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Upper bounds on the smallest size of an almost complete cap in $\mathrm{PG}(N, q)$

Alexander A. Davydov[1] adav@iitp.ru
Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Bol'shoi Karetnyi pereulok 19, GSP-4, Moscow, 127994, Russian Federation
Stefano Marcugini[2], Fernanda Pambianco[2]
{stefano.marcugini,fernanda.pambianco}@unipg.it
Dipartimento di Matematica e Informatica, Università degli Studi di Perugia
Via Vanvitelli 1, Perugia, 06123, Italy

Abstract. Conceptions of an almost complete subset of an elliptic quadric in the projective space $\mathrm{PG}(3, q)$ and an almost complete cap in the space $\mathrm{PG}(N, q)$ are proposed. Upper bounds of the smallest size of the introduced geometrical objects are obtained by probabilistic and algorithmic methods.

1 Introduction

Let $\mathrm{PG}(N, q)$ be the N-dimensional projective space over the Galois field $\mathbb{F}_q$ of order q. A cap in $\mathrm{PG}(N, q)$ is a set of points no three of which are collinear. An n-cap of $\mathrm{PG}(N, q)$ is complete if it is not contained in an $(n+1)$-cap of $\mathrm{PG}(N, q)$. Caps in $\mathrm{PG}(2, q)$ are called also arcs. A point P of $\mathrm{PG}(N, q)$ is *covered* by a cap $\mathcal{K} \subset \mathrm{PG}(N, q)$ if P lies on a bisecant of $\mathcal{K}$.

The space $\mathrm{PG}(N, q)$ contains $\theta_{N,q} = \frac{q^{N+1}-1}{q-1}$ points.

The concept of *an almost complete subset of a fixed irreducible conic in* $\mathrm{PG}(2, q)$ is considered in [6], see also [3] and the references therein. An almost complete subset of a conic covers all points of $\mathrm{PG}(2, q)$ except for the remaining points of the conic and the nucleus of the conic if q is even. Let $t(q)$ be the smallest size of an almost complete subset of a conic. In [7] it is proved that under the condition $3 \leq N \leq q + 2 - t(q)$, every normal rational curve in $\mathrm{PG}(N, q)$ is a complete $(q+1)$-arc. In [3], the following upper bound is obtained:

$$t(q) < \sqrt{q(3 \ln q + \ln \ln q + \ln 3)} + \sqrt{\frac{q}{3 \ln q}} + 4 \sim \sqrt{3q \ln q}. \qquad (1)$$

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The concept of an *almost complete arc in* PG$(2,q)$ is considered in [8] where arcs of an infinite family $\mathcal{K}(q)$ are called almost complete if

$$\lim_{q\to\infty}\frac{\#\text{points not covered by } \mathcal{K}(q)}{\#\text{points of the plane PG}(2,q)}=0. \tag{2}$$

An almost complete subset of a conic is an almost complete arc as the number of points not covered by it is smaller than q, cf. (2).

In this work, we generalize both of the aforementioned concepts.

Definition 1. (i) *In* PG$(3,q)$*, an* almost complete subset of the elliptic quadric *(ACQ-subset, for short) is a proper subset of the quadric covering all the points of* PG$(3,q)$ *except for the remaining points of the quadric .*

(ii) *In* PG(N,q)*,* $N\geq 2$*, a cap* $\mathcal{K}$ *is* almost complete *if the number of points not covered by* $\mathcal{K}$ *is not greater than* $\theta_{N-1,q}$.

Note that if caps of Definition 1(ii) form an infinite family of caps $\mathcal{K}(q)$ in the spaces PG(N,q) with growing q then it holds that (cf. (2))

$$\lim_{q\to\infty}\frac{\#\text{points not covered by } \mathcal{K}(q)}{\#\text{points of the space PG}(N,q)}\leq\frac{\theta_{N-1,q}}{\theta_{N,q}}=0.$$

An ACQ-subset is an almost complete cap as the number of points not covered by it is smaller than q^2+1.

Let $d(q)$ be *the smallest size of an ACQ-subset* in PG$(3,q)$.

Let $v(N,q)$ be *the smallest size of an almost complete cap* in PG(N,q).

This work is devoted to *upper bounds* on $d(q)$ and $v(N,q)$.

The main results of this work are presented in Theorem 1.

Theorem 1. (i) *In* PG$(3,q)$*, for the smallest size of an ACQ-subset, we have*

$$d(q)\leq(q+1)\sqrt{6\ln(q+1)}+2q+2\sim q\sqrt{6\ln q}. \tag{3}$$

(ii) *In* PG(N,q)*, for the smallest size of an almost complete cap, it holds that*

$$v(N,q)\leq\sqrt{2N\theta_{N-1,q}\ln q}+1\sim q^{\frac{N-1}{2}}\sqrt{2N\ln q},\quad N\geq 2. \tag{4}$$

Moreover, an almost complete cap of size at most $\sqrt{2N\theta_{N-1,q}\ln q}+1$ *can be constructed by a step-by-step greedy algorithm that in every step adds to the running cap a point providing the maximal possible (for the given step) number of new covered points.*

One see that the bounds (3) and (4) asymptotically coincide with each other.

For $N=2$ the bound (4) does not improve the bound (1) obtained in [3].

As far as it is known of the authors, ACQ-subsets and almost complete caps in PG(N,q), $N\geq 3$, are not considered in the literature. Therefore, it remains for us only to compare the bounds (3) and (4) with the known bounds on the smallest size $t_2(N,q)$ of a complete cap in $\mathrm{PG}(N,q)$. Of course, one should remember that these estimates are obtained for objects which are similar to the almost complete caps but not the same.

In [4], it is proved that

$$t_2(N,q) < cq^{\frac{N-1}{2}} \log^{300} q, \text{ a constant } c \text{ is independent of } q.$$

In [2], *under some probabilistic conjecture*, it is shown that

$$t_2(N,q) < \frac{1}{q-1}\sqrt{q^{N+1}(N+1)\ln q} + \frac{\sqrt{q^{N+1}}}{q-3} \sim q^{\frac{N-1}{2}}\sqrt{(N+1)\ln q}. \quad (5)$$

We see that $q^{\frac{N-1}{2}}\sqrt{2N\ln q}$ is essentially smaller than $cq^{\frac{N-1}{2}}\log^{300} q$.

In the other side, the bound $q^{\frac{N-1}{2}}\sqrt{2N\ln q}$ (that *is proved*) is greater than the *conjectural* bound (5).

In Section 2, the bound (3) is proved by probabilistic methods. In Section 3, the bound (4) is obtained by an algorithmic approach.

2 An upper bound on the smallest size of an almost complete subset of an elliptic quadric in $\mathrm{PG}(3,q)$

Let $w > 0$ be a fixed integer. Let $\mathcal{Q}$ be an elliptic quadric in $\mathrm{PG}(3,q)$. Consider a random $(w+1)$-point subset $\mathcal{K}_{w+1} \subset \mathcal{Q}$. The total number of such subsets is $\binom{q^2+1}{w+1}$. A fixed point A of $\mathrm{PG}(3,q) \setminus \mathcal{Q}$ is *covered* by $\mathcal{K}_{w+1}$ if it belongs to a bisecant of $\mathcal{K}_{w+1}$. We denote by $\mathrm{Prob}(\diamond)$ the probability of some event $\diamond$.

We estimate

$$\pi := \mathrm{Prob}(A \text{ not covered by } \mathcal{K}_{w+1})$$

as the ratio of the number of $(w+1)$-point subsets of $\mathcal{Q}$ not covering A over the total number $\binom{q^2+1}{w+1}$ of subsets of $\mathcal{Q}$ with size $(w+1)$. A set $\mathcal{K}_{w+1}$ does not cover A if and only if every line through A contains at most one point of $\mathcal{K}_{w+1}$.

There are $\frac{q(q-1)}{2}$ bisecants and $q+1$ tangents of $\mathcal{Q}$ through any point $A \in \mathrm{PG}(3,q) \setminus \mathcal{Q}$ [5]. Every bisecant has two places to put a point of $\mathcal{K}_{w+1}$ while a tangent has the only one. For simplicity of presentation, we assume that a tangent also has two places to put a point of $\mathcal{K}_{w+1}$. (This will slightly worsen our estimates.) Therefore,

$$\pi < \frac{2^{w+1}\binom{q(q-1)/2+q+1}{w+1}}{\binom{q^2+1}{w+1}} = \frac{2^{w+1}\binom{(q^2+q+1)/2}{w+1}}{\binom{q^2+1}{w+1}},$$

where the numerator estimates from above the number of $(w+1)$-point subsets of $\mathcal{Q}$ not covering A. By straightforward calculations,

$$\pi < \frac{(q^2+q+2)(q^2+q)(q^2+q-2)\cdots(q^2+q+2-2i)\cdots(q^2+q+2-2w)}{(q^2+1)(q^2)(q^2-1)\cdots(q^2+1-i)\cdots(q^2+1-w)}$$

$$= \prod_{i=0}^{w}\frac{q^2+q+2-2i}{q^2+1-i} = \prod_{i=0}^{w}\left(1-\frac{i-1-q}{q^2+1-i}\right) < \prod_{i=0}^{w}\left(1-\frac{i-1-q}{q^2+1}\right).$$

Using the inequality $1-x \le e^{-x}$ for $x \neq 0$, we obtain that under the condition $w > 2q+2+\frac{4}{2q-1}$, it holds that

$$\pi < e^{-\sum\limits_{i=0}^{w}(i-1-q)/(q^2+1)} = e^{-(w^2-(2q+1)w-2q-2)/2(q^2+1)} < e^{-(w-2q)^2/2(q+1)^2}.$$

The set $\mathcal{K}_{w+1}$ is not ACQ-subset if at least one point of $\mathrm{PG}(3,q)\setminus\mathcal{Q}$ is not covered by it. As $|\mathrm{PG}(3,q)\setminus\mathcal{Q}| = q^3+q$, we have

$$\mathrm{Prob}\,(\mathcal{K}_{w+1} \text{ is not ACQ-subset}) \le \sum_{A\in PG(3,q)\setminus\mathcal{Q}} \mathrm{Prob}(A \text{ not covered})$$

$$\le (q^3+q)\pi < (q+1)^3 e^{-(w-2q)^2/2(q+1)^2}.$$

The probability that all the points of $\mathrm{PG}(3,q)\setminus\mathcal{Q}$ are covered by $\mathcal{K}_{w+1}$ is

$$\mathrm{Prob}\,(\mathcal{K}_{w+1} \text{ is ACQ-subset}) > 1-(q+1)^3 e^{-(w-2q)^2/2(q+1)^2}.$$

This probability is larger than 0 if one takes $w-2q = \left\lceil (q+1)\sqrt{6\ln(q+1)}\right\rceil$ where the condition $w > 2q+2+\frac{4}{2q-1}$ holds. This shows that there exists an ACQ-subset $\mathcal{K}_{w+1}$ with size $w+1 \le (q+1)\sqrt{6\ln(q+1)}+2q+2$.

Theorem 1(i) is proved.

3 An upper bound on the smallest size of an almost complete cap in $\mathrm{PG}(N,q)$

Assume that in $\mathrm{PG}(N,q)$, $N \ge 2$, a cap is constructed by a step-by-step greedy algorithm (*Algorithm*, for short) which in every step adds to the cap a point providing the maximal possible (for the given step) number of new covered points. After the w-th step of Algorithm, a w-cap is obtained that does not cover exactly U_w points. For the $(w+1)$-th step of Algorithm, let Δ_w be the maximal possible number of new covered points. So,

$$U_{w+1} = U_w - \Delta_w. \tag{6}$$

Similarly to [1, Lemma 2] and [2, Lemma 4.1], we have proved Lemma 1.

Lemma 1. *It holds that*

$$\Delta_w \geq \max\left\{1, \left\lceil \frac{wU_w}{\theta_{N-1,q}+1-w} - w + 1 \right\rceil\right\}.$$

By (6) and Lemma 1,

$$U_{w+1} \leq U_w\left(1 - \frac{w}{\theta_{N-1,q}+1-w}\right) + w - 1 < U_w\left(1 - \frac{w}{\theta_{N-1,q}}\right) + w$$

whence

$$\begin{aligned} &U_{w+1} - \theta_{N-1,q} < U_w\left(1 - \frac{w}{\theta_{N-1,q}}\right) + w - \theta_{N-1,q} \\ &= U_w\left(\frac{\theta_{N-1,q} - w}{\theta_{N-1,q}}\right) - (\theta_{N-1,q} - w) = \left(1 - \frac{w}{\theta_{N-1,q}}\right)(U_w - \theta_{N-1,q}). \end{aligned} \tag{7}$$

By (7),

$$\begin{aligned} U_2 - \theta_{N-1,q} &< \left(1 - \frac{1}{\theta_{N-1,q}}\right)(U_1 - \theta_{N-1,q}); \\ U_3 - \theta_{N-1,q} &< \left(1 - \frac{2}{\theta_{N-1,q}}\right)(U_2 - \theta_{N-1,q}) \\ &= \left(1 - \frac{2}{\theta_{N-1,q}}\right)\left(1 - \frac{1}{\theta_{N-1,q}}\right)(U_1 - \theta_{N-1,q}); \\ &\dots \\ U_{w+1} - \theta_{N-1,q} &< \left(1 - \frac{w}{\theta_{N-1,q}}\right)\dots\left(1 - \frac{2}{\theta_{N-1,q}}\right)\left(1 - \frac{1}{\theta_{N-1,q}}\right)(U_1 - \theta_{N-1,q}). \end{aligned}$$

Taking into account that $U_1 = \theta_{N,q} - 1 < \theta_{N,q} = \theta_{N-1,q} + q^N$, we have

$$U_{w+1} - \theta_{N-1,q} < (U_1 - \theta_{N-1,q})\prod_{i=1}^{w}\left(1 - \frac{i}{\theta_{N-1,q}}\right);$$

$$U_{w+1} < q^N \prod_{i=1}^{w}\left(1 - \frac{i}{\theta_{N-1,q}}\right) + \theta_{N-1,q}. \tag{8}$$

Using the inequality $1 - x \leq e^{-x}$ for $x \neq 0$, we obtain

$$\prod_{i=1}^{w}\left(1 - \frac{i}{\theta_{N-1,q}}\right) < \prod_{i=1}^{w} e^{-i/\theta_{N-1,q}} = e^{-(w^2+w)/2\theta_{N-1,q}} < e^{-w^2/2\theta_{N-1,q}}. \tag{9}$$

Let

$$w = \left\lceil \sqrt{2\theta_{N-1,q} \ln q^N} \right\rceil = \left\lceil \sqrt{2N\theta_{N-1,q} \ln q} \right\rceil \sim q^{\frac{N-1}{2}} \sqrt{2N \ln q}. \tag{10}$$

Then, by (8)–(10),

$$w^2 = 2\theta_{N-1,q} \ln q^N; \quad e^{-w^2/2\theta_{N-1,q}} = \frac{1}{q^N};$$
$$U_{w+1} < \theta_{N-1,q} + 1; \quad U_{w+1} \le \theta_{N-1,q}.$$

So, the number of points of $\mathrm{PG}(N,q)$ not covered by the cap $\mathcal{K}_{w+1}$ is at most $\theta_{N-1,q}$.

We have proved Theorem 1(ii).

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