Weight Spectrum of Quasi-Perfect Binary Codes with Distance 4

Valentine B. Afanassiev
Institute for Information Transmission Problems
(Kharkevich institute), Russian Academy of Sciences
GSP-4, Moscow, 127994, Russian Federation
afanv@iitp.ru

Alexander A. Davydov
Institute for Information Transmission Problems
(Kharkevich institute), Russian Academy of Sciences
GSP-4, Moscow, 127994, Russian Federation
adav@iitp.ru

***Abstract*—We consider the weight spectrum of a class of quasi-perfect binary linear codes with code distance 4. For example, extended Hamming code and Panchenko code are the known members of this class and it is known that Panchenko code has the minimal number of weight 4 codewords. We give exact recursive formulas for the weight spectrum of quasi-perfect codes and their dual codes. As an example of application of the weight spectrum we derive a lower estimate for the conditional probability of correction of erasure patterns of high weights (equal to or greater than code distance).**

I. Introduction

Calculation or estimation of the weight spectrum of linear code is one of very old unresolved problem that gives rise a long list of other unresolved problems in coding theory. The class of binary quasi-perfect codes has a long history in investigation but with a "hole" in area of weight distribution for the most of members of the class. We caught a happy chance to find a "simple enough" solution for weight spectrum of whole class of binary quasi-perfect codes.

The other and real motivation for the research was search most effective encoding and decoding schemes for error correction and error detection in computer memory. The physical volume of a contemporary memory cells tends to "zero" but the probability of error or defect in a cell tends to be very critical for a whole memory devise. As a consequence of this trend we need more and more effective encoding schemes for correction of independent errors and their collection as a two dimensional blots.

Binary quasi-perfect code (Hamming code) is traditional choice for memory devises. We suggest as a better choice Panchenko code in original and product forms (for blot correction). The main our investment to a traditional solution is extension of decoding area for erasures with transformation of detected errors into erasures.

II. Quasi-perfect codes created by doubling construction

For a code with redundancy r we introduce the following notations: n_r is length of the code, H_r is its parity check matrix of size $r \times n_r$, and d_r is code distance.

Definition 1. *Doubling construction* creates a parity check matrix H_r of an $[n_r, n_r - r, d_r]$ code from a parity check matrix H_{r-1} of a $[\frac{1}{2}n_r, \frac{1}{2}n_r - r + 1, d_{r-1}]$ code as follows

$$H_r = \left[\begin{array}{c|c} 0\ldots0 & 1\ldots1 \\ \hline H_{r-1} & H_{r-1} \end{array}\right]. \qquad (1)$$

It is clear that in the doubling construction (1) we have $d_r \le 4$ independently of d_{r-1}.

Remind that a *quasi-perfect code* with $d = 4$ has covering radius 2 and it *is "non-lengthening"* in the sense that addition of any column to a parity check matrix decreases code distance.

Let us define matrices S and M as

$$S = \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right], \qquad M = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array}\right].$$

Theorem 2. [5] *Let $n_r \ge 2^{r-2} + 2$ and let an $[n_r, n_r - r, 4]$ code be quasi-perfect. Then a parity check matrix H_r of the code can be presented in the form* (1) *where matrix H_{r-1} is given in one of the following three variants only:*
- *H_{r-1} is a parity check matrix of a $[\frac{1}{2}n_r, \frac{1}{2}n_r - r + 1, 4]$ quasi-perfect code;*
- *$H_{r-1} = S$;*
- *$H_{r-1} = M$.*

Corollary 3. [5] *Let $n_r \ge 2^{r-2} + 2$, $r \ge 5$, and let an $[n_r, n_r - r, 4]$ code be quasi-perfect. Then length n_r can take any value exclusively from among the sequence*

$$n_r = 2^{r-2} + 2^{r-2-g} \text{ for } g = 0, 2, 3, 4, 5, \ldots, r-2. \qquad (2)$$

Now we give a general description of parity check matrix for whole class of quasi-perfect codes. Let

$$B_{k,g} = [b_k \ldots b_k]$$

be the $(r - g - 2) \times (2^g + 1)$ matrix of identical columns b_k, where r is code redundancy, b_k is the binary representation of the integer k (with the most significant bit at the top position).

Corollary 4. [5] *Let $n_r = 2^{r-2} + 2^{r-2-g}$, $r \ge 5$, $g \in \{0, 2, 3, 4, 5, \ldots, r-3\}$, and let an $[n_r, n_r - r, 4]$ code be*

quasi-perfect. Then a parity check matrix H_r of the code can be presented in the form

$$H_r = \left[\begin{array}{c|c|c|c} B_{0,g} & B_{1,g} & & B_{D,g} \\ \hline H_{g+2} & H_{g+2} & \dots & H_{g+2} \end{array}\right], \quad (3)$$

where $D = 2^{r-g-2} - 1$, H_{g+2} is a parity check matrix of a quasi-perfect $[2^g + 1, 2^g + 1 - (g + 2), 4]$ code.

Remark 5. By Corollary 4 a parity check matrix of any quasi-perfect binary code with length $2^{r-2}+2^{r-2-g}$ and redundancy r can be created by $(r - g - 2)$-fold applying of the doubling construction.

Note that an arbitrary code with $d = 4$ is either a quasi-perfect code or shortening of some quasi-perfect code. Therefore Theorem 2, Corollaries 3, 4, and Remark 5, in fact, describe all binary linear codes with $d = 4$ and length $\geq 2^{r-2} + 2$. It is why weight spectrum of codes obtained by the doubling construction (1) is an important problem.

The class of codes, say $\mathcal{D}$, obtained by the doubling construction is sufficiently wide. By (1), the $[2^r - 1, 2^r - 1 - r, 3]$ Hamming code and many its shortenings are included to $\mathcal{D}$. Directly from Theorem 2 it follows that $[2^{r-1}, 2^{r-1} - r, 4]$ extended Hamming code and Panchenko code Π_r (see below) belong to $\mathcal{D}$. Other numerous non-equivalent codes of $\mathcal{D}$ can be obtained by multiple application of doubling construction to distinct quasi-perfect $[2^g + 1, 2^g + 1 - (g + 2), 4]$ codes $\mathcal{C}_0$ with $g \in \{0, 2, 3, 4, 5, \dots, r - 3\}$, see (3). Examples of codes $\mathcal{C}_0$ can be find in [3], [5], [11] in algebraic and in geometrical form. For instance, we give a parity check matrix of a quasi-perfect $[9, 9 - 5, 4]$ code.

$$\left[\begin{array}{c|c} 00000 & 1111 \\ \hline 10001 & 0000 \\ 01001 & 1001 \\ 00101 & 0101 \\ 00011 & 0011 \end{array}\right].$$

The quasi-perfect codes Π_r were proposed by V.I. Panchenko in paper [9]. The $[n, n - r, 4]$ code Π_r has length $n = 5 \cdot 2^{r-4}$, redundancy $r \geq 5$, and code distance $d = 4$. (In paper [6] the code Π_r is denoted as Π)

The parity check $r \times 5 \cdot 2^{r-4}$ matrix P_r of Panchenko code Π_r is the matrix H_r of (3) with $g = 2$, $D = 2^{r-4} - 1$, and $H_{g+2} = S$. So,

$$P_r = \left[\begin{array}{ccccc} B_{0,2} & B_{1,2} & B_{2,2} & \dots & B_{D,2} \\ S & S & S & \dots & S \end{array}\right]. \quad (4)$$

Remind that the known and important property of Panchenko code is the following – it has the minimal number of weight 4 codewords among all other codes of the same length and redundancy (including the extended Hamming code) [1], [6], [9]. As the consequence of this property, Panchenko code has the minimal probability of undetected error. In particular, it is important for error correction in computer memory [1], [6].

Remark 6. A parity check matrix of any quasi-perfect binary code with $d = 4$ can be created by multiple application of the doubling construction. Therefore, Theorems 7 and 8 allow us to obtain weight spectrum of such code (and its dual) starting from weight spectrum of a short code.

III. Weight spectrum of codes created by the doubling construction

We use notations introduced in the previous section. Also, for a code with redundancy r we denote by $A_w^{(r)}$ the number of codewords of weight w and by $A_w^{(r)\perp}$ the number of codewords of weight w in the dual code.

Theorem 7. *Let $d_r \leq 4$. Assume that an $[n_r, n_r - r, d_r]$ code $\mathcal{C}_r$ is created from a $[\frac{1}{2}n_r, \frac{1}{2}n_r - r + 1, d_{r-1}]$ code $\mathcal{C}_{r-1}$ by doubling construction* (1)*. Then weight spectrum $\{A_w^{(r)}, w \leq n_r\}$ of $\mathcal{C}_r$ can be obtained from weight spectrum $\{A_w^{(r-1)}, w \leq \frac{1}{2}n_r\}$ of $\mathcal{C}_{r-1}$ as follows:*

$$A_{2v}^{(r)} = \delta_v^{(r)} + \sum_{j=0}^{v-2} 2^{2v-2j-1} A_{2v-2j}^{(r-1)} \binom{\frac{1}{2}n_r - 2v + 2j}{j} \quad (5)$$

where

$$\delta_v^{(r)} = \begin{cases} 0 & \text{if } v \text{ odd} \\ \binom{\frac{1}{2}n_r}{v} & \text{if } v \text{ even} \end{cases};$$

$$A_{2v+1}^{(r)} = \sum_{j=0}^{v-2} 2^{2v-2j} A_{2v+1-2j}^{(r-1)} \binom{\frac{1}{2}n_r - 2v - 1 + 2j}{j}. \quad (6)$$

Proof. We consider a structure of a set of weight w codewords and the structure of the corresponding set of w columns of a parity check matrix.

(i) Let us consider all possible structures of words of **even weight** $2v$ in the matrix H_r of (1). These words consist of the following components:

• A codeword of even weight $2v-2j$ taken from H_{r-1} and partitioned by two parts that are placed in the left and right sides of H_r.

• Two sets of the same columns of H_{r-1} placed in the left and right sides of H_r.

For $j = 0, 1, \dots, v - 2$ and for every codeword of even weight in H_{r-1}, the following executed:

– The summand $\sum_{j=0}^{v-2} 2^{2v-2j-1} A_{2v-2j}^{(r-1)} \binom{\frac{1}{2}n_r - 2v + 2j}{j}$ of (5).

A set Γ of columns corresponding to a codeword of even weight $2v - 2j$ of H_{r-1} is partitioned by two parts. Every part contains an odd (resp. even) number of columns if j is odd (resp. even). The partition is executed by all possible ways. The number of the partitions is equal to $2^{2v-2j-1}$. The obtained parts are placed in the left and right sides of H_r.

Also, in every of two matrices H_{r-1} of (1) we take the same set of j columns not belonging to Γ. The number of such j-sets is equal to $\binom{\frac{1}{2}n_r - 2v + 2j}{j}$. As a result, in the right half of H_r we always have an even number of taken columns.

– The summand $\delta_v^{(r)} = \binom{\frac{1}{2}n_r}{v}$ of (5).

If v is even then in every of two matrices H_{r-1} of (1) we take

the same set of v columns. The number of variants is equal to $\binom{\frac{1}{2}n_r}{v}$.

(ii) Let us consider all possible structures of words of **odd weight** $2v+1$ in the matrix H_r of (1). These words consist of the following components:

• A codeword of odd weight $2v+1-2j$ taken from H_{r-1} and partitioned by two parts that are placed in the left and right sides of H_r.

• Two sets of the same columns of H_{r-1} placed in the left and right sides of H_r.

For $j = 0, 1, \ldots, v-2$ and for every codeword of odd weight in H_{r-1} the following executed:

A set Γ of columns corresponding to a codeword of odd weight $2v+1-2j$ of H_{r-1} is partitioned by two parts. One part, say A_{odd}, contains an odd number of columns, another part, say B_{even}, contains an even number of columns. The partition is executed by all possible ways. The number of the partitions is equal to 2^{2v-2j}.

If j is odd then the part B_{even} (resp. A_{odd}) is placed in the left (resp. right) half of H_r.

If j is even or $j = 0$ then the part A_{odd} (resp. B_{even}) is placed in the left (resp. right) half of H_r.

Also, in every of two matrices H_{r-1} of (1) we take the same set of j columns not belonging to Γ. The number of such j-sets is equal to $\binom{\frac{1}{2}n_r-2v-1+2j}{j}$. As a result, in the right half of H_r we always have an even number of taken columns. □

As a direct corollary from the previous theorems we give the weight spectrum for duals to quasi-perfect codes.

Theorem 8. *Let $d_r \le 4$. Assume that an $[n_r, n_r - r, d_r]$ code $\mathcal{C}_r$ is created from a $[\frac{1}{2}n_r, \frac{1}{2}n_r - r + 1, d_{r-1}]$ code $\mathcal{C}_{r-1}$ by doubling construction* (1). *Then weight spectrum $\{A_w^{(r)\perp},\ w \le n_r\}$ of the $[n_r, r, d_r^{\perp}]$ code dual to $\mathcal{C}_r$ can be obtained from weight spectrum $\{A_w^{(r-1)\perp},\ w \le \frac{1}{2}n_r\}$ of the $[\frac{1}{2}n_r, r-1, d_{r-1}^{\perp}]$ code dual to $\mathcal{C}_{r-1}$ as follows:*

$$A_{2v}^{(r)\perp} = A_v^{(r-1)\perp} + \begin{cases} 0 & \text{if} \quad 2v \ne \frac{1}{2}n_r \\ 2^{r-1} & \text{if} \quad 2v = \frac{1}{2}n_r \end{cases}. \qquad (7)$$

Proof. We consider matrix (1) as a generator matrix of the dual code. If codeword of the dual code is created without inclusion the top row, then its weight is equal to the doubled weight of the corresponding word formed from rows of matrix H_{r-1}. If the top row is included into codeword, its weight is equal to $\frac{1}{2}n_r$. □

IV. On correction of erasure patterns of high weight

Remind that the known weight spectrum of a code opens a way for calculation of very important probabilities for the code, like conditional probability of correct decoding of erasure patterns, probability of undetected error and so on. Else, the number of parity bits is larger the binary code distance, so that is a good reason to investigate a total ability of a code to erasure correction.

The necessary condition for correction of weight ρ erasure patterns is the full rank of submatrix, consisting of columns of a code parity check matrix, corresponding to erased position.

Let S_ρ be the number of erasure patterns of weight ρ, which can be corrected by a code (equivalently, for a code parity check matrix – the number of distinct sets of ρ linear independent columns or the number of distinct $r \times \rho$ submatrices of the full rank).

For a code of length n, let $\delta_\rho = \frac{S_\rho}{\binom{n}{\rho}}$ be the conditional probability of correct decoding of erasure patterns of weight ρ.

In further, for $[n, n-r, d]$ code with weight spectrum $A_0, A_1, \ldots, A_n$ we introduce the function

$$\Psi(n, d, \rho) = \binom{n}{\rho} - \sum_{w=d}^{\rho} A_w \binom{n-w}{\rho-w}, \quad d \le \rho \le r. \qquad (8)$$

As we see later this function gives a lower estimate of the number S_ρ.

Theorem 9. *For an $[n, n-r, d]$ code, the conditional probability δ_ρ and the value S_ρ satisfy the following lower estimates:*

$$\delta_\rho \ge \frac{\Psi(n, d, \rho)}{\binom{n}{\rho}}, \quad S_\rho \ge \Psi(n, d, \rho), \quad d \le \rho \le r. \qquad (9)$$

In particular, the equalities hold:

$$\delta_\rho = \frac{\Psi(n, d, \rho)}{\binom{n}{\rho}}, \quad S_\rho = \Psi(n, d, \rho),$$

under condition

$$\rho \le d + \frac{d-1}{2}.$$

The proof of Theorem 9 is based on the fact that the value S_ρ is equal to difference between the total number of sets of ρ columns of a parity check matrix and the number of patterns of ρ linear dependent columns.

The following lemma allows us to improve estimates of Theorem 9 using a recursive approach.

Lemma 10. *Any set of ρ linear dependent columns of a parity check matrix is an union of w columns with the zero sum (corresponding to a weight w codeword) and a set of $\rho - w$ linear dependent columns, where $d \le w \le \rho$.*

We give a recursive form of function of type (8) :

$$\tilde{\Psi}(n, d, \rho) = \binom{n}{\rho} - \sum_{w=d}^{\rho} A_w(n) \tilde{\Psi}(n-w, d, \rho-w),$$

where $A_w(n)$ is the number of weight w words in a (shortened) code of length n.

A recursive estimate of the conditional probability of correct decoding of erasure patterns of weight ρ and the first and second steps of the recursion has the form, respectively,

$$\tilde{\delta}\,(n,d,\rho)=\frac{\tilde{\Psi}(n,d,\rho)}{\binom{n}{\rho}}$$
$$=1-\sum_{w=d}^{\rho}A_w(n)\tilde{\delta}(n-w,d,\rho-w)\frac{\binom{n-w}{\rho-w}}{\binom{n}{\rho}};$$
$$\tilde{\delta_2}\,(n,d,\rho)=1-\sum_{w_1=d}^{\rho}A_{w_1}(n)\frac{\binom{n-w_1}{\rho-w_1}}{\binom{n}{\rho}}\times$$
$$\times\left[1-\sum_{w_2=d}^{\rho-w_1}A_{w_2}\,(n-w_1)\,\frac{\binom{n-w_1-w_2}{\rho-w_1-w_2}}{\binom{n-w_1}{\rho-w_1}}\right].$$

Now we use the known binomial approximation of weight spectrum of a binary linear code [4], [7], [8], writing it as $A_w\approx 2^{-z}\binom{n}{w}$, $r-1<z\le r,\ w\ge d$, where z is a proper real value for the weight region $w\ge d$. We obtain the following approximate estimate of the function S_ρ for the region $d\le\rho\le r$.

$$S_\rho\ge\binom{n}{\rho}-\sum_{w=d}^{\rho}A_w\binom{n-w}{\rho-w}$$
$$\approx\binom{n}{\rho}-2^{-z}\sum_{w=d}^{\rho}\binom{n}{w}\binom{n-w}{\rho-w}$$
$$=\binom{n}{\rho}-2^{-z}\binom{n}{\rho}\sum_{w=d}^{\rho}\binom{\rho}{w}.$$

From here, using [8, Lemma 10.8], we obtain an estimate of the conditional probability δ_ρ of correct decoding of erasure patterns of high weight ρ.

$$\delta_\rho\ge\frac{S_\rho}{\binom{n}{\rho}}\approx 1-2^{-z}\sum_{w=d}^{\rho}\binom{\rho}{w}$$
$$\approx 1-2^{-z}\cdot 2^{\rho H(d/\rho)}\ge 1-2^{\rho-z},\ d\le\rho<z,$$

where $H(d/\rho)$ is the binary entropy.

The proposed estimate shows that for a fixed r, the probability δ_ρ decreases exponentially with growth of ρ. Therefore the reasonable extended region of correctable erasure patterns is $\rho<2d$.

V. Application to memory

An important area for application of quasi-perfect codes is computer memory (Flash or SSD). Their ability to correct a big number of erasures instead of one error and very low probability of undetected error gives us a strong incentive to investigate the conditional probability of correct decoding for erasure patterns of high weight. As an example, that is useful for application, we give two tables: one for conditional probability of correct decoding for erasure patterns of weights higher the code distance and second for the probability (unconditional) of decoding failure in memory channel with different error probability for the product of Panchenko code.

Decoding algorithm for product of Panchenko codes consists of following steps.

1) Error detection in rows and columns of the received word (in parallel).
2) Check (in parallel) of the detected row (column) list for correctability as erasure pattern.
3) Correction of the chosen erasure pattern (row or column) and output.

Check for correctability is executed in extended area up to d^+ erasures.

Table I gives a comparison between Hamming and Panchenko codes with 7 and 8 parity symbols. We can see from the table that extended decoding with correction of 4, 5, 6, 7 erasures has decreasing probability from 1 till 1/2 (approximately).

Table II demonstrates fast decreasing of the probability of decoding failure for fixed number of parity bits with extension of the decoding area for product of two Panchenko codes. We can see from the second table fast decreasing of the failure probability with extension of the decoding area from 3 till 6 erasures.

TABLE I
Conditional probability δ_ρ of correct decoding of erasure patterns of weight ρ for Hamming and Panchenko codes

code	r	$\rho=d=4$	$\rho=5$	$\rho=6$	$\rho=7$
Hamming	7	0.9836	0.9180	0.7469	0.4121
Panchenko	7	0.9870	0.9287	0.7656	0.4306
Hamming	8	0.9920	0.9600	0.8741	0.6879
Panchenko	8	0.9934	0.9647	0.8830	0.6996

TABLE II
Failure probability for product of Panchenko codes $[72, 64, 4]$

p	10^{-1}	10^{-2}	$5\cdot10^{-3}$	10^{-3}	$5\cdot10^{-4}$
$d^+=3$	1	0,996	0,250	1,1e-09	2,3e-14
$d^+=4$	1	0,988	0,092	1,6e-12	5,1e-18
$d^+=5$	1	0,967	0,027	7,0e-14	1,045e-18
$d^+=6$	1	0,926	0,008	5,8e-14	1,029e-18

Acknowledgment

The research was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project 14-50-00150).

References

[1] V.B. Afanassiev, A.A. Davydov, and D.K. Zigangirov, "Design and analysis of codes with distance 4 and 6 minimizing the probability of decoder error", *Journal of Communications Technology and Electronics*, vol. 61, 2016, pp. 1440–1455.

[2] V.B. Afanassiev, A.A. Davydov, and D.K. Zigangirov, "Estimation of the conditional probability of correct decoding of erasure patterns for linear codes", *Informatsionnye Protsessy*, vol. 16, 2016, pp. 382–404. [in Russian].

[3] A.A. Bruen and D.L. Wehlau, "Long binary linear codes and large caps in projective space", *Designs, Codes and Cryptography*, vol. 17, (1999) pp. 37–60.

[4] K.M. Cheung, *The weight distribution and randomness of linear codes*. Jet Propulsion Lab., California Inst. of Tech., Pasadena, CA. TDA Progress Report 42-97. USA. 1989, pp. 208–215. Web: https://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19890018521.pdf

[5] A.A. Davydov and L.M. Tombak, "Quasiperfect linear binary codes with minimal distance 4 and complete caps in projective geometry", *Problems of Information Transmission*, vol. 25 (4), 1989, pp. 265–275.

[6] A.A. Davydov and L.M. Tombak, "An alternative to the Hamming code in the class of SEC-DED codes in semiconductor memory", *IEEE Transactions on Information Theory*, vol. IT-37, 1991, pp. 897–902.

[7] I. Krasikov and S. Litsyn, "On spectra of BCH codes", *IEEE Transactions on Information Theory*, vol. IT-41, 1995, pp. 786–788.

[8] F.J. MacWilliams and N.J.A. Sloane, *The Theory Error-Correcting Codes*. Amsterdam, New-York: North-Holland Publ. Company, 1977.

[9] V.I. Panchenko, "On optimization of linear code with distance 4", in *Proc. 8th All-Union Conf. on Coding Theory and Communications*, Kuibyshev, 1981, Part 2: Coding Theory (Moscow, 1981), pp. 132–134 [in Russian].

[10] O.V. Popov, "On estimate of ability of linear codes to correct erasures and to detect errors when erasures are", *Electrosvyaz'*, vol. 10, 1967. [in Russian].

[11] D.L. Wehlau, "Complete caps in projective space which are disjoint from a codimension 2 subspace", In A. Blokhuis, J.W.P. Hirschfeld, D. Jungnickel and J.A. Thas (eds), *Developments in Mathematics*, vol. 3, *Finite Geometries*, Kluwer Academic Publishers, Dordrecht (2001) pp. 347–361. (Corrected version: https://arxiv.org/abs/math/0403031).