

Homogeneous locally conformally Kähler and Sasaki manifolds

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We prove various classification results for homogeneous locally conformally symplectic manifolds. In particular, we show that a homogeneous locally conformally Kähler manifold of a reductive group is of Vaisman type if the normalizer of the isotropy group is compact. We also show that such a result does not hold in the case of non-compact normalizer and determine all left-invariant lcK structures on reductive Lie groups.

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0. Introduction

Recall that the notion of a locally conformally Kähler (or shortly an lcK) manifold (M, ω, J) is a generalization of the geometric structure encountered on the Hopf manifolds [8], see Definition 2.13. The study of lcK manifolds goes beyond the framework of Kähler and symplectic geometry while still remaining within that of complex and Riemannian geometry. Ignoring the complex structure J one arrives at the more general notion of a locally conformally symplectic (or shortly an lcs) manifold (M, ω) . Such manifolds were first considered in [6]. The fundamental 2-form ω satisfies the equation

$$d\omega = \lambda \wedge \omega$$

for some closed 1-form λ , see Definition 1.11. The relation between lcK manifolds and lcs manifolds is analogous to the one existing between Kähler manifolds and symplectic manifolds.

This work started in September 2010 during a meeting in Japan with discussions about the work of Hasegawa and Kamishima on compact homogeneous lcK manifolds. And conversely, some of the results of this collaboration have influenced [5, 4] where this paper is referenced. This applies in particular to the proof of Theorem 4.10 that a homogeneous lcK manifold of a reductive group is of Vaisman type if the normalizer of the isotropy group is compact. In the special case of compact groups, this theorem has been proved in [4, 3] (c.f. [7] for a proof under additional assumptions).

Now we describe the structure of this paper and mention some of its main results. In Sec. 1 we describe some general constructions relating symplectic manifolds, contact manifolds, symplectic cones and lcs manifolds. In Sec. 2 we prove more specific results relating Kähler manifolds, Sasaki manifolds, Kähler cones and lcK manifolds. The main new object is an integrable complex structure compatible with the geometric structures considered in Sec. 1. We believe that the systematic presentation in Secs. 1 and 2 of the paper is useful although part of the material is certainly known to experts in the field. In any case, it is a basis for our investigation of homogeneous locally symplectic and lcK manifolds in Secs. 3 and 4 respectively. Under rather general assumptions, we first prove that the dimension of the center of a Lie group of automorphisms of an lcs manifold is at most 2. The main result of Sec. 3 is then a classification of all homogeneous locally symplectic manifolds $(M = G/H, \omega)$ with trivial twisted cohomology class $[\omega] \in H_\lambda^2(\mathfrak{g}, \mathfrak{h})$ (see Theorem 3.9). These assumptions are satisfied if \mathfrak{g} is reductive (see Proposition 3.11).

In the last and main section we focus on homogeneous lcK manifolds of reductive groups. As a warm up, we begin by classifying left-invariant lcK structures on four-dimensional reductive Lie groups. We find that not all of them are of Vaisman type. In Theorem 4.15 we give the classification of left-invariant lcK structures on arbitrary reductive Lie groups. The case of general homogeneous spaces G/H of reductive groups G is related to the case of trivial stabilizer H by considering the

induced lcK structure on the Lie group $N_G(H)/H$. Assuming the latter group to be compact, we prove that the initial lcK structure on G/H is necessarily of Vaisman type (see Theorem 4.10).

1. Symplectic Manifolds, Contact Manifolds and Symplectic Cones

1.1. Contactization

Definition 1.1. A symplectic manifold (M, ω) is called **A-quantizable** if there exists a principal bundle $\pi : P \rightarrow M$ with one-dimensional structure group $A = S^1$ or \mathbb{R} and connection θ such that $d\theta = \pi^*\omega$.

The closed 2-form ω gives rise to a Čech cohomology class $[c] \in \check{H}^2(M, \mathbb{R})$, which can be defined as follows. Let (U_α) be a covering of M by contractible open sets such that the intersections $U_{\alpha\beta} := U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ are also contractible. By the Poincaré Lemma, on each U_α we can choose a 1-form θ_α such that $d\theta_\alpha = \omega|_{U_\alpha}$. Similarly, the 1-form

$$\theta_{\alpha\beta} := \theta_\alpha|_{U_{\alpha\beta}} - \theta_\beta|_{U_{\alpha\beta}}$$

is closed and, hence, $\theta_{\alpha\beta} = df_{\alpha\beta}$ for some function $f_{\alpha\beta} = -f_{\beta\alpha} \in C^\infty(U_{\alpha\beta})$. Finally, the function

$$c_{\alpha\beta\gamma} := f_{\alpha\beta}|_{U_{\alpha\beta\gamma}} + f_{\beta\gamma}|_{U_{\alpha\beta\gamma}} + f_{\gamma\alpha}|_{U_{\alpha\beta\gamma}}$$

is closed and hence constant. By construction, $c = (c_{\alpha\beta\gamma})$ is a Čech 2-cocycle with values in the constant sheaf \mathbb{R} . One can check that the corresponding class $[c] \in \check{H}^2(M, \mathbb{R})$ depends only on the de Rham cohomology class $[\omega] \in H^2(M, \mathbb{R})$. We will call $[c]$ the **characteristic class** of the symplectic manifold (M, ω) . Recall that a class $[c] \in \check{H}^2(M, \mathbb{R})$ is called **integral** if it can be represented by an integral cocycle, that is a cocycle $c = (c_{\alpha\beta\gamma})$ such that $c_{\alpha\beta\gamma} \in \mathbb{Z}$.

Proposition 1.2. *A symplectic manifold (M, ω) is S^1 -quantizable if and only if its characteristic class $[c] \in \check{H}^2(M, \mathbb{R})$ is integral. It is \mathbb{R} -quantizable if and only if $[c] = 0$. In particular, any exact symplectic manifold is quantizable.*

Definition 1.3. Any such pair (P, θ) will be called a **contactization** (or, more precisely, **A-contactization**, where $A = S^1$ or \mathbb{R}) of the symplectic manifold (M, ω) . By a contact manifold we will understand a manifold P of dimension $2n + 1$ together with a globally defined contact form θ , that is $d\theta^n \wedge \theta \neq 0$. A contact manifold (P, θ) will be called **regular** if its Reeb vector field Z generates a free and proper action of $A = S^1$ or \mathbb{R} .

Proposition 1.4. *Any contactization (P, θ) of an A-quantizable symplectic manifold (M, ω) is a regular contact manifold with global contact form θ . The group $\text{Aut}(P, \theta)$ contains the one-dimensional central subgroup A , which is the kernel of the natural homomorphism $\text{Aut}(P, \theta) \rightarrow \text{Aut}(M, \omega)$.*

Proof. θ is indeed a contact form, since $d\theta = \pi^*\omega$ is non-degenerate on the horizontal distribution $\ker\theta$. The Reeb vector field Z is the generator of the principal action, which is free and proper. \square

Proposition 1.5. *There is a bijection between A -quantizable symplectic manifolds (M, ω) with $H^1(M, \mathbb{R}) = 0$ up to isomorphism and regular contact manifolds (P, θ) with Reeb action of $A = S^1$ or \mathbb{R} up to isomorphism.*

1.2. Symplectic cone over a contact manifold

Let (P, θ) be a contact manifold. We denote by $N = C(P) = \mathbb{R}^{>0} \times P$ the cone over P with the radial coordinate r .

Proposition 1.6. *For any contact manifold (P, θ) ,*

$$\omega_N := r dr \wedge \theta + \frac{r^2}{2} d\theta = d\left(\frac{r^2}{2}\theta\right)$$

is a symplectic form on the cone $N = C(P)$.

Definition 1.7. The pair (N, ω_N) is called the symplectic cone over the contact manifold (P, θ) .

Now we give an intrinsic characterization of symplectic cones in the category of symplectic manifolds.

Definition 1.8. A conical symplectic manifold (M, ω, ξ, Z) is a symplectic manifold (M, ω) endowed with two commuting vector fields ξ and Z such that

$$\omega(\xi, Z) > 0, \quad \mathcal{L}_\xi\omega = 2\omega, \quad \mathcal{L}_Z\omega = 0.$$

A global conical symplectic manifold is a conical symplectic manifold (M, ω, ξ, Z) such that ξ is complete.

Theorem 1.9. (i) *The symplectic cone over any contact manifold is a global conical symplectic manifold.*

(ii) *Conversely, any global conical symplectic manifold is a symplectic cone over a contact manifold.*

(iii) *Any conical symplectic manifold is locally isomorphic to a symplectic cone over a contact manifold.*

Proof. (i) Let $(N = C(P), \omega_N)$ be a symplectic cone over a contact manifold (P, θ) . The Reeb vector field of P can be considered as a vector field Z on N , which together with $\xi = r\partial_r$ defines a global conical structure. To prove (ii)–(iii) we need the following lemma.

Lemma 1.10. *Let (M, ω, ξ, Z) be a conical symplectic manifold. Let f be a positive smooth function defined in some open neighborhood U such that $df = -\iota_Z\omega$, i.e. f*

is the Hamiltonian of $-Z$. Then in U the symplectic form ω can be written as

$$\omega = df \wedge \theta + fd\theta = r dr \wedge \theta + \frac{r^2}{2} d\theta,$$

where

$$\theta = \frac{1}{2f}\eta, \quad \eta = \iota_\xi \omega, \quad r = \sqrt{2f}.$$

Remark. The function f is unique up to addition of a constant c such that $f+c > 0$. We can choose, for example, $f = \frac{1}{2}\omega(\xi, Z)$, which is characterized by the condition $\mathcal{L}_\xi f = 2f$.

Proof. The symplectic form is exact:

$$2\omega = d\eta, \quad \eta := \iota_\xi \omega.$$

We define

$$\theta := \frac{1}{2f}\eta.$$

Then we calculate

$$df \wedge \theta + fd\theta = \frac{df}{2f} \wedge \eta + fd \left(\frac{1}{2f} \right) \wedge \eta + \omega = \omega.$$

Now it suffices to rewrite

$$f = \frac{r^2}{2}$$

to obtain $\omega = r dr \wedge \theta + \frac{r^2}{2} d\theta$. □

The lemma proves part (iii) of the theorem. To prove (ii) we remark that using the flow of the complete vector field ξ on a global conical symplectic manifold (N, ω, ξ, Z) we get a global diffeomorphism $N \cong I \times P$, where P is some level set of $f = \frac{1}{2}\omega(\xi, Z)$ and $I = (a, b)$, where $0 \geq a = \inf f$, $b = \sup f$. We have to show that $a = 0$ and $b = \infty$. Let $\gamma : \mathbb{R} \rightarrow N$ be an integral curve of ξ . Then $\mathcal{L}_\xi f = 2f$ implies the differential equation $h' = 2h$, where $h = f \circ \gamma$. Therefore, $h(t) = ce^{2t}$ for some positive constant c , since $f > 0$. This shows that $I = \mathbb{R}^{>0}$ and that N is a symplectic cone $N = C(P)$, where $P = \{r = 1\} = \{f = 1/2\}$.

1.3. Symplectic cones and locally conformally symplectic manifolds

Definition 1.11. A lcs manifold (or shortly an lcs manifold) is a smooth manifold endowed with a non-degenerate 2-form such that $d\omega = \lambda \wedge \omega$ for some closed 1-form λ called Lee form. An lcs manifold is called **proper** if $d\omega \neq 0$. The vector field $Z := \frac{1}{2}\omega^{-1}\lambda$ is called the Reeb field.

Remark. Since ω is non-degenerate, the equation $d\omega = \lambda \wedge \omega$ implies $d\lambda = 0$ provided that $\dim M > 4$.

Proposition 1.12. *The vector field Z is an infinitesimal automorphism of (M, ω) .*

Proof.

$$\mathcal{L}_Z \omega = d\iota_Z \omega + \iota_Z d\omega = \frac{1}{2}d\lambda + \iota_Z(\lambda \wedge \omega) = 0,$$

since $\lambda(Z) = 2\omega(Z, Z) = 0$ and $\lambda \wedge \lambda = 0$. □

Let (N, ω_N) be a symplectic cone over a contact manifold (P, θ) . We define

$$\omega_{\text{lcs}} := \frac{1}{r^2} \omega_N = dt \wedge \theta + \frac{1}{2}d\theta, \quad t = \ln r.$$

Proposition 1.13. *For any nontrivial discrete subgroup $\Gamma \subset \mathbb{R}^{>0}$ the manifold $(N/\Gamma = S^1 \times P, \omega_{\text{lcs}})$ is lcs.*

2. Kähler Manifolds, Sasaki Manifolds and Kähler Cones

2.1. Contactizations of Kähler manifolds

Definition 2.1. A Sasaki manifold (S, g, Z) is a Riemannian manifold (S, g) endowed with a unit Killing vector field Z , such that $J := \nabla Z|_{\mathcal{H}}$ defines an integrable CR-structure on the distribution $\mathcal{H} := Z^\perp \subset TS$.

Let (S, g, Z) be a Sasaki manifold. Then we define the 1-form

$$\theta := g(Z, \cdot).$$

Proposition 2.2. *For any Sasaki manifold (S, g, Z) the 1-form θ is a contact form with the Reeb vector field Z and the CR-structure is strictly pseudo-convex.*

Proof. It follows from Definition 2.1 that $d\theta = g(J\cdot, \cdot)$ on $Z^\perp = \ker \theta$ is non-degenerate. Hence, θ is a contact form with positive definite Levi form. Furthermore, $\theta(Z) = 1$ and

$$0 = \mathcal{L}_Z \theta = \iota_Z d\theta,$$

which shows that Z is the Reeb vector field. □

The following theorem establishes a one-to-one correspondence between quantizable Kähler manifolds and regular Sasaki manifolds.

Theorem 2.3. *Let $A = S^1$ or \mathbb{R} .*

- (i) *The contactization of an A -quantizable Kähler manifold (M, ω, J) is a regular Sasaki manifold (S, θ, g_S, Z) , where (S, θ) , $\pi : S \rightarrow M = S/A$, is the contactization of (M, ω) with the fundamental vector field Z of the A -action and*

$$g_S = \theta^2 + \frac{1}{2}\pi^* g_M, \quad g_M = \omega(\cdot, J\cdot).$$

- (ii) *Conversely, any regular Sasaki manifold with Reeb action of A is the contactization of an A -quantizable Kähler manifold.*

2.2. Cones over Sasaki manifolds and Kähler cones

Definition 2.4. A conical Riemannian manifold (M, g, ξ) is a Riemannian manifold (M, g) endowed with a nowhere vanishing (homothetic) vector field ξ such that $\nabla\xi = \text{Id}$. If ξ is complete it is called a global conical Riemannian manifold.

Proposition 2.5. (i) *The metric cone over any Riemannian manifold is a global conical Riemannian manifold.*

(ii) *Conversely, any global conical Riemannian manifold is a metric cone.*

(iii) *Any conical Riemannian manifold is locally isometric to a metric cone.*

Definition 2.6. A Kähler cone (N, g_N, J) is a metric cone $(N = C(M), g_N = dr^2 + r^2g_M)$ over a Riemannian manifold (M, g_M) endowed with a skew-symmetric parallel complex structure J .

Proposition 2.7. *Any conical Kähler manifold is locally a Kähler cone and any global conical Kähler manifold is a Kähler cone.*

Theorem 2.8. (i) *The metric cone $(N = C(S), g_N)$ over a Sasaki manifold (S, g_S, Z) equipped with the complex structure J_N defined by*

$$J_N|_{\mathcal{H}} := J = \nabla Z|_{\mathcal{H}}, \quad J_N\xi := Z,$$

is a Kähler cone.

(ii) *Conversely, any Kähler cone is the cone over a Sasaki manifold and any conical Kähler manifold is locally isomorphic to a Kähler cone over a Sasaki manifold.*

Now we give a characterization of Sasaki manifolds in the class of strictly pseudoconvex CR-manifolds. In the same way one can characterize pseudo-Riemannian Sasaki manifolds in the class of Levi non-degenerate CR-manifolds.

Let (P, θ, J) be a strictly pseudoconvex integrable CR-structure with globally defined contact form θ , which defines the (contact) CR-distribution $\mathcal{H} = \ker \theta$. We denote by Z the Reeb vector field of θ , such that $\theta(Z) = 1$ and $d\theta(Z, \cdot) = 0$ and extend J defined on \mathcal{H} to an endomorphism field on $TP = \mathbb{R}Z \oplus \mathcal{H}$ by $JZ = 0$. Then we define a natural Riemannian metric g_P on P by

$$g_P := \theta^2 + \frac{1}{2}d\theta(\cdot, J\cdot).$$

The vector field Z preserves θ but does not preserve J and g_P in general.

Theorem 2.9. *Let (P, θ, J) be a strictly pseudoconvex integrable CR-structure with globally defined contact form θ . Then the symplectic structure ω_N of the symplectic cone (N, ω_N) over the contact manifold (P, θ) (see Definition 1.7) together with the cone metric $g_N = dr^2 + r^2g_P$ defines on $N = C(P) = \mathbb{R}^{>0} \times P$ an almost Kähler structure. It is Kähler if and only if the Reeb vector field is holomorphic, that is an infinitesimal CR-automorphism: $\mathcal{L}_Z J = 0$.*

Proof. We have to check that the skew-symmetric endomorphism $J_N = g_N^{-1} \circ \omega_N$ is an almost complex structure. Recall that

$$\begin{aligned}\omega_N &= r dr \wedge \theta + \frac{r^2}{2} d\theta, \\ g_N &= dr^2 + r^2 \theta^2 + \frac{r^2}{2} d\theta(\cdot, J\cdot).\end{aligned}$$

From these formulas we see that the decomposition $\mathcal{H} \oplus \text{span}\{\partial_r, Z\}$ is orthogonal with respect to ω_N and g_N . Hence, J_N preserves this decomposition and $J_N|_{\mathcal{H}} = J$. We check that $J_N Z = -\xi := -r\partial_r$ and $J_N \xi = Z$:

$$\begin{aligned}\omega_N(Z, \cdot) &= -r dr = -g_N(\xi, \cdot), \\ \omega_N(\xi, \cdot) &= r^2 \theta = g_N(Z, \cdot).\end{aligned}$$

Now we investigate the integrability of J_N , that is the involutivity of $T^{0,1}N \subset T^{\mathbb{C}}N$. The involutivity of $\mathcal{H}^{0,1}$ follows from the integrability of the CR-structure $J = J_N|_{\mathcal{H}}$. The involutivity of $(\mathcal{H}^\perp)^{0,1} = \mathbb{C}(Z + i\xi)$ is automatic for dimensional reasons. Finally the bracket of $Z + iJ_N Z = Z - i\xi$ with $X + iJ_N X = X + iJX$, $X \in \Gamma(P, \mathcal{H}) \subset \Gamma(N, \mathcal{H})$, is computed as follows:

$$[Z + i\xi, X + iJX] = [Z, X + iJX] = [Z, X] + i[Z, JX],$$

which is of type $(0, 1)$ if and only if $[Z, JX] = J[Z, X]$ for all X , that is if and only if $\mathcal{L}_Z J = 0$. □

As a corollary, cf. Theorem 2.8, we obtain the following (connection-free) characterization of Sasaki manifolds in terms of CR-structures.

Corollary 2.10. *A Sasaki manifold (P, g, Z) is the same as a strictly pseudoconvex CR-manifold (P, θ, J) with globally defined contact form θ such that the corresponding Reeb vector field Z is holomorphic. The metric $g = g_P$ is the natural Riemannian metric on P defined by the data (θ, J) .*

Theorem 2.11. *Let (S_i, g_i, Z_i) , $i = 1, 2$, be two Sasaki manifolds. Then the manifold $N = S_1 \times S_2$ has a two-parameter family of integrable complex structures $J = J_{a,b}$ defined by*

$$J|_{\mathcal{H}_i} = J_i, \quad JZ_1 = aZ_1 + bZ_2, \quad JZ_2 = cZ_1 - aZ_2,$$

where $a \in \mathbb{R}, b \neq 0, c = -\frac{1+a^2}{b}$ and (\mathcal{H}_i, J_i) is the CR-structure of S_i . The complex structures $J_{\text{can}} := J_{0,1}$ and $-J_{\text{can}} := J_{0,-1}$ are the only structures in the family $J_{a,b}$ for which the product metric is Hermitian.

Proof. This follows from the Newlander–Nirenberg theorem by a direct calculation. □

As a special case we obtain the famous complex structures on products of spheres, constructed by Calabi and Eckmann.

Corollary 2.12. *The product of two odd-dimensional spheres has a two-parameter family $J_{a,b}$ of integrable complex structures. The product metric is Hermitian with respect to the complex structure J_{can} .*

2.3. Kähler cones and locally conformally Kähler manifolds

Definition 2.13. A lcK manifold (or shortly an lck manifold) is a lcs manifold (M, ω) endowed with a skew-symmetric integrable complex structure J such that the metric

$$g = \omega(\cdot, J\cdot)$$

is positive definite. The Riemannian metric g is then called a lcK metric. The 1-form $\theta := \frac{1}{2}J^*\lambda$ is called the Reeb form. The (locally gradient) vector field $\xi = -\frac{1}{2}g^{-1}\lambda$ is called the Lee field. An lcK manifold (M, ω, J) is called Vaisman manifold if ξ is a parallel unit vector field.

Remark that if ξ is parallel then $\lambda(\xi)$ is constant. By rescaling ω we can always normalize $\lambda(\xi) = 2\omega(Z, \xi) = 2g(JZ, \xi) = -2g(\xi, \xi) = -2$, such that $|\xi| = 1$. Note that, as a consequence of the above definition, the Lee and the Reeb field are related by

$$Z = J\xi.$$

Similarly one defines the notion of a locally conformally pseudo-Kähler manifold and that of a pseudo-Riemannian Vaisman manifold by allowing the metric to be indefinite.

Vaisman manifolds were first studied by Vaisman, who called them *generalized Hopf manifolds*. In [8] he proved the following theorem, which relates them to Sasaki manifolds. For convenience of the reader we reprove it within the logic of our exposition.

Theorem 2.14. *Let (M, ω, J) be a complete Vaisman manifold. Then*

- (i) *the Lee field ξ and the Reeb field $Z = J\xi$ are infinitesimal automorphisms of the lcK structure (ω, J) and*
- (ii) *the universal cover of M is a Riemannian product of a line and a simply connected Sasaki manifold S .*

Proof. The de Rham theorem implies that the universal cover of a complete Vaisman manifold is a Riemannian product $M = \mathbb{R} \times S$ of a line and a simply connected manifold S , where S is a leaf of the integrable distribution $\ker \lambda = \xi^\perp$. We already know that ξ is a Killing vector field, since it is parallel. We also know that Z preserves ω by Proposition 1.12. Therefore, in order to prove (i), we only have

to show that ξ and Z are holomorphic, that is preserve the complex structure J . We recall that a (real) vector field X is holomorphic if and only if JX is holomorphic. Moreover, under this assumption, X and JX commute. Since $Z = J\xi$, it suffices to check that ξ is holomorphic. Now any lcK manifold (M, ω, J) admits a canonical torsion-free complex connection $\tilde{\nabla}$, which coincides with the Levi-Civita connection of the locally defined Kähler metric $\tilde{g} = e^{-f}g$, where f is a locally defined function such that $df = \lambda$. Indeed, since f is unique up to an additive constant, the metric \tilde{g} is unique up to a constant factor and its Levi-Civita connection is a well defined connection on M . With our conventions, the explicit expression for $\tilde{\nabla}$ is

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\lambda(X)Y - \frac{1}{2}\lambda(Y)X - g(X, Y)\xi. \tag{2.1}$$

To prove this formula, it is enough to check that the torsion-free connection on the right-hand side preserves the metric \tilde{g} . This is a straightforward calculation. Using $\nabla\xi = 0$ and (2.1), we obtain $\mathcal{L}_\xi J = \nabla_\xi J = \tilde{\nabla}_\xi J = 0$, as in [8].

It follows from (i) that $\mathcal{L}_\xi\theta = 0$. This means that θ can be considered as a 1-form on S .

Lemma 2.15. *Let (M, ω, J) be an lcK manifold. Then*

$$\mathcal{L}_\xi\omega = \lambda(\xi)\omega - \lambda \wedge \theta + d\theta.$$

Proof. We calculate

$$\mathcal{L}_\xi\omega = d\theta + \iota_\xi(\lambda \wedge \omega) = d\theta + \lambda(\xi)\omega - \lambda \wedge \theta. \quad \square$$

Under the assumptions of the theorem we have $\lambda(\xi) = -2$, $\theta(Z) = 1$ and $\mathcal{L}_\xi\omega = 0$ such that

$$\omega = -\frac{1}{2}\lambda \wedge \theta + \frac{1}{2}d\theta.$$

This implies that $d\theta|_S = 2\omega|_S$ has one-dimensional kernel $\mathbb{R}Z$ transversal to $\mathcal{H} = \ker \theta = Z^\perp$. We have shown that θ is a contact form on S with Reeb vector field Z . In order to prove that S is Sasakian, we choose a local function t such that $\lambda = -2dt$. Then we can rewrite ω and g in the form

$$\begin{aligned} \omega &= dt \wedge \theta + \frac{1}{2}d\theta, \\ g &= dt^2 + \theta^2 + \frac{1}{2}\bar{g}, \end{aligned}$$

where

$$\bar{g} = d\theta(\cdot, J\cdot) \tag{2.2}$$

is the Levi form. One can easily check that the metric $g_K = e^{2t}g$ is a Kähler metric with Kähler form $\omega_K = e^{2t}\omega = d(\frac{1}{2}e^{2t}\theta)$. The substitution $r = e^t$ yields

$$g_K = dr^2 + r^2g_S, \quad g_S = \theta^2 + \frac{1}{2}\bar{g}, \quad \xi = \partial_t = r\partial_r.$$

This is locally a Kähler cone and, hence, its covariant derivative ∇^K yields

$$\nabla^K \xi = \text{Id}, \quad \nabla^K Z = \nabla^K (J\xi) = J.$$

Notice that $g_K|_S$ and g_S are homothetic and, hence, the Levi Civita connection ∇^S of (S, g_S) coincides with the connection induced by ∇^K on the totally umbilic submanifold $S \subset (M, g_K)$. From the Gauß equation we get

$$\nabla_X^S Z = JX \quad \text{for all } X \in TS \cap Z^\perp, \quad \nabla_Z^S Z = 0.$$

This proves that (S, g_S, Z) is a Sasaki manifold. □

Remark. The isometry group of a compact Vaisman manifold does not necessarily preserve the complex structure. It suffices to consider $S^1 \times S^{2n+1}$ endowed with the product metric and the complex structure J_{can} of Theorem 2.11. This is an example of an lcK manifold as shown in the next proposition.

Let (N, ω_N, J_N) be a Kähler cone over a Sasaki manifold (S, g_S, Z) . Recall that $\omega_{\text{lcs}} = dt \wedge \theta + \frac{1}{2}d\theta$ is a conformally symplectic structure on N , where $\theta = g(Z, \cdot)$ is the contact form and $t = \ln r$.

Proposition 2.16. *For any nontrivial discrete subgroup $\Gamma \subset \mathbb{R}^{>0}$ the complex structure J_N on the Kähler cone N induces a complex structure J on $N/\Gamma = S^1 \times S$ such that $(N/\Gamma, \omega_{\text{lcs}}, J)$ is a Vaisman manifold. The group $S^1 = \mathbb{R}^{>0}/\Gamma$ acts freely, holomorphically and isometrically (with respect to the lcK metric) on the lcK manifold N/Γ and Z is an S^1 -invariant holomorphic Killing vector field on N/Γ .*

Proof. By Proposition 1.13, $(N/\Gamma, \omega_{\text{lcs}})$ is lcs. Therefore to prove that it is lcK it suffices to show that J_N is invariant under the group $\mathbb{R}^{>0}$ and, hence, induces a complex structure J on N/Γ . This follows from the equations $\mathcal{L}_\xi \omega_N = 2\omega_N$, $\mathcal{L}_\xi g_N = 2g_N$, since $J_N = g_N^{-1}\omega_N$. The group $\mathbb{R}^{>0}$ acts isometrically on N with respect to the Riemannian metric

$$\omega_{\text{lcs}}(\cdot, J_N \cdot) = dt^2 + g_S, \tag{2.3}$$

which induces the lcK metric g_{lcK} on M . In fact $\xi = \partial_t$ is an obvious Killing vector field for the metric (2.3). This shows that S^1 acts isometrically on $(N/\Gamma, g_{\text{lcK}})$. Obviously $\xi = \partial_t$ is a parallel unit field and preserves the 2-form $\omega_{\text{lcs}} = dt \wedge \theta + \frac{1}{2}d\theta$. In particular, $(N/\Gamma, \omega_{\text{lcs}}, J)$ is a Vaisman manifold. □

The above complex structure on $N/\Gamma = S^1 \times S$ coincides with the complex structure J_{can} of Theorem 2.11. The next theorem shows that the Vaisman manifolds of Proposition 2.16 admit a canonical two-parameter family of Vaisman deformations.

Theorem 2.17. *Let $(N = \mathbb{R}^{>0} \times S, \omega_N, J_N)$ be a Kähler cone over a Sasaki manifold (S, g_S, Z) endowed with the lcs structure $\omega_{\text{lcs}} = dt \wedge \theta + \frac{1}{2}d\theta$. Then $(\omega_{\text{lcs}}, J_{a,b})$, where $J_{a,b}$ is defined in Theorem 2.11, is a Vaisman lcK structure on $N/\Gamma = S^1 \times S$ if and only if $b > 0$. The Reeb vector field Z and the Lee vector field $\xi_{a,b} = -J_{a,b}^* Z$ are holomorphic Killing vector fields for all of these structures.*

Proof. $J_{a,b}$ is skew-symmetric with respect to ω_{lcs} , since

$$g_{a,b} := -\omega_{\text{lcs}}(J_{a,b}\cdot, \cdot) = bdt^2 - c\theta^2 - 2adt\theta + \frac{1}{2}\bar{g}$$

is symmetric. (Recall that \bar{g} stands for the Levi form of S , see (2.2)). The metric $g_{a,b}$ is positive definite if and only if $b > 0$. Since $J_{a,b}$ is integrable, by Theorem 2.11, we see that $(S^1 \times S, \omega_{\text{lcs}}, J_{a,b})$ is lcK if $b > 0$. The vector fields $\xi_{\text{can}} = \xi_{0,1} = \partial_t$ and Z preserve the 1-forms dt and θ and, hence, the metrics $g_{a,b}$. Since the Reeb field always preserves ω , this implies that both vector fields are holomorphic for all $J_{a,b}$. As a consequence, any linear combination of ∂_t and Z , such as $\xi_{a,b}$, is also a holomorphic Killing vector field for any of the complex structures in the two-parameter family. It remains to check that the lcK structure $(\omega_{\text{lcs}}, J_{a,b})$ is Vaisman. The Lee field $\xi_{a,b} = -\frac{1}{2}g_{a,b}^{-1}\lambda$ is given by

$$\xi_{a,b} = -c\partial_t + aZ.$$

A direct calculation using the Koszul formula for $g = g_{a,b}$ shows that for all $X, Y \in \mathcal{H} = \ker \theta \cap \ker \lambda \subset TN$ we have

$$\begin{aligned} 2g(\nabla_X Y, \partial_t) &= g([X, Y], \partial_t) = -a\theta([X, Y]) \\ 2g(\nabla_X Y, Z) &= -Zg(X, Y) + g([X, Y], Z) - g(X, [Y, Z]) \\ -g(Y, [X, Z]) &= -c\theta([X, Y]), \end{aligned}$$

since $\mathcal{L}_Z g = 0$. As consequence, we obtain

$$g(\nabla_X \xi_{a,b}, Y) = -g(\nabla_X Y, \xi_{a,b}) = \frac{1}{2}(ac - ca)\theta([X, Y]) = 0,$$

for all $X, Y \in \mathcal{H}$. Using the fact that $\xi_{a,b}$ is a holomorphic Killing vector field, proven above, we see that to prove $\nabla \xi_{a,b} = 0$ it is enough to check that $\nabla_{\xi_{a,b}} \xi_{a,b} \perp \mathcal{H}$. Let $X \in \Gamma(\mathcal{H})$ be a local section, which commutes with $\xi_{a,b}$. Then the Koszul formula yields

$$2g(\nabla_{\xi_{a,b}} \xi_{a,b}, X) = -Xg(\xi_{a,b}, \xi_{a,b}) = 0. \quad \square$$

Corollary 2.18. *The Vaisman manifold $(S^1 \times S^{2n+1}, \omega_{\text{lcs}}, J_{\text{can}})$, $n \geq 1$, admits a two-parameter deformation by Vaisman lcK manifolds $(S^1 \times S^{2n+1}, \omega_{\text{lcs}}, J_{a,b})$, $b > 0$. The group $T^2 \times \text{SU}(n+1) = S^1 \times \text{U}(n+1)$ acts transitively on $S^1 \times S^{2n+1}$ preserving all of these lcK structures. It is the maximal connected Lie group preserving any of the above lcK structures. For $b \neq 1$ this group coincides with the full connected isometry group of the lcK metric $g_{a,b}$. For $b = 1$ the full connected isometry group is strictly larger, that is $\text{Isom}_0(S^1 \times S^{2n+1}, g_{\text{can}}) = S^1 \times \text{SO}(2n+2)$.*

3. Homogeneous Locally Conformally Symplectic Manifolds

Here we give a description of homogeneous lcs manifolds.

Let $(M = G/H, \omega)$ be a homogeneous lcs manifold with Lee form λ . For all of this section we will assume that G is connected and effective and that

$d\omega \neq 0$. We will consider ω and λ as \mathfrak{h} -invariant forms on the Lie algebra \mathfrak{g} which vanish on \mathfrak{h} .

3.1. A bound on the dimension of the center

Proposition 3.1. *If λ does not vanish on the center \mathfrak{z} of \mathfrak{g} then $\dim \mathfrak{z} \leq 2$.*

Proof. As λ is closed $\mathfrak{g}^\lambda := \ker \lambda \subset \mathfrak{g}$ is an ideal. Since M is lcs we have the equation $d\omega = \lambda \wedge \omega$ on \mathfrak{g} . Let $Z_0, Z_1 \in \mathfrak{z}$, $\lambda(Z_0) = 1$, $Z_1, X \in \ker \lambda$. Then the above equation yields

$$0 = d\omega(Z_0, Z_1, X) = \omega(Z_1, X).$$

This shows that $\mathfrak{z} \cap \mathfrak{g}^\lambda \subset \ker \omega|_{\mathfrak{g}^\lambda}$, which implies $\dim \mathfrak{z} \cap \mathfrak{g}^\lambda \leq 1$ and, hence, $\dim \mathfrak{z} \leq 2$. \square

Corollary 3.2. *If \mathfrak{g} admits an ad-invariant (possibly indefinite) scalar product b such that the vector $Z_0 := b^{-1}\lambda$ is not isotropic then $\dim \mathfrak{z} \leq 2$.*

Proof. It suffices to prove that $Z_0 \in \mathfrak{z}$. For all $X, Y \in \mathfrak{g}$ we have:

$$b([Z_0, X], Y) = b(Z_0, [X, Y]) = \lambda([X, Y]) = -d\lambda(X, Y) = 0. \quad \square$$

Corollary 3.3. *If G is reductive then $\dim Z(G) \leq 2$. In particular, a reductive automorphism group of a homogeneous lcs manifold has at most two-dimensional center.*

Proposition 3.4. *Let $(M = G/H, \omega, g)$ be a homogeneous Vaisman manifold such that $G = \text{Aut}(M, \omega, g)$. Then the center \mathfrak{z} of \mathfrak{g} is two-dimensional.*

Proof. By Theorem 2.14, the Reeb vector field is an infinitesimal automorphism of (M, ω, g) , which generates a one-parameter subgroup of G . Any vector $X \in \mathfrak{g}$ defines a Killing vector field X^* on M . Let us denote by $Z \in \mathfrak{g}$ the Reeb vector, that is the vector such that Z^* is the Reeb vector field. Then the G -invariance of Z^* implies that $0 = \mathcal{L}_{X^*}Z^* = [X^*, Z^*] = -[X, Z]$ for all $X \in \mathfrak{g}$. Thus $Z \in \mathfrak{z}$, which implies $\dim \mathfrak{z} \geq 1$. The same argument applies to the Lee field $\xi = -JZ$, showing that $\dim \mathfrak{z} \geq 2$. On the other hand, Proposition 3.1 shows that $\dim \mathfrak{z} \leq 2$. \square

3.2. A construction of homogeneous lcs manifolds

Let G be a Lie group with the Lie algebra \mathfrak{g} and $Q = \text{Ad}_G^*\phi = G/K$ the coadjoint orbit of an element $\phi \in \mathfrak{g}^*$. We denote by ω_Q the (invariant) Kirillov–Kostant symplectic form in Q given by

$$(\omega_Q)_{\phi'}(X \cdot \phi', Y \cdot \phi') := \phi'([X, Y]), \quad \phi' \in Q, \quad X, Y \in \mathfrak{g},$$

where $X \cdot \phi' = -\phi' \circ \text{ad}_X \in T_{\phi'}Q$. Identifying ω_Q with an Ad_K -invariant 2-form on \mathfrak{g} vanishing on $\mathfrak{k} = \text{Lie } K$ we can simply write

$$\omega_Q(X, Y) = \phi([X, Y]), \quad X, Y \in \mathfrak{g}.$$

We will assume that the orbit Q is not conical, that is it is not invariant with respect to multiplication by positive numbers. Then the restriction $\phi|_{\mathfrak{k}}$ of the form ϕ to the stability subalgebra \mathfrak{k} is not zero and $\mathfrak{h} := \mathfrak{k} \cap \ker \phi$ is an ideal of \mathfrak{k} (see [1]). We will assume that the subalgebra \mathfrak{h} generates a closed subgroup H of G . Then we have the following.

Proposition 3.5 ([1]). *The 1-form ϕ defines an invariant contact structure ϕ in $P = G/H$ and the contact manifold $(P = G/H, \phi)$ is a quantization of the homogeneous symplectic manifold $(Q = G/K, \omega_Q)$, that is ϕ is a connection on the A -principal bundle $P = G/H \rightarrow G/K$ with the curvature form ω_Q , where $A = K/H \cong \mathbb{R}$ or $\cong S^1$.*

Let D be a derivation of the Lie algebra \mathfrak{g} and $\mathfrak{g}(D) := \mathbb{R}D + \mathfrak{g}$ the associated Lie algebra with the ideal \mathfrak{g} . We denote by λ the closed 1-form dual to D (such that $\lambda(D) = 1, \lambda(\mathfrak{g}) = 0$) and define a 2-form ω on $\mathfrak{g}(D)$ by

$$\omega = -\lambda \wedge \phi + d\phi. \tag{3.1}$$

It is an $\text{ad}_{\mathfrak{h}}^*$ -invariant 2-form with kernel \mathfrak{h} and satisfies

$$d\omega = \lambda \wedge d\phi = \lambda \wedge \omega.$$

We denote by $G(D)$ a Lie group with the Lie algebra $\mathfrak{g}(D)$ and by H its closed (connected) subgroup generated by \mathfrak{h} . Obviously, we have the following.

Proposition 3.6. *The Ad_H^* -invariant 2-form ω defines an invariant lcs structure ω on the homogeneous manifold $M = G(D)/H$, that is an invariant non-degenerate 2-form ω such that $d\omega = \lambda \wedge \omega$.*

We say that $(M = G(D)/H, \omega)$ is a homogeneous lcs manifold associated with the non-conical orbit $Q = \text{Ad}_G^*\phi$ and a derivation D of the Lie algebra \mathfrak{g} .

Remark. Let (M, ω, J) be an lcK manifold of Vaisman type with Lee form λ and Reeb form θ . Then Eq. (3.1) holds with $\phi = \frac{1}{2}\theta$.

3.3. The main result for homogeneous lcs manifolds

In this subsection we show as a main result (Theorem 3.9) that the above construction gives all homogeneous lcs manifolds satisfying a certain cohomological assumption, which we will explain now.

Let $(M = G/H, \omega)$ be a homogeneous lcs manifold with Lee form λ . We consider ω and λ as Ad_H^* -invariant forms on the Lie algebra \mathfrak{g} , which vanish on \mathfrak{h} . Then ω defines a cohomology class

$$[\omega] \in H_{\lambda}^2(\mathfrak{g}, \mathfrak{h}) := \frac{\ker(d_{\lambda} : C^2(\mathfrak{g}, \mathfrak{h}) \rightarrow C^3(\mathfrak{g}, \mathfrak{h}))}{\text{im}(d_{\lambda} : C^1(\mathfrak{g}, \mathfrak{h}) \rightarrow C^2(\mathfrak{g}, \mathfrak{h}))},$$

where

$$C^k(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in (\wedge^k \mathfrak{g}^*)^H \mid \iota_X \alpha = 0 \text{ for all } X \in \mathfrak{h}\}$$

is the vector space of Ad_H^* -invariant alternating k -forms vanishing on \mathfrak{h} and

$$d_\lambda \alpha := d\alpha - \lambda \wedge \alpha, \quad \text{for all } \alpha \in \wedge^k \mathfrak{g}^*.$$

We will assume that $[\omega] = 0$, which means that there exists $\phi \in C^1(\mathfrak{g}, \mathfrak{h})$ satisfying Eq. (3.1). Recall that $\mathfrak{g}' := \mathfrak{g}^\lambda = \ker \lambda$ is an ideal of \mathfrak{g} which contains \mathfrak{h} . We can write

$$\mathfrak{g} = \mathbb{R}D + \mathfrak{g}'$$

where $D \in \mathfrak{g}$ such that $\lambda(D) = 1$. The assumption $d\omega \neq 0$ implies that λ and ϕ are linearly independent. Therefore, adding an element of \mathfrak{g}' to D , we can assume that $\phi(D) = 0$. The restriction $\omega' = \omega|_{\mathfrak{g}'}$ is a closed 2-form on \mathfrak{g}' and its kernel \mathfrak{k} is a subalgebra which contains the codimension one subalgebra \mathfrak{h} .

Lemma 3.7. *Let $(M = G/H, \omega)$ be a homogeneous lcs manifold with Lee form λ and $d\omega \neq 0$. Assume that G contains the one-parameter subgroup generated by the Reeb vector field Z (see Proposition 1.12 and note that Z is automatically complete since it is G -invariant). If $[\omega] = 0$ in $H_\lambda^2(\mathfrak{g}, \mathfrak{h})$ then the form ω can be written as*

$$\omega = -\lambda \wedge \phi + d\phi,$$

where ϕ is an Ad_H^* -invariant 1-form on \mathfrak{g} with $\ker \phi \supset \mathbb{R}D + \mathfrak{h}$ which is not zero on \mathfrak{k} . Moreover,

$$\omega(Z, \cdot) = \phi(Z)\lambda.$$

Proof. Since $[\omega] = 0$, Eq. (3.1) holds for some Ad_H^* -invariant 1-form ϕ which vanishes on \mathfrak{h} . The inclusion $\ker \phi \supset \mathbb{R}D + \mathfrak{h}$ holds by our choice of D , as explained above. We prove that $\phi|_{\mathfrak{k}} \neq 0$. Let $Z \in \mathfrak{g}$ be the central element which corresponds to the Reeb vector field. Then $\text{ad}_Z^* \psi = 0$ for every k -form ψ on \mathfrak{g} and, in particular,

$$\iota_Z d\phi = -\text{ad}_Z^* \phi = 0. \tag{3.2}$$

Next we observe that the definition of the Reeb vector field (see Definition 1.11) implies that

$$\lambda(Z) = 0, \tag{3.3}$$

since ω is skew-symmetric. Therefore Eqs. (3.1) and (3.2) show that

$$\omega(Z, \cdot) = \phi(Z)\lambda. \tag{3.4}$$

Since ω is non-degenerate on $\mathfrak{g}/\mathfrak{h}$ this implies that

$$\phi(Z) \neq 0 \tag{3.5}$$

and, hence, $\omega(D, Z) = -\phi(Z) \neq 0$. So the plane E spanned by D and Z is ω -non-degenerate. Let $\mathfrak{m}' \subset \mathfrak{g}'$ be a subspace such that $\mathfrak{m}' \cap \mathfrak{h} = 0$ and which projects

to the ω -orthogonal complement of $\bar{E} = (E + \mathfrak{h})/\mathfrak{h} \subset \mathfrak{g}/\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$. In particular $\mathfrak{m}' \perp_{\omega} Z$ implies

$$\mathfrak{g}' = \ker \lambda = \mathfrak{h} + \mathbb{R}Z + \mathfrak{m}', \tag{3.6}$$

in view of (3.3) and (3.4). Now we see that

$$\mathfrak{k} = \ker \omega' = \mathfrak{h} + \mathbb{R}Z, \tag{3.7}$$

which, by (3.5), proves that ϕ does not vanish on \mathfrak{k} . \square

We claim that the kernel \mathfrak{k} of the exact 2-form $\omega' = \omega|_{\mathfrak{g}'} = d(\phi|_{\mathfrak{g}'})$ on \mathfrak{g}' coincides with the stabilizer of $\phi' := \phi|_{\mathfrak{g}'}$ in the coadjoint representation of \mathfrak{g}' . In fact, this is a consequence of the equation

$$\omega'(X, \cdot) = -\phi \circ \text{ad}_X|_{\mathfrak{g}'},$$

which holds for all $X \in \mathfrak{g}'$, in view of (3.1). Hence, the corresponding subgroup K of the group $G' \subset G$ is closed. By Lemma 3.7, the coadjoint orbit $Q := \text{Ad}_{G'}^* \phi' = G'/K$ is not conical and $\mathfrak{h} = \mathfrak{k} \cap \ker \phi$ generates a closed subgroup $H \subset G' \subset G$. The Ad_H^* -invariant 1-form ϕ' on \mathfrak{g}' defines a contact form on $P = G'/H$ and the contact manifold $P = G'/H$ is a quantization of the symplectic manifold $Q = G'/K$. The contact property follows from the fact that $d\phi' = \omega'$ induces a non-degenerate 2-form on $\mathfrak{g}'/\mathfrak{k}$ (see Lemma 3.7, and the next lemma).

Lemma 3.8. *Under the assumptions of Lemma 3.7, we have*

$$\ker \phi' + \mathfrak{k} = \mathfrak{g}'. \tag{3.8}$$

Proof. Since ϕ and λ are linearly independent, $\phi' = \phi|_{\mathfrak{g}'} \neq 0$ and $\ker \phi' \subset \mathfrak{g}'$ is a hyperplane. By (3.5), $Z \notin \ker \phi'$. Therefore, $\ker \phi' + \mathbb{R}Z = \mathfrak{g}'$, which implies (3.8). \square

Since $\text{ad}_D|_{\mathfrak{g}'}$ is a derivation of the Lie algebra \mathfrak{g}' , we can write $\mathfrak{g} = \mathfrak{g}'(\text{ad}_D)$ and the 2-form ω on \mathfrak{g} has the form

$$\omega = -\lambda \wedge \phi + d\phi,$$

where ϕ is the canonical extension of ϕ' to a 1-form on \mathfrak{g} . This shows the following.

Theorem 3.9. *Any homogeneous lcs manifold satisfying the assumptions of Lemma 3.7 can be obtained by the above construction, that is it is associated with a non-conical coadjoint orbit $Q = \text{Ad}_{G'}^* \phi' = G'/K$ of a Lie group G' with the standard symplectic form $\omega_Q = d\phi$ and a derivation D of the Lie algebra \mathfrak{g}' . More precisely, it has the form $(M = G'(D)/H, \omega)$ where the Lie algebra of $G'(D)$ is the D -extension $\mathfrak{g}'(D) = \mathbb{R}D + \mathfrak{g}'$ of \mathfrak{g}' , $\mathfrak{h} := \ker \phi \cap \mathfrak{k}$ and $\omega = -\lambda \wedge \phi + d\phi$.*

Now we give some sufficient conditions which ensure the cohomological assumption used in this section.

Definition 3.10. A homogeneous lcs manifold with Lee form λ is called locally splittable if the ideal $\mathfrak{g}' = \mathfrak{g}^\lambda \subset \mathfrak{g}$ has a complementary ideal, that is

$\mathfrak{g} = \mathbb{R}D \oplus \mathfrak{g}'$ ($D \in \mathfrak{g}$). It is called **splittable** if $G = A \times G^\lambda$, where $A = \mathbb{R}$ or $A = S^1$.

Proposition 3.11. *Let $(M = G/H, \omega)$ be a locally splittable homogeneous lcs manifold with Lee form λ and $d\omega \neq 0$. Then $[\omega] = 0$ in $H_\lambda^2(\mathfrak{g}, \mathfrak{h})$, $H_\lambda^1(\mathfrak{g}, \mathfrak{h}) = 0$ and $\dim Z(\mathfrak{g}') \leq 1$. In particular, this is the case if \mathfrak{g} is reductive.*

Proof. We may assume that $\lambda(D) = 1$. Then we decompose ω as

$$\omega = -\lambda \wedge \phi + \omega', \tag{3.9}$$

where ϕ and ω' are Ad_H^* -invariant forms on \mathfrak{g}' , which vanish on \mathfrak{h} . Differentiating this equation and comparing with the lcs equation, we obtain

$$d\omega = \lambda \wedge d\phi + d\omega' = \lambda \wedge \omega = \lambda \wedge \omega'.$$

This shows that

$$\omega' = d\phi.$$

Substituting this into (3.9) we get $d_\lambda \phi = \omega$. To prove $H_\lambda^1(\mathfrak{g}, \mathfrak{h}) = 0$, let $\alpha \in C^1(\mathfrak{g}, \mathfrak{h})$ be a d_λ -closed form. We decompose it as

$$\alpha = c\lambda + \alpha',$$

where c is a constant and $\alpha' \in C^1(\mathfrak{g}', \mathfrak{h}) \subset C^1(\mathfrak{g}, \mathfrak{h})$. Differentiation yields

$$0 = d_\lambda \alpha = -\lambda \wedge \alpha' + d\alpha',$$

which implies $\alpha' = 0$ and $\alpha = c\lambda = -cd_\lambda 1$, where $1 \in C^0(\mathfrak{g}, \mathfrak{h}) = \mathbb{R}$. The bound on the dimension of the center of \mathfrak{g}' follows from Proposition 3.1. \square

Corollary 3.12. *Let $Q = G/K = \text{Ad}_G^* \phi$ be a non-conical coadjoint orbit such that the normal subgroup $H \subset K$ generated by $\mathfrak{h} = \ker \phi|_{\mathfrak{k}}$ is closed. Then $(P = G/H, \phi)$ is a homogeneous contact manifold and $(M = A \times P, \omega = -dt \wedge \phi + d\phi)$ is a homogeneous lcs manifold, where $A = \mathbb{R}$ or $A = S^1$. Conversely, any splittable homogeneous proper lcs manifold $(M = G/H, \omega)$ with Lee form λ can be obtained from this construction.*

We remark that the covering $\mathbb{R} \times P$ of the lcs manifold $A \times P$ in the previous corollary, where $\mathbb{R} \rightarrow A$ is the universal covering group, is globally conformal to the symplectic cone over the contact manifold (P, ϕ) after a redefinition $t = -2\tilde{t}$: $\omega = 2(d\tilde{t} \wedge \phi + \frac{1}{2}d\phi) = \frac{2}{r^2}(rdr \wedge \phi + \frac{r^2}{2}d\phi)$, where $\tilde{t} = \ln r$.

4. Homogeneous Locally Conformally Kähler Manifolds of Reductive Groups

4.1. Left-invariant lcK structures on four-dimensional reductive groups

In this section we prepare the classification of homogeneous lcK manifolds of reductive groups, to be given in Theorem 4.10, by classifying left-invariant lcK structures

on four-dimensional reductive groups. We first describe all left-invariant complex structures J on such groups, then all left-invariant lcs structures ω and finally all left-invariant locally conformally pseudo-Kähler structures (ω, J) . In particular, we describe all lcK and Vaisman examples. This extends the results of [4, Sec. 4]. The following lemma is a well known basic fact.

Lemma 4.1. *For any Lie group G , the map*

$$J \mapsto \mathfrak{l}_J := \text{Eig}(J, i) = \ker(J - i\text{Id})$$

induces a one-to-one correspondence between left-invariant complex structures J on G and (complex) Lie subalgebras $\mathfrak{l} = \mathfrak{l}_J \subset \mathfrak{g}^{\mathbb{C}}$ such that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{l} + \rho\mathfrak{l}, \quad \mathfrak{l} \cap \rho\mathfrak{l} = 0, \tag{4.1}$$

where ρ denotes the real structure (i.e. complex anti-linear involutive automorphism) on $\mathfrak{g}^{\mathbb{C}}$ with the fixed point set \mathfrak{g} .

Let \mathfrak{g} be a four-dimensional non-commutative reductive Lie algebra, that is $\mathfrak{g} = \mathfrak{u}(2)$ or $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$, and G any connected Lie group such that $\mathfrak{g} = \text{Lie } G$. We may take $G = \text{U}(2)$ or $G = \text{GL}(2, \mathbb{R})$. Let us denote by $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s}$ the decomposition of the reductive Lie algebra \mathfrak{g} into its center $\mathfrak{z} = \mathbb{R}e_0$ and its maximal semisimple ideal $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$, which is $\mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$. We denote by e^0 the 1-form on \mathfrak{g} which vanishes on \mathfrak{s} and has the value $e^0(e_0) = 1$.

Lemma 4.2. *Let G be a (connected) four-dimensional non-commutative reductive Lie group. Up to conjugation by an element of G , every left-invariant complex structure J on G is defined by a subalgebra $\mathfrak{l}_J = \text{span}\{e_0 + e', e''\}$ such that $e', e'' \in \mathfrak{s}^{\mathbb{C}}$, $[e', e''] = \mu e''$, $\mu \in \mathbb{C}^*$. In particular, e'' belongs to the cone $\mathcal{C} \subset \mathfrak{sl}(2, \mathbb{C})$ of nilpotent elements. This is precisely the null cone with respect to the Killing form of $\mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}^3$.*

Proof. We have to describe all subalgebras $\mathfrak{l} \subset \mathfrak{g}^{\mathbb{C}} = \mathbb{C} \oplus \mathfrak{sl}(2, \mathbb{C})$ satisfying (4.1). From $\rho\mathfrak{s}^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}}$ we see that $\mathfrak{l} \not\subset \mathfrak{s}^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Therefore \mathfrak{l} admits a basis of the form $(e_0 + e', e'')$, where $e', e'' \in \mathfrak{s}^{\mathbb{C}}$. Then

$$[e_0 + e', e''] = [e', e''] \in \mathfrak{l} \cap \mathfrak{s}^{\mathbb{C}} = \mathbb{C}e''$$

shows that

$$[e', e''] = \mu e'', \quad \mu \in \mathbb{C}^*. \tag{4.2}$$

Therefore $\text{span}\{e', e''\} \subset \mathfrak{s}^{\mathbb{C}}$ is a Borel subalgebra and e'' belongs to the cone \mathcal{C} . \square

Lemma 4.3. *Given a complex structure J on \mathfrak{g} and a 1-form $\phi \in \mathfrak{s}^* \subset \mathfrak{g}^*$ such that $\omega = e^0 \wedge \phi + d\phi$ is non-degenerate (and, hence, defines an lcs structure), the structure (ω, J) is locally conformally pseudo-Kähler if and only if $\mathfrak{l}_J = \text{span}\{e_0 + e', e''\} \subset \mathfrak{g}^{\mathbb{C}}$ is isotropic with respect to ω . This is the case if and only if either $\mu = 1$ or $\phi(e'') = 0$.*

Proof. Notice first that the 2-form ω is J -invariant if and only if it is of type $(1, 1)$, which means that ι_J and $\rho\iota_J$ are isotropic. Next we evaluate $\omega = e^0 \wedge \phi + d\phi$ on the basis of \mathfrak{l}_J :

$$\omega(e_0 + e', e'') = \phi(e'') - \phi([e', e'']) = (1 - \mu)\phi(e''). \quad \square$$

The compact case

Let us first consider the case $\mathfrak{s} = \mathfrak{su}(2)$ and denote by (e_1, e_2, e_3) a basis of $\mathfrak{su}(2)$ such that $[e_\alpha, e_\beta] = -e_\gamma$ for every cyclic permutation of $(1, 2, 3)$. In the following (α, β, γ) will be always a cyclic permutation. Then the basis (e^0, e^1, e^2, e^3) of $\mathfrak{g}^* = \mathfrak{u}(2)^*$ which is dual to (e_0, e_1, e_2, e_3) has the following differentials:

$$de^0 = 0, \quad de^\alpha = e^{\beta\gamma} := e^\beta \wedge e^\gamma.$$

Proposition 4.4. *Up to conjugation by an element of $U(2)$, every left-invariant complex structure J on $U(2)$ is contained in the following Calabi–Eckmann family*

$$Je_0 = ae_0 + be_1, \quad Je_1 = ce_0 - ae_1, \quad Je_2 = -e_3, \quad Je_3 = e_2, \quad (4.3)$$

which depends on two-parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}^*$; $c = -\frac{1+a^2}{b}$.

Proof. We specialize the description of complex structures in Lemma 4.2. Since $U(2)$ acts transitively on the quadric $Q = P(\mathbb{C}) \cong \mathbb{C}P^1$ we can assume that $e'' = e_2 + ie_3$. Then Eq. (4.2) shows that $e' \equiv -i\mu e_1 \pmod{\mathbb{C}e''}$ and we can choose the above basis of \mathfrak{l} such that $e' = -i\mu e_1$. Then (4.1) is satisfied if and only if $\rho e' \neq e'$, i.e. $\mu \notin i\mathbb{R}$. This shows that the complex structure J defined by $\mathfrak{l}_J = \mathfrak{l}$ is given by (4.3), where $\mu = \mu_1 + i\mu_2$ is related to a, b, c by

$$a = \frac{\mu_2}{\mu_1}, \quad b = \frac{|\mu|^2}{\mu_1}, \quad c = -\frac{1}{\mu_1}. \quad (4.4) \quad \square$$

Proposition 4.5. *Up to scale, every left-invariant lcs form on $U(2)$ is of the form*

$$\omega = e^0 \wedge \phi + d\phi, \quad (4.5)$$

where $\phi = \sum a_\alpha e^\alpha \in \mathfrak{s}^*$ is any nonzero form. All these structures are equivalent up to conjugation in $U(2)$.

Proof. Let ω be an lcs structure on $\mathfrak{g} = \mathfrak{u}(2)$. Since e^0 is the only closed 1-form on \mathfrak{g} , up to scale, we can assume that the Lee form of ω is given by $\lambda = -e^0$. The canonical 1-form of ω is given by a nonzero element $\phi \in \mathfrak{s}^*$ and any such element defines an lcs structure ω by the formula (4.5). □

Theorem 4.6. *Let $J = J_{a,b}$ be any of the left-invariant complex structures on $G = U(2)$, as defined in (4.3).*

- (i) *If $(a, b) \neq (0, 1)$ then, up to scale, there is a unique left-invariant lcs structure ω on $U(2)$ such that (ω, J) is locally conformally pseudo-Kähler. It is given by*

$\omega = e^{01} + e^{23}$. All these structures are of Vaisman type. The locally conformally pseudo-Kähler metric $g = -\omega \circ J$ is definite if and only if $b < 0$.

- (ii) If $(a, b) = (0, 1)$ then (ω, J) is locally conformally pseudo-Kähler for every left-invariant lcs structure ω on $U(2)$. The metric is always indefinite and the structure (ω, J) is of Vaisman type if and only if ω is proportional to $e^{01} + e^{23}$.

Proof. The pair (ω, J) defines a locally conformally pseudo-Kähler structure on G if and only if $\mathfrak{l}_J = \text{span}\{e_0 + e', e''\} \subset \mathfrak{g}^{\mathbb{C}}$ is isotropic with respect to ω , where $e' = -i\mu e_1$, $e'' = e_2 + ie_3$. To check this property we evaluate (4.5),

$$\omega = -\lambda \wedge \phi + d\phi = \sum a_\alpha e^{0\alpha} + \sum a_\alpha e^{\beta\gamma} \tag{4.6}$$

on the above basis of \mathfrak{l}_J :

$$\begin{aligned} \omega(e_0 + e', e'') &= a_2 + ia_3 + a_2 e^{31}(-i\mu e_1, ie_3) + a_3 e^{12}(-i\mu e_1, e_2) \\ &= a_2 + ia_3 - \mu a_2 - i\mu a_3 \\ &= (1 - \mu)(a_2 + ia_3). \end{aligned}$$

So we see that \mathfrak{l}_J is ω -isotropic if and only if either

- (i) $a_2 = a_3 = 0$, that is $\omega = e^{01} + e^{23}$, up to scale, or
(ii) $\mu = 1$, that is $(a, b) = (0, 1)$.

In case (i) we compute

$$2\xi = \omega^{-1} J^* \lambda = -\omega^{-1}(ae^0 + ce^1) = -(-ae_1 + ce_0) = ae_1 - ce_0$$

and

$$2Z = 2J\xi = a(ce_0 - ae_1) - c(ae_0 + be_1) = (-a^2 - cb)e_1 = e_1.$$

This shows that $X = 2(\xi - aZ) = -ce_0 \in \mathfrak{z}$ and, hence, defines a (nonzero) Killing vector field. On the other hand, $\mathcal{L}_v \omega = 0$ for all $v \in \text{span}\{e_0, e_1\} = \text{span}\{Z, \xi\}$, since $e_0, e_1 \in \ker d\phi = e^{23}$, where

$$\mathcal{L}_v := d \circ \iota_v + \iota_v \circ d : \wedge^k \mathfrak{g}^* \rightarrow \wedge^k \mathfrak{g}^*$$

is the linear map induced by the Lie derivative in direction of the left-invariant vector field associated with the vector $v \in \mathfrak{g}$. In particular, $\mathcal{L}_X \omega = 0$. These two properties of X show that X and, therefore, JX define (real) holomorphic vector fields. Writing ξ as a linear combination of X and JX we see that also ξ defines a holomorphic vector field. On the other hand, by the same argument as for X we see that $\mathcal{L}_\xi \omega = 0$, since ξ is a linear combination of e_0 and e_1 . Therefore ξ defines a Killing vector field. Now it suffices to remark that a locally conformally pseudo-Kähler manifold is Vaisman if and only if the Lee field is Killing. In fact, the Lee field is locally a gradient vector field (due to $d\lambda = 0$) and a gradient vector field is

Killing if and only if it is parallel. To finish the proof of (i) we have to check when the metric $g = -\omega \circ J$ is definite. We compute

$$\begin{aligned}\omega \circ J &= J^*e^0 \otimes e^1 - J^*e^1 \otimes e_0 + J^*e^2 \otimes e^3 - J^*e^3 \otimes e^2 \\ &= (ae^0 + ce^1) \otimes e^1 - (be^0 - ae^1) \otimes e^0 + e^3 \otimes e^3 + e^2 \otimes e^2 \\ &= -b(e^0)^2 + 2ae^0e^1 + c(e^1)^2 + (e^3)^2 + (e^2)^2,\end{aligned}$$

which is definite if and only if $b < 0$. To prove (ii) we compute $\omega \circ J$ for ω given in (4.6) and $J = J_{0,1}$:

$$\begin{aligned}\omega \circ J &= \sum a_\alpha (J^*e^0 \otimes e^\alpha - J^*e^\alpha \otimes e^0) + \sum a_\alpha (J^*e^\beta \otimes e^\gamma - J^*e^\gamma \otimes e^\beta) \\ &= -\sum a_\alpha e^1 \otimes e^\alpha - a_1(e^0)^2 - a_2e^3 \otimes e^0 + a_3e^2 \otimes e^0 + a_1((e^2)^2 + (e^3)^2) \\ &\quad - a_2(e^2 \otimes e^1 + e^0 \otimes e^3) + a_3(e^0 \otimes e^2 - e^3 \otimes e^1) \\ &= -a_1(e^1)^2 - a_1(e^0)^2 + a_1(e^2)^2 + a_1(e^3)^2 - 2a_2e^1e^2 \\ &\quad - 2a_3e^1e^3 - 2a_2e^3e^0 + 2a_3e^2e^0.\end{aligned}$$

This metric is always of signature $(2, 2)$. Now suppose that (ω, J) is of Vaisman type. Then the Lee vector ξ satisfies $\mathcal{L}_\xi \phi = \iota_\xi d\phi = 0$. This implies that ξ is a linear combination $c_0e_0 + c_1\vec{a}$ of e_0 and $\vec{a} = \sum a_\alpha e_\alpha$. Since $g(\xi, \cdot) = -\frac{1}{2}\lambda$ applying $\omega \circ J$ to $c_0e_0 + c_1\vec{a}$ should be a multiple of $\lambda = -e^0$. We calculate

$$\begin{aligned}\omega J(c_0e_0 + c_1\vec{a}) &= c_0(-a_1e^0 - a_2e^3 + a_3e^2) + c_1a_1(-a_1e^1 - a_2e^2 - a_3e^3) \\ &\quad + c_1a_2(a_1e^2 - a_2e^1 + a_3e^0) + c_1a_3(a_1e^3 - a_3e^1 - a_2e^0).\end{aligned}$$

The coefficient of e^1 is

$$-c_1 \sum a_\alpha^2$$

and has to vanish. Since $\vec{a} \neq 0$ this shows that $c_1 = 0$ and that ξ is proportional to e_0 . Then

$$\omega J e_0 = -a_1e^0 - a_2e^3 + a_3e^2,$$

which is proportional to e^0 only if $a_2 = a_3 = 0$. This implies $\omega = e^{01} + e^{23}$ up to a factor, as claimed. \square

The non-compact case

Let us now consider the case $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ and denote by (h, e_+, e_-) a basis of $\mathfrak{sl}(2, \mathbb{R})$ such that $[h, e_\pm] = \pm 2e_\pm$, $[e_+, e_-] = h$. Then the basis (e^0, h^*, e^+, e^-) of $\mathfrak{g}^* = \mathfrak{gl}(2, \mathbb{R})^*$ which is dual to (e_0, h, e_+, e_-) has the following differentials:

$$de^0 = 0, \quad dh^* = -e^+ \wedge e^-, \quad de^\pm = \mp 2h^* \wedge e^\pm.$$

We denote by ρ the standard real structure on $\mathfrak{g}^{\mathbb{C}}$ associated with the real form $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$.

Proposition 4.7. *Up to conjugation by an element of $GL(2, \mathbb{R})$, every left-invariant complex structure J on $GL(2, \mathbb{R})$ belongs to one of the following two families depending on $\mu = \mu_1 + i\mu_2 \in \mathbb{C} \setminus i\mathbb{R}$.*

(i)

$$\begin{aligned} J e_0 &= \frac{\mu_2}{\mu_1} e_0 - \frac{|\mu|^2}{2\mu_1} (e_+ - e_-) \\ J h &= e_+ + e_- \\ J e_{\pm} &= \pm \frac{1}{\mu_1} e_0 \mp \frac{\mu_2}{2\mu_1} (e_+ - e_-) - \frac{1}{2} h. \end{aligned}$$

(ii)

$$\begin{aligned} J e_0 &= \frac{\mu_2}{\mu_1} e_0 + \frac{|\mu|^2}{2\mu_1} (e_+ - e_-) \\ J h &= -(e_+ + e_-) \\ J e_{\pm} &= \mp \frac{1}{\mu_1} e_0 \mp \frac{\mu_2}{2\mu_1} (e_+ - e_-) + \frac{1}{2} h. \end{aligned}$$

These two families are related by the outer automorphism of $\mathfrak{gl}(2, \mathbb{R})$ which maps (e_0, h, e_{\pm}) to $(e_0, h, -e_{\pm})$. (See remark below for a description of these complex structures in a basis which is orthonormal with respect to a suitably normalized bi-invariant scalar product on $\mathfrak{gl}(2, \mathbb{R})$.)

Proof. As before, any complex structure is defined by a subalgebra $\mathfrak{l} \subset \mathfrak{g}^{\mathbb{C}}$ satisfying (4.1). The latter admits a basis $(e_0 + e', e'')$, where $e', e'' \in \mathfrak{s}^{\mathbb{C}}$. Then $[e', e''] = \mu e''$, $\mu \in \mathbb{C}^*$, and $e'' \in \mathfrak{C}$. The group $SL(2, \mathbb{R})$ has three orbits on the quadric $Q = P(\mathbb{C})$. As representatives e'' of these orbits we choose

$$e_+, \quad ih + e_+ + e_-, \quad h + i(e_+ + e_-).$$

The first case is excluded, since $\rho e_+ = e_+$. The elements e' corresponding to $e'' = ih + e_+ + e_-$ and $e'' = h + i(e_+ + e_-)$ are given by

$$\frac{i\mu}{2}(e_+ - e_-), \quad -\frac{i\mu}{2}(e_+ - e_-).$$

Again $\mu \notin i\mathbb{R}$ by (4.1). This gives the two families (i) and (ii). □

Using the Killing form we can identify \mathfrak{s}^* with \mathfrak{s} . Since the Killing form of $\mathfrak{s} = \mathfrak{sl}(2, \mathbb{R})$ is Lorentzian we can further identify \mathfrak{s} with a Lorentzian vector space $\mathbb{R}^{2,1}$.

Remark. Putting $e_1 := (e_+ - e_-)/2$, $e_2 = h/2$, $e_3 := (e_+ + e_-)/2$ and using the abbreviations (4.4) we can rewrite the complex structures in Proposition 4.7 in a

form similar to (4.3):

(i)

$$Je_0 = ae_0 - be_1, \quad Je_1 = -ce_0 - ae_1, \quad Je_2 = e_3, \quad Je_3 = -e_2.$$

(ii)

$$Je_0 = ae_0 + be_1, \quad Je_1 = ce_0 - ae_1, \quad Je_2 = -e_3, \quad Je_3 = e_2.$$

Proposition 4.8. *Up to scale, every left-invariant lcs form on $GL(2, \mathbb{R})$ is of the form*

$$\omega = e^0 \wedge \phi + d\phi, \tag{4.7}$$

where $\phi = \sum a_\alpha e^\alpha \in \mathfrak{s}^* \cong \mathfrak{s} = \mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}^{2,1}$ is any non-isotropic 1-form.

Proof. It suffices to check that ω is non-degenerate if and only if ϕ is space-like or time-like. \square

Next we describe all left-invariant lcs structures which are compatible with any of the complex structures J_μ on $G = GL(2, \mathbb{R})$, as described in Proposition 4.7. It is sufficient to consider the family (i), since it is equivalent to (ii) by an automorphism of G .

Theorem 4.9. *Let $J = J_\mu$ be any of the left-invariant complex structures on $G = GL(2, \mathbb{R})$, as defined in Proposition 4.7(i).*

(i) *If $\mu \neq 1$ then, up to scale, there is a unique left-invariant lcs structure ω on $GL(2, \mathbb{R})$ such that (ω, J) is locally conformally pseudo-Kähler. It is given by*

$$\omega = e^0 \wedge (e^+ - e^-) - 2h^* \wedge (e^+ + e^-) = e^0 \wedge e^1 - e^2 \wedge e^3,$$

where (e^0, e^1, e^2, e^3) denotes the basis dual to (e_0, e_1, e_2, e_3) . All these structures are of Vaisman type with (positive or negative) definite metric.

(ii) *If $\mu = 1$ then (ω, J) is locally conformally pseudo-Kähler for every left-invariant lcs structure $\omega = e^0 \wedge \phi + d\phi$ on $GL(2, \mathbb{R})$. The locally conformally pseudo-Kähler metric $g = -\omega \circ J$ associated with a non-isotropic 1-form $\phi = a_h h^* + a_+ e^+ + a_- e^- \in \mathfrak{s}^*$ is given by*

$$\begin{aligned} g = & -\frac{1}{2}(a_+ - a_-)(e^0)^2 - 2(a_+ - a_-)(h^*)^2 \\ & + 2(a_+ + a_-)e^0 h^* - 2a_+(e^+)^2 + 2a_-(e^-)^2 \\ & - a_h e^0(e^+ + e^-) - 2a_h h^*(e^+ - e^-). \end{aligned} \tag{4.8}$$

It is of Vaisman type if and only if $a_h = 0$ and $a_+ = -a_- \neq 0$, in which case the metric is definite. In particular, the locally conformally pseudo-Kähler metric g is non-Vaisman and positive definite if and only if, first, $a_h \neq 0$ or $a_+ \neq -a_-$ and, second, $-a_h^2 > 4a_+a_-$ and $a_- > 0 > a_+$.

Proof. According to Proposition 4.8 any lcs structure on \mathfrak{g} is of the form $\omega = e^0 \wedge \phi + d\phi$, where $\phi = a_h h^* + a_+ e^+ + a_- e^- \in \mathfrak{s}^*$ is any non-isotropic 1-form. It is of type $(1, 1)$ with respect to J if and only if either (i) $\phi(e'') = a_h + i(a_+ + a_-) = 0$ or (ii) $\mu = 1$ (see Lemma 4.3). In the first case, we have, up to scale, $\phi = e^+ - e^-$, which implies $\omega = e^0 \wedge (e^+ - e^-) - 2h^* \wedge (e^+ + e^-)$. The corresponding locally conformally pseudo-Kähler metric g is definite and Vaisman (the above basis of \mathfrak{g} is g -orthogonal). In the second case, a straightforward calculation of the metric yields the above formula (4.8), depending on the parameters a_h, a_{\pm} . Assuming that this metric is Vaisman, we see that

$$\xi \in \ker d\phi = \text{span} \left\{ e_0, \vec{a} = \frac{a_h}{2} h + a_+ e_- + a_- e_+ \right\}.$$

So $\xi = \alpha e_0 + \beta \vec{a}$ for some $(\alpha, \beta) \in \mathbb{R}^2 \setminus \{0\}$. Then using (4.8) we see that $g(\xi, \cdot)$ is proportional to $\lambda = -e^0$ if and only if the following equations hold

$$\begin{aligned} \alpha(a_+ + a_-) &= 0, \\ \alpha a_h &= 0, \\ \beta \left(\frac{a_h^2}{2} + 2a_+ a_- \right) &= 0. \end{aligned}$$

Since ϕ is not light-like, we see that $\frac{a_h^2}{2} + 2a_+ a_- \neq 0$. Therefore $\beta = 0$ and $\alpha \neq 0$, which shows that $a_h = a_+ + a_- = 0$. In that case, $g = -a_+(e^0)^2 - 4a_+(h^*)^2 - 2a_+(e^+)^2 - 2a_+(e^-)^2$, which is definite. Now it suffices to check that the metric (4.8) is always definite if $a_h = 0$ and $a_+ a_- < 0$. (In the case $a_+ < 0$ it is positive definite.) Now that we have characterized the Vaisman case in (ii), it follows that the metric is non-Vaisman if and only if $a_h \neq 0$ or $a_+ \neq -a_-$. So it only remains to check that the metric is positive definite if and only if $-a_h^2 > 4a_+ a_-$ and $a_- > 0 > a_+$. This is obtained from a calculation of principal minors. \square

4.2. Classification of homogeneous lcK manifolds of reductive groups

In this subsection we prove the following main theorem.

Theorem 4.10. *Every homogeneous proper lcK manifold $(M = G/H, \omega, J)$ of a connected reductive Lie group G such that H is connected and $N_G(H)$ is compact is of Vaisman type.*

Proof. We assume without restriction of generality that G is effective. As before we consider the fundamental form ω , the Lee form λ and the Reeb form $\theta = \frac{1}{2} J^* \lambda$ as H -invariant forms on \mathfrak{g} which vanish on \mathfrak{h} . By Proposition 3.11 we know that there exists $\phi \in C^1(\mathfrak{g}, \mathfrak{h})$ such that (3.1) is satisfied and that the 1-form ϕ is unique up to addition of a multiple of λ . Let $\mathfrak{m} \subset \mathfrak{g}$ be an H -invariant complement of \mathfrak{h} containing the center \mathfrak{z} of \mathfrak{g} . Let us denote by $Z, \xi \in \mathfrak{m}$ the linearly independent

H -invariant vectors which correspond to the Reeb and Lee vector fields on M . We choose ϕ such that $\phi(\xi) = 0$. Together with Eq. (3.1) this makes ϕ unique. We will call ϕ the canonical 1-form. \square

Proposition 4.11. *Under the assumptions of Theorem 4.10, the canonical 1-form coincides with the Reeb form θ up to a factor $1/2$:*

$$\phi = \frac{1}{2}\theta.$$

Proof. The proof of Proposition 4.11 is based on the following key lemma, the proof of which is given below.

Lemma 4.12. *Under the assumptions of Theorem 4.10, we have $Z, \xi \in \ker d\phi$.*

Using Lemma 4.12, we compute

$$\mathcal{L}_\xi \phi = \iota_\xi d\phi = 0,$$

where, for any Ad_H -invariant $v \in \mathfrak{m}$,

$$\mathcal{L}_v := d \circ \iota_v + \iota_v \circ d : C^k(\mathfrak{g}, \mathfrak{h}) \rightarrow C^k(\mathfrak{g}, \mathfrak{h}).$$

\mathcal{L}_v is the linear map induced by the Lie derivative in direction of the G -invariant vector field X_v which extends v . Since also $\mathcal{L}_\xi \lambda = \iota_\xi d\lambda = 0$, Eq. (3.1) implies

$$\mathcal{L}_\xi \omega = -\lambda \wedge \mathcal{L}_\xi \phi + d\mathcal{L}_\xi \phi = 0. \tag{4.9}$$

Now Lemma 2.15 shows that

$$\omega = -\frac{1}{\lambda(\xi)} d_\lambda \theta = \frac{1}{2} d_\lambda \theta.$$

Since $\omega = d_\lambda \phi$ and $H_\lambda^1(\mathfrak{g}, \mathfrak{h}) = 0$, this proves that $\phi = \frac{1}{2}\theta \pmod{\mathbb{R}\lambda}$. Finally, for the canonical 1-form we have $\phi(\xi) = 0$, such that $\phi = \frac{1}{2}\theta$. This finishes the proof of Proposition 4.11. \square

Proof of Lemma 4.12. Let us denote by G_0 the maximal connected subgroup of the normalizer of H in G . Since H is compact, G_0 is reductive. The Lie algebra \mathfrak{g}_0 of G_0 is decomposed as

$$\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{m}_0,$$

where $\mathfrak{m}_0 = Z_{\mathfrak{m}}(\mathfrak{h})$ contains \mathfrak{z} , Z and ξ . Since J is H -invariant, the maximal trivial H -submodule $\mathfrak{m}_0 \subset \mathfrak{m}$ is J -invariant. This implies that ω is non-degenerate on \mathfrak{m}_0 , because $g = -\omega \circ J$ is positive definite. Therefore the restriction of (ω, J) to \mathfrak{m}_0 defines an invariant lcK structure on $M_0 = G_0/H$ with the Lee form $\lambda_0 = \lambda|_{\mathfrak{m}_0}$. Notice that $\lambda_0 \neq 0$, since $\xi \in \mathfrak{m}_0$. Therefore, the lcK structure on M_0 is not Kähler, unless $\dim M_0 = 2$. From the fact that H is normal in G_0 , we see that M_0 is a Lie group. In the Kähler case, the Lie group M_0 is two-dimensional and thus Abelian. So, in that case, $d\phi = 0$ and the assertion of Lemma 4.12 follows. Otherwise M_0

is at least four-dimensional and the lcK structure is non-Kähler. Therefore, we can assume from the beginning that H is trivial. This reduces the proof of Lemma 4.12 to the following special case.

Lemma 4.13. *Under the assumptions of Theorem 4.10 and the additional assumption that H is trivial, we have $Z, \xi \in \ker d\phi$.*

Proof. Let B be a non-degenerate Ad_G -invariant symmetric bilinear form on \mathfrak{g} . Then there exist endomorphisms $A_\omega, A_g, A_{d\phi}, A_{\lambda \wedge \phi} \in \text{End } \mathfrak{g}$ and a vector $v = v_\phi \in \mathfrak{g}$ such that

$$\omega = B \circ A_\omega, \quad g = B \circ A_g, \quad d\phi = B \circ A_{d\phi}, \quad \lambda \wedge \phi = B \circ A_{\lambda \wedge \phi}, \quad \phi = Bv.$$

We claim that

$$A_{d\phi} = -\text{ad}_v, \quad A_{\lambda \wedge \phi} = \lambda \otimes v + 2\phi \otimes A_g \xi.$$

In fact,

$$d\phi = -\phi \circ [\cdot, \cdot] = -B(v, [\cdot, \cdot]) = B([\cdot, v], \cdot) = -B \circ \text{ad}_v,$$

$$\lambda \wedge \phi = \lambda \otimes \phi - \phi \otimes \lambda = \lambda \otimes Bv - \phi \otimes (-2g\xi) = B \circ (\lambda \otimes v + 2\phi \otimes A_g \xi).$$

The equation $\omega = -\lambda \wedge \phi + d\phi$ can now be rewritten as

$$A_\omega = -A_{\lambda \wedge \phi} - \text{ad}_v = -\lambda \otimes v - 2\phi \otimes A_g \xi - \text{ad}_v.$$

Since λ and ϕ are linearly independent ($d\omega \neq 0$), the skew-symmetric endomorphism $A_{\lambda \wedge \phi}$ has rank two. More precisely,

$$\text{im } A_{\lambda \wedge \phi} = \text{span}\{v, A_g \xi\}.$$

Notice that $-2(B \circ A_g)\xi = -2g\xi = \lambda$. Therefore, the equation $d\lambda = 0$ shows that $A_g \xi \in \mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]^{\perp B}$. In particular, $\mathfrak{z} \neq 0$. Since A_ω has maximal rank, we see that the image of ad_v is complementary to $\text{span}\{v, A_g \xi\}$ in \mathfrak{g} and of codimension one in the semisimple Lie algebra $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] \supset \text{im } \text{ad}_v$. This implies that the centralizer $Z_{\mathfrak{s}}(v)$ of v in \mathfrak{s} is one-dimensional.

This shows that the rank of \mathfrak{s} is one and $\dim \mathfrak{s} = 3$. Since the dimension of \mathfrak{g} is even, the inequality $1 \leq \dim \mathfrak{z} \leq 2$ implies that $\dim \mathfrak{z} = 1$. Therefore, $\mathfrak{g} = \mathfrak{u}(2)$, because \mathfrak{g} is compact.

We have proven in Sec. 4.1 that all lcK structures on $\mathfrak{g} = \mathfrak{u}(2)$ are of Vaisman type and, hence, satisfy $Z, \xi \in \ker d\phi$. This finishes the proof of Lemmas 4.13 and 4.12, and thus completes the proof of Proposition 4.11. \square

The following proposition finishes the proof of Theorem 4.10.

Proposition 4.14. *Let $(M = G/H, \omega, J)$ be a homogeneous proper lcK manifold of a reductive Lie group G such that $N_G(H)$ is compact and such that the canonical 1-form is given by $\phi = \frac{1}{2}\theta$. Then $(M = G/H, \omega, J)$ is of Vaisman type.*

Proof. Using the assertion $\xi \in \ker d\phi$ in Lemma 4.12, we have shown in (4.9) that $\mathcal{L}_\xi\omega = 0$. Similarly, $Z \in \ker d\phi$ implies

$$\mathcal{L}_Z\phi = \iota_Z d\phi = 0$$

and, hence,

$$\mathcal{L}_Z\omega = -\lambda \wedge \mathcal{L}_Z\phi + d\mathcal{L}_Z\phi = 0.$$

We claim that

$$\text{span}\{Z, \xi\} \cap \mathfrak{z} \neq 0. \tag{4.10}$$

Since Z, ξ and \mathfrak{z} are contained in the normalizer $\mathfrak{g}_0 = N_{\mathfrak{g}}(\mathfrak{h})$ of \mathfrak{h} in \mathfrak{g} , it is sufficient to prove this in the case $\mathfrak{g} = \mathfrak{u}(2)$, $\mathfrak{h} = 0$. Recall that any element $X \in \mathfrak{g}$ defines a Killing vector field X^* on $M = G/H$ and that any Ad_H -invariant element $X \in \mathfrak{m}$ extends as a G -invariant vector field \tilde{X} on $M = G/H$. If $X \in \mathfrak{z} \subset \mathfrak{m}$ then $\tilde{X} = X^*$, that is $\mathcal{L}_{\tilde{X}}g = 0$. If $0 \neq X \in \text{span}\{Z, \xi\} \cap \mathfrak{z}$, then $\mathcal{L}_Z\omega = \mathcal{L}_\xi\omega = 0$ imply $\mathcal{L}_X\omega = 0$ and, hence, $\mathcal{L}_{\tilde{X}}\omega = 0$. Combining these equations, we see that $\mathcal{L}_{\tilde{X}}J = 0$, which implies that the Reeb and the Lee vector fields are both holomorphic. Since the Lee field is a gradient vector field ($d\lambda = 0$) this shows that the Lee field is parallel. This proves the proposition. \square

Example. Note that the normalizer $N_G(H) = T^2 = S^1 \times S^1$ of $H = \text{SO}(2) \subset \text{SL}(2, \mathbb{R})$ in $T^2 \times \text{SL}(2, \mathbb{R})$ is compact. Therefore, Theorem 4.10 shows that every G -invariant lcK structure on $M = G/H = T^2 \times \text{SL}(2, \mathbb{R})/\text{SO}(2)$ is of Vaisman type. This should be contrasted with the fact that $S^1 \times \text{SL}(2, \mathbb{R})$ admits left-invariant non-Vaisman lcK structures by Theorem 4.9.

4.3. Left-invariant lcK structures on reductive Lie groups

In this section we specialize to the case of left-invariant lcK structures on Lie groups G . We will not assume that G is compact and will allow the pseudo-Kähler metric to be indefinite.

Theorem 4.15. *Let (G, ω, J) be a Lie group endowed with a left-invariant (proper) locally conformally pseudo-Kähler structure.*

- (i) *If $\mathfrak{g} = \text{Lie } G$ admits a bi-invariant (possibly indefinite) scalar product B with non-isotropic $B^{-1}\lambda$, then the dimension of the centralizer of v (as defined in Lemma 4.13) in \mathfrak{g} is at most 2.*
- (ii) *If \mathfrak{g} is reductive, then we have either $\mathfrak{g} = \mathfrak{u}(2)$ or $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$, and (ω, J) is one of the locally conformally pseudo-Kähler structures classified in Theorems 4.6 and 4.9. In both cases there exist locally conformally pseudo-Kähler structures that are not of Vaisman type and in the case $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ there even exist such structures that are not of Vaisman type with positive definite metric.*

Proof. We keep the same notation as in the proof of Lemma 4.13. We first note that since $B^{-1}\lambda$ is non-isotropic, \mathfrak{g} is splittable; and thus $[\omega] = 0$ in $H_\lambda^2(\mathfrak{g})$. The equation

$$\mathrm{ad}_v = -A_\omega - \lambda \otimes v - 2\phi \otimes A_g \xi$$

proven there (without using the compactness assumption of Lemma 4.13) shows that the rank of ad_v is at least $\mathrm{rk} \omega - 2 = \dim \mathfrak{g} - 2$. This implies that $Z_{\mathfrak{g}}(v)$ is at most two-dimensional. This proves (i). Now we prove (ii). If \mathfrak{g} is reductive the image of ad_v is necessarily a proper subspace of \mathfrak{s} . To see this it is sufficient to decompose v according to the decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$. This proves that the image of ad_v in \mathfrak{s} is a hyperplane and that $Z_{\mathfrak{s}}(v)$ is one-dimensional, since $0 \neq A_g \xi \in \mathfrak{z}$. Since the nilpotent part as well as the semisimple part of $\mathrm{ad}_v|_{\mathfrak{s}}$ belongs to $Z_{\mathfrak{s}}(v) \subset \mathfrak{s} \cong \mathrm{ad}(\mathfrak{s})$, it follows that $\mathrm{ad}_v|_{\mathfrak{s}}$ is either semisimple or nilpotent. It is clear that the dimension of the centralizer of a semisimple element in a semisimple Lie algebra \mathfrak{s} is bounded from below by the rank of \mathfrak{s} . The same is true for a nilpotent element. In fact, by a theorem of de Siebenthal, Dynkin and Kostant [2, Theorem 4.1.6], the dimension of the centralizer of a nilpotent element in a semisimple Lie algebra \mathfrak{s} is bounded from below by the rank of \mathfrak{s} [2]. This proves that $\mathrm{rk} \mathfrak{s} = 1$ and $\mathfrak{g} = \mathfrak{u}(2)$ or $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$, since $\dim \mathfrak{z} \leq 2$ and $\dim \mathfrak{g}$ is even. \square

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