1$ reduces to $\beta\omega$. It is easy to directly show that, for any A and B such that $\Psi(A),\Psi(B)$ are defined, we have $\Psi(A)\leqslant\Psi(B)$ if and only if there exists a partial recursive function f such that its domain of definition is A, $f(z)\in B$ and $\beta(z)=\beta f(z)$ for all $z\in A$. In particular, if $A\subseteq B$ then

$\psi(A) \preccurlyeq \psi(B)$. Let $a = \psi(A)$. The sequence $\{(D_i, \preccurlyeq_i)\}$ and recursive functions g_i will be defined in steps.

<u>Step 0.</u> $D_0 = \{0,1\}$, $x \preccurlyeq_0 y \longleftrightarrow (x=0 \vee y=1) \,\&\, x, y \in D_0$, $g_0(0), g_0(1)$ are the indices of $\emptyset, N$, respectively, in the standard numbering of the class of all r.e. sets. For the remaining x we put $g_0(x) = 0$.

We define the relation P_i: $(x,y) \in P_i \longleftrightarrow (x,y) \in \Gamma_{f_i} \,\&\, \forall j\, \forall k\, \exists \ell\, \exists t\, ((j,k) \in \Gamma_{f_i}^{x} \longrightarrow \beta^{x}(j) \subseteq \beta^{\ell}(k) \,\&\, \beta^{x}(k) \subseteq \beta^{t}(j))$. The basic properties of P_i are: $\forall x\, \forall y\, ((x,y) \in \Gamma_{f_i} \longrightarrow \beta(x) = \beta(y)) \longrightarrow P_i = \Gamma_{f_i}$, $\exists x \exists y\, ((x,y) \in \Gamma_{f_i} \,\&\, \beta(x) \neq \beta(y)) \longrightarrow P_i$ is finite. Let $P^i = \bigcup_{x \leq i} P_x$ and let Q_i be the least-equivalence relation containing P^i. Obviously, Q_i is recursively enumerable. We call the pair (x,y) true if $\beta(x) = \beta(y)$; otherwise, it is false. Also $\overline{W}^{Q_i} = \{x \mid \exists y\, (y \in W \,\&\, (x,y) \in Q_i\}$. Let W be an r.e. set such that $\{\beta(x) \mid x \in W\} = \mathfrak{A}$. We shall show that $\psi(\overline{W}^{Q_i}) = \psi(W)$. Since $W \subseteq \overline{W}^{Q_i}$, we have $\psi(W) \leq \psi(\overline{W}^{Q_i})$. It can easily be seen that P^i contains only a finite number of false pairs. Let $y_0, \ldots, y_k$ be all the elements of false pairs of P^i, $a_0, \ldots, a_k$, such that $\beta(a_j) = \beta(y_j)$ for $j \leq k$. For any $x \in \overline{W}^{Q_i}$ we can effectively find a sequence $x_0, x_1, \ldots, x_m$, such that $x = x_0$, (x_j, x_{j+1}) or (x_{j+1}, x_j) is a true pair of P^i for $j < m$ and $x_m \in W$ or x_m is an element of a false pair of P^i. We define

$$\sigma(x) = \begin{cases} a_j, & \text{if } x_m \in \{y_0, \ldots, y_k\},\ x_m = y_j, \\ x_m & \text{otherwise.} \end{cases}$$

Then, for all $x \in \overline{W}^{Q_i}$ the value of $\sigma(x)$ is defined, $\sigma(x) \in W$ and $\beta(x) = \beta\sigma(x)$. Hence, $\psi(\overline{W}^{Q_i}) \leq \psi(W)$.

<u>Step $i+1$.</u> Let k be the least element of $N - D_i$. As the value of $g_{i+1}(k)$ we take the index $W_i \cup A$, which can be effectively found with respect to i; for $x \in D_i$ as the value of $g_{i+1}(x)$ we take the index $\overline{W}^{Q_i}_{g_i(x)}$. We number all the classes, consisting of subsets $D_i \cup \{k\}$, which cannot be compared with respect to inclusion; for this we use the necessary initial segment of $N - (D_i \cup \{k\})$. Let D be this segment and C_x the class with number x. We define $D_{i+1} = D_i \cup \{k\} \cup D$, and for $x \in D$ we take as the value of $g_{i+1}(x)$ the index $\bigcup\{\bigcap\{W_{g_{i+1}(y)} \mid y \in X\} \mid X \in C_x\}$. For all remaining x we define $g_{i+1}(x) = 0$. We define

$$x \preccurlyeq_{i+1} y \longleftrightarrow W_{g_{i+1}(x)} \subseteq W_{g_{i+1}(y)} \,\&\, x, y \in D_{i+1}.$$

Clearly, the partially ordered set associated with $(D_{i+1}, \preccurlyeq_{i+1})$ is isomorphic to the distributive lattice generated by the sets $W_{g_i(x)}$ for $x \in D_i \cup \{k\}$. Let $x, y, z \in D_i$ and $W_{g_i(x)} \cup W_{g_i(y)} = W_{g_i(z)}$; then, $\overline{W}^{Q_i}_{g_i(x)} \cup \overline{W}^{Q_i}_{g_i(y)} = \overline{W}^{Q_i}_{g_i(z)}$, i.e., $W_{g_{i+1}(x)} \cup W_{g_{i+1}(y)} = W_{g_{i+1}(z)}$. This means that condition 3) is satisfied. Also, $W_{g_i(x)} \subseteq W_{g_i(y)}$ is a $\forall\exists$ relation, so that condition 5) is satisfied. It can easily be shown that all the other conditions likewise hold, and $D_\omega = \bigcup_{i \geq 0} D_i = N$. Let us show that the partially ordered set associated with $(D_\omega, \preccurlyeq_\omega)$ is isomorphic to the segment $(L(\mathfrak{A})_{ab}, \preccurlyeq_{ab})$. We define $\chi(x) = \psi(W_{g_i(x)})$ where i is any number such that $x \in D_i$. From what has been proved above, the value of $\chi(x)$ is independent of the choice of i; here, $a \preccurlyeq \chi(x) \preccurlyeq b$ for all x.

Let $x \preccurlyeq_\omega y$. Then $W_{g_i(x)} \subseteq W_{g_i(y)}$ for all sufficiently large i, whence $\psi(W_{g_i(x)}) \leqslant \psi(W_{g_i(y)})$. Hence, $\chi(x) \leqslant \chi(y)$.

Let $\chi(x) \leqslant \chi(y)$. Then there exist i and t such that $x, y \in \mathcal{D}_i$; for any $z \in W_{g_i(x)}$ the value of $f_t(z)$ is defined, $f_t(z) \in W_{g_i(y)}$, and all pairs Γ_{f_t} are true. Let $j \geqslant i, t$. Then $\Gamma_{f_t} = P_t \subseteq Q_j$, so that

$$W_{g_{j+1}(x)} = \overline{W}^{Q_j}_{g_i(x)} \subseteq \overline{W}^{Q_j}_{g_i(y)} = W_{g_{j+1}(y)},$$

i.e., $x \preccurlyeq_{j+1} y$. Hence, $x \preccurlyeq_\omega y$. Let $x \in L(\mathfrak{A})_{ab}$. Then, $x = \psi(W_i) = \psi(W_i \cup A)$ for some i. By construction, for $k \in \mathcal{D}_{i+1} - \mathcal{D}_i$ we have $W_{g_{i+1}(k)} = W_i \cup A$. Then, $\chi(k) = \psi(W_{g_{i+1}(k)}) = x$. Hence, it follows that the mapping χ induces the necessary isomorphism.

QED.

Let $(L, \leqslant)$ be an upper semilattice of r.e. m-degrees, and let $a \cup b$ be the least upper bound of elements a and b in $(L, \leqslant)$.

<u>THEOREM 2.</u> Let a be an incomplete r.e. m-degree, and let $\{(\mathcal{D}_i, \preccurlyeq_i)\}$ be an increasing sequence of finite preordered lattices, satisfying conditions 1-6) of Theorem 1, with the limit $(\mathcal{D}_\omega, \preccurlyeq_\omega)$. Then there exists an r.e. m-degree u such that:

1) $a \leqslant u$;

2) for any r.e. m-degree x, if $x \leqslant u$, then $x \leqslant a$ or $a \leqslant x$;

3) the segment $(L_{au}, \leqslant_{au})$ of the upper semilattice of the r.e. m-degree $(L, \leqslant)$ is isomorphic to the partially ordered set associated with $(\mathcal{D}_\omega, \preccurlyeq_\omega)$

<u>Proof.</u> Let $\{(\mathcal{D}_i, \preccurlyeq_i)\}$ be an increasing sequence of finite preordered lattices, satisfying the conditions 1-6) of Theorem 1, and let $(\mathcal{D}_\omega, \preccurlyeq_\omega)$ be its limit. If $1 \preccurlyeq_\omega 0$ then the partially ordered set associated with $(\mathcal{D}_\omega, \preccurlyeq_\omega)$ has a single element, and $u = a$ satisfies the condition of the theorem. Henceforth we shall assume that $1 \not\preccurlyeq_\omega 0$. Let $A \in a$ and let K be a creative set, $K \not\leqslant_m A$.

In order to describe the construction, we shall repeat some definitions of [3] (see [3] for more details).

A family of nonempty nonintersecting subsets of a set F, the union of which is equal to F, is called a division of F. Let $\mathcal{P}$ and Q be divisions of the same set; then $\mathcal{P}$ is finer than Q, if every element of Q is the union of several elements of $\mathcal{P}$.

A tower of order n is a collection $(\mathcal{P}_n, \dots, \mathcal{P}_0, \varphi_n, \dots, \varphi_0)$, where $\mathcal{P}_n, \dots, \mathcal{P}_0$ are divisions of the finite set F, called the base of the tower, and φ_i are mappings of $\mathcal{P}_i$ into the set of all subsets $\mathcal{D}_i$ such that:

(T1) $\mathcal{P}_n$ contains a single element, and $\mathcal{P}_i$ is finer than $\mathcal{P}_{i+1}$ for $i < n$;

(T2) for any $i < n$, $P \in \mathcal{P}_{i+1}$, $A = \varphi_{i+1}(P)$ and $P_0, \dots, P_k \in \mathcal{P}_i$ such that $\bigcup_{j \leqslant k} P_j = P$, the class $\{\varphi_i(P_0), \dots, \varphi_i(P_k)\}$ consists of sets $\bigcup_{j \leqslant k} \varphi_i(P_j) = A \cap \mathcal{D}_i$ pairwise incomparable with respect to inclusion, and this class is the unique subclass of $\varphi_i(\mathcal{P}_i)$, possessing these properties.

A basis of order n is a collection $(\mathcal{A}_n,\dots,\mathcal{A}_0)$, where $\mathcal{A}_i$ is a class of subsets $\mathcal{D}_i$ such that:

(F1) $\mathcal{A}_n$ consists of a single element;

(F2) for any $i<n$ and $A\in\mathcal{A}_{i+1}$ there exists a unique subclass $\mathcal{A}_i(A)$ in $\mathcal{A}_i$, consisting of sets which are incomparable with respect to inclusion, the union of which is equal to $A\cap\mathcal{D}_i$;

(F3) $\mathcal{A}_i = \bigcup\{\mathcal{A}_i(A)\mid A\in\mathcal{A}_{i+1}\}$ for $i<n$.

An associated basis of a tower $(\mathcal{P}_n,\dots,\mathcal{P}_0,\varphi_n,\dots,\varphi_0)$ is defined as a basis $(\varphi_n(\mathcal{P}_n),\dots,\varphi_0(\mathcal{P}_0))$ (it is easily verified that this collection satisfies (F1) - (F3)).

It was shown in [3] how, given any basis $(\mathcal{A}_n,\dots,\mathcal{A}_0)$, we can construct a tower for which this basis is an associated basis. This is done as follows. Let

$$S=\{(A_n,\dots,A_0)\mid A_n\in\mathcal{A}_n,\ A_i\in\mathcal{A}_i(A_{i+1})\}\qquad \text{for } i<n,$$

let P be the number of elements of S, and let $F=\{a_1,\dots,a_p\}$ be any finite set of natural numbers containing p elements. We enumerate S with the numbers from 1 to p. Let $(A_n^j,\dots,A_0^j)$ be the element of S with the number j. For $i\leq n$ we define

$$\mathcal{P}_i = \{\{a_x\mid \forall y(i\leq y\leq n \to A_y^x = A_y)\}\mid (A_n,\dots,A_0)\in S\}$$

and

$$\varphi(\{a_x\mid \forall y(i\leq y\leq n \to A_y^x = A_y)\})=A_i \quad \text{for } (A_n,\dots,A_0)\in S.$$

It is easily shown that F is the base of the tower $(\mathcal{P}_n,\dots,\mathcal{P}_0,\varphi_n,\dots,\varphi_0)$ while $(\mathcal{A}_n,\dots,\mathcal{A}_0)$ is an associated basis.

We say that the tower $\mathcal{T}=(\mathcal{P}_m,\dots,\mathcal{P}_0,\varphi_m,\dots,\varphi_0)$ is compatible with the tower $\mathcal{U}=(Q_n,\dots,Q_0,\psi_n,\dots,\psi_0)$, if their bases do not intersect, and for arbitrary $i\leq m,n$, $P\in\mathcal{P}_i$, $Q\in Q_i$ such that $\psi_i(Q)\subseteq\varphi_i(P)$, there exists a tower

$$\mathcal{T}^*=(\mathcal{P}_m^*,\dots,\mathcal{P}_0^*,\varphi_m^*,\dots,\varphi_0^*)$$

such that:

(M1) the base of $\mathcal{T}^*$ is $F\cup Q$, where F is the base of $\mathcal{T}$;

(M2) $\mathcal{P}_i^* = (\mathcal{P}_i-\{P\})\cup\{P\cup Q\}$;

(M3) for arbitrary $j\leq m$ and $X\in\mathcal{P}_j$ there exists a unique $X^*\in\mathcal{P}_j^*$ such that $X\subseteq X^*$ and $\varphi_j(X)=\varphi_j^*(X^*)$, and for arbitrary $X^*\in\mathcal{P}_j^*$ there exists a unique $X\in\mathcal{P}_j$ such that $X\subseteq X^*$;

(M4) for arbitrary $j<i$ and $X\in Q_j$ such that $X\subseteq Q$ there exists $X^*\in\mathcal{P}_j^*$ such that $X\subseteq X^*$ and $\varphi_j(X)\subseteq\varphi_j^*(X^*)$.

Given any two towers $\mathcal{T}$ and $\mathcal{U}$, we can effectively discover whether $\mathcal{T}$ is comparible with $\mathcal{U}$ and given the i,P,Q described above, we can find the tower $\mathcal{T}^*$, satisfying (M1-M4), in the case when $\mathcal{T}$ is compatible with $\mathcal{U}$.

Let n, u be recursive functions satisfying the hypotheses of the theorem. Given any $A \subseteq D_i$

$$u(A,i) = \begin{cases} 0, & \text{if } A=\emptyset, \\ a_1, & \text{if } A=\{a_1\}, \\ u(i,a_1,u(i,a_2,\dots u(i,a_{k-1},a_k)\dots), & \text{if } A=\{a_1,\dots,a_k\}, k>1. \end{cases}$$

We define $n(A,i)$ similarly. For $A \subseteq D_i$

$$mod_s(A,i) = min(\{z \mid (\forall w<s) \neg R(i,n(A,i),u(D_i - A,i),z,w)\} \cup \{s\}).$$

Let p_n be the n-th prime number. Given any basis $\mathcal{T}=(\mathcal{A}_n,\dots \mathcal{A}_0)$, its modulus at the step s is $mod_s \mathcal{T} = p_n^{t_n+1}$ where $t_n = \mu x [p_n^x > max\{mod_s(A,i) \mid A \in \mathcal{A}_i \ , \ i \leq n\}]$. The modulus of the tower at the step s is defined as the modulus of the basis associated with it at this step.

We shall assume that an effective enumeration of the set $\{(m,n,e) \mid m,n \in D_\omega\}$ is given, in which the triple with number k is denoted by (m_k,n_k,e_k) and which is such that, for all k, $m_k, n_k \in D_k$. We are also given an effective unique enumeration of all pairs, where $c(i,j)$ is the number of the pair (i,j) in this enumeration.

CONSTRUCTION

Step 0. We put $f(x,0) = 0$ for all x, $U^0 = D^0 = \emptyset$. After the instructions of the step $s-1$ $(s \geq 1)$ have been performed, there may exist a finite number of towers, the bases of which do not intersect. The base of each of these towers contain only even numbers, and $\subseteq U^{s-1}$ or $\subseteq N - U^{s-1}$. Some towers may have equal orders; for any two such towers, the associated bases are identical. All towers of the same order are numbered by the numbers forming the initial segment of the natural series. A number is said to be used at the step s if it is located in the base of the tower constructed at a step $<s$; otherwise, it is said to be unused at the step s. A tower exists at the step s if it was constructed at a step $<s$ and was not destroyed at a step $<s$. A number is discarded at the step s if it was used at a step s and is not located in the base of the tower existing at the step s. The finite set $F(i,j)$ is defined at the step s if the value of $F(i,j)$ was defined in the construction at a step $<s$. Let D^{s-1} be the set of all numbers discarded at the step s, and let U^{s-1} be the set of all numbers enumerated in U after $<s$ steps of the construction. Denote by K^s, A^s the finite parts of the r.e. sets K, A, evaluated after s steps.

Let $p_k = min\{i \mid m_k, n_k \in D_i\}$, $p_k \leq k$.

The number k requires attention at the step s if the function $f_{e_k,s}$ is defined on the set $\{0,1,\dots f(k,s-1)\} \cap U\{P \mid$ at the step s there exists the tower $\mathcal{T}=(\mathcal{P}_m,\dots \mathcal{P}_0,\varphi_m,\dots,\varphi_0)$ such that $P \in \mathcal{P}_{p_k}$ and $m_k \in \varphi_{p_k}(P)\}$ and for all x of this set $f_{e_k,s}(x) \in U\{P \mid$ at the step s there exists the tower $\mathcal{T}=(\mathcal{P}_m,\dots,\mathcal{P}_0,\varphi_m,\dots,\varphi_0)$ such that $P \in \mathcal{P}_{p_k}$ and $n_k \in \varphi_{p_k}(P)\} \cup N^- \cup D^{s-1} \cup U^{s-1}$ and $x \in U^{s-1} \longleftrightarrow f_{e_k,s}(x) \in U^{s-1}$.

The tower $\mathcal{T} = (\mathcal{P}_m,\dots,\mathcal{P}_0,\varphi_m,\dots,\varphi_0)$ requires attention at the step s from the side of i if it exists at the step s, and there exists at the step s the tower $\mathcal{U}=(Q_n,\dots,Q_0,\psi_n,\dots,\psi_0)$ and there exist $P\in\mathcal{P}_i$, $Q\in Q_i$, such that

(1) $mod_s\,\mathcal{T} < mod_s\,\mathcal{U}\;\&\;\psi_i(Q)\subseteq\varphi_i(P)\;\&\;P\subseteq N-\mathcal{U}^{s-1}\;\&\;Q\subseteq N-\mathcal{U}^{s-1}\;\&\;P\cap W_i^s=\emptyset\;\&\;Q\cap W_i^s\neq\emptyset\;\&$ ($\mathcal{U}$ was constructed later than $\mathcal{T}$) & (for every tower $\mathcal{L}$ existing at the step s, if $mod_s\,\mathcal{L} < mod_s\,\mathcal{U}$, then $\mathcal{L}$ is compatible with $\mathcal{U}$).

<u>Step $s\,(s\geqslant 1)$.</u> One of the following five cases must always hold.

<u>Case 1.</u> At the step s there exists at any rate one tower $\mathcal{T}$ for which $mod_s\,\mathcal{T} > mod_{s-1}\,\mathcal{T}$ or $mod_s\,\mathcal{T} > mod_{s-1}\,\mathcal{T}^-$, if $\mathcal{T}^-$ was modified into $\mathcal{T}$ at the step $s-1$. In this case we destroy each such tower.

<u>Case 2.</u> Case 1 does not hold, and there exists at least one number k, requiring attention at the step s, and such that, at the step s, there exist precisely $f(k,s-1)+1$ towers of order k. Let k_0 be the k for which $mod_s\,\mathcal{T}$ is least, where $\mathcal{T}$ is a tower of order k, existing at the step s. We define $f(k_0,s)=f(k_0,s-1)+1$ and destroy each tower $\mathcal{U}$ existing at the step s, for which $mod_s\,\mathcal{U} > mod_s\,\mathcal{T}_0$, where $\mathcal{T}_0$ is a tower of order k_0, existing at the step s.

<u>Case 3.</u> Cases 1 and 2 do not hold, and there exists a tower requiring attention at the step s. We choose such a tower $\mathcal{T}$ at the least $mod_s\,\mathcal{T}$, and let i be the least number, from the side of which $\mathcal{T}$ requires attention. Let $\mathcal{T}=(\mathcal{P}_m,\dots,\mathcal{P}_0,\varphi_m,\dots,\varphi_0)$, and let $\mathcal{U}=(Q_n,\dots,Q_0,\psi_n,\dots,\psi_0)$ be a tower, existing at the step s, while $P\in\mathcal{P}_i$, $Q\in Q_i$ are such that (1) is satisfied, and in particular, $\mathcal{T}$ is compatible with $\mathcal{U}$. We find the tower $\mathcal{T}^*=(\mathcal{P}_m^*,\dots,\mathcal{P}_0^*,\varphi_m^*,\dots,\varphi_0^*)$, satisfying (M1-M4).

We replace $\mathcal{T}$ by $\mathcal{T}^*$, and call this replacement the modification of $\mathcal{T}$ by the number i at the step s. Henceforth, as in [3], we shall allow some inaccuracy for the sake of simplicity, and regard $\mathcal{T}$ and $\mathcal{T}^*$ as the same tower, undergoing change at the step s. We destroy all the towers of order n and say that they are destroyed by the number i.

<u>Case 4.</u> Cases 1-3 do not hold, and there exists at least one pair (i,j) such that $F(i,j)$ is not defined at the step s, and such that there exists, at the step s, a tower $\mathcal{T}$ of order greater than $c(i,j)$ with base F such that $F\cap W_i^s\neq\emptyset$ and $F\subseteq N-\mathcal{U}^{s-1}$. In this case we choose a pair (i,j) with the least number and a tower $\mathcal{T}$ of the least order, and we define $F(i,j)=F$, where F is the base of $\mathcal{T}$. We destroy every tower existing at the step s, the order of which is equal to the order of $\mathcal{T}$.

<u>Case 5.</u> Cases 1-4 do not hold. In this case, among all k and the bases of $\mathcal{T}=(\mathcal{A}_k,\dots,\mathcal{A}_0)$ such that $m_k\in A, n_k\notin A$, where $\mathcal{A}_k=\{A\}$, and at the step s there exist less than $f(k,s-1)+1$ towers of order k, we find those k and the basis of $\mathcal{T}$ of order k such that $mod_s\,\mathcal{T}$ is minimal (the fact that such k and $\mathcal{T}$ exist is easily proved by using Lemma 1 (see below) and the fact that $1\not\leq_k 0$ for all k). It can easily be seen that, by definition of $mod_s\,\mathcal{T}$ just a finite number of bases needs

to be considered in order to find the required k and $\mathcal{T}$.

If no tower of order s exists at the step k, we construct $f(k,s-1)+1$ towers on the basis $\mathcal{T}$ in the way indicated above, using as their bases the necessary nonintersecting segments, the union of which is the initial segment of the even numbers unused at the step s. If there exists at the step s at least one tower $\mathcal{U}$ of order k, then we construct the towers that are missing (so as to get a total of $f(k,s-1)+1$ towers) in the basis associated with $\mathcal{U}$ choosing sets of bases for them in the same way as was done in Case 1. Suppose that there existed $j \geqslant 0$ towers of order k at the step S. We number the constructed towers by the numbers $j, j+1, \dots, f(k,s-1)$.

After analyzing all the cases, we proceed as follows. We take each k, and each k-th order tower existing at the step s and not destroyed at this step, and let j be the order number of this tower among those of order k existing at step s; if $j \in K^s$, we enumerate all the elements of the base of the tower in $\mathcal{U}$. For every pair (i,j) such that $F(i,j)$ is defined at the step s, if $j \in A^s$, we refer all the elements of $F(i,j)$ to $\mathcal{U}$. For every $x \in A^s$ we refer $2x+1$ elements to $\mathcal{U}$. For all z, if $f(z,s)$ is not yet defined, we define it as $f(z,S)=f(z,s-1)$. We pass to step $s+1$.

The instructions of step s are not precise (single-valued) in some places. However, it is easy to make them precise in some suitable way. We shall assume that this has been done.

We have

$$\mathcal{U} = \bigcup_{S \geqslant 0} \mathcal{U}^s, \quad \mathcal{D} = \bigcup_{s \geqslant 0} \mathcal{D}^s.$$

In the light of what has been said $\mathcal{U}, \mathcal{D}$ are r.e. sets. Also, let Ψ be the operator defined in [3]: $\Psi(\emptyset)=0$ (a recursive m-degree), and for any nonempty r.e. set W, the set $\Psi(W)$ is an m-degree of the r.e. set $\{x \mid \omega(x) \in \mathcal{U}\}$, where ω is any recursive function, enumerating W. By definition of $\mathcal{U}$, $\Psi(N^-)=a$. Let us show that $\Psi(\mathcal{D}) \leq a$. For any $x \in \mathcal{D}$, we find S such that $x \in \mathcal{D}^s - \mathcal{D}^{s-1}$. If, at the $(s+1)$-th step, $x \in F(i,j)$ for certain i,j (there can only be one such pair (i,j)), then we define $g(x)=2j+1$; otherwise, we define $g(x)=2a+1$ if $x \in \mathcal{U}^s$, $g(x)=2b+1$, if $x \notin \mathcal{U}^s$ where $2a+1 \in \mathcal{U}$, $2b+1 \notin \mathcal{U}$ (we assume that $A \neq N$ and $A \neq \emptyset$). Then, for all $x \in \mathcal{D}$, the value of $g(x)$ is defined, $g(x) \in N^-$ and $x \in \mathcal{U} \longleftrightarrow g(x) \in \mathcal{U}$. Hence, $\Psi(\mathcal{D}) \leq \Psi(N^-)=a$.

We shall slow by means of Lemmas 1-9 below that the m-degree u of the set $\mathcal{U}$ satisfies the condition of our theorem. Most of the lemmas are proved in the same way as the corresponding propositions in [3], so that their proofs will be omitted.

Let $m,n \in \mathcal{D}_k$. The basis $\mathcal{T}=(\mathcal{A}_k, \dots, \mathcal{A}_0)$ separates m from n, if $m \in A$, $n \notin A$, where $\mathcal{A}_k=\{A\}$.

<u>LEMMA 1.</u> A basis $\mathcal{T}=(\mathcal{A}_k, \dots, \mathcal{A}_0)$ separating m_k from n_k and such that $\lim_x mod_x \mathcal{T} < \infty$, exists if and only if $m_k \not\leq_k n_k$.

The proof is the same as that of Proposition 2 of [3].

LEMMA 2. Every tower may be modified at most at a finite number of steps.

The proof is the same as that of Proposition 3 of [3].

A tower is said to be constant if it is constructed at a certain step and is not later destroyed. A tower is called final if it is constant and is not further modified during the construction. By Lemma 2, every constant tower becomes a final tower after a finite number of modifications. If $\mathcal{U}$ is a constant tower, existing at the step s_0, then $mod_s\,\mathcal{U} = mod_{s_0}\mathcal{U}$ for all $s \geqslant s_0$. Assume that, in this case, $mod\,\mathcal{U} = \lim_s mod_s\,\mathcal{U}$.

LEMMA 3. For any k, if $m_k \not\leqslant_k n_k$, then $\lim_s f(k,s) < \infty$, while there exist $\lim_s f(k,s) + 1$ constant towers of order k, and at all sufficiently large steps, k does not require attention.

Proof. Let us show as a preliminary that Case 5 holds at an infinite number of steps. Assume the contrary. Then only a finite number of towers will be constructed during the construction. Hence, only a finite number of them can be destroyed. If, at some step, one of Cases 1, 3, 4 holds, then at least one tower will be destroyed at this step. Hence, each of these cases can only hold at a finite number of steps. This means that Case 2 must hold at an infinite number of steps. But it can easily be seen that in that case, there must exist an infinite number of towers in the course of the construction. This contradiction shows that Case 5 holds at an infinite number of steps.

We define $\beta(k) = \inf\{\lim_x mod_x\,\mathcal{T} \mid \mathcal{T}$ is a basis of order k separating m_k from $n_k\}$ (we assume that $\inf\varnothing = \infty$). Then, by Lemma 1, $m_k \not\leqslant_k n_k \longleftrightarrow \beta(k) < \infty$. If $\beta(k) < \infty$, then $\beta(k) = p_k^t$ for some $t > 0$, so that, if $\beta(x), \beta(y) < \infty$ and $x \neq y$, then $\beta(x) \neq \beta(y)$. Notice that, since the basis with least modulus is chosen in Case 5, we have $mod\,\mathcal{T} = \beta(k)$ for every constant tower $\mathcal{T}$ of order k.

The lemma will be proved by induction on $\beta(k)$. Let $\beta(k) < \infty$ and let the lemma hold for all x such that $\beta(x) < \beta(k)$. Let t be such that, for all x such that $\beta(x) < \beta(k)$, and all $s \geqslant t$, we have $f(x,s) = f(x,t)$; then all $f(x,t) + 1$ constant towers of order x will be in existence and will not be modified at the step s, and x requires no attention at the step s.

Let y be such that $\beta(y) > \beta(k)$. Every tower $\mathcal{U}$ of order y, existing at the step s such that $mod_s\,\mathcal{U} \leqslant \beta(k)$, will henceforth be destroyed in accordance with Case 1. For any basis of $\mathcal{U}$ of order y, we have $mod_s\,\mathcal{U} > \beta(k)$ for all sufficiently large s. There are only a finite number of bases of order y. For all bases $\mathcal{U}$ of sufficiently large orders, $mod_s\,\mathcal{U} > \beta(k)$ for all s. It follows from all this that t can be increased in such a way that $\forall y\,[\beta(y) > \beta(k) \longrightarrow (\forall s \geqslant t)$ (at the step s there exists no tower $\mathcal{U}$ of order y for which $mod_s\,\mathcal{U} \leqslant \beta(k)$]. The number t can also be increased so that, for all bases $\mathcal{U}$ of order k, we have $mod_s\,\mathcal{U} > \beta(k)$ for all $s \geqslant t$ or $mod_s\,\mathcal{U} = \beta(k)$ for all $s \geqslant t$, the second being true for at any rate one of these bases. We also increase t so that, for any pair

(i,j) such that $c(i,j) < k$, the quantity $F(i,j)$ is defined at the step t or else $F(i,j)$ is never defined at all during the construction. Now suppose that, at some step $s > t$, there exists no tower of order k, and let s' be the first step $\geqslant s$ at which Case 5 holds. Then, in accordance with the choice of t, at least one tower $\mathcal{T}$ of order k will be constructed at this step, for which $mod_\ell \mathcal{T} = \beta(k)$ for all $\ell \geqslant s'$. Clearly, $\mathcal{T}$ cannot be destroyed at steps $\ell \geqslant s'$ in accordance with Case 1.

By the choice of t, $\mathcal{T}$ cannot be destroyed in accordance with Case 4 at these steps. Also, $\mathcal{T}$ cannot be destroyed at a step $\ell \geqslant s'$, in accordance with Case 2, since, for this, there must exist at the step ℓ a tower $\mathcal{U}$ of order x, for which $mod_\ell \mathcal{U} < mod_\ell \mathcal{T} = \beta(k)$, and x must require attention at the step ℓ, which contradicts our inductive assumption, since it follows from this inequality and $\ell > t$ that $\beta(x) < \beta(k)$. Further, the tower $\mathcal{T}$ cannot be destroyed in accordance with Case 3 at a step $\ell \geqslant s'$, since, for this, some tower $\mathcal{U}$ of order x must exist and be modified at the step ℓ, for which $mod_\ell \mathcal{U} < mod_\ell \mathcal{T}$, which is impossible, since $\beta(x) < \beta(k)$. In short, the tower $\mathcal{T}$ constructed at the step s' is constant. It follows from these arguments that at least one constant tower of order k exists.

Suppose that a tower $\mathcal{T}^1$ of order k is constructed for the first time at the step s_1. Then, at steps $s \geqslant s_1$, towers of order k will not be destroyed. As we pointed out, $mod\, \mathcal{T}^1 = \beta(k)$. If, at a step $s > s_1, t$, at which Case 5 holds, there exist less than $f(k, s-1) + 1$ towers of order k, then, since $s > t$, all the missing towers will be constructed at this step. It is easily seen from this that, if $\lim_s f(k,s) < \infty$ then there exist $\lim_s f(k,s) + 1$ constant towers of order k, while if $\lim_s f(k,s) = \infty$, then there will exist infinitely many such towers. Let k require attention at a step $s > t$, and let s' be the step $\geqslant s$ at which Case 1 does not hold for the first time. Then Case 2 will hold at the step s' and $f(k,s') \neq f(k,s'-1)$. It is easily seen from this that k requires attention at only a finite number of steps if and only if $\lim_s f(k,s) < \infty$. It thus remains to show that $\lim_s f(k,s) < \infty$. Assume the contrary. Then there exist infinitely many constant towers of order k. For an infinite number of distinct s, we have $f(k,s) \neq f(k,s-1)$, so that, with this k, Case 2 holds at an infinite number of steps. Hence, there cannot exist a constant tower $\mathcal{U}$ for which $mod\, \mathcal{U} > mod\, \mathcal{T}^1 = \beta(k)$. Let $\mathcal{T}_1, \dots, \mathcal{T}_p$ be all the final towers $\mathcal{T}$ for which $mod\, \mathcal{T} < \beta(k)$ (by the induction assumption, there is a finite number of them). Every tower of order k, constructed at a step $\geqslant s_1$, is constant. Each such tower receives a certain number. Let $\mathcal{T} = (\mathcal{P}_k, \dots, \mathcal{P}_0, \varphi_k, \dots, \varphi_0)$ be such a tower with the number i, constructed at a step $s_i \geqslant s_1$, and let x_i be the least x such that, at the step s_i, there exists $P \in \mathcal{P}_{\rho_k}$ such that $m_k \in \varphi_{\rho_k}(P)$ and $x \in P$. It is easily seen that the function $\lambda_i x_i$ is recursive. By construction, for all i we have $i \in K \leftrightarrow x_i \in \mathcal{U}$. Since $\lim_s f(k,s) = \infty$ and k requires attention at an infinite number of steps, then $f_{e_k}(x_i)$ is defined and $x_i \in \mathcal{U} \leftrightarrow f_{e_k}(x_i) \in \mathcal{U}$ for all i. In addition, if $f_{e_k}(x_i) \notin \mathcal{U}$, then $f_{e_k}(x_i)$ cannot be located in the base of a final tower of order k, since, for any such tower

$\mathcal{U} = (\mathcal{Q}_k, \ldots, \mathcal{Q}_0, \psi_k, \ldots, \psi_0)$ and for any $Q \in \mathcal{Q}_{p_k}$ we have $n_k \notin \psi_{p_k}(Q)$. It is easily shown that every even number is discarded or is located in the base of some final tower. Hence, $f_{e_k}(x_i)$ is odd, or is discarded, or belongs to $\mathcal{U}$, or is located in the base of one of the towers $\mathcal{T}_1, \ldots, \mathcal{T}_p$. Let $a \in A$, for $t \leq p$ let j_t be such that $j_t \in A \longleftrightarrow$ the base of $\mathcal{T}_t$ is contained in $\mathcal{U}$, and let g be a partial recursive function such that for all $x \in \mathcal{D}$, the value of $g(x)$ is defined, $g(x) \in N^-$ and $x \in \mathcal{U} \longleftrightarrow g(x) \in \mathcal{U}$, where the $\mathcal{U}$ was defined above. We fix certain enumerations of $\mathcal{D}$ and $\mathcal{U}$. We define

$$z(i) = \begin{cases} j_t, & \text{if } f_{e_k}(x_i) \text{ belongs to the base } \mathcal{T}_t,\ t \leq p; \\ \frac{1}{2}(f_{e_k}(x_i) \dot{-} 1), & \text{if } f_{e_k}(x_i) \text{ is odd.} \\ \frac{1}{2}(g f_{e_k}(x_i) \dot{-} 1), & \text{if the first two cases do not hold and } f_{e_k}(x_i) \text{ is enumerated in } \mathcal{D} \text{ earlier than in } \mathcal{U}; \\ a & \text{otherwise.} \end{cases}$$

It is easily shown that, for all i, we have $i \in K \longleftrightarrow x_i \in \mathcal{U} \longleftrightarrow f_{e_k}(x_i) \in \mathcal{U} \longleftrightarrow z(i) \in A$, and hence $K \leq_m A$. This is a contradiction, and hence $\lim_s f(k,s) < \infty$.

QED.

For all x, i such that $x \in \mathcal{D}_i$, we define $R_{x,i} = \mathcal{U} \cup \mathcal{D} \cup \bigcup \{ P \mid$ there exists the final tower $\mathcal{T} = (\mathcal{P}_m, \ldots, \mathcal{P}_0, \varphi_m, \ldots, \varphi_0)$ such that $P \in \mathcal{P}_i$ and $x \in \varphi_i(P) \}$.

<u>LEMMA 4.</u> For any x, i such that $x \in \mathcal{D}_i$, $R_{x,i}$ is an r.e. set.

The proof is the same as that of Proposition 8 of [3]. Notice that, to prove this proposition, and also Propositions 9 and 12 of [3], use was made in [3] of Propositions 5-7. These latter also apply in our case. There are some slight inaccuracies in the statements of these propositions in [3], and also in the relevant definitions. They are easily eliminated if, in the statements of Propositions 5 and 6, and also in the relevant definitions, we replace the phrase "final tower" by "final tower with base $\subseteq N - \mathcal{U}$." Further, in order for Proposition 7 to perform its task, it should be stated as follows: "There is at most a finite number of steps s such that some tower, i, satisfied at some step $\leq s$, is modified at the step s by the number i."

All these refinements to the statements of the proportions in [3] demand only obvious refinements to the proofs, which may be left to the reader.

<u>LEMMA 5.</u> If $x \in \mathcal{D}_{i-1}$, then $\Psi(R_{x,i-1}) = \Psi(R_{x,i})$.

The proof is the same as that of Proposition 9 of [3].

It follows from Lemmas 4 and 5 that we can correctly define $u_x = \Psi(R_{x,i} \cup N^-)$ for any $x \in \mathcal{D}_\omega$, where i is any number such that $x \in \mathcal{D}_i$. For any $x \in \mathcal{D}_\omega$ we have $a \leq u_x \leq u$.

<u>LEMMA 6.</u> Let $m, n, p \in \mathcal{D}_\omega$ and let p represent the union in the semilattice, associated with $(\mathcal{D}_\omega, \preceq_\omega)$ of elements represented by m and n. Then $u_m \cup u_n = u_p$.

The proof is the same as that of Proposition 10 of [3].

LEMMA 7. If $m,n \in \mathcal{D}_\omega$ and $m \not\leq_\omega n$, then $u_m \not\leq u_n$.

Proof. Let $m,n \in \mathcal{D}_\omega$ and $m \not\leq_\omega n$.

$$u_m = \Psi(R_{mp} \cup N^-),\ u_n = \Psi(R_{np} \cup N^-),$$

where $p = \min\{i \mid m,n \in \mathcal{D}_i\}$.

Suppose that $u_m \leq u_n$. Then there exists a partial recursive function f such that, for all $x \in R_{mp} \cup N^-$, the value of $f(x)$ is defined, $f(x) \in R_{np} \cup N^-$ and $x \in \mathcal{U} \longleftrightarrow f(x) \in \mathcal{U}$. Let k be such that $m_k = m$, $n_k = n$, $f_{e_K} = f$. Since $m \not\leq_K n$, we have, by Lemma 3, $\lim_s f(k,s) < \infty$ and at all sufficiently large steps, k does not require attention. Let $f(k,s) = f(k,s_0)$ and suppose that k does not require attention at step s for all $s \geq s_0$. At a sufficiently large step, any even number will be discarded or will be enumerated in the base of a final tower. Hence, for all sufficiently large s, $\{0,1,\dots,f(k,s_0)\} \cap U\{Q|$ at step s, there exists the tower $\mathcal{U} = (Q_t,\dots,Q_0,\psi_t,\dots,\psi_0)$ such that $Q \in Q_p$ and $m \in \psi_p(Q)\} = \{0,1,\dots,f(k,s_0)\} \cap U\{Q|$, there exists the final tower $\mathcal{U} = (Q_t,\dots,Q_0,\psi_t,\dots,\psi_0)$ such that $Q \in Q_p$ and $m \in \psi_p(Q)\} = P \subseteq R_{mp}$ and $\{0,1,\dots,c\} \cap (U\{Q|$ at step s, there exists the towers $\mathcal{U} = (Q_t,\dots,Q_0,\psi_t,\dots,\psi_0)$ such that $Q \in Q_p$ and $n \in \psi_p(Q)\} \cup N^- \cup \mathcal{D}^{s-1} \cup U^{s-1}) = \{0,1,\dots,c\} \cap (U\{Q|$, there exists the final tower $\mathcal{U} = (Q_t,\dots,Q_0,\psi_t,\dots,\psi_0)$ such that $Q \in Q_p$ and $n \in \psi_p(Q)\} \cup N^- \cup \mathcal{D} \cup \mathcal{U}) = P_1 \subseteq R_{np} \cup N^-$, where $c = \max\{f(x) \mid x \in P\}$. By the property of f, for all sufficiently large s and all $x \in P$, the value of $f_{e_K,s}(x)$ is defined, $f_{e_K,s}(x) \in P_1$ and $x \in \mathcal{U}^{s-1} \longleftrightarrow f_{e_K,s}(x) \in \mathcal{U}^{s-1}$, i.e., k requires attention at sufficiently large steps. This contradiction proves the lemma.

LEMMA 8. For any i such that $a \leq \Psi(W_i)$, there exists $x \in \mathcal{D}_i$, such that $\Psi(W_i \cap N^+) = \Psi(R_{x,i})$.

The proof is the same as the proof of Proposition 12 of [3].

Let i be such that $a \leq \Psi(W_i)$. By Lemma 8, there exists $x \in \mathcal{D}_i$ such that $\Psi(R_{x,i}) = \Psi(W_i \cap N^+)$. We have $a = \Psi(N^-)$. Then,

$$u_x = \Psi(R_{x,i} \cup N^-) = \Psi(R_{x,i}) \cup \Psi(N^-) \leq \Psi(W_i \cap N^+) \cup \Psi(W_i) = \Psi(W_i),$$

$$\Psi(W_i) = \Psi(W_i \cap N^+) \cup \Psi(W_i \cap N^-) \leq \Psi(R_{x,i}) \cup \Psi(N^-) = \Psi(R_{x,i} \cup N^-) = u_x.$$

Hence, $u_x = \Psi(W_i)$. It follows from this and Lemmas 6 and 7 that the mapping $\lambda x u_x$ induces an isomorphic correspondence between the partially ordered set associated with $(\mathcal{D}_\omega, \leq_\omega)$ and the segment $(L_{au}, \leq_{au})$ of the r.e. m-degree.

LEMMA 9. For any r.e. m-degree $\boldsymbol{x}$, if $\boldsymbol{x} \leq \boldsymbol{u}$ then $\boldsymbol{a} \leq \boldsymbol{x}$ or $\boldsymbol{x} \leq \boldsymbol{a}$.

Proof. Let $\boldsymbol{x} \leq \boldsymbol{u}$. Then, for some i, we have $\boldsymbol{x} = \Psi(W_i)$. We first take the case when $F(i,j)$ is defined for all j. Then, by construction, $j \in A \longleftrightarrow F(i,j) \subseteq \mathcal{U}$ for all j. Let s_j be the least step at which $F(i,j)$ is defined. Then $F(i,j) \cap W_i^{s_j} \neq \emptyset$. Let $k(i,j)$ be the least element of this set; $\lambda j k(i,j)$ is a recursive function. By definition, $k(i,j) \in W_i$ and $j \in A \longleftrightarrow k(i,j) \in \mathcal{U}$ for all j, whence it easily follows that $a \leq \Psi(W_i)$.

Now suppose that $F(i,j)$ is not defined for some j. Every even number is located in the base of a final tower or in $\mathcal{D}$. Hence, $W_i - (N^- \cup \mathcal{D} \cup U)$ is finite, since otherwise $F(i,j)$ would be defined. Hence, we obtain $x = \Psi(W_i) \leq \Psi(N^- \cup \mathcal{D}) = a$.

QED.

It can be shown that this lemma remains in force if $(L, \leq)$ is replaced by the upper semilattice of computable numerations of the class $\{\emptyset, \{0\}, \ldots, \{n\}\}$ for arbitrary n. It is not known if the lemma remains in force if $(L, \leq)$ is replaced by an arbitrary upper semilattice of computable numerations.

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