ON DISCRETE FAMILIES OF RECURSIVELY ENUMERABLE SETS

V. V. V'yugin

UDC 517.11:518.5

All the enumerations considered below are primitive enumerations of families of recursively enumerable sets and are assumed to be computable. $\langle x, y\rangle$ denotes the numeral of the pair of the natural numbers x and y in Cantor's enumeration of all the pairs of natural numbers, and $\ell(x)$ and $r(x)$ are the left and the right enumeration functions, respectively: $\ell(\langle x, y\rangle) = x$, $r(\langle x, y\rangle) = y$. In what follows we shall not distinguish between a pair of natural numbers and its numeral. An enumeration α of a family of recursively enumerable sets is called positive if the set $\{\langle x, y\rangle \mid \alpha(x) = \alpha(y)\}$ is recursively enumerable. An enumeration of a family is called minimal if it is equivalent to every enumeration of this family which is reducible to it. Every positive enumeration is minimal [3]. All these definitions and properties can be found in [1].

A family $\mathfrak{A}$ of sets is called discrete if there exists a family $\mathfrak{A}_F$ of finite sets such that:

1) for every $A \in \mathfrak{A}$ there exists $P \in \mathfrak{A}_F$, such that $P \subseteq A$;

2) for every $A, B \in \mathfrak{A}$ and $P \in \mathfrak{A}_F$, if $P \subseteq A$ and $P \subseteq B$, then $A = B$.

If we introduce the natural topology on the power set of the set of natural numbers [5], then every element of the discrete family $\mathfrak{A}$ would be an isolated point, and every finite set $\mathfrak{A}_F$ would determine a neighborhood not containing more than one element of $\mathfrak{A}$. If a discrete family is such that there exists a strictly computable family of finite sets which satisfies the conditions 1) and 2) (i.e., there exist a computable enumeration β of the family $\mathfrak{A}_F$ and a general recursive function g such that for all x, $g(x)$ = the number of the elements of $\beta(x)$), then $\mathfrak{A}$ is called effectively discrete (finitely separable in [3]). If α is a computable enumeration of an effectively discrete family of recursively enumerable sets, then α is positive since $\alpha(x) = \alpha(y) \Longleftrightarrow \exists z\, (\beta(z) \subseteq \alpha(x) \cap \alpha(y))$, where β is a strictly computable enumeration of the family $\mathfrak{A}_F$. Let us weaken the condition of the family $\mathfrak{A}$. We shall call a family of sets $\mathfrak{A}$ weakly effective discrete if there exists a computable family of finite sets satisfying the properties 1) and 2). The following theorem shows that the property of having a positive enumeration is preserved in this case.

THEOREM 1. Every weakly effectively discrete computable family of recursively enumerable sets has a positive enumeration.

Proof. Let $\mathfrak{A}$ be a weakly effective discrete family, α be its computable enumeration, and let $\mathfrak{A}_F$ be a computable family of finite sets satisfying 1) and 2), β be its computable enumeration. Let us construct the enumerations τ and ν and the set P. $\alpha^s(x)$ is a finite or empty subset of the set $\alpha(x)$ constructed after s steps of computation of the set $\alpha(x)$ by the method which gives the computable enumeration α. $\tau^{s+1}(x)$ and $\nu^{s+1}(x)$ are finite or empty subsets of the sets $\tau(x)$ and $\nu(x)$ respectively, constructed at the step s of the construction described below. To start with, let us put $\nu^0(x) = \tau^0(x) = \varnothing$ for all x.

Step s $(s \geq 0)$. 1. We make the first number which is not already a successor a successor of s. Let $\ell\ell(s) = n$ and $r\ell(s) = m$. If $m = n$, then we pass to the point 2. Let therefore $m \neq n$. It follows from the properties of the functions $\ell(x)$ and $r(x)$ that $m, n \leq s$, which means that m and n have successors. Let $n < m$. In this case let us verify the condition $(\exists z \leq s)(\beta^s(z) \subseteq \tau^s(m) \cap \tau^s(n))$. If such a z does not exist, then we pass to the point 2. If the condition is satisfied, then we find the small-

Translated from Algebra i Logika, Vol. 11, No. 3, pp. 243–256, May–June, 1972. Original article submitted February 17, 1972.

est z which satisfies it (let us denote it by z'); then we find $s' > s$ such that $\tau^s(m) \subseteq \alpha^{s'}(n) \vee \beta^{s'}(z') \not\subseteq \tau^s(n) \cap \tau^s(m)$. Such an s' can always be found; for if we assume that $(\forall s' > s)(\beta^{s'}(z') \subseteq \tau^s(m) \cap \tau^s(n))$ and hence that $\beta(z')$ is finite, then it follows that $\beta(z') \subseteq \tau(m) \cap \tau(n)$. Hence, by virtue of discreteness, the first term of the disjunction is satisfied by $\tau(n) = \tau(m)$ (it is obvious from the sequel that $\tau(x) = \alpha(x)$). If in the process of finding such an s' the left hand term of the disjunction is satisfied first of all, then we set m free from all its successors, we make all former successors of m successors of n and make the first number which is not already a successor, a successor of m. Let, in this case, s_0 be the smallest such s', then we put $\tau^{s+1}(n) = \tau^s(n) \cup \alpha^{s_0}(n)$. We now pass to the point 2. In the contrary case we directly pass to the point 2.

2. Let us put

$$\tau^{s+1}(m) = \tau^s(m) \cup \alpha^{s+1}(m), \quad \tau^{s+1}(n) = \tau^s(n) \cup \alpha^{s+1}(n),$$

if they have already been constructed at the point 1. For each x such that x is a succesor of n, we put $\nu^{s+1}(x) = \nu^s(x) \cup \tau^{s+1}(n)$. Similarly we construct $\nu^{s+1}(x)$ for all x which are successors of m. For all the remaining x and k we put $\nu^{s+1}(x) = \nu^s(x)$, $\tau^{s+1}(k) = \tau^s(k)$.

For all x, y which are successors of n, let the pairs $\langle x, y\rangle, \langle y, x\rangle$ belong to P. Next we do the same for all the successors of m. Let the pair $\langle s, s\rangle$ belong to P. We pass to the step $s+1$. The construction is now complete.

Let us prove that ν is an essential positive enumeration. It is obvious from the construction that ν is a computable enumeration and P is a recursively enumerable set such that for every x, $\alpha(x) = \tau(x)$.

a) For every x, $\nu(x) \in \mathfrak{A}$.

It is obvious from the construction that if x is a constant successor of n, then $\nu(x) = \alpha(n)$. Therefore, it is sufficient to prove that x becomes a constant successor of some number at a certain step. Since the construction can be continued infinitely long, x becomes a successor of some m. If x becomes free, then at this step it would become a successor of a lesser number. Hence it follows that there exists $n_0 \leq m$ of which x is a constant successor.

b) For every $A \in \mathfrak{A}$ there exists an x such that $\nu(x) = A$. Let $A = \alpha(n_0) = \tau(n_0)$, where n_0 is the smallest numeral of A in the enumeration α. It is sufficient to prove that n_0 acquires a constant successor at some step. If n_0 loses a successor at some step, then it acquires a new one at the same step. Consequently, it is sufficient to prove that n_0 cannot lose successors at an infinite number of steps. Let us assume the contrary. Then there exist infinitely many steps s_i $(s_0 < s_1 < \dots)$, and numbers s_i', m_i (not necessarily distinct) such that $s_i' > s_i$, $m_i < n_0$, $\tau^{s_i}(m_i) \subseteq \alpha^{s_i'}(n_0) \subseteq \alpha(n_0) = A$ (the first term of the disjunction at the point 1 is satisfied at the step s_i). It follows from $m_i < n_0$ that there exists an m such that $m < n_0$ and $m_i = m$ for infinitely many i. Then $\tau(m) \subseteq \alpha(n_0) = \tau(n_0)$ and from the discreteness of $\mathfrak{A}$, we have $\tau(m) = \alpha(m) = \alpha(n_0) = A$. This is a contradiction since n_0 is the smallest numeral of A in the enumeration α.

c) $P = \{\langle x, y\rangle \mid \nu(x) = \nu(y)\}$. If $\langle x, y\rangle \in P$, then x and y are constant successors of the same number from some moment. In this case it is easy to see that $\nu(x) = \nu(y)$. Let $\nu(x) = \nu(y)$. At a sufficiently great step in a), x and y become constant successors. Let us assume that x and y are constant successors of different numbers n and m, $n < m$. Then $\nu(x) = \tau(n)$, $\nu(y) = \tau(m)$, and $\tau(n) = \tau(m)$. We now find a step s and a number z such that $\ell\ell(s) = n$, $r\ell(s) = m$, $z \leq s$, $\beta^s(z') = \beta(z')$ for all $z' \leq z$, and $\beta(z) \subseteq \tau^s(m) \cap \tau^s(n)$. The right hand term of the disjunction at the point 1 cannot hold good. So we set y free. This is a contradiction. Hence, x and y become successors of the same number at some step, and the pair $\langle x, y\rangle$ belongs to P. The theorem is proved.

<u>THEOREM 2</u>. There exists a computable weakly effectively discrete family of recursively enumerable sets which is not discrete.

<u>Proof.</u> Let A be an enumerable unsolvable set and $a \in \bar{A}$. Let us define $\mathfrak{A} = \{A \cup \{x\} \mid x \in \bar{A}\}$ and its enumeration

$$\alpha_a(x) = \begin{cases} A \cup \{x\}, & x \notin A, \\ A \cup \{a\}, & x \in A. \end{cases}$$

Then $\alpha_a^{-1}(A \cup \{a\}) = A \cup \{a\}$ is unsolvable and if $x \neq a$, $x \in \bar{A}$, then $\alpha_a^{-1}(A \cup \{x\}) = \{x\}$. Hence it follows that if $a \neq b$ and $a, b \in \bar{A}$, then the enumerations α_a and α_b are not comparable. Consequently, $\mathcal{A}$ is not effectively discrete according to [3]. However, $\mathcal{A}$ is a weakly effectively discrete family. We construct the family $\mathcal{A}_F$ of finite sets satisfying the conditions 1) and 2) and its computable enumeration ν in the following manner. Let α be a computable enumeration of $\mathcal{A}$. Let us define $\nu^{-1}(x) = \emptyset$ for all x.

Step n. $\ell(n) = x$. If $\nu^{n-1}(x) \subseteq \alpha^n(x)$ and there exists $y \in \alpha^n(x) - A^n$, then we define $\nu^n(x) = \nu^{n-1}(x) \cup \{y_0\}$, where y_0 is the smallest such y. For every $k \neq x$ (for all k, if the condition is not satisfied) we define $\nu^n(k) = \nu^{n-1}(k)$. We now pass to step $n+1$. The theorem is proved.

By using the diagonal method used in the construction of Sec. 4 in [2] we can construct a weakly effectively discrete computable family of finite sets which is not effectively discrete. Let us now construct a computable discrete family which is not weakly effectively discrete. In this construction we use ideas from [2]. We need the following lemma for further use.

LEMMA 1. Let $\mathcal{A}$ be a computable weakly effectively discrete family of recursively enumerable sets. Then there exists a computable family $\mathcal{L}$ of finite sets satisfying the conditions 1) and 2) of the definition and the condition 3): For every $D \in \mathcal{L}$ there exists an $A \in \mathcal{A}$ such that $D \subseteq A$.

Proof. Let α be a computable enumeration of $\mathcal{A}$ and β be a computable enumeration of the family $\mathcal{A}_F$ of finite sets satisfying the conditions 1) and 2). Let us construct the enumeration ν. We define $\nu^{-1}(x) = \emptyset$ for all x.

Step n ($n \geq 0$). $\ell(n) = x$. If there exists an i such that $i \leq n$ and $\beta^n(i) \subseteq \alpha^n(x)$, then we define $\nu^n(x) = \nu^{n-1}(x) \cup \beta^n(i_0)$, where i_0 is the smallest such i.

For every $y \neq x$ (for all y if such an i does not exist) we define $\nu^n(y) = \nu^{n-1}(y)$. We now pass to the step $n+1$. Thus the construction has been accomplished.

Let us define $\nu(x) = \bigcup_{n \geq 0} \nu^n(x)$, $\mathcal{L} = \{\nu(x)\}_{x \geq 0}$. It is easily seen that $\nu(x) \subseteq \alpha(x)$. Let i_0 be the smallest i such that $\beta(i) \subseteq \alpha(x)$. Then if $\ell(n) = x$ for all n, starting from a certain one, we have $\nu^n(x) = \nu^{n-1}(x) \cup \beta(i_0)$, and if $\ell(n) \neq x$ for all such n, then we have $\nu^n(x) = \nu^{n-1}(x)$; whence it is obvious that $\nu(x)$ is finite and $\beta(i_0) \subseteq \nu(x)$. If $\nu(x) \subseteq \alpha(y) \cap \alpha(z)$, then $\beta(i_0) \subseteq \alpha(y) \cap \alpha(z)$. In this case, since the family $\mathcal{A}_F$ satisfies the condition 2), we have $\alpha(y) = \alpha(z)$. Hence $\mathcal{L}$ satisfies the conditions 1), 2), and 3). The lemma is proved.

Let $\pi(i,j)$ be a computable enumeration of the family of all recursively enumerable sets such that for every computable enumeration α of an arbitrary family of enumerable sets there exists an i such that for all j, we have $\alpha(j) = \pi(i,j)$ [1].

THEOREM 3. There exists a discrete family of recursively enumerable sets which is not weakly effectively discrete and which has a positive enumeration.

Proof. We construct an enumeration α of some family of recursively enumerable sets. At the same time we construct the sets M, P and the functions $c(e,n)$, $f(e,n)$, $g(e,n)$, $h_1(e,n)$, $h_2(e,n)$. M^n, P^n, and $\alpha^n(x)$ are the finite or empty subsets of the sets M, P and $\alpha(x)$, respectively, obtained at the step n of the construction. $\pi^n(e,i)$ is the finite or the empty subset of the set $\pi(e,i)$ computed after n steps of the computation of the set $\pi(e,i)$ by the same method which gives the computable enumeration of π. It will be obvious from what follows that $P = \{\langle x,y\rangle \mid \alpha(y) = \alpha(x)\}$, and that every set $\alpha(x)$ is of one of the forms $M \cup \{f(e,n)\}$, $M \cup \{f(e,n), h_1(e,n)\}$, $M \cup \{f(e,n), h_2(e,n)\}$ for some e, n, where $f(e,n)$, $h_1(e,n)$, $h_2(e,n) \notin M$. During the construction every number becomes a successor of some number. An even number has successors of both the left and right types simultaneously. If a number x is a successor of e at all steps from a certain step onwards, then x is called a constant sucsor of e. A number n is said to be unused at the step x if $x \notin \bigcup_{i \geq 0} \alpha^{n-1}(i)$ (for every n this set is finite) and a number k is called free if k is not a successor. We define $M^{-1} = \emptyset$ and $\alpha^{-1}(x) = \emptyset$ for every x.

Step n $(n \geq 0)$. If n is even, then we pass to the point 1; in the contrary case, we pass to the point 2.

1. $n = 2k$, $\ell(k) = e$. If $c(e, n-1)$ has been defined, then we pass to $(*)$. Let $c(e, n-1)$ be not defined. Then if e is even, we pass to the point 1.1; and if e is odd, we pass to the point 1.2.

1.1. $e = 2m$. We verify the condition

$$(\exists i \leq n)(\exists y)(\forall e' < e)(y \in \pi^n(m,i) \,\&\, y \notin M^{n-1} \,\&\, (c(e', n-1)$$

is defined $\Longrightarrow$ (e' is even $\Longrightarrow \{f(e',n-1), h_1(e',n-1), h_2(e',n-1)\} \cap \pi^n(m,i) = \emptyset$) & (e' is odd $\Longrightarrow$ $f(e', n-1) \notin \pi^n(m,i)$))). If this condition is not fulfilled, then we pass to $(*)$. If it is fulfilled, then let i_0, y_0 be the smallest numbers satisfying it. We define $f(e,n) = y_0$, $g(e,n) = i_0$, and

$$M^n = M^{n-1} \cup \{x \mid x \in \pi^n(m, i_0) \cup \bigcup_{\substack{e' \text{ even}, e' > e, \\ c(e', n-1) \text{ defined}}} \{f(e',n-1), h_1(e',n-1), h_2(e',n-1)\} \cup \bigcup_{\substack{e' > e, e' \text{ odd} \\ c(e', n-1) \text{ defined}}} \{f(e',n-1)\} \,\&\, x \neq y_0\}$$

For every $e' > e$ we let $c(e',n), f(e',n), g(e',n), h_1(e',n), h_2(e',n)$ remain undefined. Let u_1, u_2 be the two smallest unused numbers not belonging to $M^n \cup \{y_0\}$ and k_1, k_2 be the two smallest free numbers. We define $c(e,n) = \langle k_1, k_2 \rangle$, $h_1(e,n) = u_1$, $h_2(e,n) = u_2$, where we call k_1 as a left successor, and k_2 as a right successor, of e. For every $e' > e$ we set e' free from all its successors (if it has them) and we make all former successors of e' left successors of e. We define $\alpha^n(k_1) = M^n \cup \{f(e,n), h_1(e,n)\}$, $\alpha^n(k_2) = M^n \cup \{f(e,n), h_2(e,n)\}$. (Let us observe that $\pi^n(m, i_0) \subseteq \alpha^n(k_1) \cap \alpha^n(k_2)$). For each remaining left successor of e, we define $\alpha^n(x) = \alpha^{n-1}(x) \cup \alpha^n(k_1)$. (Let us observe that for every left successor of e at this step, $\alpha^{n-1}(x) \subseteq \alpha^n(k_1)$.) We now pass to $(*)$.

1.2. $e = 2m+1$. Let y be the smallest unused number. We define $f(e,n) = y$, $c(e,n) = k_1$, where k_1 is the smallest free number. We call k_1 as a successor of e. We define $\alpha^n(k_1) = M^{n-1} \cup \{f(e,n)\}$. We now pass to $(*)$.

2. $n = 2k+1$, $\ell(k) = e$. If $c(e, n-1)$ is not defined or e is odd, then we pass to $(*)$. Let therefore $e = 2m$ and $c(e,n-1)$ be defined, $c(e,n-1) = \langle k_1, k_2 \rangle$, $g(e,n) = i$. One of the following cases always occurs:

Case 2.1. $(\exists e')(e' \neq e \,\&\, c(e', n-1)$ is defined $\&\, ((e'$ is even $\&\, \{f(e',n-1), h_1(e',n-1), h_2(e',n-1)\} \cap \pi^n(m,i) \neq \emptyset) \vee (e'$ is odd $\&\, f(e', n-1) \in \pi^n(m,i)))$.

In this case we pass to $(*)$.

Case 2.2. The case 2.1 does not occur, and $\{h_1(e,n-1), h_2(e,n-1)\} \cap \pi^n(m,i) \neq \emptyset$. In this case we define $M^n = M^{n-1} \cup \{x \mid x \in \pi^n(m,i) \cup \{h_1(e,n-1)\, h_2(e,n-1)\} \,\&\, x \neq f(e,n-1)\}$, $f(e,n) = f(e,n-1)$. Let us redefine $h_1(e,n) = u_1$, $h_2(e,n) = u_2$, where u_1, u_2 are the two smallest unused numbers not belonging to M^n. Let k_3 be the smallest free number. Let us set e free from the right successor k_2 and make k_2 a left successor, and k_3 a right successor, of e. We define $c(e,n) = \langle k_1, k_3 \rangle$, $\alpha^n(k_1) = \alpha^{n-1}(k_1) \cup M^n \cup \cup \{h_1(e,n)\} = M^n \cup \{f(e,n), h_1(e,n)\}$, $\alpha^n(k_3) = M^n \cup \{f(e,n), h_2(e,n)\}$. (We observe that $\alpha^{n-1}(k_2) \subseteq \alpha^n(k_1)$, $\pi^n(m,i) \subseteq \alpha^n(k_1) \cap \alpha^n(k_3)$). For each remaining left successor of e we define $\alpha^n(x) = \alpha^{n-1}(x) \cup \alpha^n(k_1)$. We now pass to $(*)$.

Case 2.3. The cases 2.1 and 2.2 do not occur. We define, in this case, $M^n = M^{n-1} \cup \{x \mid x \in \pi^n(m,i) \,\&\, x \neq f(e,n-1)\}$. We then pass to $(*)$. (We observe that after $\alpha^n(k_1)$ and $\alpha^n(k_2)$, have been constructed as in $(*)$, we should have $\pi^n(m,i) \subseteq \alpha^n(k_1) \cap \alpha^n(k_2)$).

(*). If, at the step n, M^n is not defined, we put $M^n = M^{n-1}$, and if for every k, $\alpha^n(k)$ has not already been defined, we put $\alpha^n(k) = \alpha^{n-1}(k) \cup M^n$. If for each of the following equations the left hand side has not yet been defined and was not defined at the point 1.1 and the right hand side has been defined, then we put $c(e',n) = c(e',n-1)$, $f(e',n) = f(e',n-1)$, $g(e',n) = g(e',n-1)$, $h_1(e',n) = h_1(e',n-1)$, $h_2(e',n) = h_2(e',n-1)$ for every e'.

If x and y are both left successors of the same number, then we let the pair $<x,y>$ belong to P. We also let the pair $<n,n>$ belong to P. We then pass to the step $n+1$. Thus the construction is complete.

It is obvious that α is a computable enumeration, and M and P are enumerable sets. Let $\mathcal{O}\!\mathcal{L} = \{\alpha(x)\}_{x\geq 0}$. We observe that $c(e,n)$ and $f(e,n)$ are either both defined or are both not defined; if e is odd, then $g(e,n)$, $h_1(e,n)$, and $h_2(e,n)$ are not defined for all n; and if e is even, then $c(e,n), g(e,n)$, $h_1(e,n)$, and $h_2(e,n)$ are either all defined or all not defined.

a) For every e there exists an n_0 such that either $c(e,n)$ is defined, or it is not defined for all $n \geq n_0$. For $e=0$ this statement is obvious. Let this statement be true for all $e' < e$ $(e > 0)$, and let n_0 be same for all $e' < e$. In this case if $c(e,n-1)$ is defined and $c(e,n)$ is not defined for $n > n_0$, then this can be so only at the point 1.1 when $n = 2k$, $\ell(k) = e'$, $e' < e$, $c(e',n-1)$ is not defined and $c(e',n)$ is defined, which contradicts the induction hypothesis. We observe that if for every $n \geq n_0$, $c(e,n)$ is defined, then for every $n \geq n_0$, we have $f(e,n) = f(e,n_0)$.

b) Let us assume that e is such that $c(e,n)$ is defined and $f(e,n) = f(e,n_0)$ for all $n \geq n_0$, for some number n_0. If e is odd, then for every $n \geq n_0$, we have $c(e,n) = k$, where k is the only successor of e, $\alpha(k) = M \cup \{f(e,n_0)\}$, and for every $x \neq k$, we have $f(e,n_0) \notin \alpha(x)$. Let e be even. Then for every $n \geq n_0$, we have $\ell(c(e,n)) = k_1$ and $g(e,n) = i_0$ for some k_1 and i_0, and e does not lose left successors. If the case 2.2 occurs for e, and i_0 at infinitely many steps, then every right successor of e becomes a left successor at a sufficiently great step, the functions $h_1(e,n)$, and $h_2(e,n)$ are redefined infinitely many times, and for every constant successor x of e we have $\alpha(x) = M \cup \{f(e,n_0)\}$ at a sufficiently great step. Also for every x which is not a constant successor of e, we have $f(e,n_0) \notin \alpha(x)$. If the case 2.2 occurs for e and i_0 not more than a finite number of times, then there exist n_1 and k_2 $(n_1 \geq n_0)$ such that for every $n \geq n_1$, we have $c(e,n) = <k_1, k_2>$, k_2 is a constant right successor of e, $h_1(e,n) = h_1(e,n_1)$, $h_2(e,n) = h_2(e,n_1)$, $\alpha(k_1) = M \cup \{f(e,n_1), h_1(e,n_1)\}$, $\alpha(k_2) = M \cup \{f(e,n_1), h_2(e,n_1)\}$; for each x which is not a constant of e, left or right, we have $f(e,n_1) \notin \alpha(x)$; for every x which is not a constant left successor of e, we have $h_1(e,n_1) \notin \alpha(x)$, and for every $x \neq k_2$, we have $h_2(e,n_1) \notin \alpha(x)$. Let e and n_0 be such that $n \geq n_0$ is not defined for every $c(e,n)$. Then e does not have a successor at any step after the step n_0. All these facts are immediately obvious from the construction.

c) $\mathcal{O}\!\mathcal{L}$ is an infinite family. It is easily seen that if e is odd, then there exist n_0 and k such that $c(e,n)$ is defined and is equal to k for every $n \geq n_0$. According to b), $\alpha(k)$ is distinct from $\alpha(x)$ for $x \neq k$.

d) For every x there exists an e such that x is a constant successor of e (left or right, if e is even). Every number becomes a successor, only if at the point 1.2, at some step. Moreover, if e is even and x is a right successor of e, then, according to the case 2.2, x can become a left successor of e. If e is odd and x is successor of e, or if e is even and x is a right or a left succesor of e, then x can become a left successor of some $e' < e$ at an even step. All possible transpositions of x have been described above. Hence it follows that there exists an $e_0' \leq e$ such that x is a constant successor of e_0', right or left, if e_0' is even.

e) $\mathcal{O}\!\mathcal{L}$ is a discrete family. $A \in \mathcal{O}\!\mathcal{L}$, which means that $A = \alpha(x)$. According to d), x is a constant successor of e. We observe that if x becomes a constant successor of e at the step n_0, then $c(e,n)$ is defined for every $n \geq n_0$. Hence according to b) there exists a number y which is the value of one of the functions f, h_1, and h_2 such that $y \in A$ and for every $B \in \mathcal{O}\!\mathcal{L}$, $B \neq A$ $y \notin B$ $(B = \alpha(z))$, z cannot be a constant successor of e, right or left if e is even, as is x since $B \neq A$).

f) α is a positive enumeration.

If e is odd, then e has a unique successor, and if e is even, then e cannot have more than one constant right successor. Let $x \neq y$. If $<x, y> \in P$, then x and y are left successors of the same number at some step. If they are set free, then they again become the left successors of one and the same number at the same step. According to the construction, $\alpha(x) = \alpha(y)$.

Let $\alpha(x) = \alpha(y)$. According to d), x and y are constant successors, and according to b), x and y can be constant successors of only one number; moreover both are left successors since $x \neq y$. Then, according to the construction, the pair $<x, y>$ belongs to P. Hence, $\{<x, y> \mid \alpha(x) = \alpha(y)\} = P$ is enumerable.

g) $\mathcal{O}\!\mathcal{L}$ is not a weakly effectively discrete family. According to b) and d) every set of $\mathcal{O}\!\mathcal{L}$ has one of the forms

$$M \cup \{f(e, n_0)\},\ M \cup \{f(e, n_0), h_1(e, n_0)\},\ M \cup \{f(e, n_0), h_2(e, n_0)\},$$

where $f(e, n_0), h_1(e, n_0), h_2(e, n_0) \notin M$. Let us assume that $\mathcal{O}\!\mathcal{L}$ is a weakly effectively discrete family. Then according to Lemma 1 there exists a computable family of finite sets satisfying the conditions 1), 2), and 3). Let e be such that $\{\pi(e, i)\}_{i \geq 0}$ is this family. Then from the form of the elements of $\mathcal{O}\!\mathcal{L}$, we have $\pi(e, i) \cap \overline{M} \neq \phi$ for every i. We shall prove that there exists an n_0 such that $c(2e, n)$ is defined for every $n \geq n_0$. Let us assume the contrary. Then there exists an n_0 such that for every $e' < 2e$ either $c(e', n)$ is defined for every $n \geq n_0$, or neither $c(e', n)$ nor $c(2e, n)$ is defined for every $n \geq n_0$. Let $A = \{e' \mid e' < 2e$ and $c(e', n)$ is defined for every $n \geq n_0\}$, $B = \{e' \mid e' \in A$, e' is even, the case 2.2 occurs for e' and i_0' not more than a finite number of times$\}$, $C = \{e' \mid e' \in A$, e' is even, the case 2.2 occurs for e', and i_0' is an infinite number of times$\}$, where $g(e', n_0) = i_0'$.

According to a) and b) there exists an n_1 $(n_1 \geq n_0)$ such that for every $n \geq n_1$, $f(e', n) = f(e', n_1)$, for every $e' \in A$, and $h_1(e', n) = h_1(e', n_1)$, $h_2(e', n) = h_2(e', n_1)$ for every $e' \in B$. If $e' \in A$, then according to b) $f(e', n_1)$ is an element of not more than two sets of $\mathcal{O}\!\mathcal{L}$; and if $e' \in B$, then $h_1(e', n_1)$ is an element of exactly one set of $\mathcal{O}\!\mathcal{L}$ and the same is true of $h_2(e', n_1)$. Since $\mathcal{O}\!\mathcal{L}$ is infinite, the family $\{\pi(e, i)\}_{i \geq 0}$ is also infinite. Hence, we can find an i such that $f(e', n_1) \notin \pi(e, i)$ for every $e' \in A$ and $h_1(e', n_1)$, $h_2(e', n_1) \notin \pi(e, i)$ for every $e' \in B$. There exists $y \in \pi(e, i)$, $y \notin M$. Let us choose an n_2 such that $n_2 > n_1$, $n_2 = 2k$, $\ell(k) = 2e$, $i < n_2$, $\pi^{n_2}(e, i) = \pi(e, i)$ and $h_1(e', n_2 - 1), h_2(e', n_2 - 1) \notin \pi(e, i)$ for every $e' \in C$. Such an n_2 exists since the set $\pi(e, i)$ is finite, and since the case 2.2 occurs for every $e' \in C$ on an infinite number of steps, the functions $h_1(e', n)$ and $h_2(e', n)$ are redefined infinitely many times, every number which was formerly a value of one of these functions and then ceased to be so according to the case 2.2 cannot be a value of any of them at a sufficiently great step. Hence for $n = n_2$, $\pi(e, i)$ satisfies the condition 1) of the construction and $c(2e, n_2)$ is defined. This contradicts our supposition.

This implies that there exists an n_0 such that $c(2e, n)$, $f(2e, n)$, and $g(2e, n)$ are defined for every $n \geq n_0$. Let $g(2e, n) = i_0$. The case 2.2 cannot occur an infinite number of times for e and i_0 since $\pi(e, i_0)$ is a finite set. This means that there exist n_1, k_1 and k_2 such that $n_1 \geq n_0$, $c(2e, n) = <k_1, k_2>$ for every $n \geq n_1$, k_1 is a constant left successor of e, and k_2 is a constant right successor of e. If the case 2.2 occurs for e and i_0 an infinite number of steps, then from the observations made during the construction we have $\pi(e, i_0) \subseteq \alpha(k_1) \cap \alpha(k_2)$. But $\alpha(k_1) \neq \alpha(k_2)$, which contradicts the condition 2). This implies that there exists an n_2 such that $n_2 \geq n_1$ and for every $n \geq n_2$, we have $n = 2k + 1$, $\ell(k) = 2e$ and the case 2.1 occurs. Let us assume that at an infinite number of steps n the corresponding $e' \neq 2e$ are even and $\{f(e', n-1), h_1(e', n-1), h_2(e', n-1)\} \cap \pi(e, i_0) \neq \phi$. (The case where at an infinite number of steps the corresponding e' are odd and $f(e', n-1) \in \pi(e, i_0)$ is considered analogously.) Then there exist infinitely many steps n such that there exists an e' for each of them such that $e' \neq 2e$, e' is even, and $f(e', n-1) \in \pi(e, i_0)$, or there exist infinitely many steps n such that there exists an e' for each of them such that $e' \neq 2e$, e' is even, and $h_1(e', n-1) \in \pi(e, i_0)$, or the same is true of $h_2(e', n-1)$.

Let the first case occur. Then, since $\pi(e, i_0)$ is finite, there exists $y \in \pi(e, i_0)$ such that y is the value of the function f at an infinite number of steps. The function f has the property: If $x = f(e, n)$, then $x \neq f(e', m)$ for every $e' > e$, $m > n$. This implies that there exists an e_0' such that $e_0' \neq 2e$ and $f(e_0', n) = y$ for infinitely many n. According to a) the function $f(e_0', n)$ is either not defined for all sufficiently large n, or is defined and is constant for them. Consequently, there exists an n_3 such that $n_3 \geq n_0$, $f(e_0', n) = f(e_0', n_3)$ for every $n \geq n_3$, and $f(e_0', n_3) \in \pi(e, i_0)$. According to the condition 3) of Lemma 1, $\pi(e, i_0)$ is contained in some set of $\mathfrak{A}$. For every $n \geq n_0$ we have $f(2e, n) = f(2e, n_0)$ and according to b) $f(2e, n_0) \in \pi(e, i_0)$ implies that either $\pi(e, i_0) \subseteq \alpha(k_1)$ or $\pi(e, i_0) \subseteq \alpha(k_2)$. Hence, $f(e_0', n_3) \in \alpha(k_1)$ or $f(e_0', n_3) \in \alpha(k_2)$. But k_1 and k_2 are not constant successors of e_0' since $e_0' \neq 2e$. This contradicts b). The remaining two cases are considered analogously. Hence $\mathfrak{A}$ is not a weakly effectively discrete family. The theorem is proved.

A family of sets is called normal if with every finite set all its nonempty subsets belong to it [4]. It has been proved in [4] that every minimal enumeration of an infinite normal family is equivalent to an enumeration in which every finite set has a finite set of numerals. The following theorem states that every minimal enumeration of a discrete family also has this property.

THEOREM 4. Every minimal enumeration of an infinite computable discrete family of recursively enumerable sets is equivalent to an enumeration in which every finite set has a finite set of numerals.

Proof. Let τ be a minimal computable enumeration of the family $\mathfrak{A}$ and γ be a strictly computable enumeration of the family of all finite sets. We shall construct the enumeration β and the function h. As a preliminary we put $\beta^0(x) = \emptyset$ for all x.

Step 0. We put $h(0) = 0$, $\beta^1(0) = \tau^1(0)$.

Step $s+1$ $(s \geq 0)$. Let $h(0), h(1) \ldots h(i)$, $i < s+1$, be already defined at the previous steps. Let $\ell\ell(s+1) = x$, $r\ell(s+1) = z$ [$\ell(x)$ and $r(x)$ are the left and the right enumeration functions in Cantor's enumeration of all the pairs of natural numbers].

(1) If $\gamma(z) \subseteq \tau^{s+1}(x)$ and $(\forall j \leq i)(\gamma(z) \nsubseteq \beta^{s+1}(j))$, then we put $h(i+1) = x$. In the contrary case we have $h(i+1)$ undefined. For every $j \leq i+1$, if $h(j)$ is defined, we put $\beta^{s+2}(j) = \beta^{s+1}(j) \cup \tau^{s+2}h(j)$. We put $\beta^{s+2}(j) = \beta^{s+1}(j)$ for all the remaining j. We now pass to the step $s+2$.

The construction is complete. It is obvious from the construction that β is a computable enumeration and h is a computable function. The family $\mathfrak{A}$ is infinite and discrete; therefore the condition 1) will be satisfied at an infinite number of steps. Hence h is defined everywhere. It is easily seen that $\beta(x) = \tau h(x)$ for all x; consequently, $\beta(x) \in \mathfrak{A}$. Let us assume that $A \in \mathfrak{A}$ and that $\beta(x) \neq A$ for every x. Let $A = \tau(n)$. As $\mathfrak{A}$ is discrete, there exists a step $s+1$ such that $\ell\ell(s+1) = n$, $r\ell(s+1) = z$, $\gamma(z) \subseteq \tau^{s+1}(n)$ and $\gamma(z) \nsubseteq B$ for every $B \in \mathfrak{A}$, $B \neq A$. Then we put $h(i+1) = n$ at this step for the corresponding i. We obtain $\beta(i+1) = \tau(n)$, which contradicts the assumption. Hence β is an enumeration of the family $\mathfrak{A}$.

Let A be finite and $A \in \mathfrak{A}$. Also let $A = \tau(n)$. By virtue of what has been proved above, we have $\beta(x) = A$ for some x. Let $\beta^{s_0+1}(x) = \tau^{s_0+1}(x) = A$ at the step s_0+1, then at every subsequent step $s+1$, such that $\ell\ell(s+1) = m$, $\tau(m) = A$, the condition 1) would not be satisfied and A would not get more than one numeral in the enumeration β. Thus A has a finite set of numerals in the enumeration β. According to the construction, $\beta(x) = \tau h(x)$ for every x, i.e., β reduces to τ. If τ is a minimal enumeration, then β is equivalent to τ. The theorem is proved.

COROLLARY. Every infinite computable discrete family of finite sets is computable with finite repetitions (i.e., has an enumeration in which every set has a finite set of numerals).

Let us observe that an example of a computable family of finite sets which cannot be computable with finite repetitions is given in [4].

I take this opportunity to express my gratitude to S. S. Marchenkov and V. A. Uspenskii for taking interest in this paper.

LITERATURE CITED

1. Yu. L. Ershov, Theory of Enumerations, Part 1 [in Russian], Novosibirsk Gos. Univ. (1969).
2. Yu. L. Ershov, "On computable enumerations," Algebra i Logika, 7, No. 5, 71-99 (1968).
3. A. I. Mal'tsev, "Positive and negative enumerations," Dokl. Akad. Nauk SSSR, 160, No. 2, 278-280 (1965).
4. S. S. Marchenkov, "On minimal enumerations of systems of recursively enumerable sets," Dokl. Akad. Nauk SSSR, 198, No. 3, 530-532 (1971).
5. V. A. Uspenskii, "On computable operations," Dokl. Akad. Nauk SSSR, 103, No. 5, 773-776 (1955).