Upper bounds on the smallest size of a complete cap in PG(N, q), $N \geq 3$, under a certain probabilistic conjecture*

Alexander A. Davydov

Institute for Information Transmission Problems (Kharkevich institute)

Russian Academy of Sciences, Bol'shoi Karetnyi per. 19, GSP-4, Moscow, 127994

Russian Federation. E-mail: adav@iitp.ru

Giorgio Faina, Stefano Marcugini and Fernanda Pambianco

Dipartimento di Matematica e Informatica, Università degli Studi di Perugia,

Via Vanvitelli 1, Perugia, 06123, Italy.

E-mail: {giorgio.faina,stefano.marcugini,fernanda.pambianco}@unipg.it

Abstract

In the projective space PG(N, q) over the Galois field of order q, $N \geq 3$, an iterative step-by-step construction of complete caps by adding a new point on every step is considered. It is proved that uncovered points are evenly placed on the space. A natural conjecture on an estimate of the number of new covered points on every step is done. For a part of the iterative process, this estimate is proved rigorously. Under the conjecture mentioned, new upper bounds on the smallest size $t_2(N, q)$ of a complete cap in PG(N, q) are obtained, in particular,

$$t_2(N, q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2 \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}, \quad N \geq 3.$$

A connection with the Birthday problem is noted. The effectiveness of the new bounds is illustrated by comparison with sizes of complete caps obtained by computer in wide regions of q.

*The research of A.A. Davydov was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project 14-50-00150). The research of G. Faina, S. Marcugini and F. Pambianco was supported in part by Ministry for Education, University and Research of Italy (MIUR) (Project "Geometrie di Galois e strutture di incidenza") and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM).

Mathematics Subject Classification (2010). Primary 51E21, 51E22; Secondary 94B05.

Keywords. Small complete caps, projective spaces, upper bounds on the smallest size of a complete cap, quasi-perfect codes

1 Introduction

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the Galois field $\mathbb{F}_q$ of order q. A k-cap in $\mathrm{PG}(N,q)$ is a set of k points no three of which are collinear. A k-cap $\mathcal{K}$ is complete if it is not contained in a $(k+1)$-cap or, equivalently, if every point of $\mathrm{PG}(N,q)\setminus\mathcal{K}$ is collinear with two points of $\mathcal{K}$. Caps in $\mathrm{PG}(2,q)$ are also called arcs and they have been widely studied by many authors in the past decades, see [4, 5, 7, 8, 20, 28, 30–33, 41] and the references therein. Let $\mathrm{AG}(N,q)$ be the N-dimensional affine space over $\mathbb{F}_q$. If $N>2$ only few constructions and bounds are known for small complete caps in $\mathrm{PG}(N,q)$ and $\mathrm{AG}(N,q)$, see [1–3, 6, 10–14, 20–32, 37, 38, 40, 41] for survey and results.

Caps have been intensively studied for their connection with Coding Theory [30,31,34]. A linear q-ary code with length n, dimension k, and minimum distance d is denoted by $[n,k,d]_q$. If a parity-check matrix of a linear q-ary code is obtained by taking as columns the homogeneous coordinates of the points of a cap in $\mathrm{PG}(N,q)$, then the code has minimum distance 4 (with the exceptions of the complete 5-cap in $\mathrm{PG}(3,2)$ and 11-cap in $\mathrm{PG}(4,3)$ giving rise to the $[5,1,5]_2$ and $[11,6,5]_3$ codes). Complete n-caps in $\mathrm{PG}(N,q)$ correspond to non-extendable $[n,n-N-1,4]_q$ quasi-perfect codes of covering radius 2 [17,19]. If $N=2$ these codes are Minimum Distance Separable (MDS); for $N=3$ they are Almost MDS since their Singleton defect is equal to 1. For fixed N, the covering density of the mentioned codes decreases with decreasing n. So, small complete caps have a better covering quality than the big ones.

Note also that caps are connected with quantum codes; see e.g. [15, 42].

In general, a central problem concerning caps is to determine the spectrum of the possible sizes of complete caps in a given space; see [30, 31] and the references therein. Of particular interest for applications to Coding Theory is the lower part of the spectrum as small complete caps correspond to quasi-perfect linear codes with small covering density.

Let $t_2(N,q)$ be the *smallest size* of a complete cap in $\mathrm{PG}(N,q)$.

A hard open problem in the study of projective spaces is the determination of $t_2(N,q)$. The exact values of $t_2(N,q)$, $N\geq 3$, are known only for very small q. For instance, $t_2(3,q)$ is known only for $q\leq 7$; see [20, Tab. 3].

This work is devoted to *upper bounds* on $t_2(N,q)$, $N\geq 3$.

The trivial lower bound for $t_2(N,q)$ is $\sqrt{2}q^{\frac{N-1}{2}}$. Constructions of complete caps whose size is close to this lower bound are known only for the following cases: $q=2$ and N arbitrary; $q=2^m>2$ and N odd; q is even square [14, 20, 21, 25, 27, 37, 40]. Using a modification of the approach of [33] for the projective plane, the probabilistic upper bound

$$t_2(N,q) < cq^{\frac{N-1}{2}} \log^{300} q,$$

where c is a constant independent of q, has been obtained in [13]. Computer assisted results on small complete caps in $\mathrm{PG}(N,q)$ and $\mathrm{AG}(N,q)$ are given in [6, 10–12, 20, 22, 24, 38].

The main result of the paper is given by Theorem 1.1 based on Theorem 4.5.

Theorem 1.1. (***the main result***) *Let $t_2(N,q)$ be the* smallest size *of a complete cap in the projective space* $\mathrm{PG}(N,q)$*. Let $D \geq 1$ be a constant independent of q.*

(i) *Under Conjecture* 3.3(i)*, in* $\mathrm{PG}(N,q)$*, it holds that*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{D}\sqrt{(N+1)\ln q}+1\right)+2 \sim \sqrt{D}q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}, \quad N \geq 3. \tag{1.1}$$

(ii) *Under Conjecture* 3.3(ii)*, in* $\mathrm{PG}(N,q)$*, the bound* (1.1) *with $D = 1$ holds, i.e.*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2 \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}, \quad N \geq 3. \tag{1.2}$$

Conjecture 1.2. *In* $\mathrm{PG}(N,q)$*, $N \geq 3$, the upper bound* (1.2) *holds for all q without any extra conditions and conjectures.*

This work can be treated as a development of the paper [4].

Some results of this work were briefly presented in [9].

The paper is organized as follows. In Section 2, we describe the iterative step-by-step process constructing caps. In Section 3, probabilities of events, that points of $\mathrm{PG}(N,q)$ are not covered by a running cap, are considered. It is proved that uncovered points are evenly placed on the space. A natural Conjecture 3.3 on an estimate of the number of new covered points on every step of the iterative process is done. In Section 4, under the conjecture of Section 3 we give new upper bounds on $t_2(N,q)$. In Section 5, we illustrate the effectiveness of the new bounds comparing them with the results of computer search from the papers [10, 11]. A rigorous proof of Conjecture 3.3 for a part of the iterative process is given in Section 6. In Section 7, the *reasonableness of Conjecture* 3.3 is discussed. It is shown that in the steps of the iterative process when the rigorous estimates give not good results, actually these estimates do not reflect the real situation effectively. The reason is that the rigorous estimates assume that the number of uncovered points on unisecants is the same for all unisecants. However, in fact, there is a dispersion of the number of uncovered points on unisecants, see Fig. 3. Moreover, this dispersion grows in the iterative process. In Conclusion, the obtained results are briefly discussed.

2 An iterative step-by-step process

Assume that in PG(N, q), $N \geq 3$, a complete cap is constructed by a step-by-step algorithm (*Algorithm* for short) which adds one new point to the cap in each step. As an example, we can mention the greedy algorithm that in every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points; see [7, 8, 20, 22].

Recall that a *point* of PG(N, q) is *covered by* a *cap* if the point lies on a bisecant of the cap, i.e. on a line meeting the cap in two points. Clearly, all points of the cap are covered.

The space PG(N, q) contains

$$\theta_{N,q} = \frac{q^{N+1} - 1}{q - 1} = q^N + q^{N-1} + \ldots + q + 1$$

points.

Assume that after the w-th step of Algorithm, a w-cap is obtained that does not cover exactly U_w points. Let $\mathbf{S}(U_w)$ be the set of all w-caps in PG(N, q) each of which does not cover exactly U_w points. Evidently, the group of collineations $P\Gamma L(N + 1, q)$ preserves $\mathbf{S}(U_w)$.

Consider the $(w + 1)$-st step of Algorithm. This step starts from a w-cap $\mathcal{K}_w$ with $\mathcal{K}_w \in \mathbf{S}(U_w)$. The choice $\mathcal{K}_w$ from $\mathbf{S}(U_w)$ can be done by distinct ways.

One way is to choose randomly a w-cap of $\mathbf{S}(U_w)$ so that for every cap of $\mathbf{S}(U_w)$ the probability to be chosen is equal to $\frac{1}{\#\mathbf{S}(U_w)}$. In this case, the set $\mathbf{S}(U_w)$ is considered as an *ensemble of random objects* with the uniform probability distribution. Anywhere where we say on probabilities and mathematical expectations, the such random choice is supposed.

On the other side, sometimes we study some values average or maximum by all caps of $\mathbf{S}(U_w)$ without a random choice. Also, we can consider some properties that hold for all caps of $\mathbf{S}(U_w)$.

Finally, for practice calculations (in particular, for the illustration of investigations) we use the same cap adding to it an one point in the each step of the iterative process.

Denote by $\mathcal{U}(\mathcal{K})$ the set of points of PG(N, q) that are not covered by a cap $\mathcal{K}$. By the definition,

$$\#\mathcal{U}(\mathcal{K}_w) = U_w.$$

Let the cap $\mathcal{K}_w$ consist of w points $A_1, A_2, \ldots, A_w$. Let $A_{w+1} \in \mathcal{U}(\mathcal{K}_w)$ be the point that will be included into the cap in the $(w + 1)$-st step.

Remark 2.1. Below we introduce a few point subsets, depending on A_{w+1}, for which we use the notation of the type $\mathcal{M}_w(A_{w+1})$. Any uncovered point may be added to $\mathcal{K}_w$. So, there exist U_w distinct subsets $\mathcal{M}_w(A_{w+1})$. When a particular point A_{w+1} is not relevant, one may use the short notation $\mathcal{M}_w$. The same concerns to quantities $\Delta_w(A_{w+1})$ and Δ_w introduced below.

A point A_{w+1} defines a bundle $\mathcal{B}(A_{w+1})$ of w unisecants to $\mathcal{K}_w$ which are denoted as $\overline{A_1A_{w+1}}, \overline{A_2A_{w+1}}, \dots, \overline{A_wA_{w+1}}$, where $\overline{A_iA_{w+1}}$ is the unisecant connecting A_{w+1} with the cap point A_i. Every unisecant contains $q+1$ points. Except for $A_1, \dots, A_w$, all the points on the unisecants in the bundle are **candidates** to be new covered points in the $(w+1)$-st step. Denote by $\mathcal{C}_w(A_{w+1})$ the point *set of the candidates*. By the definition,

$$\mathcal{C}_w(A_{w+1}) = \mathcal{B}(A_{w+1}) \setminus \mathcal{K}_w,$$
$$\#\mathcal{C}_w = w(q-1) + 1.$$

We call $\{A_{w+1}\}$ and $\mathcal{B}(A_{w+1}) \setminus (\mathcal{K}_w \cup \{A_{w+1}\})$, respectively, the *head* and the *basic part* of the bundle $\mathcal{B}(A_{w+1})$. For a given cap $\mathcal{K}_w$, in total, there are $\#\mathcal{U}(\mathcal{K}_w) = U_w$ distinct bundles and, respectively, U_w distinct sets of the candidates.

Let $\Delta_w(A_{w+1})$ be the number of **new covered points** in the $(w+1)$-st step, i.e.

$$\Delta_w(A_{w+1}) = \#\mathcal{U}(\mathcal{K}_w) - \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = \#\{\mathcal{C}_w(A_{w+1}) \cap \mathcal{U}(\mathcal{K}_w)\}. \quad (2.1)$$

In future, we consider continuous approximations of the discrete functions $\Delta_w(A_{w+1})$, $\#\mathcal{U}(\mathcal{K}_w)$, $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\})$, and some other ones keeping the same notations.

3 Probabilities of uncovering. Conjectures on the number of new covered points in every step

Let $n_w(H)$ be the number of caps of $\mathbf{S}(U_w)$ that do not cover a point H of $\mathrm{PG}(N,q)$. Each point $H \in PG(N,q)$ will be considered as a random object that is not covered by a randomly chosen w-cap $\mathcal{K}_w$ with some probability $p_w(H)$ defined as

$$p_w(H) = \frac{n_w(H)}{\#\mathbf{S}(U_w)}.$$

Lemma 3.1. *The value $n_w(H)$ is the same for all points $H \in \mathrm{PG}(N,q)$.*

Proof. Let $\mathbf{K}_w(H) \subseteq \mathbf{S}(U_w)$ be the subset of w-caps in $\mathbf{S}(U_w)$ that do not cover H. By the definition, $n_w(H) = \#\mathbf{K}_w(H)$. Let H_i and H_j be two distinct points of $\mathrm{PG}(N,q)$. In the group $P\Gamma L(N+1,q)$, denote by $\Psi(H_i, H_j)$ the subset of collineations taking H_i to H_j. Clearly, $\Psi(H_i, H_j)$ embeds the subset $\mathbf{K}_w(H_i)$ in $\mathbf{K}_w(H_j)$. Therefore, $\#\mathbf{K}_w(H_i) \leq \#\mathbf{K}_w(H_j)$. Vice versa, $\Psi(H_j, H_i)$ embeds $\mathbf{K}_w(H_j)$ into $\mathbf{K}_w(H_i)$, and we have $\#\mathbf{K}_w(H_j) \leq \#\mathbf{K}_w(H_i)$. Thus, $\#\mathbf{K}_w(H_i) = \#\mathbf{K}_w(H_j)$, i.e. $n_w(H_i) = n_w(H_j)$. □

So, $n_w(H)$ can be considered as n_w. This means that the *probability* $p_w(H)$ is *the same for all points* H; it may be considered as

$$p_w = \frac{n_w}{\#\mathbf{S}(U_w)}.$$

In turn, since the probability to be uncovered is independent of a point, we conclude that, for a w-cap $\mathcal{K}_w$ randomly chosen from $\mathbf{S}(U_w)$, the *fraction* $\#\mathcal{U}_w(\mathcal{K}_w)/\theta_{N,q}$ of uncovered points of $\mathrm{PG}(N,q)$ *is equal to the probability* p_w that a point of $\mathrm{PG}(N,q)$ is not covered. In other words,

$$p_w = \frac{\#\mathcal{U}_w(\mathcal{K}_w)}{\theta_{N,q}} = \frac{U_w}{\theta_{N,q}}. \tag{3.1}$$

Equality (3.1) can also be explained as follows. By Lemma 3.1, the multiset consisting of all points that are not covered by all caps of $\mathbf{S}(U_w)$ has cardinality $n_w \cdot \#PG(N,q)$, where $\#PG(N,q) = \theta_{N,q}$. This cardinality can also be written as $U_w \cdot \#\mathbf{S}(U_w)$. Thus, $n_w\theta_{N,q} = U_w \cdot \#\mathbf{S}(U_w)$, whence

$$\frac{n_w}{\#\mathbf{S}(U_w)} = \frac{U_w}{\theta_{N,q}}.$$

Let $s_w(h)$ be the number of ones in a sequence of h random and independent 1/0 trials each of which yields 1 with the probability p_w. For the random variable $s_w(h)$ we have the *binomial* probability *distribution*; the *expected value* of $s_w(h)$ is

$$\mathbf{E}[s_w(h)] = hp_w = h\frac{U_w}{\theta_{N,q}}. \tag{3.2}$$

Remark 3.2. One can consider also the *hypergeometric* probability *distribution*, which describes the probability of $s'_w(h)$ successes in h random and independent draws without replacement from a finite population of size $\theta_{N,q}$ containing exactly U_w successes. The *expected value* of $s'_w(h)$ again is

$$\mathbf{E}[s'_w(h)] = h\frac{U_w}{\theta_{N,q}} = \mathbf{E}[s_w(h)].$$

Note also that the *average number* of uncovered points among h points of $\mathrm{PG}(N,q)$ calculated over all $\binom{\theta_{N,q}}{h}$ combinations of h points is

$$\frac{1}{\binom{\theta_{N,q}}{h}}\sum_{i=1}^{h} i\binom{\theta_{N,q}-U_w}{h-i}\binom{U_w}{i} = \frac{U_w}{\binom{\theta_{N,q}}{h}}\sum_{i=1}^{h}\binom{\theta_{N,q}-U_w}{h-i}\binom{U_w-1}{i-1} = \frac{U_w\binom{\theta_{N,q}-1}{h-1}}{\binom{\theta_{N,q}}{h}}$$
$$= h\frac{U_w}{\theta_{N,q}} = \mathbf{E}[s_w(h)].$$

Denote by $\mathbf{E}_{w,q}$ the **expected value** of the number of uncovered points among $w(q-1)+1$ *randomly* taken points in $\mathrm{PG}(N,q)$, if the events to be uncovered are *independent*. By Lemma 3.1, taking into account (3.1), (3.2), we have

$$\mathbf{E}_{w,q} = \mathbf{E}[s_w(w(q-1)+1)] = (w(q-1)+1)p_w = \frac{(w(q-1)+1)U_w}{\theta_{N,q}}. \qquad (3.3)$$

In (2.1), we defined $\Delta_w(A_{w+1})$ as the number of new covered points on the $(w+1)$-st step. Since all candidates to be new covered points lie on some bundle, they cannot be considered as randomly taken points for which the events to be uncovered are independent. So, in the general case, the expected value $\mathbf{E}[\Delta_w]$ is not equal to $\mathbf{E}_{w,q}$.

On the other side, there is a large number of random factors affecting the process, for instance, the relative positions and intersections of bisecants and unisecants. These factors especially act for growing q, when the volume of the ensemble $\mathbf{S}(U_w)$ and the number of distinct bundles $\mathcal{B}(A_{w+1})$ are relatively large. Therefore, the variance of the random variable Δ_w, in principle, implies the existence of bundles $\mathcal{B}(A_{w+1})$ providing the inequality $\Delta_w(A_{w+1}) > \mathbf{E}[\Delta_w]$. By these arguments (see also Section 7), Conjecture 3.3 seems to be reasonable and founded.

Conjecture 3.3. **(i)** (***the generalized conjecture***) *In* $\mathrm{PG}(N,q)$, *for q large enough, in every $(w+1)$-st step of the iterative process, considered in Section 2, there exists a w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$ such that one can find an uncovered point A_{w+1} providing the inequality*

$$\Delta_w(A_{w+1}) \geq \frac{\mathbf{E}_{w,q}}{D} = \frac{1}{D}\cdot\frac{(w(q-1)+1)U_w}{\theta_{N,q}}, \qquad (3.4)$$

where $D \geq 1$ is a constant independent of q.

(ii) (***the basic conjecture***) *In* (3.4) *we have $D = 1$.*

4 Upper bounds on $t_2(N,q)$

We denote

$$Q = \frac{\theta_{N,q}}{q-1} = \frac{q^{N+1}-1}{(q-1)^2}. \qquad (4.1)$$

By Conjecture 3.3, taking into account (2.1), (3.3), (3.4), we obtain

$$\begin{aligned}\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) &= \#\mathcal{U}(\mathcal{K}_w) - \Delta_w(A_{w+1}) \qquad (4.2)\\ &\leq U_w\left(1 - \frac{w(q-1)+1}{D\theta_{N,q}}\right) < U_w\left(1 - \frac{w(q-1)}{D\theta_{N,q}}\right) < U_w\left(1 - \frac{w}{DQ}\right).\end{aligned}$$

Clearly, $\#\mathcal{U}(\mathcal{K}_1) = U_1 = \theta_{N,q} - 1$. Using (4.2) iteratively, we have

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq (\theta_{N,q}-1)f_q(w;D) < \theta_{N,q}f_q(w;D) \qquad (4.3)$$

where

$$f_q(w;D) = \prod_{i=1}^{w}\left(1 - \frac{i}{DQ}\right). \qquad (4.4)$$

Remark 4.1. The function $f_q(w;D)$ and its approximations, including (4.8), appear in distinct tasks of Probability Theory, e.g. in the *Birthday problem* (or the Birthday paradox) [16,18,39]. Really, let the year contain DQ days and let all birthdays occur with the same probability. Then $P^{\neq}_{DQ}(w+1) = f_q(w;D)$ where $P^{\neq}_{DQ}(w+1)$ is the probability that no two persons from $w+1$ random persons have the same birthday. Moreover, if birthdays occur with different probabilities we have $P^{\neq}_{DQ}(w+1) < f_q(w;D)$ [18].

In further, we consider a *truncated iterative process*. The iterative process ends when $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi$ where $\xi \geq 1$ is some value chosen to improve estimates. Then a few (at most ξ) points are added to $\mathcal{K}_w$ in order to get a complete k-cap. The size k of an obtained complete cap is as follows:

$$w+1 \leq k \leq w+1+\xi \text{ under condition } \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi. \qquad (4.5)$$

Theorem 4.2. *Let $f_q(w;D)$ be as in* (4.4)*. Let ξ be a constant independent of w with $\xi \geq 1$. Under Conjecture* 3.3*, in* $\mathrm{PG}(N,q)$ *it holds that*

$$t_2(N,q) \leq w+1+\xi \qquad (4.6)$$

where the value w satisfies the inequality

$$f_q(w;D) \leq \frac{\xi}{\theta_{N,q}}. \qquad (4.7)$$

Proof. By (4.3), to provide the inequality $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi$ it is sufficient to find w such that $\theta_{N,q} f_q(w;D) \leq \xi$. Now (4.6) follows from (4.5). □

We find an upper bound on the smallest possible solution of inequality (4.7).

The Taylor series of $e^{-\alpha}$ implies $1-\alpha < e^{-\alpha}$ for $\alpha \neq 0$, whence

$$\prod_{i=1}^{w}\left(1 - \frac{i}{DQ}\right) < \prod_{i=1}^{w} e^{-i/DQ} = e^{-(w^2+w)/2DQ} < e^{-w^2/2DQ}. \qquad (4.8)$$

Lemma 4.3. *Let ξ be a constant independent of w with $\xi \geq 1$. The value*

$$w \geq \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + 1 \qquad (4.9)$$

satisfies the inequality (4.7)*.*

Proof. By (4.4),(4.8), to provide (4.7) it is sufficient to find w such that

$$e^{-w^2/2DQ} \leq \frac{\xi}{\theta_{N,q}}.$$

As w should be an integer, in (4.9) one is added. $\square$

Theorem 4.4. *Let $D \geq 1$ be a constant independent of q. Under Conjecture* 3.3(i)*, in $PG(N,q)$ it holds that*

$$t_2(N,q) \leq \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + \xi + 2, \quad \xi \geq 1, \tag{4.10}$$

where ξ is an arbitrarily chosen constant independent of w.

Proof. The assertion follows from (4.6) and (4.9). $\square$

We should choose ξ so to obtain a relatively small value in the right part of (4.10). We consider the function of ξ of the form

$$\phi(\xi) = \sqrt{2DQ}\sqrt{\ln\frac{\theta_{N,q}}{\xi}} + \xi + 2.$$

Its derivative by ξ is

$$\phi'(\xi) = 1 - \frac{1}{\xi}\sqrt{\frac{DQ}{2\ln\frac{\theta_{N,q}}{\xi}}}\ .$$

Put $\phi'(\xi) = 0$. Then

$$\xi^2 = \frac{DQ}{2\ln\theta_{N,q} - 2\ln\xi} = \frac{D\theta_{N,q}}{2(q-1)(\ln\theta_{N,q} - \ln\xi)}\ . \tag{4.11}$$

We find ξ in the form $\xi = \sqrt{\frac{\theta_{N,q}}{c\ln\theta_{N,q}}}$. By (4.11),

$$c = \frac{q-1}{D\ln\theta_{N,q}}(\ln\theta_{N,q} + \ln c + \ln\ln\theta_{N,q}) = \frac{q-1}{D}\left(1 + \frac{\ln c + \ln\ln\theta_{N,q}}{\ln\theta_{N,q}}\right).$$

So, for growing q one could take

$$c = \frac{q-1}{D}, \quad \xi = \sqrt{\frac{D\theta_{N,q}}{(q-1)\ln\theta_{N,q}}} = \sqrt{\frac{D(q^{N+1}-1)}{(q-1)^2\ln\theta_{N,q}}}\ .$$

For simplicity of the presentation, we put

$$\xi = \frac{\sqrt{q^{N+1}}}{q-1}\,. \tag{4.12}$$

Theorem 4.5. *Let $D \geq 1$ be a constant independent of q. Under Conjecture* 3.3(i)*, the following upper bound on the smallest size $t_2(N,q)$ of a complete cap in* $\mathrm{PG}(N,q)$*, $N \geq 3$, holds:*

$$t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{D}\sqrt{(N+1)\ln q}+1\right)+2 \sim \sqrt{D}q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}. \tag{4.13}$$

Proof. In (4.10), we take Q and ξ from (4.1) and (4.12) and obtain

$$t_2(N,q) < \sqrt{2D\frac{q^{N+1}-1}{(q-1)^2}\cdot \ln\frac{\frac{q^{N+1}-1}{q-1}}{\frac{q^{\frac{N+1}{2}}}{q-1}}}+\frac{\sqrt{q^{N+1}}}{q-1}+2$$

whence the relation (4.13) follows directly as $q^{N+1}-1 < q^{N+1}$. □

From Theorem 4.5 we obtain Theorem 1.1.

5 Illustration of the effectiveness of the new bounds

In the works [10, 11], for $\mathrm{PG}(N,q)$, $N = 3,4$, $q \in L_N$, complete caps are obtained by computer search. Here

$$L_3 := \{q \leq 4673,\ q \text{ prime}\} \cup \{5003, 6007, 7001, 8009\},$$
$$L_4 := \{q \leq 1361,\ q \text{ prime}\} \cup \{1409\}.$$

All obtained complete caps ***satisfy*** *bound* (4.13) *with* D = 1 (equivalently, *bound* (1.2)).

Let $\overline{t}_2(N,q)$ be the smallest known size of complete caps in $\mathrm{PG}(N,q)$; these sizes can be found in [10].

In Fig. 1 we compare the upper bound of (1.2) with the sizes $\overline{t}_2(N,q)$. The top dashed-dotted red curve, corresponding to the bound of (1.2), is ***strictly higher*** than the bottom black curve $\overline{t}_2(N,q)$.

6 A rigorous proof of Conjecture 3.3 for a part of the iterative process

In further, we take into account that *all points that are not covered* by a cap *lie on unisecants* to the cap.

In total there are $\theta_{N-1,q}$ lines through every point of $\mathrm{PG}(N,q)$. Therefore, through every point A_i of $\mathcal{K}_w$ there is a pencil $\mathcal{P}(A_i)$ of $\theta_{N-1,q}-(w-1)$ unisecants to $\mathcal{K}_w$, where $i = 1, 2, \ldots, w$. The total number T_w^Σ of the unisecants to $\mathcal{K}_w$ is

$$T_w^\Sigma = w(\theta_{N-1,q}+1-w). \tag{6.1}$$

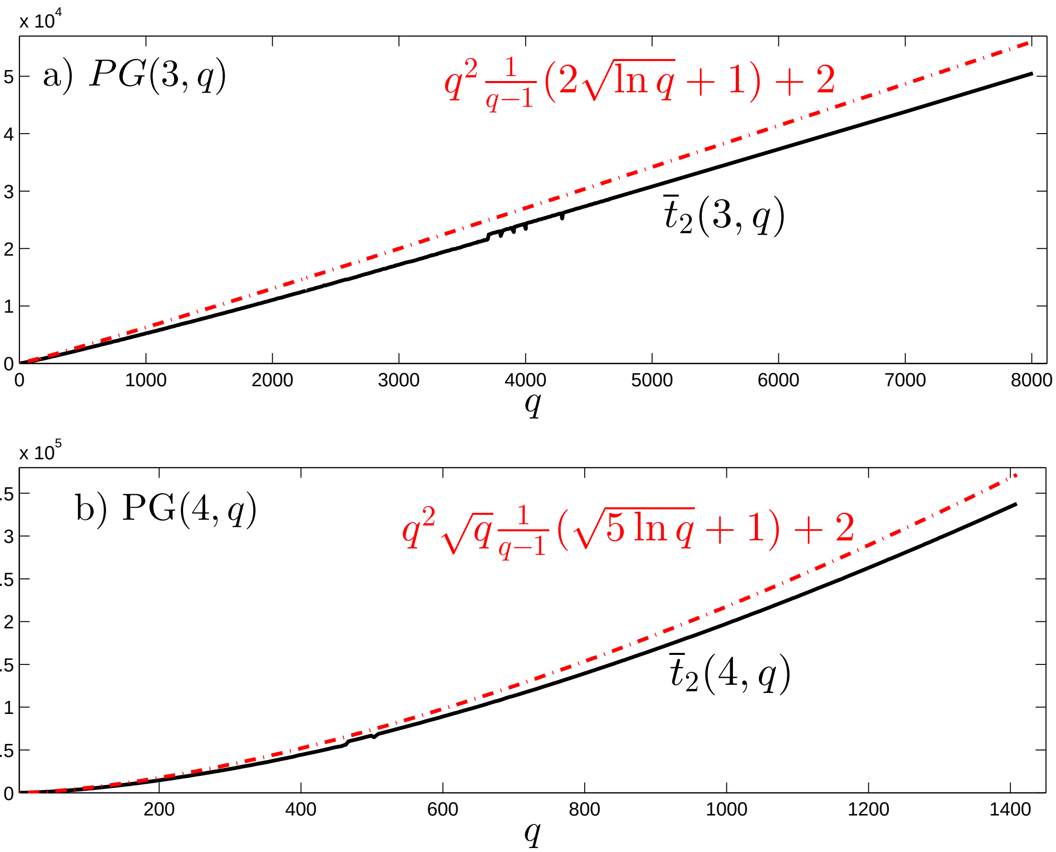

Figure 1: **Bound** $t_2(N,q) < \frac{\sqrt{q^{N+1}}}{q-1}\left(\sqrt{(N+1)\ln q}+1\right)+2$ (*top dashed-dotted red curve*) **vs the smallest known sizes** $\overline{t}_2(N,q)$ of complete caps , $q \in L_N$, $N = 3,4$ (*bottom black curve*). a) PG$(3,q)$ b) PG$(4,q)$

Let $\gamma_{w,j}$ be the number of uncovered points on the j-th unisecant $\mathcal{T}_j$, $j = 1, 2, \ldots, T_w^\Sigma$.

Observation 6.1. *Every unisecant to $\mathcal{K}_w$ belongs to one and only one pencil $\mathcal{P}(A_i)$, $i \in \{1, 2, \ldots, w\}$. Every uncovered point belongs to one and only one unisecant from every pencil $\mathcal{P}(A_i)$, $i = 1, 2, \ldots, w$. Every uncovered point A lies on exactly w unisecants which form the bundle $\mathcal{B}(A)$ with the head $\{A\}$. All unisecants from the same bundle belong to distinct pencils. A unisecant $\mathcal{T}_j$ belongs to $\gamma_{w,j}$ distinct bundles.*

Every uncovered point lies on exactly w unisecants; due to this *multiplicity*, on all unisecants there are in total Γ_w^Σ uncovered points, where

$$\Gamma_w^\Sigma = \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w. \tag{6.2}$$

By (6.1), (6.2), the average number γ_w^{aver} of uncovered points on a unisecant is

$$\gamma_w^{\text{aver}} = \frac{\Gamma_w^\Sigma}{T_w^\Sigma} = \frac{U_w}{\theta_{N-1,q} + 1 - w}. \tag{6.3}$$

A *unisecant $\mathcal{T}_j$ belongs to $\gamma_{w,j}$ distinct bundles*, as every uncovered point on $\mathcal{T}_j$ may be the head of a bundle. Moreover, $\mathcal{T}_j$ provides $\gamma_{w,j}(\gamma_{w,j} - 1)$ uncovered points to the basic parts of all these bundles. The noted points are counted with *multiplicity*.

Taking into account the multiplicity, in all U_w the bundles there are

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}(\gamma_{w,j} - 1) \tag{6.4}$$

uncovered points, where U_w is the total numbers of all the heads. By (6.2), (6.4),

$$\sum_{A_{w+1}} \Delta_w(A_{w+1}) = U_w + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 - \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = U_w(1 - w) + \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2.$$

For a cap $\mathcal{K}_w$, we denote by $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ the average value of $\Delta_w(A_{w+1})$ by all $\#\mathcal{U}(\mathcal{K}_w)$ uncovered points A_{w+1}, i.e.

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{\#\mathcal{U}(\mathcal{K}_w)} = \frac{\sum_{A_{w+1}} \Delta_w(A_{w+1})}{U_w} = \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w} - w + 1 \geq 1 \tag{6.5}$$

where the inequality is obvious by sense; also note that

$$\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \geq \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j} = wU_w. \tag{6.6}$$

We denote a lower estimate of $\Delta_w^{\text{aver}}(\mathcal{K}_w)$, see Lemma 6.2 below, as follows:

$$\Delta_w^{\text{rigor}}(\mathcal{K}_w) := \max\left\{1, \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1\right\} = \tag{6.7}$$
$$= \begin{cases} \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 & \text{if} \quad U_w \geq \theta_{N-1,q}+1-w, \\ 1 & \text{if} \quad U_w < \theta_{N-1,q}+1-w. \end{cases}$$

Lemma 6.2. *For any w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$, the following holds:*

- *This inequality always fulfills*
$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w). \tag{6.8}$$
- *In* (6.8)*, we have the equality*
$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 \tag{6.9}$$
if and only if every unisecant contains the same number $\frac{U_w}{\theta_{N-1,q}+1-w}$ of uncovered points where $\frac{U_w}{\theta_{N-1,q}+1-w}$ is integer.
- *In* (6.8)*, the equality*
$$\Delta_w^{\text{aver}}(\mathcal{K}_w) = \Delta_w^{\text{rigor}}(\mathcal{K}_w) = 1 \tag{6.10}$$
holds if and only if each unisecant contains at most an one uncovered point.

Proof. By Cauchy–Schwarz–Bunyakovsky inequality, it holds that

$$\left(\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}\right)^2 \leq T_w^\Sigma \sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2 \tag{6.11}$$

where equality holds if and only if all $\gamma_{w,j}$ coincide. In this case $\gamma_{w,j} = \frac{U_w}{\theta_{N-1,q}+1-w}$ for all j and, moreover, the ratio $\frac{U_w}{\theta_{N-1,q}+1-w}$ is integer. Now, by (6.1), (6.2), we have

$$\frac{wU_w}{\theta_{N-1,q}+1-w} \leq \frac{\sum_{j=1}^{T_w^\Sigma} \gamma_{w,j}^2}{U_w}$$

that together with (6.2), (6.5), (6.6), (6.7) gives (6.8)–(6.10). □

Remark 6.3. One can treat the estimate (6.8), (6.9) as follows. A bundle contains w unisecants having a common point, its head. Therefore the average number of uncovered points in a bundle is $w\gamma_w^{\text{aver}} - (w-1)$ where γ_w^{aver} is defined in (6.3) and the term $w-1$ takes into account the common point.

It is clear that for any w-cap $\mathcal{K}_w \in \mathbf{S}(U_w)$ we have

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) \geq \lceil \Delta_w^{\text{aver}}(\mathcal{K}_w) \rceil . \qquad (6.12)$$

Corollary 6.4. *It hold that*

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) \geq \max\left\{1, \left\lceil \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 \right\rceil\right\}.$$

Remark 6.5. The results and approaches, connected with estimates of line-point incidences (see e.g. [35, 36] and the references therein) could be useful for estimates and bounds considered in this paper.

Let $D \geq 1$ be a constant independent of q. Throughout the paper we denote

$$\Phi_{w,q}(D) = \frac{D(w-1)\theta_{N,q}(\theta_{N-1,q}+1-w)}{Dw\theta_{N,q} - (\theta_{N-1,q}+1-w)(w(q-1)+1)},$$
$$\Upsilon_{w,q}(D) = \frac{D\theta_{N,q}}{w(q-1)+1}.$$

Lemma 6.6. *Let $D \geq 1$ be a constant independent of q. Let an one of the following two conditions hold:*

$$U_w \geq \Phi_{w,q}(D), \quad \Upsilon_{w,q}(D) \geq U_w.$$

Then, for any cap $\mathcal{K}_w$ of $\mathbf{S}(U_w)$, it holds that

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \frac{\mathbf{E}_{w,q}}{D}.$$

Proof. By (6.7), (6.8), we have

$$\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w) \geq \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1.$$

It is easy to see that under condition $U_w \geq \Phi_{w,q}(D)$ it holds that

$$\frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 - \frac{(w(q-1)+1)U_w}{D\theta_{N,q}} \geq 0.$$

If $U_w \leq \Upsilon_{w,q}(D)$ then $\frac{\mathbf{E}_{w,q}}{D} \leq 1$. On the other side, by (6.7), (6.8), we always have $\Delta_w^{\text{aver}}(\mathcal{K}_w) \geq \Delta_w^{\text{rigor}}(\mathcal{K}_w) \geq 1$. ∎

From Lemmas 6.2 and 6.6 we obtain the corollary.

Corollary 6.7. *Let $D \geq 1$ be a constant independent of q. Let an one of the following two conditions hold:*

$$U_w \geq \Phi_{w,q}(D), \quad \Upsilon_{w,q}(D) \geq U_w.$$

Then, for any cap $\mathcal{K}_w$ of $\mathbf{S}(U_w)$, there exists an uncovered point A_{w+1} providing the inequality

$$\Delta_w(A_{w+1}) \geq \frac{\mathbf{E}_{w,q}}{D} = \frac{(w(q-1)+1)U_w}{D\theta_{N,q}}.$$

Proof. By the definition of the average value (6.5), always there is an uncovered point A_{w+1} providing the inequality $\Delta_w(A_{w+1}) \geq \Delta_w^{\text{aver}}(\mathcal{K}_w)$, see also (6.12). □

7 On reasonableness of Conjecture 3.3

In this section we show (by reflections, calculations and figures) that in the steps of the iterative process when the rigorous estimates give not good results, actually these estimates do not reflect the real situation effectively.

• In the first we will illustrate the following: when the rigorous bound (6.7)–(6.8) is smaller than the expectation $\mathbf{E}_{w,q}$, in fact, the average value $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ of (6.5) is greater (and the maximum value $\max\limits_{A_{w+1}} \Delta_w(A_{w+1})$ is essentially greater) than $\mathbf{E}_{w,q}$, see Fig. 2.

We have calculated the values $\Delta_w(A_{w+1})$, defined in (2.1), for numerous concrete iterative processes in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$. It is important that *for all the calculations have been done*, it holds that

$$\max_{A_{w+1}} \Delta_w(A_{w+1}) > \mathbf{E}_{w,q}.$$

Moreover, the ratio $\max\limits_{A_{w+1}} \Delta_w(A_{w+1})/\mathbf{E}_{w,q}$ has the increasing trend when w grows. Thus, the variance of the random value Δ_w helps to get good results.

The existence of points A_{w+1} providing $\Delta_w(A_{w+1}) > \mathbf{E}_{w,q}$ is used by the greedy algorithms to obtain complete caps smaller than the bounds following from Conjecture 3.3.

An illustration of the aforesaid is shown on Fig. 2 where for complete k-caps in $\mathrm{PG}(3,101)$, $k = 415$, and in $\mathrm{PG}(4,31)$, $k = 706$, obtained by the greedy algorithm, the values

$$\delta_w^{\min} = \frac{\min\limits_{A_{w+1}} \Delta_w(A_{w+1})}{\mathbf{E}_{w,q}}, \quad \delta_w^{\max} = \frac{\max\limits_{A_{w+1}} \Delta_w(A_{w+1})}{\mathbf{E}_{w,q}},$$
$$\delta_w^{\text{aver}} = \frac{\Delta_w^{\text{aver}}(\mathcal{K}_w)}{\mathbf{E}_{w,q}}, \quad \delta_w^{\text{rigor}} = \frac{\Delta_w^{\text{rigor}}(\mathcal{K}_w)}{\mathbf{E}_{w,q}},$$

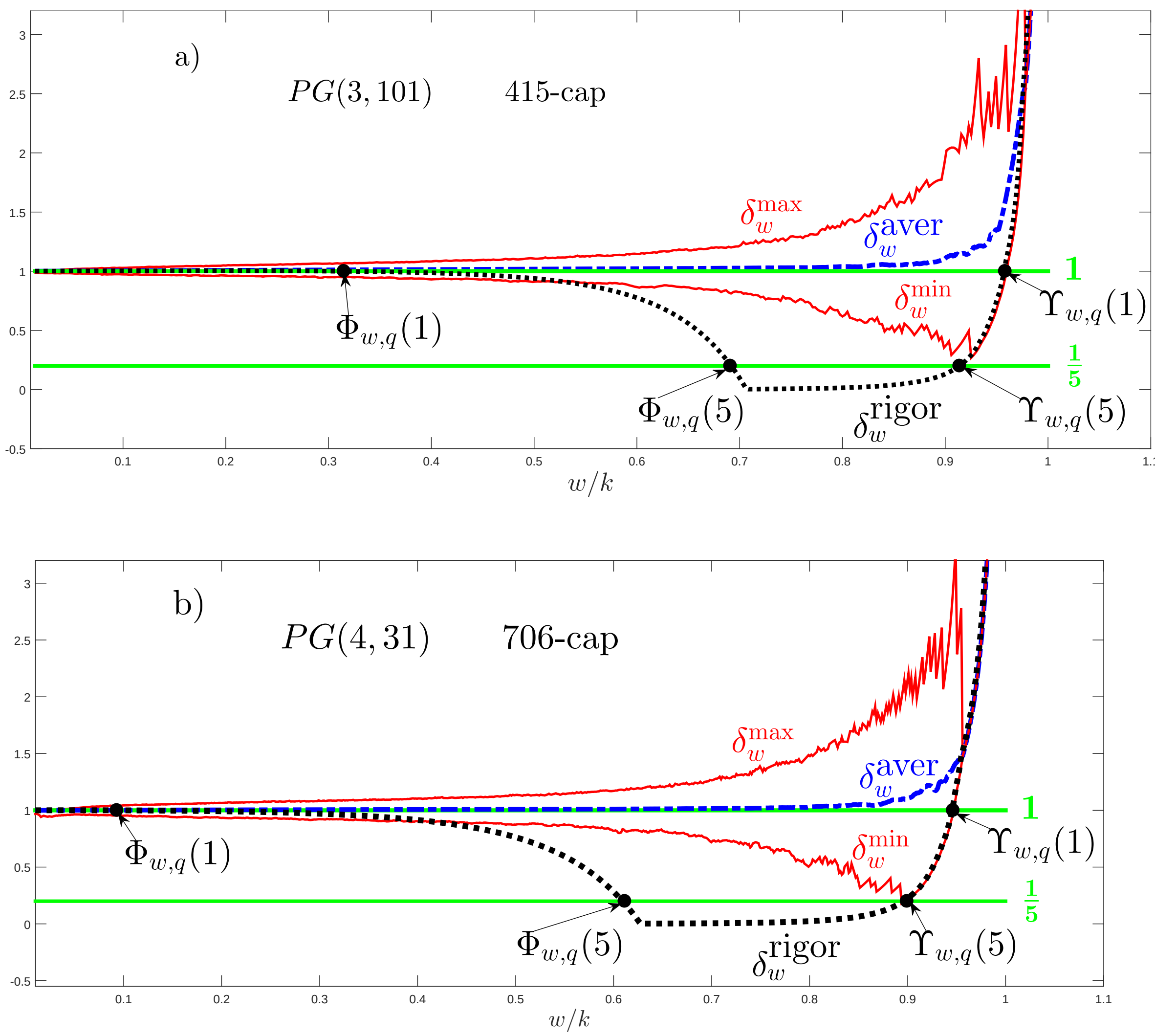

Figure 2: **Illustration of reasonableness of Conjecture 3.3.** Values $\delta_w^\bullet$ for a complete k-cap in PG(N,q). a) $N=3$, $q=101$, $k=415$; b) $N=4$, $q=31$, $k=706$: $\delta_w^{\max}$ (*top solid red curve*), δ_w^{aver} (*the* 2-*nd dashed-dotted blue curve*), $\delta_w^{\min}$ (*the* 3-*rd solid red curve*), δ_w^{rigor} (*bottom dotted black curve*), *green lines* $y=1$ (for $D=1$) and $y=\frac{1}{5}$ (for $D=5$). The region where Conjecture 3.3 is rigorously proved lies on the left of $\Phi_{w,q}(D)$ and on the right of $\Upsilon_{w,q}(D)$

are presented. The horizontal axis shows the values of $\frac{w}{k}$. The final region of the iterative process when $U_w \leq \Upsilon_{w,q}(D)$ and $\frac{\mathbf{E}_{w,q}}{D} \leq 1$ is shown not completely. The green lines $y = 1$ and $y = \frac{1}{5}$ correspond, respectively, to Conjecture 3.3(ii), where $D = 1$, and Conjecture 3.3(i) with $D = 5$. The signs • correspond to the values $\Phi_{w,q}(D)$ and $\Upsilon_{w,q}(D)$ with $D = 1$ and $D = 5$. It is interesting (and expected) that, for all the steps of the iterative process, we have $\Delta_w^{\text{aver}}(\mathcal{K}_w) > \mathbf{E}_{w,q}$, i.e. $\delta_w^{\text{aver}} > 1$.

In Fig. 2, the region where we rigorously prove Conjecture 3.3 lies on the left of $\Phi_{w,q}(D)$ and on the right of $\Upsilon_{w,q}(D)$. This region in $\mathrm{PG}(3, 101)$ takes $\sim 35\%$ of the whole iterative process for $D = 1$ and $\sim 75\%$ for $D = 5$.

Note that the forms of curves $\delta_w^{\max}$ and δ_w^{aver} are similar for all q's and N's for which we calculated these values.

• Now we consider *the dispersion of the number of uncovered points on unisecants.*

The lower estimate in (6.8) based on (6.11) is attained in two cases: either *every unisecant contains the same number of uncovered points* or *each unisecant contains at most an one uncovered point.*

The 1-st situation holds in the first steps of the iterative process only. Then the differences $\gamma_{w,j} - \gamma_{w,i}$ become nonzero. But, while the inequality $U_w(D) \geq \Phi_{w,q}(D)$ holds, these differences are relatively small and estimate (6.8) works "well". When U_w decreases, the differences relatively increase, and the estimate becomes worse in the sense that actually $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\Delta_w^{\text{rigor}}(\mathcal{K}_w)$.

The 2-nd situation is possible, in principle, when $U_w \leq \theta_{N-1,q} + 1 - w$ and the average number γ_w^{aver} of uncovered points on an unisecant is smaller than one, see (6.3). But on this stage of the iterative process variations in the values $\gamma_{w,j}$ are relatively big; and again $\Delta_w^{\text{aver}}(\mathcal{K}_w)$ is considerably greater than $\Delta_w^{\text{rigor}}(\mathcal{K}_w)$.

In the final region of the iterative process, where $U_w \leq \Upsilon_{w,q}(D)$ and $\frac{\mathbf{E}_{w,q}}{D} \leq 1$, estimate (6.8) becomes reasonable once more. Thus, in the region

$$\Phi_{w,q}(D) > U_w > \Upsilon_{w,q}(D)$$

the lower estimate (6.8) does not reflect the real situation effectively. This leads the necessity to formulate Conjecture 3.3 as a (plausible) hypothesis.

Let γ_w^{aver} be defined in (6.3). Let $\gamma_w^{\max}$ and $\gamma_w^{\min}$ be, respectively, the maximum and minimum of the number $\gamma_{w,j}$ of uncovered points on an unisecant, i.e.

$$\gamma_w^{\max} = \max_j \gamma_{w,j}, \quad \gamma_w^{\min} = \min_j \gamma_{w,j}.$$

An illustration of the fact that the numbers $\gamma_{w,j}$ of uncovered points on unisecants lie in a relatively wide region is shown on Fig. 3, where for complete k-caps in $\mathrm{PG}(3, 101)$, $k = 415$, and in $\mathrm{PG}(4, 31)$, $k = 706$, obtained by the greedy algorithm, the values $\gamma_w^{\max}/\gamma_w^{aver}$ and $\gamma_w^{\min}/\gamma_w^{aver}$ are presented. The horizontal axis shows the values of $\frac{w}{k}$. The such curves were obtained for numerous concrete iterative processes in $\mathrm{PG}(3, q)$ and $\mathrm{PG}(4, q)$. It is important that *for all the calculations have been done*, the forms of the curves are similar. Moreover, the value $\gamma_w^{\max}/\gamma_w^{aver}$ increases when the ratio $\frac{w}{k}$ grows; in the region $0.78 < \frac{w}{k} < 0.95$ (it is not shown in Fig. 3); the value $\gamma_w^{\max}/\gamma_w^{aver}$ increases from 20 to 590 for the 415-cap in PG(3, 101) and from 36 to 1400 for the 706-cap in PG(4, 31).

Remark 7.1. It can be proved rigorously (using Observation 6.1) that if in some step of the iterative process every unisecant contains the same number of uncovered points then in the next step this situation does not hold.

The calculations mentioned in this section and Figs. 2, 3 illustrate the soundness of the key Conjecture 3.3.

8 Conclusion

In the present paper, we make an attempt to obtain a theoretical upper bound on $t_2(N, q)$ with the main term of the form $cq^{\frac{N-1}{2}}\sqrt{\ln q}$, where c is a small constant independent of q. The bound is based on explaining the mechanism of a step-by-step greedy algorithm for constructing complete caps in PG(N, q) and on quantitative estimations of the algorithm. For a part of steps of the iterative process, these estimations are proved rigorously. We make a natural (and wellfounded) conjecture that they hold for other steps too. Under this conjecture we give new upper bounds on $t_2(N, q)$ in the needed form, see (1.1), (1.2). We illustrate the effectiveness of the new bounds comparing them with the results of computer search from the papers [10, 11], see Fig. 1.

We did not obtain a rigorous proof for precisely the part of the process where the variance of the random variable $\Delta_w(A_{w+1})$ determining the estimates implies the existence of points A_{w+1} which are considerably better than what is necessary for fulfillment of the conjecture (see the curve $\delta_w^{\max}$ in Fig. 2). In other words, in the steps of the iterative process when the rigorous estimates give not well results, in fact, these estimates do not reflect the real situation effectively. The reason is that the rigorous estimates assume that the number of uncovered points on unisecants is the same for all unisecants. However, in fact, there is a dispersion of the number of uncovered points on unisecants, see Section 7. Moreover, this dispersion grows in the iterative process. So, Conjecture 3.3 seems to be reasonable.

References

[1] N. Anbar, D. Bartoli, M. Giulietti, I. Platoni, Small complete caps from singular cubics. *J. Combin. Des.* **22**, 409–424 (2014)

[2] N. Anbar, D. Bartoli, M. Giulietti, I. Platoni, Small complete caps from singular cubics, II. *J. Algebraic Combin.* **41**, 185–216 (2015)

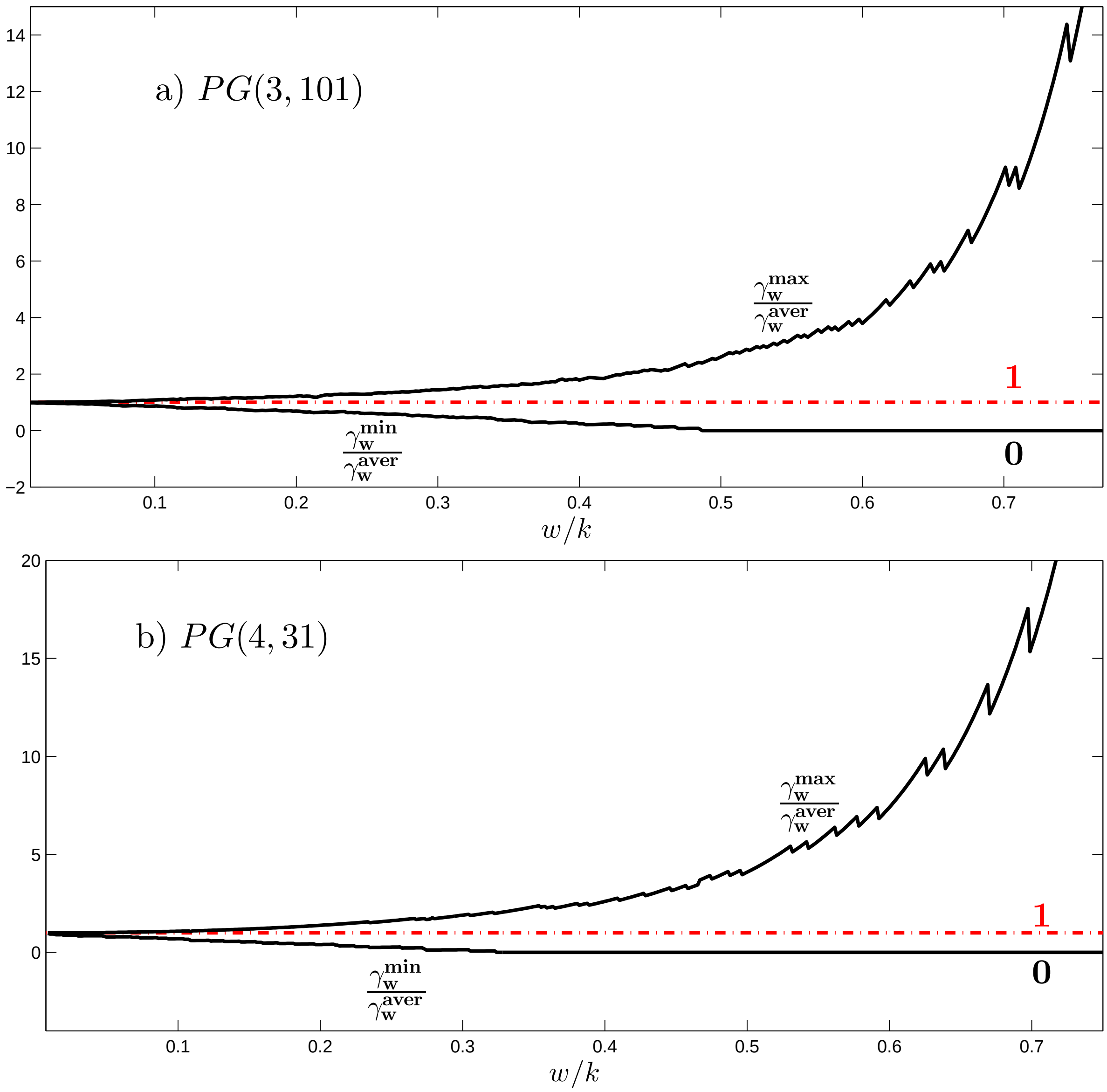

Figure 3: **Dispersion of the number $\gamma_{w,j}$ of uncovered points on unisecants.** Values $\gamma_w^{\max}/\gamma_w^{aver}$ (*top solid black curve*) and $\gamma_w^{\min}/\gamma_w^{aver}$ (*bottom solid black curve*) and *dashed-dotted red line* $y = 1$ for a complete k-cap in $\mathrm{PG}(N, q)$. a) $N = 3$, $q = 101$, $k = 415$; b) $N = 4$, $q = 31$, $k = 706$

[3] N. Anbar, M. Giulietti, Bicovering arcs and small complete caps from elliptic curves, *J. Algebraic. Combin.* **38**, 371-392 (2013)

[4] D. Bartoli, A.A. Davydov, G. Faina, A.A. Kreshchuk, S. Marcugini, F. Pambianco, Upper bounds on the smallest size of a complete arc in PG$(2, q)$ under a certain probabilistic conjecture. *Problems Inform. Transmission* **50**, 320–339 (2014)

[5] D. Bartoli, A.A. Davydov, G. Faina, A.A. Kreshchuk, S. Marcugini, F. Pambianco, Upper bounds on the smallest size of a complete arc in a finite Desarguesian projective plane based on computer search. *J. Geom.* **107**, 89–117 (2016)

[6] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, New upper bounds on the smallest size of a complete cap in the space PG$(3, q)$. In: *Proc. VII Int. Workshop on Optimal Codes and Related Topics, OC2013*, Albena, Bulgaria, pp. 26–32 (2013) http://www.moi.math.bas.bg/oc2013/a4.pdf

[7] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, On sizes of complete arcs in PG$(2, q)$, *Discrete Math.* **312**, 680–698 (2012)

[8] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, New types of estimates for the smallest size of complete arcs in a finite Desarguesian projective plane, *J. Geom.* **106**, 1–17 (2015)

[9] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, Conjectural upper bounds on the smallest size of a complete cap in PG(N, q), $N \geq 3$, *Electron. Notes Discrete Math.* **57**, 15–20 (2017)

[10] D. Bartoli, A.A. Davydov, A.A. Kreshchuk, S. Marcugini, F. Pambianco, Tables, bounds and graphics of the smallest known sizes of complete caps in the spaces PG(3,q) and PG(4,q), arXiv:1610.09656[math.CO] (2016) http://arxiv.org/abs/1610.09656

[11] D. Bartoli, A.A. Davydov, A.A. Kreshchuk, S. Marcugini, F. Pambianco, Upper bounds on the smallest size of a complete cap in PG$(3, q)$ and PG$(4, q)$, *Electron. Notes Discrete Math.* **57**, 21–26 (2017)

[12] D. Bartoli, G. Faina, M. Giulietti, Small complete caps in three-dimensional Galois spaces, *Finite Fields Appl.* **24**, 184–191 (2013)

[13] D. Bartoli, G. Faina, S. Marcugini, F. Pambianco, A construction of small complete caps in projective spaces, *J. Geom.*, **108**, 215-246 (2017)

[14] D. Bartoli, M. Giulietti, G. Marino, O. Polverino, Maximum scattered linear sets and complete caps in Galois spaces, *Combinatorica*, to appear.

[15] D. Bartoli, S. Marcugini, F. Pambianco, New quantum caps in PG(4, 4), *J. Combin. Des.* **20**, 448–466 (2012)

[16] D. Brink, A (probably) exact solution to the birthday problem, *The Ramanujan J.* **28**, 223–238 (2012)

[17] R.A. Brualdi, S. Litsyn, V.S. Pless, Covering radius. In: Pless, V.S., Huffman, W.C., Brualdi, R.A. (eds) *Handbook of Coding Theory,* Vol. 1, pp. 755–826. Elsevier, Amsterdam, The Netherlands (1998)

[18] M.L. Clevenson, W. Watkins, Majorization and the Birthday inequality, *Math. Magazine* **64**, 183–188 (1991)

[19] G.D. Cohen, I.S. Honkala, S. Litsyn, A.C. Lobstein, *Covering Codes*, Elsevier, Amsterdam, The Netherlands (1997))

[20] A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, On sizes of complete caps in projective spaces PG(n, q) and arcs in planes PG($2, q$), *J. Geom.* **94**, 31–58 (2009)

[21] A.A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, New inductive constructions of complete caps in PG(n, q), q even. *J. Combin. Des.* **18**, 177–201 (2010)

[22] A.A. Davydov, S. Marcugini, F. Pambianco, Complete caps in projective spaces PG(n, q), *J. Geom.* **80**, 23–30 (2004)

[23] A.A. Davydov, P.R.J. Östergård, Recursive constructions of complete caps, *J. Statist. Planning. Infer.* **95**, 167–173 (2001)

[24] G. Faina, F. Pasticci, L. Schmidt, Small complete caps in Galois spaces. *Ars Combin.* **105**, 299–303 (2012)

[25] E.M. Gabidulin, A.A. Davydov, L.M. Tombak, Linear codes with covering radius 2 and other new covering codes, *IEEE Trans. Inform. Theory* **37**, 219–224 (1991)

[26] M. Giulietti, Small complete caps in Galois affine spaces, *J. Algebraic Combin.* **25**(2), 149–168 (2007)

[27] M. Giulietti, Small complete caps in PG(n, q), q even, *J. Combin. Des.* **15**, 420–436 (2007)

[28] M. Giulietti, The geometry of covering codes: small complete caps and saturating sets in Galois spaces, *Surveys in Combinatorics* 2013 - London Mathematical Society Lecture Note Series 409, Cambridge University Press, 2013, pp. 51–90.

[29] M. Giulietti, F. Pasticci, Quasi-Perfect Linear Codes with Minimum Distance 4, *IEEE Trans. Inform. Theory* **53**(5), 1928–1935 (2007)

[30] J.W.P. Hirschfeld, L. Storme, The packing problem in statistics, coding theory and finite projective spaces, *J. Statist. Planning Infer.* **72**, 355–380 (1998)

[31] J.W.P. Hirschfeld, L. Storme, The packing problem in statistics, coding theory and finite geometry: update 2001. In: A. Blokhuis, J.W.P. Hirschfeld, et al. (eds.) *Finite Geometries, Developments of Mathematics*, vol. 3, Proc. of the Fourth Isle of Thorns Conf., Chelwood Gate, 2000, pp. 201–246. Kluwer Academic Publisher, Boston (2001)

[32] J.W.P. Hirschfeld, J.A. Thas, Open problems in finite projective spaces, *Finite Fields Their Appl.* **32**, 44–81 (2015)

[33] J.H. Kim, V. Vu, Small complete arcs in projective planes. *Combinatorica* **23**, 311–363 (2003)

[34] I. Landjev, L. Storme, Galois geometry and coding theory. In: *Current Research Topics in Galois geometry,* J. De Beule, L. Storme, Eds., Chapter 8, Nova Science Publisher, (2011) pp. 185–212.

[35] B. Murphy, G. Petridis, A point-line incidence identity in finite fields, and applications, *Moscow J. Combinatorics Number Theory* **6**, 64–95 (2016)

[36] B. Murphy, G. Petridis, O. Roche-Newton, M. Rudnev, I.D. Shkredov, New results on sum-product type growth over fields, arXiv:1702.01003v2[math.CO] (2017)

[37] F. Pambianco, L. Storme, Small complete caps in spaces of even characteristic, *J. Combin. Theory Ser. A* **75**, 70–84 (1996)

[38] I. Platoni, Complete caps in $\mathrm{AG}(3,q)$ from elliptic curves, *J. Alg. Appl.* **13**, 1450050 (8 pages) (2014)

[39] M. Sayrafiezadeh, The Birthday problem revisited, *Math. Magazine* **67**, 220–223 (1994)

[40] B. Segre, On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, *Acta Arith.* **5**, 315–332 (1959)

[41] T. Szőnyi, Arcs, caps, codes and 3-independent subsets. In: G. Faina et al. (eds.) *Giornate di Geometrie Combinatorie*, Università degli studi di Perugia, pp. 57–80. Perugia (1993)

[42] V.D. Tonchev, Quantum codes from caps, *Discrete Math.* **308**, 6368–6372 (2008)