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Asymptotic Stability of Stationary States in the Wave Equation Coupled to a Nonrelativistic Particle

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Abstract. We consider the Hamiltonian system consisting of a scalar wave field and a single particle coupled in a translation invariant manner. The point particle is subjected to an external potential. The stationary solutions of the system are a Coulomb type wave field centered at those particle positions for which the external force vanishes. It is assumed that the charge density satisfies the Wiener condition, which is a version of the "Fermi Golden Rule." We prove that in the large time approximation, any finite energy solution, with the initial state close to the some stable stationary solution, is a sum of this stationary solution and a dispersive wave which is a solution of the free wave equation.

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1. INTRODUCTION

Our paper deals with nonlinear field-particle interaction. We consider a scalar real-valued wave field $\phi(x)$ in $\mathbb{R}^3$ coupled to a nonrelativistic particle with position q and momentum p governed by

$$\begin{cases} \dot\phi(x,t) = \pi(x,t), \quad \dot\pi(x,t) = \ \Delta\phi(x,t) - \rho(x-q(t)), \\ \dot q(t) \ = p(t), \qquad\quad \dot p(t) \ = -\nabla V(q(t)) + \displaystyle\int \phi(x,t)\nabla\rho(x-q(t))\,dx. \end{cases} \tag{1.1}$$

This is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\phi,\pi,q,p) = \frac{1}{2}\int \Big(|\pi(x)|^2 + |\nabla\phi(x)|^2\Big)dx + \int \phi(x)\rho(x-q)dx + \frac{1}{2}p^2. \tag{1.2}$$

The first two equations in (1.1) for the fields are equivalent to the wave equation with the source $\rho(x-q)$. The form of the last two equations is determined by the choice of the nonrelativistic kinetic energy $p^2/2$ in (1.2).

It is easy to find stationary solutions to the system (1.1). For $q \in \mathbb{R}^3$, we set

$$s_q(x) = -\int \frac{d^3y}{4\pi|y-x|}\rho(y-q). \tag{1.3}$$

Let $Z = \{q \in \mathbb{R}^3 : \nabla V(q) = 0\}$ be the set of critical points for V. Then the set $\mathcal{S}$ of stationary solutions is given by

$$\mathcal{S} = \{(\phi,\pi,q,p) = (s_q, 0, q, 0) =: S_q|\ q \in Z\}. \tag{1.4}$$

We assume that $V \in C^2(\mathbb{R}^3)$ and set

$$V_0 := \inf_{q\in\mathbb{R}^3} V(q) > -\infty. \tag{1.5}$$

For the charge distribution ρ, we assume that

$$\rho \in C_0^\infty(\mathbb{R}^3), \quad \rho(x) = 0 \text{ for } |x| \geqslant R_\rho,\ \rho(x) = \rho_r(|x|). \tag{1.6}$$

We also assume that the Wiener condition is satisfied:

$$\hat\rho(k) = \int d^3x\, e^{ikx}\rho(x) \neq 0, \quad k \in \mathbb{R}^3. \tag{1.7}$$

This is an analog of the Fermi Golden Rule: the coupling term $\rho(x-q)$ is not orthogonal to the eigenfunctions e^{ikx} of the continuous spectrum of the linear part of the equation (cf. [10]).

Finally, we assume that some $q^* \in Z$ is a stable critical point of V.

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Definition 1.1. A *point* $q^* \in Z$ *is stable* if $d^2V(q^*) > 0$ as a quadratic form.

Our main results are as follows.

For solutions to the system (1.1) with initial data close to $S_{q^*} = (s_{q^*}, 0, q^*, 0)$, we prove the asymptotics

$$\|\phi(\cdot,t) - s_{q^*}\|_{\dot H^1_{-\sigma}} + \|\pi(\cdot,t)\|_{L^2_{-\sigma}} + |q(t) - q^*| + |p(t)| = \mathcal{O}(t^{-\sigma}), \quad t \pm \infty, \quad \sigma > 1, \tag{1.8}$$

in weighted Sobolev norms (see (2.1)). Such asymptotics in global energy norm do not hold in general since the field components may contain a dispersive term, whose energy radiates to infinity as $t \to \pm\infty$ but its norm does not converge to zero. Namely, in global energy norms we obtain the following scattering asymptotics::

$$(\phi(x,t), \pi(x,t)) \sim (s_{q^*}, 0) + W_0(t)\Phi_\pm, \quad t \to \pm\infty. \tag{1.9}$$

Here $W_0(t)$ is the dynamical group of the free wave equation, and $\Phi_\pm$ are the corresponding asymptotic scattering states.

Asymptotics similar to (1.8) in local energy semi-norms was obtained in [7] in the case of compactly supported difference $\phi(x,0) - s_{q^*}(x)$. We get rid of this restriction in the present paper.

For the proof, we establish long-time decay of the linearized dynamics using our results [8] on the dispersion decay for the wave equation in weighted Sobolev norms. Then we apply the method of majorants.

Let us comment on previous results in these directions. The asymptotic stability of solitons was proved in [2] for systems of type (1.1) with the Klein–Gordon equation instead of the wave equations. This result was extended in [3-6] to similar system with the Schrödinger, Dirac, wave, and Maxwell equations. A survey of these results can be found in [1].

2. MAIN RESULTS

To formulate our results precisely, we introduce a suitable phase space. Let L^2 be the real Hilbert space $L^2(\mathbb{R}^3)$ with scalar product $\langle \cdot\,, \cdot\rangle$. Denote by $\dot H^1$ the completion of real space $C_0^\infty(\mathbb{R}^3)$ with norm $\|\nabla\phi(x)\|_{L^2}$. Equivalently, by Sobolev's embedding theorem (see [9]),

$$\dot H^1 = \{\phi(x) \in L^6(\mathbb{R}^3) : \ |\nabla\phi(x)| \in L^2\}.$$

Introduce the weighted Sobolev spaces L^2_α and $\dot H^1_\alpha$, $\alpha \in \mathbb{R}$, with the norms

$$\|\psi\|_{L^2_\alpha} := \|(1+|x|)^\alpha \psi\|_{L^2}, \quad \|\psi\|_{\dot H^1_\alpha} := \|(1+|x|)^\alpha \psi\|_{\dot H^1}. \tag{2.1}$$

Definition 2.1. (i) The phase space $\mathcal{E}$ is the real Hilbert space $\dot H^1 \oplus L^2 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ of states $Y = (\psi, \pi, q, p)$ equipped with the finite norm

$$\|Y\|_{\mathcal{E}} = \|\nabla\psi\|_{L^2} + \|\pi\|_{L^2} + |q| + |p|.$$

(ii) $\mathcal{E}_\alpha$ is the space $\dot H^1_\alpha \oplus L^2_\alpha \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$ equipped with its norm

$$\|Y\|_{\mathcal{E}_\alpha} = \|\psi\|_{\dot H^1_\alpha} + \|\pi\|_{L^2_\alpha} + |q| + |p|. \tag{2.2}$$

(iii) $\mathcal{F}_\alpha$ is the space $\dot H^1_\alpha \oplus L^2_\alpha$ of fields $F = (\psi, \pi)$ equipped with the finite its norm

$$\|\,F\|_{\mathcal{F}_\alpha} = \|\psi\|_{\dot H^1_\alpha} + \|\pi\|_{L^2_\alpha}. \tag{2.3}$$

We consider the Cauchy problem for the Hamiltonian system (1.1)

$$\dot Y(t) = F(Y(t)), \quad t \in \mathbb{R}, \quad Y(0) = Y_0. \tag{2.4}$$

All derivatives are understood in the sense of distributions. Here,

$$Y(t) = (\phi(t), \pi(t), q(t), p(t)), \quad Y_0 = (\phi_0, \pi_0, q_0, p_0) \in \mathcal{E}.$$

Lemma 2.2. (cf. [7, Lemma 2.1]) *Let* (1.5) *and* (1.6) *be satisfied. Then the following assertions hold.*

(i) *For every $Y_0 \in \mathcal{E}$, the Cauchy problem* (2.4) *has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{E})$.*

(ii) *For every $t \in \mathbb{R}$, the map $Y_0 \mapsto Y(t)$ is continuous on $\mathcal{E}$.*

(iii) *The energy is conserved, i.e.,*

$$\mathcal{H}(Y(t)) = \mathcal{H}(Y_0) \ for \ t \in \mathbb{R}. \tag{2.5}$$

(iv) *The energy is bounded below, and*

$$\inf_{Y \in \mathcal{E}} \mathcal{H}(Y) = V_0 + \frac{1}{2}(\rho, \Delta^{-1}\rho). \tag{2.6}$$

Our first result is the following long-time convergence in $\mathcal{E}_{-\sigma}$ to the stationary stable state.

Theorem 2.3. *Let conditions* (1.5)–(1.7) *hold, and let $Y(t)$ be a solution to the Cauchy problem* (2.4) *with initial state $Y_0 \in \mathcal{E}$ close to $S_{q^*} = (s_{q^*}, 0, q^*, 0)$ with stable $q^* \in Z$:*

$$d_0 := \|\nabla(\phi_0 - s_{q^*})\|_{L^2_\sigma} + \|\pi_0\|_{L^2_\sigma} + |q_0 - q^*| + |p_0| \ll 1, \tag{2.7}$$

where $\sigma > 1$. Then, for sufficiently small d_0,

$$\|Y(t) - S_{q^*}\|_{\mathcal{E}_{-\sigma}} \leqslant C(d_0)(1 + |t|)^{-\sigma}, \quad t \in \mathbb{R}. \tag{2.8}$$

Our second result is the following scattering long-time asymptotics in global energy norms for the field components of the solution.

Theorem 2.4. *Let the assumptions of Theorem* 2.3 *hold. Then, for sufficiently small d_0,*

$$(\phi(x,t), \pi(x,t)) = (s_{q^*}, 0) + W_0(t)\Phi_\pm + r_\pm(x,t), \qquad t \to \pm\infty, \tag{2.9}$$

where $W_0(t)$ is the dynamical group of the free wave equation, $\Phi_\pm \in \dot{H}^1 \oplus L^2$, and

$$\|r_\pm(t)\|_{\dot{H}^1 \oplus L^2} = \mathcal{O}(|t|^{-\sigma+1}), \qquad t \to \pm\infty. \tag{2.10}$$

It suffices to prove (2.9) for positive $t \to +\infty$ since the system (1.1) is time reversible.

3. LINEARIZATION AT A STATIONARY STATE

For notational simplicity, we also assume isotropy, which means that

$$\partial_i \partial_j V(q^*) = \omega_0^2 \delta_{ij}, \quad i,j = 1,2,3, \quad \omega_0 > 0\,. \tag{3.1}$$

Without loss of generality, we take $q^* = 0$.

Let $S_q = S_0 = (s_0, 0, 0, 0)$ be the stationary state of (1.1) corresponding to $q^* = 0$, and let $Y_0 = (\phi_0, \pi_0, q_0, p_0) \in \mathcal{E}$ be an arbitrary initial data satisfying (2.7). Let us consider $Y(t) = (\phi(x,t), \pi(x,t), q(t), p(t)) \in \mathcal{E}$, the solution to (1.1) with $Y(0) = Y_0$.

To linearize (1.1) at S_0, we set $\phi(x,t) = \psi(x,t) + s_0(x)$. Then (1.1) becomes

$$\begin{cases} \dot{\psi}(x,t) &= \pi(x,t), \\ \dot{\pi}(x,t) &= \Delta\psi(x,t) + \rho(x) - \rho(x - q(t)), \\ \dot{q}(t) &= p(t), \\ \dot{p}(t) &= -\nabla V(q(t)) + \displaystyle\int d^3x\, \psi(x,t)\, \nabla\rho(x - q(t)) + \int d^3x\, s_0(x)[\nabla\rho(x - q(t)) - \nabla\rho(x)]. \end{cases} \tag{3.2}$$

Introducing $X(t) = Y(t) - S_0 = (\psi(t), \pi(t), q(t), p(t)) \in C(\mathbb{R}, \mathcal{E})$, we rewrite the nonlinear system (3.2) in the form

$$\dot{X}(t) = AX(t) + B(X(t)). \tag{3.3}$$

Here A is the linear operator defined by

$$A \begin{pmatrix} \psi \\ \pi \\ q \\ p \end{pmatrix} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta & 0 & \nabla\rho\cdot & 0 \\ 0 & 0 & 0 & E \\ \langle\cdot, \nabla\rho\rangle & 0 & -\omega_0^2 - \omega_1^2 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \pi \\ q \\ p \end{pmatrix} \tag{3.4}$$

with

$$\omega_1^2\delta_{ij} = \frac{1}{3}\,\|\rho\|^2_{L^2}\delta_{ij} = -\int d^3x\,\partial_i s_0(x)\partial_j\rho(x)\,. \tag{3.5}$$

Here, the factor 1/3 is due to a spherical symmetry of $\rho(x)$ (cf. (1.6)).

The nonlinear part is given by

$$B(X) = (0, \pi_1, 0, p_1), \tag{3.6}$$

where

$$\pi_1 = \rho(x) - \rho(x-q) - \nabla\rho(x)\cdot q \tag{3.7}$$

and

$$\begin{aligned} p_1 = -\nabla V(q) + \omega_0^2 q + \int d^3x\,\psi(x)[\nabla\rho(x-q) - \nabla\rho(x)] \\ + \int d^3x\,\nabla s_0(x)[\rho(x) - \rho(x-q) - \nabla\rho(x)\cdot q]. \end{aligned} \tag{3.8}$$

Let us consider the Cauchy problem for the linear equation

$$\dot Z(t) = AZ(t), \quad Z = (\Psi, \Pi, Q, P), \quad t\in\mathbb{R}, \tag{3.9}$$

with initial condition

$$Z|_{t=0} = Z_0. \tag{3.10}$$

System (3.9) is a formal Hamiltonian system with the quadratic Hamiltonian

$$\mathcal{H}_0(Z) = \frac{1}{2}\Big(P^2 + \omega^2 Q^2 + \int d^3x\,(|\Pi(x)|^2 + |\nabla\Psi(x)|^2 - 2\Psi(x)\nabla\rho(x)\cdot Q)\Big),$$

which is the formal Taylor expansion of $\mathcal{H}(Y_0 + Z)$ up to second order at $Z = 0$.

Lemma 3.1. *Let condition* (1.6) *be satisfied. Then the following assertions hold.*

(i) *For every $Z_0 \in \mathcal{E}$, the Cauchy problem* (3.9), (3.10) *has a unique solution $Z(\cdot)\in C(\mathbb{R},\mathcal{E})$.*

(ii) *For every $t\in\mathbb{R}$, the map $U(t): Z_0 \mapsto Z(t)$ is continuous on $\mathcal{E}$.*

(iii) *For $Z_0\in\mathcal{E}$, the energy $\mathcal{H}_0$ is finite and is conserved, i.e.,*

$$\mathcal{H}_0(Z(t)) = \mathcal{H}_0(Z_0) \quad \textit{for} \quad t\in\mathbb{R}. \tag{3.11}$$

(iv) *For $Z_0\in\mathcal{E}$,*

$$\|Z(t)\|_{\mathcal{E}} \leqslant C \ \textit{for}\ t\in\mathbb{R} \tag{3.12}$$

where C depending only on the norm $\|Z_0\|_{\mathcal{E}}$.

4. DECAY OF LINEARIZED DYNAMICS

We prove the following long-time decay of the solution $Z(t)$ to (3.9).

Proposition 4.1. *Let conditions* (1.6) *and* (1.7) *hold, and let $Z_0\in\mathcal{E}$ be such that*

$$\|\nabla\Psi_0\|_{L^2_\sigma} + \|\Pi_0\|_{L^2_\sigma} < \infty$$

with some $\sigma > 1$. Then for $Z(t) = U(t)Z_0$,

$$\|Z(t)\|_{\mathcal{E}_{-\sigma}} \leqslant C(\rho,\sigma)(1+|t|)^{-\sigma+1}(\|\nabla\Psi_0\|_{L^2_\sigma} + \|\Pi_0\|_{L^2_\sigma}). \tag{4.1}$$

To prove this assertion, we apply the Fourier–Laplace transform

$$\tilde Z(\lambda) = \Lambda Z(t) = \int_0^\infty e^{-\lambda t} Z(t)dt, \quad \mathrm{Re}\lambda > 0 \tag{4.2}$$

to (3.9). We expect that the solution $Z(t)$ is bounded in the norm $\|\cdot\|_{\mathcal{E}}$. Then the integral (4.2) converges and is analytic for $\mathrm{Re}\lambda > 0$, and

$$\|\tilde Z(\lambda)\|_{\mathcal{E}} \leqslant \frac{C}{\mathrm{Re}\lambda}, \quad \mathrm{Re}\lambda > 0. \tag{4.3}$$

Applying the Fourier–Laplace transform to (3.9), we obtain

$$\lambda\tilde Z(\lambda) = A\tilde Z(\lambda) + Z_0, \quad \mathrm{Re}\lambda > 0. \tag{4.4}$$

Hence, the solution $Z(t)$ is given by

$$\tilde Z(\lambda)=-(A-\lambda)^{-1}Z_0,\quad \mathrm{Re}\lambda>0. \tag{4.5}$$

By (4.3), the resolvent $R(\lambda)=(A-\lambda)^{-1}$ exists and is analytic in $\mathcal{E}$ for $\mathrm{Re}\lambda>0$.

Let us construct the resolvent for $\mathrm{Re}\lambda>0$. Equation (4.4) becomes

$$\lambda\begin{pmatrix}\tilde\Psi\\ \tilde\Pi\\ \tilde Q\\ \tilde P\end{pmatrix}=\begin{pmatrix}\tilde\Pi\\ \Delta\tilde\Psi+\tilde Q\cdot\nabla\rho\\ \tilde P\\ -\langle\nabla\tilde\Psi,\rho\rangle-\omega^2\tilde Q\end{pmatrix}+\begin{pmatrix}\Psi_0\\ \Pi_0\\ Q_0\\ P_0\end{pmatrix}, \tag{4.6}$$

where $\omega^2=\omega_0^2+\omega_1^2$.

Step (i). We consider the first two equations of (4.6):

$$\begin{cases}-\lambda\tilde\Psi+\tilde\Pi= & -\Psi_0,\\ \Delta\tilde\Psi-\lambda\tilde\Pi= & -\Pi_0-\tilde Q\cdot\nabla\rho\end{cases} \tag{4.7}$$

A solution to system (4.7) admit the convolution representation

$$\begin{cases}\tilde\Psi &= \quad \lambda g_\lambda*\Psi_0+g_\lambda*\Pi_0+(g_\lambda*\nabla\rho)\cdot\tilde Q,\\ \tilde\Pi &= \quad \Delta g_\lambda*\Psi_0+\lambda g_\lambda*\Pi_0+\lambda(g_\lambda*\nabla\rho)\cdot\tilde Q,\end{cases} \tag{4.8}$$

where

$$g_\lambda(z)=(-\Delta+\lambda^2)^{-1}=\frac{e^{-\lambda|z|}}{4\pi|z|}. \tag{4.9}$$

Step (ii). We consider the last two equations of (4.6):

$$\begin{cases}-\lambda\tilde Q+\tilde P &= \quad -Q_0,\\ -\omega_2\tilde Q-\langle\nabla\tilde\Psi,\rho\rangle-\lambda\tilde P &= \quad -P_0.\end{cases} \tag{4.10}$$

Let us write the first equation of (4.8) in the form $\tilde\Psi(x)=\tilde\Psi_1(\tilde Q)+\tilde\Psi_2(\Psi_0,\Pi_0)$, where

$$\tilde\Psi_1(\tilde Q)=\tilde Q\cdot(g_\lambda*\nabla\rho),\quad \tilde\Psi_2(\Psi_0,\Pi_0)=\lambda g_\lambda*\Psi_0+g_\lambda*\Pi_0. \tag{4.11}$$

Then the second equation in (4.10) becomes

$$-\omega^2\tilde Q-\langle\nabla\tilde\Psi_1,\rho\rangle-\lambda\tilde P=-P_0+\langle\nabla\tilde\Psi_2,\rho\rangle=:-P_0'.$$

Now we compute the term $\langle\nabla\tilde\Psi_1,\rho\rangle$:

$$\langle\nabla\tilde\Psi_1,\rho\rangle=-\langle\tilde\Psi_1,\partial_i\rho\rangle=-\Big\langle\sum_j(g_\lambda*\partial_j\rho)\tilde Q_j,\partial_i\rho\Big\rangle=-\sum_j\langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle\tilde Q_j=-\sum_j H_{ij}(\lambda)\tilde Q_j,$$

where

$$H_{ij}(\lambda):=\langle g_\lambda*\partial_j\rho,\partial_i\rho\rangle=\langle i\hat g_\lambda(k)k_j\hat\rho(k),ik_i\hat\rho(k)\rangle=\Big\langle\frac{ik_j\hat\rho(k)}{k^2+\lambda^2},ik_i\hat\rho(k)\Big\rangle=\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+\lambda^2}. \tag{4.12}$$

The matrix H with entries H_{jj}, $1\leqslant j\leqslant 3$, is well defined for $\mathrm{Re}\lambda>0$ since the denominator does not vanish. The matrix H is diagonal; moreover,

$$H_{11}(\lambda)=H_{22}(\lambda)=H_{33}(\lambda)=h(\lambda). \tag{4.13}$$

Finally, the system (4.10) takes the form

$$M(\lambda)\begin{pmatrix}\tilde Q\\ \tilde P\end{pmatrix}=\begin{pmatrix}Q_0\\ P_0'\end{pmatrix},\quad\text{where } M(\lambda)=\begin{pmatrix}\lambda E & -E\\ \omega^2E-H(\lambda) & \lambda E\end{pmatrix}. \tag{4.14}$$

Lemma 4.2. *The matrix-valued function $M(\lambda)$ $(M^{-1}(\lambda))$ admits an analytic (meromorphic) continuation to the entire complex plane $\mathbb{C}$.*

Proof. The Green function g_λ admits an analytic continuation in λ to the entire complex plane $\mathbb{C}$. Then an analytic continuation of $M(\lambda)$ exists in view of (4.12) since the function $\rho(x)$ is compactly supported because of (1.6). Then the inverse matrix is meromorphic, since it exists for large $\mathrm{Re}\lambda$. This fact follows from (4.14), since $H(\lambda)\to 0$, $\mathrm{Re}\lambda\to\infty$ in view of (4.12).

Since the matrix $H(\lambda)$ is diagonal, the matrix $M(\lambda)$ is equivalent to three independent 2×2-matrices. Indeed, let us transpose the columns and rows of the matrix $M(\lambda)$ in the order (142536).

Then we obtain a matrix with three 2×2-blocks on the main diagonal. Therefore, the determinant of $M(\lambda)$ is the product of determinants of three matrices. Namely,

$$\det M(\lambda) = (\lambda^2 + \omega^2 - h(\lambda))^3 = (\lambda^2 + \omega_0^2 + \omega_1^2 - h(\lambda))^3, \tag{4.15}$$

where

$$\omega_1^2 = \int \frac{k_1^2 |\hat\rho(k)|^2 dk}{k^2}, \quad h(\lambda) = \int \frac{k_1^2 |\hat\rho(k)|^2 dk}{k^2 + \lambda^2}.$$

Proposition 4.3. *The matrix $M^{-1}(i\nu + 0)$ is analytic in $\nu \in \mathbb{R}$.*

Proof. It suffices to prove that the limit matrix $M(i\nu+0)$ is invertible for $\nu \in \mathbb{R}$ if ρ satisfies the Wiener condition (1.7). Formula (4.15) implies $\det M(0) = \omega_0^2 > 0$. For $\nu \neq 0$, $\nu \in \mathbb{R}$, we consider

$$h(i\nu + \varepsilon) = \int \frac{k_1^2 |\hat\rho(k)|^2 dk}{k^2 - (\nu - i\varepsilon)^2}, \quad \varepsilon > 0. \tag{4.16}$$

The denominator $D(\nu, k) = k^2 - \nu^2$ vanishes on $T_\nu = \{k : k^2 = \nu^2\}$. Denote by dS the surface area element. Then it follows from the Sokhotsky–Plemelj formula for C^1-functions that

$$\Im h(i\nu + 0) = -\frac{\nu}{|\nu|}\pi \int_{T_\nu} \frac{k_1^2 |\hat\rho(k)|^2}{|\nabla D(\nu, k)|} dS \neq 0, \tag{4.17}$$

since the integrand in (4.17) is positive by the Wiener condition (1.7). Now, the invertibility of $M(i\nu)$ follows from (4.15).

4.1. Time Decay

Here we prove Proposition 4.1. First, we obtain the decay (4.1) for the vector components $Q(t)$ and $P(t)$ of $Z(t)$. By (4.15), the components are given by the Fourier integral

$$\begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \frac{1}{2\pi} \int e^{i\nu t} M^{-1}(i\nu + 0) \begin{pmatrix} Q_0 \\ P_0' \end{pmatrix} d\nu = \mathcal{L}(t) \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} + \mathcal{L}(t) * \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \tag{4.18}$$

where

$$\mathcal{L}(t) = \frac{1}{2\pi} \int e^{i\nu t} M^{-1}(i\nu + 0) d\nu = \lambda^{-1} M^{-1}(i\nu + 0),$$

$$f(t) = \Lambda^{-1}[\langle \Psi_2(\Psi_0, \Pi_0), \nabla\rho \rangle] = \Lambda^{-1}[\langle i\nu g_{i\nu} * \Psi_0 + g_{i\nu} * \Pi_0, \nabla\rho \rangle] = \langle W_0(t)[(\Psi_0, \Pi_0)], \nabla\rho \rangle. \tag{4.19}$$

We write out the nonzero entries of the matrix $M(i\nu + 0)$,

$$\frac{i\nu}{-\nu^2 + \omega^2 - h(i\nu + 0)}, \quad \frac{1}{-\nu^2 + \omega^2 - h(i\nu + 0)}, \quad \frac{-\omega^2 + h(i\nu)}{-\nu^2 + \omega^2 - h(i\nu + 0)}.$$

Hence,

$$|M^{-1}(i\nu + 0)| \leqslant \frac{C}{|\nu|}, \quad |\partial^k M^{-1}(i\nu + 0)| \leqslant \frac{C_k}{|\nu|^2}, \quad \nu \in \mathbb{R}, \quad |\nu| \geqslant 1, \quad k \in \mathbb{N}.$$

Therefore, $\mathcal{L}(t)$ is continuous in $t \in \mathbb{R}$ and

$$\mathcal{L}(t) = \mathcal{O}(|t|^{-N}), \quad t \to \infty, \quad \forall N > 0. \tag{4.20}$$

For the solutions of the free wave equation, the following dispersion decay holds.

Lemma 4.4. (cf. [8, Proposition 2.1]) *Let $(\Psi_0, \Pi_0) \in \mathcal{F}_0$ be such that $\|(\Psi_0, \Pi_0)\|_{\mathcal{F}_\sigma} < \infty$ with some $\sigma > 1$. Then*

$$\|W(t)[(\Psi_0, \Pi_0)]\|_{\mathcal{F}_{-\sigma}} \leqslant C(1 + |t|)^{-\sigma} \|(\Psi_0, \Pi_0)\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{4.21}$$

Lemma 4.4 and the definition (4.19) imply

$$|f(t)| \leqslant C(\sigma, \rho)(1 + |t|)^{-\sigma} \|(\Psi_0, \Pi_0)\|_{\mathcal{F}_\sigma}, \quad t \in \mathbb{R}. \tag{4.22}$$

Therefore, (4.18), (4.20), and (4.22) imply

$$|Q(t)| + |P(t)| \leqslant C(\sigma, \rho)(1 + |t|)^{-\sigma} \|(\Psi_0, \Pi_0)\|_{\mathcal{F}_\sigma}. \tag{4.23}$$

Then (4.1) holds for the vector components $Q(t)$ and $P(t)$.

Now we prove (4.1) for the field components of $Z(t)$. The first two equations of (3.9) have the form

$$\begin{pmatrix} \dot\Psi(t) \\ \dot\Pi(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \Psi(t) \\ \Pi(t) \end{pmatrix} + \begin{pmatrix} 0 \\ Q(t)\cdot\nabla\rho \end{pmatrix}. \tag{4.24}$$

The integrated version of (4.24) reads

$$(\Psi(t),\Pi(t)) = W(t)[(\Psi_0,\Pi_0)] + \int_0^t W(t-s)[0, Q(s)\cdot\nabla\rho]ds, \quad t \geqslant 0. \tag{4.25}$$

By (4.21) and (4.23), $\|(\Psi(t),\Pi(t))\|_{\mathcal{F}_{-\sigma}} \leqslant C(\rho,\sigma)(1+|t|)^{-\sigma+1}(\|\nabla\Psi_0\|_{L^2_\sigma} + \|\Pi_0\|_{L^2_\sigma})$, $t\in\mathbb{R}$. Proposition 4.1 is proved.

5. ASYMPTOTIC STABILITY OF STATIONARY STATES

Here we prove Theorem 2.3. First, we obtain bounds for the nonlinear part $B(X(t)) = (0,\pi_1,0,p_1)$ defined in (3.6)–(3.8). We have

$$\|\pi_1(t)\|_{L^2_\sigma} \leqslant \mathcal{R}(|q|)|q(t)|^2, \qquad |p_1(t)| \leqslant \mathcal{R}(|q|)[|q(t)|\|\psi(t)\|_{\dot H^1_{-\sigma}} + |q(t)|^2],$$

where $R(A)$ denotes a positive function that remains bounded for sufficiently small A. Hence,

$$\|B(t)\|_{\mathcal{E}_\sigma} \leqslant \mathcal{R}(|q|)\|X(t)\|^2_{\mathcal{E}_{-\sigma}}. \tag{5.1}$$

We introduce the majorant

$$m(t) = \sup_{0\leqslant s\leqslant t}(1+s)^{-\sigma}\|X(s)\|_{\mathcal{E}_{-\sigma}}. \tag{5.2}$$

We fix $\varepsilon > \|X(0)\|_{\mathcal{E}_{-\sigma}}$ and introduce the *exit time*

$$t_* = \sup\{t>0 : m(t)\leqslant\varepsilon\}. \tag{5.3}$$

The integrated version of (3.3) can be written as

$$X(t) = e^{At}X(0) + \int_0^t ds\, e^{A(t-s)}B(X(s)). \tag{5.4}$$

Proposition 4.1 implies the integral inequality

$$\|X(t)\|_{\mathcal{E}_{-\sigma}} \leqslant C\,(1+|t|)^{-\sigma}\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + C\int_0^t ds\,(1+|t-s|)^{-\sigma}\|X(s)\|^2_{\mathcal{E}_{-\sigma}} \tag{5.5}$$

for $t<t_*$. We multiply both sides of (5.4) by $(1+t)^{-\sigma}$, and take the supremum over $t\in[0,t_*]$. Then

$$m(t) \leqslant C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + C\sup_{t\in[0,t_*]}\int_0^t \frac{(1+t)^\sigma}{(1+|t-s|)^\sigma}\frac{m^2(s)}{(1+s)^{2\sigma}}ds$$

for $t<t_*$. Since $m(t)$ is a monotone increasing function, we obtain

$$m(t) \leqslant C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + Cm^2(t)I(t), \quad t\leqslant t_*, \tag{5.6}$$

where

$$I(t) = \int_0^t \frac{(1+t)^\sigma}{(1+|t-s|)^\sigma}\frac{m^2(s)}{(1+s)^{2\sigma}}ds \leqslant \overline{I} < \infty, \qquad t\geqslant 0, \quad \sigma>1.$$

Therefore, (5.6) takes the form

$$m(t) \leqslant C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} + C\overline{I}m^2(t), \quad t\leqslant t_*, \tag{5.7}$$

which implies that $m(t)$ is bounded for $t<t_*$; moreover,

$$m(t) \leqslant C\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma}, \quad t<t_*, \tag{5.8}$$

since $m(0) = \|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma}$ is sufficiently small by (2.7).

The constant C in the estimate (5.8) is independent of t_*. We choose d_0 in (2.7) so small that $\|(\psi_0,\pi_0)\|_{\mathcal{F}_\sigma} < \varepsilon/(2C)$, which is possible due to (2.7). Then the estimate (5.8) implies $t_*=\infty$, and (5.8) holds for all $t>0$ if d_0 is small enough.

6. SCATTERING ASYMPTOTICS

Here we prove Theorem 2.4. From the first two equations of (1.1), we obtain the inhomogeneous wave equation for the difference $F(x,t) = (\psi(x,t), \pi(x,t)) = (\phi(x,t), \pi(x,t)) - (s_{q^*}, 0)$:

$$\dot\psi(x,t) = \pi(x,t), \qquad \dot\pi(x,t) = \Delta\psi(x,t) + \rho(x-q^*) - \rho(x-q(t)). \tag{6.1}$$

Then

$$F(t) = W_0(t)F(0) - \int_0^t W_0(t-s)[(0, \rho(x-q^*) - \rho(x-q(s)))]ds. \tag{6.2}$$

To obtain the asymptotics (2.9), it suffices to prove that $F(t) = W_0(t)\Phi_+ + r_+(t)$ for some function $\Phi_+ \in \dot H^1 \oplus L^2$ and $\|r_+(t)\|_{\dot H^1\oplus L^2} = \mathcal{O}(t^{-\sigma+1})$. This fact is equivalent to the asymptotics

$$W_0(-t)F(t) = \Phi_+ + r'_+(t), \qquad \|r'_+(t)\|_{\dot H^1\oplus L^2} = \mathcal{O}(t^{-\sigma+1}), \tag{6.3}$$

since $W_0(t)$ is a unitary group on $\dot H^1 \oplus L^2$ by the energy conservation law for the free wave equation. Finally, the asymptotics (6.3) hold since (6.2) implies

$$W_0(-t)F(t) = F(0) - \int_0^t W_0(-s)R(s)ds, \quad R(s) = (0, \rho(x-q^*) - \rho(x-q(s))). \tag{6.4}$$

We set

$$\Phi_+ = F(0) - \int_0^\infty W_0(-s)R(s)ds, \;\; r'_+(t) = \int_t^\infty W_0(-s)R(s)ds \tag{6.5}$$

The integral on the right hand side of the first equation in (6.5) converges in $\dot H^1 \oplus L^2$ with the rate $\mathcal{O}(t^{-\sigma+1})$ because

$$\|W_0(-s)R(s)\|_{\dot H^1\oplus L^2} = \mathcal{O}(s^{-\sigma})$$

by the unitarity of $W_0(-s)$ and the decay rate $\|R(s)\|_{\dot H^1\oplus L^2} = \mathcal{O}(s^{-\sigma})$ which follows from the conditions (1.6) on ρ and the asymptotics (2.8). Hence, $\Phi_+ \in \dot H^1 \oplus L^2$ and (2.10) holds.

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