Light, Eye, Brain. Geometrical problems in vision

Dmitri Alekseevsky
Institute for Information Transmission Problems, Moscow, Russia

October 24, 2016

Contents

Abstract

We discuss geometrical problems which arise in investigation of systems of early vision of mammals. In particular, we consider two questions:
1) Which visual information comes to eye (retina) and how does it change under movement of eye and head?
2) How do eyes and brain extract invariant information about the external geometry from the subjective (dependant on position etc.) input information which light brings to retina?

The aim of vision is to obtain information about (Euclidean) geometry of the external world from light which falls to the retina. It must be objective, i.e. independent from position of observer (that is invariant with respect to change of position of eyes, head, velocity etc.)
In the first part of the paper, we analyze which visual information comes to retina and how it changes due to eye movements. We show that under some assumptions, visual information which brain use for reconstruction of black-white picture is encoded into a function I of the energy of light falling on retina R (which is a part of the eye sphere S^2). The differential dI of this input function defines the 1-dimensional distribution on retina. Integral curves of this distributions ("contours", that is the level set $I = const$) are the main geometric objects which are detected in early vision. We show that rotation of eye induces a conformal change of the input function I. One of the main problem (called in neurophysiology "stability problem") is to describe a mechanism of compensation of such transformations of image in retina. We review known fact about information processing in retina and primary visual cortex VI and functional architecture of VI cortex, including models of Petitot, Citti, Sarti of VI cortrex and model of Bressloff and Covan of hypercolumns and propose an unification of these models . We consider an application of this unified model to problem of stability. [1]

[1]This work is written at the IITP and is supported by an RNF grant (project n.14-50-00150).

1 Geometry of light

1.1 Light in approximation of geometric optics

In the approximation of geometric optics light is described in terms of the space $L(E^3) \simeq TS^2$ of straight lines. Light travels along a line ℓ and has energy density (the average value of the square norm of electric filed). In geometric optics one assumes that this energy density $I(\ell)$ is constant along ℓ and does not depend on time. The wave length (color) and polarization of light are ignored . (It seems that polarization is important only for birds and insects which use it for navigation). Then all information which is available for eye is encoded in the density or energy of light $I : L(E^3) \ni \ell \to \mathbb{R}^{\geq 0}$.

1.2 Light in approximation of classical electrodynamics (CED)

Let $(M^{1,3}, g)$ be the Minkowski space and $(M^{1,3} = \mathbb{R}t \times E^3,\ g = -dt^2 + g_E)$ its decomposition into the time and the space associated with an inertial observer. According to Maxwell electrodynamics, light is described as a radiation solution of the Maxwell equation ($dF = d * F = 0$) that is a harmonic 2-form F which is a superposition of null (or plane-wave) solutions F such that $(g(F,F) = g(F, *F) = 0)$. Such plane-wave 2-form F can be written as $F = p \wedge e$ where p is an isotropic geodesic vector field ($\nabla_p p = 0$) and $e \in \Gamma(E(p))$ is a section of the screen bundle $E(p) = p^{\perp}/\mathbb{R}p$. If $p = e_0 + e_1$, $e = e_2$ then $F = p \wedge e_2 = e_0 \wedge e_2 + e_1 \wedge e_2$ and F describes an electro-magnetic wave with electric vector $E = e_2$ and magnetic vector $H = *(e_1 \wedge e_2) = e_3$ which is propagated in direction of the vector $e_1 = e_2 \times e_3$.

1.2.1 The space $L^0(M^{1,3}) = S^2 \times E^3 = S(E^3) \subset TE^3$ of isotropic lines

Any isotropic line has the form $\ell = x + \mathbb{R}p$ where $x = \ell \cap E^3$ and $p \in C_x = C(T_x M^{1,3})$ an isotropic vector. So the space $L^0(M^{1,3})$ of isotropic lines is naturally identified with
$L^0(M^{1,3}) = S^2 \times E^3 = S(E^3) = \{(\mathbb{R}p, x)\}$ where $S^2 = PC_x$ is the celestial sphere i.e. the projectivization of the light cone C_x.
Note that the space $L^0(M^{1,3})$ of isotropic lines can be identified with the incident space $Inc(E^3, L(E^3)) = \{(x, \ell),\ x \in \ell\} \subset E^3 \times L(E^3)$. Also $L(M^{1,3})$ is identified with the unit sphere bundle $S(E^3)$ of the Euclidean space E^3: a unite vector $v \in S_x(E^3) \subset T_x(E^3)$ corresponds to isotropic line $\ell(x, v) = x + \mathbb{R}(\partial_t + v)$.

1.2.2 Geodesic congruences of isotropic lines

An isotropic vector field $M^{1,3} \ni x \to p(x) \in C_x$ is called geodesic field if its orbits are (isotropic) geodesics (equivalently, $\nabla_p p = 0$). It is determined by its restriction to E^3 which is a section $p_{E^3} : E^3 \to L^0(M^{1,3}) = S^2 \times E^3$ of the bundle

$$\pi : L^0(M^{1,3}) = S^2 \times E^3 = S(E^3) \to E^3.$$

The 3-dimensional submanifold $M(p) := p_{E^3}(E^3) \subset L^0(M^{1,3})$ of the manifold of isotropic lines associated with an isotropic geodesic vector field p is called a **congruence of isotropic lines**.

The section p_{E^3} can be identified with a unit vector field $p_{E^3} : E^3 \ni x \mapsto v_x \in T_xE^3 = E^3$, where $p(x) = \partial_t + v_x$.

1.2.3 Shear free congruences of isotropic lines

Denote by φ_t the (local) flow generated by an isotropic geodesic vector field p. The field p and associated congruence $M(p)$ is called **shear-free** if the isomorphism

$$(\varphi_t)_* : T_{x(0)}M^{1,3} \to T_{x(t)}M^{1,3}$$

induced by the shift φ_t along a geodesic $\ell = \{x(t) = \varphi_t x(0)\}$ preserves the orthogonal complement $p^\perp$ to the tangent vector $p(x(t)) = \dot{x}(t)$ and the map

$$(\varphi_t)_* : p(x(0))^\perp \to p(x(t))^\perp$$

is a conformal map with respect to the (degenerate) metric $g|_{p^\perp}$ induced by the Minkowski metric of $M^{1,3}$.

1.2.4 Robertson and Kerr theorems

Robertson theorem states that an isotropic geodesic vector field $p(x) \in \mathcal{X}(M^{1,3})$ can be extended to a null solution $F = p(x) \wedge e(x) \in \Omega(M^{1,3})$ of the Maxwell equation if and only if the associated geodesic congruence is shear-free.
The Kerr theorem gives a description of shear-free congruence in terms of complex surfaces of the Penrose twistor space $\mathbb{C}P^3$ of the Minkowski space $M^{1,3}$.

Let $\mathbb{R}^4 = \mathbb{R}e_0 \oplus \mathbb{R}^3$ be an orthogonal decomposition of the Eucledean space $(\mathbb{R}^4, g =< .,. >)$.
$\mathbb{R}^{1,3} = \mathbb{R}ie_0 \oplus \mathbb{R}^3$ associated Minkowski vector space and $\mathbb{C}^4 = \mathbb{R}^4 \otimes \mathbb{C}$.
Recall the direct sum decomposition of orthogonal Lie algebra $\mathfrak{so}_4 = \mathfrak{sp}_1 \oplus \mathfrak{sp}'_1$. The set of complex structures in $\mathfrak{sp}_1$ is parametrised by unit vectors $e_1 \in \mathbb{R}^3$:
$J = J^{e_1} = e_0 \wedge e_1 + e_2 \wedge e_3$ where e_1, e_2, e_3 is an oriented orthonormal basis of $\mathbb{R}^3$.
It defines the eigenspace decomposition $\mathbb{C}^4 = \Pi \oplus \bar{\Pi}$, where the J-holomorphic complex isotropic 2-plane $\Pi = \Pi_J = \mathrm{span}(e_0 + ie_1, e_2 + ie_3)$ is called an α-plane or selfdual planes.

1.2.5 An algebraic lemma

Lemma 1 *[1] There is 1-1 correspondence between*
i) complex structure $J \in \mathfrak{sp}_1$ in $\mathbb{R}^4$,

ii) α-planes Π_J,
iii) unit vectors $e_1 = Je_0 \in \mathbb{R}^3$
iv) isotropic lines $\mathbb{R}p$, $p = e_0 + e_1$ in the Minkowski space $\mathbb{R}^{1,3}$.

1.2.6 CR structures on E^3 and the canonical CR structure on $L^0(M^{1,3}) = S^2 \times E^3$.

A unit vector field $V : E^3 \to S^2$ defines a CR structure $(\mathcal{H}, J^V) = (V^\perp, R^{\pi/2})$ in E^3.

The canonical integrable CR structure in $L^0(M^{1,3}) = S^2 \times E^3$ is defined as

$$(\mathcal{H}, J)_{x,V} = (V_x^\perp \oplus T_V S^2, R^{\pi/2} \oplus J^{S^2}).$$

This CR structure is induced by the embedding $L^0(M^{1,3}) = S(E^3) \subset TS^2$ of the space of isotropic line as a unite sphere bundle into the tangent bundle TS^2 of the conformal sphere S^2 with the natural complex structure defined by the conformal structure of S^2.

1.2.7 Conformal 1-dimensional foliations of E^3.

The foliation of E^3 defined by a unit vector field V is called to be **conformal** if one of the equivalent conditions holds:
1) The map $V : E^3 \to S(E^3) = L^0(E^3)$ is a CR map;
2) $((\mathcal{L}_V J^V)X)^\perp = 0,\ X \in V^\perp$, where $\mathcal{L}_X$ stands for the Lie derivative.
4)$L_V \circ J^V = J^V \circ L_V|_{V^\perp}$ where $L_V = \nabla V \in \mathrm{Hom}(TE^3, V^\perp)$
Locally any conformal foliation has the form $\ker(df)$ for some complex function $f : E^3 \to \mathbb{C}$ with $(\mathrm{grad} f)^2 = 0$.

1.2.8 A description of shear free congruences (P. Baird, J.Wood)

Locally the following objects are equivalent :
i) A complex structure J in a domain $D \subset \mathbb{R}^4$ s.t. (g_{can}, J) is a Hermitian structure;
ii) Foliation of the domain $D^{\mathbb{C}} \subset \mathbb{C}^4$ by α-planes;
iii)The conformal foliation of the domain $D \cap \mathbb{R}^3$;
iv) Shear-free congruence of the domain $D^{\mathbb{C}} \cap \mathbb{R}^{1,3}$.

1.3 Geometry of lines in E^3

The space $L(E^3)$ of lines $\ell_{x,e} = \{x + \mathbb{R}e, x \perp e\}$ in $E^3 = \mathbb{R}^3$ is a 4-dimensional manifold which is naturally identified with the (co)tangent bundle $L(E^3) = T^*S^2 = TS^2$ of the standard unit sphere $S^2 \subset \mathbb{R}^3$.

The group $E(3)$ of Euclidean motions acts transitively on $L(E^3)$ with the stability subgroup $SO_2 \times \mathbb{R}$.
The homogeneous manifold $L(E^3) = E(3)/(SO_2 \times \mathbb{R})$ carries the natural $E(3)$-invariant symplectic structure ω, the complex structure J and the metric $g = \omega \circ J$ of neutral signature $(2, 2)$.
Moreover, (g, J, ω) is an invariant pseudo-Kähler structure and $L(E^3) =$

$E(3)/SO_2 \times \mathbb{R}$ is a pseudo-Kähler symmetric space. The symmetry at a point ℓ is defined by the Euclidean reflection w.r.t. ℓ.

A point $x \in E^3$ defines the projective plane $L(x)$ of lines going through x. It is a 2-dimensional Lagrangian submanifold of $L(E^3)$.

A line $\ell \subset E^3$ defines a 3-dimensional submanifold of lines which intersect ℓ.

A unit vector field $V : E^3 \to S^2$ (which is a section of the bundle $L^0(M^{1,3}) = E^3 \times S^2 \to E^3$) defines a 3-dimensional submanifolds in $L^0(M^{1,3})$ and $L(E^3)$.

The manifold $L(E^3)$ can be considered as the supersphere (2-sphere over the Grassman algebra of dual numbers) (Study-Kotelnikov).

1.3.1 Embedding of $L(E^3)$ into the manifold $L(P^3) = Gr_2(V^4) \subset PC(\Lambda^2 V)$ of projective lines

The natural compactification of $L(E^3)$ is the space $L(P^3) = Gr_2(V^4)$ of projective lines, that is the Grassmannian of two-planes in the 4-dimensional space $V^4 = \{(t, x, y, z)\}$.
The projective group $PGL(V) = SL_4(\mathbb{R})/\pm 1$ and its subgroup SO_4 act transitively on $L(P^3)$ and act locally on $L(E^3)$. In particular

$$L(P^3) = SO_4/S(O_2 \times O_2) = (SO_3 \times SO_3)/\mathbb{Z}_2)/S(O_2 \times O_2) = (S^2 \times S^2)/\mathbb{Z}_2.$$

Also the group $SO_{2,2}$ acts locally transitively on $L(P^3)$ with four open orbits and two codimension one orbits.

The subgroup of $PGL(V)$ which preserves a ball $B^3 \subset E^3 \subset P^3$ is the Lorentz group $SO_{1,3}$. (In terms of homogeneous coordinates, B^3 is defined by $\{x^2 + y^2 + z^2 \leq t^2\}$.)
It acts in the interior of the ball B^3 as the group of isometries of the Lobachevski metric and on the boundary "celestial" sphere S^2 as the conformal Möbius group.

In particular, a transformation $A \in SO_{1,3}$ preserves the Euclidean angle between curves on S^2 and transforms circles (i.e. intersection of S^2 with planes of E^3), but do not preserves the length of curves.

1.3.2 Space of lines as the Klein quadric

The space of lines $L(P^3)$ has a natural embedding

$$i : \Pi = \mathrm{span}(a, b) \mapsto \mathbb{R}(a \wedge b) \in Q = PC(\Lambda^2 V) \subset P(\Lambda^2 V) = P^5$$

as the Klein quadric Q into the projective space $P\Lambda^2 V$. The Klein quadric is the projectivisation of the cone $C(\Lambda^2 V)$ of isotropic vectors in the space $\Lambda^2 V$ of bivectors w.r.t. the natural scalar product

$$g_\Lambda(a \wedge b, c \wedge d) := (a \wedge b \wedge c \wedge d)/\mathrm{vol}$$

of signature $(3,3)$. Hence the group $SO_{3,3}$ acts transitively in $L(P^3) = SO_{3,3}/P$ and acts locally in $L(E^3)$. The set $L(x)$ of lines through a point corresponds to the projective plane $i(L(x)) \subset P^5$ which belongs to the Klein quadric $Q = SO_{3,3}/P = L(P^3) = P(C(\Lambda^2 V))$. The group $SO_{3,3}$ is the group of conformal transformations of Q w.r.t. the conformal structure induced by the metric g_Λ.

2 Eye

2.1 Eye as an optical device

We will consider now an eye as an optical device in the framework of geometric optic. Of course, a description of light as electromagnetic waves in the framework of Maxwell electrodynamics or in terms of quantum electrodynamics would be more relevant.

Eye as an optical system is a transparent ball B^3 with a system of lenses which consists of cornea and lens. They focus light to retina R which is back part of the boundary sphere S^2. The inner part of retina contains receptors (cones and rodes) which work as photoelements and transform light into electric impulses.

The cumulative effect of the cornea and lens is equivalent to action of a lens with center at a point F of the eye sphere S^2 , which focuses light rays to the retina. So we will assume that there is only a lens with center at a point $F \in S^2$.

2.1.1 The function of energy of light falling on retina

Consider a point $A \in E^3$ which is a source of light, going along rays with constant energy density. The light beam $\ell = (AF)$ which emits from a point A and passes through the center F of the eye lens has no refraction and is registered by a receptor at the point $\bar{A} := \ell \cap S^2$ of the retina $R \subset S^2$. Any other beam from A which goes through the lens L is focused and enters the same point $\bar{A}$. So the energy of light $I(\bar{A})$ at the point $\bar{A}$ of retina is given by the integral of the density of light $I((AX)) = I(A)$ emits from A in direction (AX) and incident upon the lens :

$$I(\bar{A}) := \int_{Y \in D} I(AY) d\sigma = \Omega(A) \cdot I(A)$$

where $D = \{(AX) \cap S^2(A),\, X \in L\}$ is the intersection of the cone with vertex A of beams incident upon the lens L, $d\sigma$ is the standard measure of this sphere and $\Omega(A) = \int_D d\sigma$ is the solid angle. The last equality holds since we assume that the density of energy $I(AX) = I(A)$ does not depend of the direction of a beam.

Let $M \subset E^3$ be a surface and each point $A \in M$ emits light along any ray with energy density $I_M(A)$ (which does not depend on direction). We consider the **central projection** $\pi_F : M \to S^2$ with the center at

$F \in S^2$ which maps the surface to the eye sphere by

$$\pi_F : M \ni A \mapsto \bar{A} := (AF) \cap S^2$$

where $(AF) \cap S^2$ the point of intersection , different from F.
It is a local diffeomorphism near any point $A \in M$ where the line (FA) does not belong to the tangent planes $T_A M$ and $T_F S^2$. Then the energy function of light enters the retina is given by

$$I(\bar{A}) = \Omega(A) \cdot I_M(\pi^{-1}(\bar{A})) = \Omega(A) \cdot I_M(A).$$

We assume that the solid angle $\Sigma = \Omega(A)$ associated with any point $A \in M$ is constant. Then the energy function of light

$$I : R \to \mathbb{R},\ \bar{A} \mapsto I(\bar{A})$$

enter to retina and detected by receptors is proportional to the pull back of the density energy function I_M via the inverse map $\pi^{-1} : R \to M$. More precisely,

$$I(\bar{A}) = \Omega \cdot I_M(A) = \Omega \cdot I_M(\pi^{-1}(\bar{A})).$$

Even if the surface M is stationary, the energy function I of light falling on retina depends on time, since the eye is always rotates around the center O. So we have to consider the energy function

$$I : S^2 \times \mathbb{R} \supset R \times \mathbb{R} \to \mathbb{R},\ (\bar{A}, t) \mapsto I(\bar{A}, t) = I(z, t)$$

of light falling on retina as function of three variables, where $t \in \mathbb{R}$ is the time and $z = (x, y)$ are coordinates fixed with respect to eye.

The brain extracts all visual information about external world from this energy function of light falling on retina.
(Recall that we consider only black-white vision and ignore the color).

2.1.2 Remark about input function on retina

As we mention above, all information on black-white vision is extracted from the energy function I of light falling on retina. Since we discuss black-white vision, the wave length (color) is not important as well as the polarization of light. Note that the polarisation of light play an important role for birds and insects, which use it for orientation.
Geometrically, the function I is described by level sets $I = c = const$ together with indication of its value c on a level set. However the values of the energy function I depends on luminosity and not important for perception of images. For example, it changes dozens of times when we turn on the light.
So it seems that the visual system detects information mostly from the 1-dimensional distribution (Pfaff system) whose integral curves are contours (level sets of I). They are determined by the differential dI and even by the conformal class $[dI]$ of 1-form dI. I thanks Valentin Lychagin for this remark.

2.2 Eye as a rigid body. Donder's and Listing's laws

Eye is a rigid ball B_O^3 which can rotate around the center O w.r.t. three mutually orthogonal axis $\vec{i}, \vec{j}, \vec{k}$.
For a fixed position of head, there is a privilege initial position $B(OF_0)$ of the eye ball corresponding to the standard (frontal) direction (OF_0).
Donder's law (1846)(No twist). If the head is fixed, the result of a movement of eye from position $B(OF_0)$ to a new position $B(OF)$ is uniquely determined by the line of sight OF and do not depend on previous movements.
This means that the curve on the sphere, which is described by the rotation of the line of sight OF, determines the curve in the orthogonal group SO_3, which describe the eye movement. Mathematically, this defines a section of the frame bundle $SO_3 \to S^2 = SO_3/SO_2$.
Listing's law (1845) The movement from the initial position $B(OF_0)$ to other position $B(OF)$ is obtained by rotation with respect to the axis $\overrightarrow{OF_0} \times \overrightarrow{OF}$.
The curve in SO_3 is the parallel lift of the arc $F_0F \subset S^2$.

2.3 Fixation eyes movements. Tremor, drift and microsaccades

Eyes participates in different involuntary types of movements. Even when the gaze is "fixed", the eye participates in so called "fixation eye movements"

Fixation eye movements include: **tremor, drifts** and **microsaccades**.

Tremor is an aperiodic, wave-like motion of the eyes of high frequency but very small amplitude.
Drifts occur simultaneously with tremor and are slow motions of eyes, in which the image of the fixation point for each eye remains within the fovea Drifts occurs between the fast, jerk-like, linear microsaccades.

2.3.1 Characteristics of fixation eye movements

	Amplitude	Duration	Frequency	Speed
Tremor	20-40 sec	-	30-100 Hz	Max 20 min/s
Drift	1-9 min	0.2-0.8 s	95-97% of time	1-30 min/s
Micsac	1-50 min	0.01-0.02 s	0.1-5 Hz	$10-50°/s$

Per 1 s tremor moves on 1-1.5 diameters of the fovea cone
drift moves on 10-15 diameters
microsaccades moves on 15-300 diameters.
Under tremor the axis of eye draws a cone for 0.1 s.
Many microsaccades arise when the image of the fixing point is in the center of fovea or near the border of fovea. One of the aim of microsaccades is to control that the projection of the fixed point of the gaze will be inside the fovea. It is a typical example of on-off control.

2.3.2 Model of eye movements by R.Engbert, K. Mergenthaler, P. Sinn, A. Pikovsky: "Self-avoiding random walk in a swamp on paraboloid"

R.Engbert, K. Mergenthaler, P. Sinn, A. Pikovsky propose a model of involuntary eye movement described as a self-avoiding random walk on the square lattice $\mathbb{Z}^2$ with quadratic potential ("Walk in a swamp on a paraboloid".
The physiological aim of such random walk is that when the gaze is fixed at a point A, the images of this point on retina must be homogeneously distributed between all receptors of the fovea.
The model is defined by the cost function ("depth of the swamp")

$$h : \mathbb{Z}^2 \to \mathbb{R}, (i,j) \mapsto h_{ij}.$$

The walker at point (i,j) moves to the neighbor point (i',j') which is one of the four points $(i \pm 1, j), (i, j \pm 1)$ which has the smallest cost. After this, the cost $h(i',j')$ increases by 1 (the swamp at (i',j') becomes deeper)

$$h(i',j') \to h(i',j') + 1$$

and the cost at all other points relax by the law

$$h_{kl} \to (1-\epsilon)h_{kl}.$$

The mean square displacement $D^2(t)$ for time $t \in \mathbb{Z}$ can be locally approximated by the function t^α. For classical random walk $\alpha = 1$. The authors find parameters which gives good correspondence with experiments. The generic path has less selfintersections then the classical random walk and it demonstrates persistent behavior ($\alpha > 1$) on a short timescale and antipersistent behavior ($\alpha < 1$) on a long timescale, which is consistent with experimental results.

2.3.3 Why eyes must always rotate?

In the framework of geometric optic ,the information carried by the light is encoded in the density energy function on the 4-dimensional space $L(E^3)$ of lines. When the position of the eye is fixed, the receptors in the retina can detect only restriction of this function to the 2-dimensional (Lagrangian) submanifolds L_F of lines , which pass through the center of lens F. Using fixation eye movements, humans are able to detect information about energy function I on some 4-dimensional neighbourhood of the Lagrangian submanifold $L(F)$ in $L(E^3)$. The cost is that the organization of such movements is a very complicated problem and many brains and sensomotor structures are involved in it.
A neurophysiological reason for fixation eye movements will be discussed later.

3 Central projection. Euclidean case

We give formula for the cental projection $\pi_F : M \to S^2$ of a parametrized surface $M = \{A = (x(u,v), y(u,v), z(u,v))\} \subset \mathbb{R}^3$ of the Euclidean space

$E^3 = \mathbb{R}^3$ into the sphere

$$S^2 = \{A, \overrightarrow{OA}^2 = r_0^2\}$$

with center $O = (0,0,0)$ at the origin with respect to the point $F = (r,0,0)$.
It maps a point $A \in M$ to the second point of intersection $(AF) \cap S^2$ of the ray (AF) with the sphere S^2.

3.0.4 Formula for central projection

We identify a point $A = (x,y,z)$ with the position vector $\overrightarrow{OA} = (x,y,z)$. The central projection is given by

$$\pi_F : M \ni A \mapsto \bar{A},\ \overrightarrow{O\bar{A}} = \overrightarrow{OF} - f(A)\overrightarrow{FA} = -f(A)\overrightarrow{OA} + (1 - f(A))\overrightarrow{OF}$$

where the function $f(A)$ is the positive solution of the quadratic equation

$$(FA)^2 f^2 - 2(\overrightarrow{OF} \cdot \overrightarrow{FA})f + OF^2 - r_0^2 = 0.$$

Consider the case when $F \in S^2$, i.e. $r_0 = r$. Then the equation becomes linear and we get

$$f(A) = \frac{2\overrightarrow{OF} \cdot \overrightarrow{FA}}{(FA)^2} = \frac{2r(x-r)}{(x-r)^2 + y^2 + z^2}$$

Finally,

$$\pi_F : A \mapsto \bar{A},\ \overrightarrow{O\bar{A}} = \quad \overrightarrow{OF} - 2\tfrac{\overrightarrow{OF} \cdot \overrightarrow{FA}}{(FA)^2}\overrightarrow{FA} = r(\overrightarrow{i} - 2\cos\phi\,\overrightarrow{e})$$

where $\overrightarrow{i}, \overrightarrow{j}, \overrightarrow{k}$ are basic orts , ϕ is the angle between $\overrightarrow{i}$ and $\overrightarrow{FA}$ and $\overrightarrow{e}$ is the unit vector in direction of $\overrightarrow{FA}$. In terms of coordinates,

$$\pi_F : (x,y,z) \mapsto \frac{r}{R^2}(-(x-r)^2 + y^2 + z^2, 2(x-r)y, 2(x-r)z),$$

where $R^2 := (A-F)^2 = (x-r)^2 + y^2 + z^2$.

3.0.5 Relation between metrics on M and S^2

We calculate the metric $g_{S^2} = (d\bar{A})^2$ of the sphere S^2 in coordinates u, v, where $\bar{A}(u,v) = \pi_F A(u,v)$. We have

$$d\bar{A} = -d(f(A-F)) = -df(A-F) - f dA,$$

$$\begin{aligned} g_{S^2} &= (d\bar{A})^2 = f^2(dA)^2 + f df d(R^2) + R^2 df^2 \\ &= f^2(dA)^2 + df(d(R^2 f)) \\ &= f^2(dA)^2 + 2r dx df. \end{aligned}$$

Corollary 1 *The cental projection $\pi_F : M \to S^2$ is a conformal map if and only if $dx = 0$ i.e. M is a part of the plane orthogonal to $\overrightarrow{i} = (1;0;0)$ or $df = 0$ i.e. M is a part of the sphere*

$$(x - r(1+\lambda))^2 + y^2 + z^2 = \lambda^2 r^2$$

with the center on the coordinate line $0x$ which contains the point $F = (r,0,0)$.

3.0.6 Rotation lemma

Let $R \in SO_3$ be a rotation around the center of the sphere S^2. It preserves the sphere S^2 and transform the point F into $F' = RF$. We denote by $\pi_{RF} : M \to RS^2 = S^2$ the central projection of the surface into the sphere RS^2. We identify the image $\bar{A} = \pi_F(A)$ of a point A on the sphere S^2 with the point $R\bar{A} \in RS^2 = S^2$.

Lemma 2 *The central projections $\pi_F : M \to S^2$ and $\pi_{RF} : M \to RS^2$ are related by*

$$\pi_{RF} = R\pi_F \circ R^{-1} : M \to S^2.$$

In other words, the rotation R of the sphere around the center O is equivalent to the rotation on the surface M by the inverse transformation R^{-1}.

Proof. We set $f_F(A) = 2\frac{\overrightarrow{FA}\cdot\overrightarrow{OF}}{(FA)^2}$. Then $f_{RF}(A) = f_F(R^{-1}A)$ and

$$\pi_{RF}(A) = (\overrightarrow{O(RF)} + f_{RF}(A) = R[\overrightarrow{OF} + f_F(R^{-1}A)\overrightarrow{F(R^{-1}A)}] = R\pi_F(R^{-1}A)$$

□

3.0.7 Central projection of a plane to sphere

We consider special case when M is a plane. We may assume that $M = \Pi_n^\rho = \{A, n \cdot A = \rho\}$ is the plane with the normal vector $n = (\cos\varphi, \sin\varphi, 0)$ where ρ is the distance from M to the center of S^2. Then

$$\Pi_n^\rho = \{A = \rho n + (\sin\varphi y, -\cos\varphi y, z) = (\rho\cos\varphi + \sin\varphi y, \rho\sin\varphi - \cos\varphi y, z)\}$$

Proposition 1 *The induces metric g_{S^2} of the sphere S^2 w.r.t. the local coordinates y, z s.t. $\bar{A}(y,z) = \pi_F A(y,z)$ is given by*

$$g_{S^2} = d\bar{A}^2 = f^2 dA^2 - 2r\sin\varphi dydf$$

where $dA^2 = dy^2 + dz^2$ is the metric of the plane Π_ρ^n

$$f = -\frac{2\overrightarrow{OF}\cdot\overrightarrow{FA}}{\overrightarrow{FA}^2} = -\frac{2r(\sin\varphi y + \beta)}{R^2},$$

$$R^2 = (FA)^2 := (y - r\sin\varphi)^2 + (\rho - r\cos\varphi)^2 + z^2, \quad \beta = \rho\cos\varphi - r.$$

3.0.8 When the central projection of a plane is a conformal map ?

Corollary 2
1. *The central projection $\pi_i : \Pi_\rho^n \to S^2$ is a conformal map if and only if the plane is frontal , i.e. it is orthogonal to the frontal direction $\overrightarrow{OF}$ (i.e. $\overrightarrow{n} = \overrightarrow{i} = (1,0,0)$).*
2. *For the plane Π_ρ^n which is obtained from a central plane by rotation R_{0z}^φ w.r.t. the axis $0z$ the deviation from conformality $2r\sin\varphi dydf$ is small if the angle φ is small.*

3. *Let $\pi : \Pi \to S^2$, $\pi' : \Pi' \to S^2$ be the central projections with respect to $F = (r, 0, 0)$ where Π is a frontal plane and $\Pi' = R\Pi$ is obtained from Π by a rotation R on angle φ. Then the local diffeomorphism $\Phi = \pi' \circ R \circ \pi^{-1} : S^2 \to S^2$ (which describe the transformation of the image $\pi(\Pi)$)) is not a conformal transformation, but it is closed to a conformal transformation if the angle φ is small.*

4 Central projection. Projective case

4.1 Minkowski space and conformal sphere

Let $(M^{1,3}, g)$ be the Minkowski space with metric g of signature $(-,+,+,+)$. We fix a pseudo-orthogonal frame (O, e_0, e_1, e_2, e_3) and identify $M^{1,3}$ with Minkowski vector space $V = \mathbb{R}^{1,3} = \mathbb{R}e_0 \oplus \mathbb{R}^3$ with coordinates (t, x, y, z). We denote by $SO(V) = \mathrm{Aut}(V, g)^{con}$ the connected Lorentz group. It acts naturally into the projective space $P^3 = PV$ with three orbits :
i)the open orbit $B^3 = PV_- = SO(V)/SO(V)_{e_0} = SO_{1,3}/SO_3$ which is the projectivisation of the set $V_- = \{X \in V, g(X,X) < 0\}$ of timelike vectors.
ii)the open orbit $B^3_+ = PV_+ = SO(V)/SO(V)_{e_3} = SO_{1,3}/SO_{1,2}$, which is the projectivization of the set $V_+ = \{X \in V, g(X,X) > 0\}$ of spacelike vectors,
iii) the closed codimension one orbit $Q = PV_0 = \{[p] := \mathbb{R}p,\ p \in V_0\} \simeq S^2$, which is the projectivisation of the cone of isotropic vectors V_0.
The metric g induces a conformal structure on $Q \simeq S^2$. The submanifold $Q = \{p = (t, x, y, z), -t^2 + x^2 + y^2 + z^2 = 0\} \subset PV$ is a projective quadric. The connected Lorentz group $SO(V) = SO_{1,3}$ acts transitively on Q as the conformal group (the Möbius group). The stability subgroup $P = SO(V)_{[p]}$ of the point $[p]$, $p = \frac{1}{\sqrt{2}}(e_3 + e_0)$ is isomorphic to the group $Sim(E^2) = \mathbb{R}^+ \cdot SO_2 \cdot \mathbb{R}^2$ of similarity transformations of the Euclidean plane. With respect to the basis $p = \frac{1}{\sqrt{2}}(e_3 + e_0), e_1, e_2, q = \frac{1}{\sqrt{2}}(e_3 - e_0)$, the stability subgroup P consists from the matrices of the form

$$\{A = \begin{pmatrix} a & -X^T & 0 \\ 0 & A_0 & X \\ 0 & 0 & a^{-1} \end{pmatrix} \tag{1}$$

where $a \in \mathbb{R}^+ . A_0 \in SO_2$, $X \in \mathbb{R}^2$.

Lemma 3 *The subgroup $P = SO(V)_{[p]}$ acts transitively on $Q \setminus \{[p]\}$ with stability subgroup $P_{[q]} = \{\mathrm{diag}(a, A_0, a^{-1})\} \simeq CO_2$ and on $B^3 = PV_-$.*

Proof: A non zero isotropic vector which is not proportional to p can be written as $p' = up + Z + vq$, where $uv + Z^2 = 0, u \neq 0, v \neq 0, Z = Z_1e_1 + Z_2e_2 \neq 0$. Using transformation of the form (1) with $a = 1, A = \mathrm{id}$, we can transform p' into a vector $p'' = (0, Z', v)$ with zero first coordinate. Since p'' is an isotropic vector, $Z' = 0$ and $p'' = vq$. This shows that any element $[p'] \in Q \setminus \{[p]\}$ can be transform into element $[q]$ and $Q \setminus \{[p]\}$ consists of one orbit of P.Now we may transform any line $[n] \in PV_-$ into a line $[n'] \subset \mathrm{span}(p, q)$ and then the last result follows from (1). □

4.1.1 Graded Lie algebra of the group $SO(V)$

Let p, q be isotropic vectors with $g(p,q) = 1$ and $E := \mathrm{span}(p,q)^{\perp}$.
The gradation $V = V_{-1} + V_0 + V_1 = \mathbb{R}q + E^2 + \mathbb{R}p$
induces the gradation of the Lie algebra $\mathfrak{so}(V)$ which we identify with the space of bivectors $\Lambda^2(V)$:

$$\mathfrak{so}(V) = q \wedge E^2 + (\mathbb{R}p \wedge q + \Lambda^2(E^2)) + p \wedge E^2 = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1.$$

The stability subalgebra of the point $[p] \in Q$ is

$$\mathfrak{so}(V)_{[p]} = \mathfrak{g}_0 + \mathfrak{g}_1 = (\mathbb{R}p \wedge q + \mathfrak{so}(E^2)) + p \wedge E^2.$$

The gradation of $\mathfrak{so}(V)$ defines a (local) decomposition of the Möbius group $SO(V)$ into a product

$$SO(V) = G_{-1} \cdot G_0 \cdot G_1 = \mathbb{R}^2_{-} \cdot CO(\mathbb{R}^2) \cdot \mathbb{R}^2_{+}$$

of three subgroups $G_{\pm 1} \simeq \mathbb{R}^2, G_0 \simeq CO_2 = \mathbb{R}^+ \cdot SO_2$. Subgroups $G_0 \cdot G_{\pm 1}$ are isomorphic to the similarity group $\mathrm{Sim}(E^2) = \mathbb{R}^+ \cdot SO_2 \cdot \mathbb{R}^2$ of the plane.

4.1.2 Riemann model of conformal sphere

The stereographic projection of the sphere S^2 with respect to the "north pole" N is a conformal diffeomorphism of the sphere without the pole $S^2 \setminus \{N\}$ onto the Euclidean plane E^2. We can identify E^2 with $\mathbb{C}$ and the sphere S^2 with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with holomorphic coordinate z. Then the conformal Möbius group $SO(V)$ is identified with the group $SL_2(\mathbb{C})$ (more precisely, with its quotient $SL_2(\mathbb{C})/\pm \mathrm{id}$ by the subgroup $\mathbb{Z}_2 = \{\pm \mathrm{id}\}$ which acts trivially) of fractional linear transformations

$$SL_2(\mathbb{C}) \ni A : z \mapsto \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C}.$$

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is identified with the Lie algebra of quadratic holomorphic vector fields with the natural gradation

$$\mathfrak{sl}_2(\mathbb{C}) = \{X = (a+bz+cz^2)\partial_z\} = \mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1 = \{a\partial_z\}+\{bz\partial_z\}+\{cz^2\partial_z\}.$$

The corresponding (local) decomposition $SL_2(\mathbb{C}) = G_{-1} \cdot G_0 \cdot G_1$ is the standard Gauss decomposition into upper triangular unipotent subgroup , diagonal subgroup and law triangular unipotent subgroup. In particular, $G_{-1} = \mathbb{R}^2_{-}$ consists of parallel translations $z \mapsto z + b$ and
the dual subgroup $G_1 = \mathbb{R}^2_{+}$ consists of transformations $z \mapsto \frac{z}{cz+1}$.

4.1.3 Generators of the stability subgroup $P = G_0 \cdot G_+$

In term of the Riemann model $\hat{\mathbb{C}}$ of the conformal sphere, we describe the standard generators of the stability subgroup $P = SL_2(\mathbb{C})_0$ of the origin $0 \in \hat{\mathbb{C}}$. Let (x,y) be the Euclidean coordinates of $\mathbb{C}$ associated with the holomorphic coordinate $z = x + iy$ and (r, φ) corresponding polar coordinates.

The stability subalgebra $\mathfrak{p} = (\mathfrak{sl}_2(\mathbb{C})_0 = \mathfrak{g}_0 + \mathfrak{g}_1$ of the origin 0 is identifies with the Lie algebra of conformal vector fields in $\mathbb{R}^2 = \mathbb{C}$ which vanish at the origin. It has the following basis:
$E = r\partial_r = x\partial_x + y\partial_y$ (the Euler field or dilatation)
$R = x\partial_y - y\partial_x$ (rotation)
$Y^1 = 2xe - r^2\partial_x = 2xr\partial_r - r^2\partial_x$, $Y^2 = 2ye - r^2\partial_y = 2xr\partial_r - r^2\partial_y$
The fields Y^1, Y^2 form a basis of $\mathfrak{g}_1$ and generate the commutative subgroup $G_1 = \mathbb{R}^2_+ \subset P$ which acts trivially on the tangent space.

4.1.4 Tits group model of conformal sphere and Cartan connection

Besides the projective model $Q = PV_0 = SO(V)/P$ and Riemann model $\hat{C} = SL_2(\mathbb{C})/P$ of the conformal sphere S^2, we consider purely group description of S^2 as the set of subgroups H of the group $G = SL_2(\mathbb{C})$ isomorphic to $Sim(E^2)$.
The conformal sphere $S^2 = G/P = SL_2(\mathbb{C})/Sim(E^2)$ is an asystatic manifold, i.e. the stability subgroup P coincides with its normalizer : $P = N_G(P)$. This implies that different points of S^2 have different (but conjugated in G) stabilizers. Moreover, any subgroup H isomorphic to $Sim(E^2)$ and any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ isomorphic to $\mathfrak{sim}(E^2)$ is the stability subgroup (respectively, stability subalgebra) of unique point $q \in S^2$. This allows to describe the conformal sphere as the set of subgroups isomorphic to the subgroup $Sim(E^2)$ (or subalgebra isomorphic to $\mathfrak{sim}(E^2)$):

$$S^2 = \{H \subset G = SL_2(\mathbb{C}), H \simeq Sim(E^2)\} = \{\mathfrak{h} \subset \mathfrak{sl}_2(\mathbb{C}), \mathfrak{h} \simeq \mathfrak{sim}(E^2)\}.$$

We call this model the **Tits model**. We will see that this model is relevant for vision.

The left invariant Maurer-Cartan form of $G = SL_2(\mathbb{C})$ is defined by

$$\mu = g^{-1}dg : T_gG \to \mathfrak{g} = \mathfrak{so}(V),\ \dot{g} \mapsto g^{-1}\dot{g}$$

where $\dot{g} \in T_gG$ and we use the following physical notation for the action of the left translation $L_g : a \mapsto ga$ on tangent vector $\dot{g}$:

$$g^{-1}\dot{g} := (L_{g^{-1}})_*\dot{g}.$$

The Maurer-Cartan form μ defines a Cartan connection on the principal bundle $\pi : G \to S^2 = G/P = SL_2(\mathbb{C})/G_0 \cdot G_1$. This means that it is a $\mathfrak{g}$-valued 1-form which defines an isomorphism $\mu_g : T_gG \to \mathfrak{g}$ of any tangent space T_gG to $\mathfrak{g}$ with two properties:
i) it is an extension of the canonical vertical parallelism

$$T^vG \to \mathfrak{p},\ T_gG \ni g\dot{h} \mapsto \dot{h} \in \mathfrak{p}$$

(isomorphism of the vertical (tangent to the fibre $\pi^{-1}x = gP$ space) to $\mathfrak{p}$;
ii) it is P -equivariant, i.e.

$$\mu(R_h\dot{g}) = \mu(\dot{g}h) = (gh)^{-1}(\dot{g}h) = \mathrm{Ad}_{h^{-1}}\mu(\dot{g}),\ \dot{g} \in T_gG.$$

Recall that Cartan connection is the main tool for investigation of differential geometric structures , construction their invariants and solution of equivalence problem. A conformal structure on a manifold of dimension greater then 2 admits a canonical Cartan connection, which is invariant with respect to (local) conformal transformation, but it is not true for (non conformally flat) 2-dimensional manifolds.

4.2 Central projection of a plane $\Pi \subset P^3$ into the quadric Q

Now we return to the projective model $S^2 = Q = PV_0 \subset P^3$ of the conformal sphere. A point $F \in \bar{B}^3 = P(V_T \cup V_0) \subset P^3$ of the closed ball with the boundary $\partial\bar{B}^3 = Q$ defines the central projection with center F of a plane $\Pi^n,\ n \in SV_S$ into Q, given as in Euclidean case by :

$$\pi^n_F = \pi_F : \Pi^n \to Q,\ A \mapsto \hat{A} := (AF) \cap Q$$

It associates with a point $A \in \Pi^n$ the second point $\hat{A}$ of intersection of the oriented projective line (AF) with Q.

4.2.1 Consistency of Euclidean and projective central projection

A unit timelike vector e_0 (”inertial observer”) defines a decomposition

$$V = \mathbb{R}e_0 + E^3 = \mathbb{R}e_0 + e_0^\perp$$

of the Minkowski vector space into time and space.

We denote by $E^3_{e_0} = e_0 + E^3$ the Euclidean vector space with the origin at $0_{E^3} = e_0$ and we identify the quadric Q with the cone of isotropic lines. Then we have the following correspondence between projective objects and associated Euclidean objects in the Euclidean space $E^3_{e_0}$.

$$\begin{array}{lcl} Q & \leftrightarrow & S^2 := Q \cap E^3_{e_0} \\ Q \ni [p] & \leftrightarrow & p := [p] \cap E^3_{e_0} \in S^2 \\ \Pi^n = Pn^\perp & \leftrightarrow & \Pi^{n_0}_{\vec{n}} = \{\vec{x},\ \vec{n} \cdot \vec{x} = n_0\},\ n = (n_0, \vec{n}) \in V_- \\ \ell = P(\mathrm{span}(u,v)) & \leftrightarrow & \bar{\ell} := (\mathrm{span}(u,v)) \cap E^3_{e_0} \end{array}$$

if span$(u, v))$ is not parallel to E^3.

Lemma 4 *Under above correspondence, the central projection*

$$\pi_{[f]} : \Pi^n \to Q$$

with center $[f] \in B^3 \cup Q$ of a projective plane to the quadric Q corresponds to the central projection

$$\pi_{\bar{F}} : \Pi^{n_0}_{\vec{n}} \to S^2 = Q \cap E^3_{e_0},\ n = (n_0, \vec{n})$$

with the center $\bar{F} = [f] \cap E^3_{e_0}$ of the corresponding 2-plane in $E^3_{e_0}$ into the sphere. The natural bijection $\chi : Q \to S^2 \subset E^3(e_0)$ is a conformal diffeomorphism of Q onto the unit Euclidean sphere $S^2 \subset E^3_{e_0}$.

4.2.2 Change of observer

Let e_0' be another unit timelike vector ("observer") and $(S^2)' = Q \cap E^3_{e_0'} \subset E^3_{e_0'}$ the unit Euclidean sphere for the observer e_0'. Denote by $\chi' : Q \to (S^2)'$ the natural conformal diffeomorphism. Obviously, we get

Lemma 5 *The map*

$$\chi' \circ (\chi)^{-1} : S^2 \to (S^2)'$$

is a conformal diffeomorphism between two unit Euclidean spheres.
Any transformation $L \in SO(V)$ with $Le_0 = e_0'$ induces an isometry $L : S^2 \to LS^2 = (S^2)'$ of unit spheres with induced metrics.

Remark Any timelike unit vector $e_0 \in V_T$ is the center of a sphere which represent Q in the Euclidean space $E^3_{e_0}$.

4.2.3 Change the central projection under a Lorentz transformation

A Lorentz transformation $L \in G = SO(V)$ maps the hyperplane $n^\perp$ isometrically onto $Ln^\perp$ and the projective plane Π^n isomorphically onto $L\Pi^n = \Pi^{Ln}$. Also it preserves the quadric Q and maps lines in Π^n onto circles of Q. In particular, we have

$$\pi^{Ln}_{LF}(LA) = L(\pi^n_F A), \ \forall A \in \Pi^n.$$

If we identify corresponding points $A \sim LA$ of the planes $\Pi = \Pi^n$ and $L\Pi = \Pi^{Ln}$, then the images $\pi^n_F : \Pi \to Q$ and $\pi^{Ln}_{LF} : L\Pi \to Q$ are related by the transformation $\Phi_L : Q \to Q$ given by

$$\Phi_L := (\pi^{Ln}_{LF})L(\pi^n_F)^{-1} = L(\pi^n_F) \circ (\pi^n_F)^{-1} = L$$

We get

Lemma 6 *If we identify the corresponding points of the planes Π and $L\Pi$, then the images of the central projections $\pi_F : \Pi \to Q$ and $\pi_{LF} : L\Pi \to Q$ are related by the conformal transformation $\Phi_L = L|_Q$.*
In particular, if $LF = F$ then the change the plane Π by a Lorentz transformation L produces the conformal transformation $L|_Q$ of the image $\pi_F(\Pi)$.

Let $F = [p] \in Q = PV_0$ be a point of the conformal sphere. We proved that the stabilizer $P = SO(V)_{[p]}$ acts transitively on the ball PV_- of timelike lines. Since there is a natural one-to-one correspondence between timelike lines $[n]$ and projective planes Π^n which does not intersects the quadric Q, the group P acts transitively on the set of planes of the projective space PV which does not intersect the quadric Q. Now the above lemma implies

Proposition 2 *Let $\Pi^n, \Pi^{n'}$ be two projective planes which does not intersect the quadric Q and $F = [p] \in Q$. Then there exists a linear transformation $L \in SO(V)$ (defined up to a rotation $R \in SO(E^2)$ in the Euclidean plane $E^2 = \mathrm{span}(n,p)^\perp$) which transforms Π^n to $\Pi^{n'} = \Pi^{Ln}$. Then the change of the image on Q of central projection $\pi := \pi_F|_{\Pi^n} \to Q$ to $\pi' := \pi_F|_{\Pi^{n'}} \to Q$ induces a conformal transformation $L|_Q$.*

Remark The result is not true if the center $F = [e]$ of the central projection belongs to the ball $B_3 = PV_-$. In this case, the action of the stability subgroup $SO(V)_{[e]} \simeq SO_3$ on the set of planes Π^n which does not intersect Q is not transitive.
The center of lens in human eye is inside the eye ball, but since the light is refracted by cornea, the full refraction is equivalent to the refraction by a lens with center at the boundary sphere.

5 Multiscale Differential Geometry

5.1 Sigma-approximation of differential geometry (following Jan Koenderink and Luc Florack)

From classical point of view, the basic object of geometry is a point. Points form a space (manifold) and geometry studies geometrical object on this manifold of points.
From quantum point of view, the basic object is an algebra of functions. In terms of functions, a point $z \in M$ can be defined as a special linear functional (called "Dirac delta function")
$\delta_{z_0} : f \mapsto \delta_{z_0}(f) = f(z_0)$. Tangent vector at z_0 is a linear functional $v : C^\infty(M) \to \mathbb{R}$ which satisfies the Leibnitz rule

$$v(fg) = f(z_0)v(g) + g(z_0)v(f).$$

Moreover, such functional can be consider as a partial derivative of the delta function due to the formula $(\partial_x \delta_{z_0})(f) = -\delta_{z_0}(\partial_x f)$ where (x, y) are local coordinates of a point $z \in M^2$.
A linear functional (on an appropriate class of test functions) is called a generalized function or a distribution.
A function $F(z) = F(x, y)$ (say, on $M = \mathbb{R}^2$) defines a generalized function T_F

$$T_F(f) := \int F(z)f(z)dxdy.$$

called in neurophysiology " the linear filter with receptive profile (RP) F".

Note that if we assume that the retina has an affine structure, using shift $F_{z_0}(z) := F(z-z_0)$ we can define a new function $(F*f)(z_0) := T_{F_{z_0}} f$ which is called the convolution of F and f .

5.1.1 Visual neurons as filters

Many visual neurons in retina and visual cortex can be considered as "filter" i.e. a functional which associate with the input function I on retina R a number which measure the degree of excitation of the neuron. We will consider only linear neurons, which acts as linear functionals (generalized functions) on the space of input functions I, and have the form

$$T_F : I(z) \mapsto T_F(I) = \int_D F(z)I(z)dxdy.$$

Here F is a function (called the **receptive profile of the neuron (RP)**) with support $D \subset R$ (called the **receptive field (RF)**).

Roughly speaking, such filter associates with an input function I the average value I calculated with weight F in a small domain D.

5.1.2 Gauss filter

Since the receptive fields of visual neurons are small, for simplicity, we may assume that the retina R is a part of Euclidean plane $\mathbb{R}^2$ with Cartesian coordinares $z = (x, y)$.

The Dirac functional δ_{z_0} can be approximated by the Gauss functionals $T_{G^{\sigma}_{z_0}}$ where

$$G = G^{\sigma}_{z_0}(z) = \frac{1}{2\pi\sigma^2} \exp(-\frac{|z - z_0|^2}{2\sigma^2})$$

is the Gauss function with means z_0 and small standard deviation σ. More precisely,

$$\delta_{z_0} = \lim_{\sigma \to 0} T_{G^{\sigma}_{z_0}}.$$

We call $T_{G^{\sigma}_{z_0}}$ the sigma-approximation of the Dirac functional.

More generally, for the functional associated with the derivative $X \cdot G$ of the Gauss function in direction of vector field X, we have

$$\begin{array}{rcc} T_{X \cdot G}(f) = & \int X \cdot GIdxdy = -\int G(X \cdot Idydz) + \int X \cdot (GI)dxdy) \\ = & -\int G(X \cdot f + \mathrm{div}X)dydz \end{array}.$$

Assume that $\mathrm{div}X = 0$. Then the functional $T_{X \cdot G}$ acts on I as

$$T_{X \cdot G} : I \mapsto -\int G(x, y)(X \cdot I)(x, y)dxdy.$$

So it can be considered as sigma- approximation of the functional

$$I \mapsto -(X \cdot I)(z_0)$$

which is identified with the vector $-X_{z_0}$. Similarly, iterated directional derivatives $X_k \cdots X_2 \cdot X_1 \cdot G$ of the Gauss function defines functionals, which can be considered as sigma-approximations of higher order linear differential opetators at the point z_0.

5.1.3 Ganglions as Marr filters (Kuffler and Marr)

It was experimentally found by Stiven Kuffler (1950) that receptive field of many ganglion neurons in retina is a disc D and the receptive profile F is rotationally invariant. The disc D contains a concentric small disc D' such that F is positive (resp., negative) in D' and negative (resp. positive) in $D \setminus D'$ for ON-cells (respectively, OFF-cells). David Marr (late 70) showed that the receptive profile of such cells can be approximated by the Marr function $F = \pm\Delta G^{r_0}_{\sigma}(r)$. Sign $-$ gives ON cell, and singe $+$ OFF cell.

Marr filter is defined by receptive profile "Laplacian of Gauss"

$$M^{\sigma}_{z_0}(z) := \Delta G^{\sigma}_{z_0}.$$

Like Gauss filter , Marr filter is isotropic i.e. invariant under the rotatrion of the plane with center at z_0.

5.1.4 Gabor filters

Non-formally, an even (resp., an odd) Gabor filter is an anisotropic Gauss filter modulated by cos (resp., sin).
More precisely, the even and odd mother Gabor filters are defines by RP which is the imaginary and real part of the function

$$Gab := G^1_0(z) \exp \sqrt{2} iy = \frac{1}{2\pi} \exp \frac{-|z|^2}{\sqrt{2}} \exp iy,\ z = x + iy.$$

A general Gabor filter is obtained from the mother Gabor filter $Gab(z)$ by application a general transformation $A = A_{z_0,c} = T_{z_0} \cdot L_c$ from the similarity group $Sim(E^2) = T_{\mathbb{C}} \cdot \mathbb{C}^*$ acts on vectors by

$$A = A_{z_0,c} : z \mapsto z_0 + cz$$

and on functions by

$$A = A_{z_0,c} : f \mapsto A^* f := (\det A)^{-1} f(Az) = |c|^{-2} f(c^{-1} z - z_0).$$

This action preserves the integral of a function :

$$\int_{\mathbb{R}^2} A^*(f)(z) dxdy = \int_{\mathbb{R}^2} f(z) dxdy.$$

In particular, it preserves the density of probability measure and transforms the standard Gauss function G^1_0 into

$$A^* G^1_0 = G^{\sigma}_{z_0} = \frac{1}{2\pi\sigma^2} e^{\frac{|z-z_0|^2}{2\sigma^2}},$$

where we set $c = \sigma e^{i\theta}$. The RP of the general Gabor filter depends on 4 parameters $\sigma, \theta, z_0 = (x_0, y_0)$ and it is parametrized by points of the group $Sim(E^2) = T_{\mathbb{C}} \cdot \mathbb{C}^*$. More precisely, the RP of a general Gabor filter has the form

$$Gab^{\sigma}_{\phi,Z_0} = A^* Gab^1_0 = G^{\sigma}_{z_0} \mathrm{exp} i\sqrt{2}\sigma^{-1}(y \cos\phi - (x_0 \cos\phi + y_0 \sin\phi)).$$

For $z_0 = 0, \phi = 0$, the Gabor filter has RP

$$Gab^{\sigma}_0 = G^{\sigma}_0 \cdot \exp(\sqrt{2} iy)/\sigma = G^{\sigma}_0 \cdot (\cos\sqrt{2} y/\sigma) + i(\sin\sqrt{2} y/\sigma).$$

Since

$$\partial_y G^{\sigma}_0 = -\frac{y}{\sigma^2} G^{\sigma}_0,\ (\partial_y)^2 G^{\sigma}_0 = -\frac{1}{\sigma^2}(1 - \frac{y^2}{\sigma^2}) G^{\sigma_0}$$

we conclude that for small $\frac{y}{\sigma}$ the odd and even Gabor filters approximately have RP proportional to first and second derivative of Gauss function:

$$\begin{aligned} (Gab^{odd})^{\sigma}_0 \approx \quad & G^{\sigma}_0 \cdot \tfrac{\sqrt{2}y}{\sigma} = & -\sqrt{2}\sigma\partial_y G^{\sigma}_0 \\ (Gab^{ev})^{\sigma}_0 \approx \quad & G^{\sigma}_0 \cdot (1 - \tfrac{y^2}{\sigma^2}) = & -(\sigma\partial_y)^2 G^{\sigma}_0. \end{aligned}$$

So the odd Gabor filters correspond to tangent vectors and even Gabor filters correspond to second order tangent vectors i.e. second jets of curves.

5.1.5 Functionals associated with conformal vector fields

We describe the receptive profiles $X \cdot G$ where $G = G_0^\sigma = \frac{1}{2\pi\sigma^2} exp(-\frac{r^2}{2\sigma^2})$ and X is one of the basic conformal vector fileds of $\mathfrak{sl}_2(\mathbb{C})$. We will use polar coordinates r, φ such that $x = r\cos\varphi, y = r\sin\varphi$.

$$\partial_x G = -\frac{x}{\sigma^2} G = -\frac{r}{\sigma^2}\cos\varphi G,$$

$$\partial_y G = -\frac{y}{\sigma^2} G = -\frac{r}{\sigma^2}\sin\varphi G,$$

$$E \cdot G = r\partial_r G = -\frac{r}{\sigma^2} G, \ \ \mathrm{div} E = 2$$

$$Y^1 \cdot G = (2xe - r^2\partial_x)G = (2xr\partial_r - r^2\partial_x)G = -\frac{r^2 x}{\sigma^2} G = -\frac{r^3}{\sigma^2}\cos\varphi G,$$

$$Y^2 \cdot G = (2ye - r^2\partial_y)G = (2yr\partial_r - r^2\partial_y)G = -\frac{r^2 y}{\sigma^2} G = -\frac{r^3}{\sigma^2}\sin\varphi G,$$

$$\Delta G = (\partial_x^2 + \partial_y^2)G = -\frac{1}{\sigma^2}(1 - \frac{1}{\sigma^2})G = -\frac{1}{\sigma^2}(G + E\cdot G).$$

Note that $Y^1 \cdot G = r^2\partial_x G, \ Y^2 \cdot G = r^2 \partial_y G$ and that $\mathrm{div} Y^1 = 2x$, $\mathrm{div} Y^2 = 2y$.

6 Architecture of the retina and retinotopic map to primary visual cortex

The brain extracts all visual information from the retina R which occupies the big part of the eye sphere S^2.
The bottom layer of the retina consists of receptors (rods and three types of cones), that is photoelements which transform the light energy into electricity.
They measure the energy function I of light falling to the retina $R \subset S^2$.
The information about energy function I is sent to the external layer of retina, which consists of ganglion cells.

There are two types of vision : central color vision and peripheral black-white vision.
During day-time central color vision, most visual information comes from fovea which is a yellow spot on retina of diameter approx. 0,35 mm. The most cones, which are responsible for color vision, are concentrated there.
One cone in fovea is connected with 1 or 2 ganglions which send the visual information to V1 cortex.
During peripheral low light black-white vision, the information comes from the rods situated in the periphery of the retina. Here one rode is connected with $10^2 - 10^3$ ganglions.
There are 1 million of ganglions and $125 - 150$ millions of receptors.
There are two types of paths from receptors to ganglions:
Direct path : receptors-bipolars-ganglions and
indirect path: receptors -(sometimes horizonal cells)-bipolar cells-(sometimes amacrine cells) -ganglions.

About functions of all all there types of retinal neurons see [11]. Note that the number of different types of neurons in human retina is larger then 80 (and in rabbit retina is 55).

6.0.6 Aim of the information processing in retina is the regularization and contourization of the input function

The input function I of the retina is very irregular. The aim of the information processing in retina is to prepare the input function for decoding, do it more regular and highlight the contours - level set of the input function with big gradient. It is done by a system of cells with isotropic (i.e. rotational invariant) receptive fields, which are working as Marr filters. Roughly speaking, system of Marr filters reduces the complexity of the picture on retina and transform it into graphics (system of contours) .

6.0.7 Conformal ("topographic") map from retina to primary visual cortex VI

There is a conformal map from retina R to the lateral geniculate nucleus (LGN) (a part of the thalamus). and then to primary visual cortex VI of the form

$$z = x + iy \to k \log \frac{z+a}{z+b}.$$

Physiologically, the path goes from R through optic chiasm (where each of visual nerve splits in two part , one remains in the same side of the brain, other goes to another side) and then come to LGN.
LGN consists of 6 layers. Layers 1,4,6 get information from the opposite w.r.t. the hemisphere of LGN eye, layers 2,3,5 from the eye from the same side. Then the information is sent to VI cortex.

7 Architecture of primary visual cortex VI

7.1 Pinwheel structure

Cortex VI is a layered structure (1,8 mm thick) which consists of 6 horizontal layers, most important is sublayer 4C, where most of the fibres from LGN projects.
Visual cells of VI are organized in columns and columns are combined in hypercolumns. Neurons of columns work as filters (functionals) with small RF.
Hubel and Wiesel classified visual cells into simple cells (25%) and complex cells (75%).

7.1.1 Simple cells as Gabor filters

Simple cells work as Gabor filters.

The odd Gabor functions in approximation of Gauss optics ($\sin y \simeq y$) are proportional to directional derivatives of the Gauss function $G^{\sigma}_{z_0}$. Hence for small standard deviation σ, the associated Gabor filters act on input function as tangent vectors v and they detects contours orthogonal to v. It was shown in A. Sarti , G. Citti and J. Petitot , see [(S-C-P)], that even simple cells detect the distance to the nearby contour.

7.1.2 Field of 1-distributions with singularities (pinwheels) in VI

It was discovered by D. Hubel and T. Wiesel (Nobel Prize 1981) that VI cortex of mammals (tree shrew = tupaia, cat, monkey, human etc.) has a 1-dimensional distribution (field of directions or " orientation")

$$\Gamma : z \mapsto \Gamma_z = \ker \omega_z$$

with isolated singularities (called pinwheels) where the 1-form ω, which defines the distribution, vanishes.
All simple cells of a regular column (at a point z with $\Gamma_z \neq 0$) are excited only when the contour at z has direction (or orientation) Γ_z. A singular column (which corresponds to a singular point z which $\Gamma_z = 0$) contains simple cells which detect contour of any orientation.
When eye rotates, the contours on retina intersect centres of pinwheels and more cells detect the contour.This is one of the aims of fixation eye movements.

7.1.3 Problem of formation of pinwheel structure

The singular points (pinwheels) of the distribution Γ in VI cortex form a rather dense finite set of points. Mechanism of creation and evolution of field of direction Γ with such singularities is proposed by F.Wolf and his group, see (K-Sh-W). Like in quantum physics, spontaneous symmetry breaking plays an important role in this process.

7.2 J.Petitot's model

The history of physics shows that the estimated dimension of the physical world is increasing from 3 to 4,5, ... 10,11. Similar situation is in neurogeometry. D.H. Hubel proposed an idea, that the primary visual cortex can be modeled by a fiber bundle over the surface whose fibre depends on many parameters. He called this "engrafting of variables".
W. Hoffman expressed an idea that the primary visual cortex is a contact bundle. This idea was realized by J. Petitot. He considers primary visual cortex VI as a surface V with a field of directions Γ. J. Petitot notices that if we will parametrized simple cells according to their function (as Gabor filters), they may be parametrized by points of the surface $\tilde{V}$ which is obtained from V by blow up at all centers z_0 of pinwheels. Recall that the projection $\pi : \tilde{V} \to V$ is a bijection for any regular point $z \in V$ (which corresponds to column z of simple cell which detect only direction Γ_z) and the preimage $\pi^{-1}(z_0)$ of the center of pinwheel is a circle (which

corresponds to the circle of direction measured by cells from the column z_0.

7.2.1 Petitot's model: primary cortex as a contact bundle

Under approximation that all points are centers of pinwheel, J.Petitot concludes that points of VI cortex are parametrized by the S^1-bundle over the surface V which is naturally identified with the contact bundle in the sense of S. Lie, that is the projectivized (co)tangent bundle $PT(V) = PT^*V$ (the space of directions) with the natural contact structure.(For 2-dimensional manifold there is a canonical identification $PTM = PT^*M$). Simple cells of V detect not only points of a contour C, but also its direction T_zC . So they determine the lift of the contour to a horizontal curve $\bar{C} \subset PT(V)$. (Such lift is called in geometry the Legendrian lift).

So, according to Petotot's model, VI cortex is the contact bundle PTV with the canonical contact structure and simple cells determine the Legendrian lift of contours in V to PTV.
If (x, y) are coordinates in V such that contours are described as $y = y(x)$, then the contact manifold PTV can be locally identified with the manifold $J^1(\mathbb{R})$ of 1-jets of functions with coordinates $(x, y, p = \frac{dy}{dx})$ and the contact form $\theta = dy - pdx$. The contact manifold $J^1(\mathbb{R})$ is identified with the Heisenberg group $Heis_3$ or with the group $E(2) = SO_2 \cdot \mathbb{R}^2$ of Euclidean motions of the plane with left invariant contact structure.

7.3 Sarti-Citti-Petitot's model: VI as a principal CO_2 bundle

A generalization of the Petitot's model was proposed by A.Sarti, G.Citti and J.Petitot. They assume that the set of simple cells (i.e. the set of Gabor filters) in VI cortex are parametrized by points of the similarity group $Sim(E^2) = G_0 \cdot G_- = CO_2 \cdot \mathbb{R}^2_-$.
The parameters $(\sigma, \theta, z = (x, y))$ associated with a Gabor filter has the following interpretations : $\sigma \in \mathbb{R}^+$ (scaling) is the intensity of reply of the Gabor filter on stimulus, θ is the orientation (the angle between a fixed direction and the direction , detected by Gabor filter), and $z = (x, y) \in G_- = \mathbb{R}^2_-$ is the position of the center of filter , which is identified with the parallel translation from a fixed point of the cortex (considered as a plane).
Note that one may identify the space of simple cells for this model with the total space $Sim(E^2)$ of the principal $CO_2 = \mathbb{R}^+ \cdot SO_2$-bundle $Sim(E^2) = G_0 \cdot G_- \to Sim(E^2)/CO_2$ of conformal frames in the retina.

7.4 Hypercolumns of VI cortex (Hubel-Wiezel)

Hubel and Wiezel proposed a deep idea that columns in VI cortex are grouped into hypercolumns or modules, which detect local information about the image. It is based on their fundamental discovery that any simple neuron which measure orientation of the contour is excited only

when the orientation take a certain value (up to 10-15%). They suggested that this is a general principle, valid for visual neurons, which measure some parameter of an image, and gave a very general definition of hypercolumns, which is now applied to different parameters in VI and VII cortex:
A **hypercolumn** in a neighborhood of a given point of cortex associated with some local characteristics (orientation, ocular dominance, spatial frequency, temporal frequency, curvature, color etc) is a minimal system of columns containing neurons which measure all possible values of these parameters. In other words, it is a system (module) which detects local structure of the image in a neighborhood of given point. We conjecture that the retina fields of cells of columns which form a hypercolumn cover a domain in retina which contains images of the point of line of sight under fixation eye movements.

The basic observation by Hubel and Wiezel during neurophysiological experiments was that when the electrode in monkey's VI cortex moved from column to next column, the direction of orientation smoothly rotates with period approximately 12 columns and after 12 columns (approximately 1mm) it turns on 180^o. Similar 12 column (1 mm) periodicity they found for other local parameter - ocular dominance. The lines of isodominancy and isoorientation are approx. orthogonal (and form locally an orthogonal system). Analysing these results, they proposed the famous " ice cube" model of hypercolumn associated with orientation and ocular dominance. Ocular dominance is $\mathbb{Z}_2$-valued function on VI cortex which indicates the eye (left or right) which provides main information about value of input function I at the corresponding points of receptive field.

7.4.1 Spherical model of hypercolumns by Paul Bressloff and Jack Cowan

Together with orientation θ , one of the most important characteristic of image is **spatial frequency.** The spatial frequency measures how often a periodic components of the structure repeat per unit of distance. More precisely, spatial frequency p of a grating is defined as number of lines of grating per millimeter. For more complicated image it is applied to the main component of the Fourier decomposition of the image.

Paul Bressloff and Jack Cowan proposed a spherical model of hypercolumns, associated with two parameters : orientation θ and a logarithmic function of the spatial frequency p. Assume that the spatial frequency changes in the interval $[p_L, p_H]$. Then Bressloff and Cowan choose as the second parameter the following normalized logarithmic function of p: $\varphi = \pi \frac{\log(p/p_L)}{\log(p_H/p_L)} - \pi/2$. It varies in the interval $[-\pi/2, \pi/2]$. They proposed a model of hypercolumn as a sphere with spherical coordinates $\theta \in (0, 2\pi)$ (longitude) and $\varphi \in (-\pi/2, \pi/2)$ (latitude). They assume that north and south pole where $\varphi = +\pi/2$ and respectively $\varphi = -\pi/2$ corresponds to two centres of pinwheels where the orientation θ is not defined since the simple cells of the associated singular points measure all values of the orientation. This model is a generalization of so called " ring model".

7.4.2 Evolution of an excitation in a hypercolumn

Evolution of an excitation between cells of the hypercolumn is described by the famous Wilson-Cowan's equation

$$\partial_t u(\theta, \varphi, t) = -u(\theta, \varphi, t) + \int_0^\pi \int_0^\pi W(\theta, \varphi|\theta', \varphi')\sigma(u(\theta', \varphi', t))d\nu + h(\theta, \varphi).$$

Here $u(\theta, \varphi, t)$ denotes the activity of a local population of cells on the sphere with spherical coordinates (θ, φ). W is the weight of interaction between two cells, σ is a sigmoidal function , h is a stimulus from LGN. Bressloff and Cowan assumed that the weight function $W \in C^\infty(S^2 \times S^2)$ is SO_3-invariant. The simplest example of such function is the function

$$W(\varphi, \theta|\varphi, \theta) = W_0 + W_1(cos\varphi \cos \varphi' + sin\varphi \sin \varphi' cos(2[\theta - \theta']))$$

(where W_0, W_1 are constant), associated with Riemannian distance on the sphere. More general form is described in terms of spherical harmonics.

7.4.3 Unification of Bressloff-Cowan and Petitot-Citti-Sarti models

A weak point of the Petitot- Citti- Sarti model of VI cortex is that it uses the scale parameter σ which is not known in neurophysiology and which does not correspond to any characteristic of the image. We propose to change the parameter σ to the spatial frequency , more precisely to the the logarithmic function φ of spatial frequency, defined by Bressloff and Cowan. Since the excitation of simple cells depends also on value of spatial frequency, they have different intensity of reply to visual stimuli with the same orientation but different spatial frequencies. Due to this, the parameter φ is similar to the scaling parameter σ considered by Petitot-Citti-Sarti.

According to Petitot-Citti-Sarti model, simple cells of VI cortex are locally parametrized by points of the subgroup $P_- = G_0 \cdot G_- \simeq Sim(E^2)$ of the Möbius group $G = SL_2(\mathbb{C}) = G_- \cdot G_0 \cdot G_+$.
According to Hubel and Wiezel, hypercolumn detects information about the local structure of image near a point $z \in R$, We assume that simple cells of a hypercolumn are parametrized by the points of the stability subgroup $P_+ = G_0 \cdot G_+ \simeq Sim(E^2)$ of the point z. The coordinates $(\lambda, \theta) \in G_0 = \mathbb{R}^+ \times SO_2$ corresponds to generators $r\partial_r = x\partial_x + y\partial_y$ and $x\partial_y - y\partial_x$ of the subgroup $G_0 = CO_2$ and can be identified with the spherical coordinates (φ, θ) of Bressloff and Cowan. Note that the 1-parameter subgroup generated by $r\partial_r$ look like a homothety in the neighbourhood of the fixed point z. Instead of coordinates $z = (x, y)$ which corresponds to generators ∂_x, ∂_y, we propose to consider two new parameters, associated with generators $(x^2 - y^2)\partial_x + 2xy\partial_y)$, $(x^2 - y^2)\partial_y - 2xy\partial_x)$ of the group G_+. They correspond to some local characteristics of second order of the image, probably , the components of the gradient of the spatial frequency

p. Then the space of all simple cells of VI cortex is identified with total space $G = SL_2(\mathbb{C})$ of the bundle $\pi : SL_2(\mathbb{C}) \to SL_2(\mathbb{C})/P = S^2$, In other words, the system of simple cells of VI cortex realize the Tits model of the conformal sphere. The Maurer-Cartan form gives a natural Cartan connection in S^2 which determines conformal structure on S^2.

Note that this model is consistent with Bressloff and Cowan model. If we remove two second order new parameters, we get parametrization of simple cells of the hypercolumn by spherical coordinates θ and φ. They are not working in two centers of pinwheels which correspond to north and south poles of the sphere. Geometrically the Bressloff-Cowan sphere may be identified with the (universal cover) of the projectivization of the tangent space of the Petitot contact space PTP associated with the retina.

7.4.4 Application to stability problem

We start from the remark how the description of some objects may be done invariant with respect to a group of transformations. Let G be a group of transformation of a manifold M, (for example $G = SO_2$ is the group of rotation of the plane $\mathbb{R}^2$). If observers are distributed along an orbit Gx , (for example, a circle, which is an orbit of $G = SO_2$) then information which they send to some center is G- invariant.

In particular, the information, which simple cells of a hypercolumn (parametrized by the points of the stability subgroup $P = G_0 \cdot G_+ \simeq Sim(E^2)$) send to complex cells or to the next level VII will be invariant with respect to the 4-dimensional stability subgroup $P = G_0 \cdot G_+$ of the Möbius group $SL_2(\mathbb{C}) = G_- \cdot G_0 \cdot G_+$. We know that fixation eye movement corresponds to conformal transformation of the image on retina. Hence, the stability means that perception of the images by the brain is invariant under (local) conformal transformation of image in retina. It remains to do perception invariant with respect to the subgroup G_-, which consists of translations in Riemann model. We suggest that it is done on the next level of the visual system, probably, in cortex VII.
This conjecture is consistent with fact that the level of invariancy increase when we go to the next level, see [16].
It is supported also by the following experimental fact. One of the principal difference between simple and complex cells is that the excitation of simple cells is not invariant with respect to the shift of the contour, but the excitation of complex cells is invariant with respect to such shift, (see [8]).

References

[1] P. Baird and J.C. Wood , Harmonic morphisms and shear-free ray congruences, Bull. Belg. Math. Soc. Simon Stevin v. 5, n 4 (1998), 549-564. arXiv:math/0306390.

[2] P.C. Bressloff, J.D. Cowan, A spherical model for orientation as spatial-frequency tuning in a cortical hypercolumn, Phil. Trans. Royal Soc. London. B, 2002, 1-22.
[3] Paul C. Bressloff, Jack D. Cowan,The visual cortex as a crystal, Physica D 173 (2002) 226–258.
[4] G. Citti, A. Sarti (ed.) Neuromathematics of Vision, 2014, Lecture Notes in Morphogenesis.
[5] G. Citti, A. Sarti, Models of the visual cortex in Lie groups, April 2009, 49p., INTERNET.
[6] R.Engbert, K. Mergenthaler, P. Sinn, A. Pikovsky, An integrated model of fixational eye movements and microsaccades, Proc. Nat. Acad. Sci. USA,2011, 108, (39).
[7] L.M.J. Florack, B.M. Ter Haar Romeny, J.J. Koenderink, M.A. Viergever, Scale and the differential structure of images, Image and Vision Computing, 1992, 10 (6), 376-388
[8] M. Hansard, R. Heraud , A differentail model of complex cells, Neural Computation 23, 2324–2357, 2011 , MIT.
[9] W. Hoffman, Visual cortex as a contact bundle, Applied mathematics and computetion, 1989,32, 137-167.
[10] David Hubel, Eye, Brain and Vision, 1995.
[11] E.Kaplan, E. Benardete, The dynamics of primate retinal ganglion cells, Progr. in Brain Research, 2001, 1-22 .
[12] M. Kaschube, M. Schnabel, F. Wolf, Self-organization and the selection of pinwheel density in visual cortical development, New Journal of Physics, 10, 015009, 2008, 20p.
[13] J.Koenderink, The structure of images, Biological Cybernetics, 50, 363-370, 1984.
[14] E. Kowler, Eye movements: The past 25 years. Vis. Res., 51, 2011, 1457-83.
[15] J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, J. Physiol. Paris, 2003, 97, 265-309.
[16] T. Poggio, J. Leibo, J. Mutch, L., S. Ullman, Learning Generic Invariances in Object Recognition: Translation and Scale, 2010.
[17] R.Penrose, Twistor geometry of light rays, Class. Quant. Gravity, 1997, 14, A299-A323.
[18] V.S.N. Prasad, J. Domke, Gabor filter visualization, Marilend University,
[19] A. Sarti, G. Citti, J. Petitot, The symplectic structure of the primary visual cortex, Biol. Cybern., 2008, 98, 33-48.
[20] A. Sarti, G. Citti, J. Petitot, Functional geometry of the horizontal connectivity in the primary visual cortex, J. Physiol. Paris, 2009. 103, 37-45.
[21] Westheimer D., The third dimension in the primary visual cortex, J. Phys., 587, 12, 2807-16
[22] R.H.Wurtz Neuronal mechanisms of visual stability, Vision Research,2008,48, 2070-2089.