New Bounds for Linear Codes of Covering Radius 2*

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Abstract. The length function $\ell_q(r, R)$ is the smallest length of a q-ary linear code of covering radius R and codimension r. New upper bounds on $\ell_q(r, 2)$ are obtained for odd $r \geq 3$. In particular, using the one-to-one correspondence between linear codes of covering radius 2 and saturating sets in the projective planes over finite fields, we prove that

$$\ell_q(3, 2) \leq \sqrt{q(3 \ln q + \ln \ln q)} + \sqrt{\frac{q}{3 \ln q}} + 3$$

and then obtain estimations of $\ell_q(r, 2)$ for all odd $r \geq 5$. The new upper bounds are smaller than the previously known ones. Also, the new bounds hold for all q, not necessary large, whereas the previously best known estimations are proved only for q large enough.

Keywords: Covering codes · Saturating sets · The length function · Upper bounds · Projective spaces

1 Introduction

Let F_q be the Galois field with q elements. Let F_q^n be the n-dimensional vector space over F_q. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is, a subspace of F_q^n of dimension $n - r$. The sphere of radius R with center c in F_q^n is the set $\{v : v \in F_q^n, d(v, c) \leq R\}$ where $d(v, c)$ is the Hamming distance between vectors v and c.

Definition 1. (i) *The* covering radius *of a linear $[n, n-r]_q$ code is the least integer R such that the space F_q^n is covered by spheres of radius R centered at codewords.*

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(ii) *A linear* $[n, n-r]_q$ *code has* covering radius R *if every column of* F_q^r *is equal to a linear combination of at most* R *columns of a parity check matrix of the code, and* R *is the smallest value with such property.*

Definitions 1(i) and 1(ii) are equivalent. Let an $[n, n-r]_q R$ code be an $[n, n-r]_q$ code with covering radius R. For an introduction to coverings of vector Hamming spaces over finite fields, see [3, 4].

The covering density μ of an $[n, n-r]_q R$-code is defined as

$$\mu = \frac{1}{q^r} \sum_{i=0}^{R} (q-1)^i \binom{n}{i} \geq 1.$$

The covering quality of a code is better if its covering density is smaller. For fixed q, r, and R the covering density of an $[n, n-r]_q R$ code decreases with decreasing n.

Definition 2. [3, 4] *The* length function $\ell_q(r, R)$ *is the smallest length of a* q*-ary linear code with covering radius* R *and codimension* r.

Codes investigated from the point view of the covering quality are usually called *covering codes*; see an online bibliography in [13].

In this paper we consider covering codes with radius $R = 2$.

The known lower bound on $\ell_q(r, 2)$, based on Definition 1(ii), is

$$\ell_q(r, 2) > \sqrt{2} q^{(r-2)/2}. \tag{1}$$

Really, in a parity check matrix of an $[n, n-r]_q 2$ code, one can take $\binom{n}{2}$ distinct pair of columns and then form q^2 linear combinations from every pair. By Definition 1(ii), it holds that $\binom{n}{2} q^2 \geq q^r$ whence (1) follows.

For arbitrary q, covering codes of length close to this lower bound are known only for r even [5, 7, 9, 10]. In particular, the following bounds are obtained by algebraic constructions [7, Sect. 4.3, eq. (4.6)], [9, Th. 9]:

$$\ell_q(r, 2) \leq 2q^{(r-2)/2} + q^{(r-4)/2}, \; q \geq 7, \; q \neq 9, \; r = 2t \geq 4, \; t = 2, 3, 5, \text{ and } t \geq 7.$$

$$\ell_q(r, 2) \leq 2q^{(r-2)/2} + q^{(r-4)/2} + q^{(r-6)/2} + q^{(r-8)/2}, \; q \geq 7, \; q \neq 9, \; r = 8, 12.$$

If r is *odd*, covering codes of length close to lower bound (1) are known only when q is an *even power of a prime*, i.e. more exactly when $q = (q')^2$ and $q = (q')^4$, where q' is a prime power, and when $q = p^6$ with prime $p \leq 73$ [5–7,10,12]. In particular, the following bounds are obtained by algebraic constructions, see [5, Ex. 6, eq. (33)], [6], [7, Sect. 4.4, eqs. (4.12),(4.13),(4.15)], [12], and the references therein:

$$\ell_q(r, 2) \leq \left(3 - \frac{1}{\sqrt{q}}\right) q^{(r-2)/2} + \left\lfloor q^{(r-5)/2} \right\rfloor, \; q = (q')^2 \geq 16, \; r = 2t+1 \geq 3.$$

$$\ell_q(r, 2) \leq \left(2 + \frac{2}{\sqrt[4]{q}} + \frac{2}{\sqrt{q}}\right) q^{(r-2)/2} + \left\lfloor q^{(r-5)/2} \right\rfloor, \; q = (q')^4, \; r = 2t+1 \geq 3.$$

$$\ell_q(r,2) \le \left(2 + \frac{2}{\sqrt[6]{q}} + \frac{2}{\sqrt[3]{q}} + \frac{2}{\sqrt{q}}\right) q^{(r-2)/2} + 2\left\lfloor q^{(r-5)/2}\right\rfloor, \; q = (q')^6,$$
$$q' \le 73 \text{ prime}, \; r = 2t+1 \ge 3, \; r \ne 9, 13.$$

The *goal of this work* is to obtain new upper bounds on the length function $\ell_q(r, 2)$ with r *odd* and *arbitrary* q, not necessarily having the form $q = (q')^2$ where q' is a prime power. It is a *hard open problem.* The first and the most important step in this problem is finding of upper bounds on $\ell_q(3, 2)$. It is usually considered as a separate open problem.

Let $\mathrm{PG}(N, q)$, $N \ge 2$, be the N-dimensional projective space over the field F_q; see [11] for an introduction to the projective spaces over finite fields. Effective methods obtaining upper bounds on $\ell_q(r, 2)$ with r odd, in particular on $\ell_q(3, 2)$, are connected with saturating sets in $\mathrm{PG}(N, q)$, $N \ge 2$.

Definition 3. *A point set* $\mathcal{S} \subset \mathrm{PG}(N, q)$ *is* saturating *if any point of* $\mathrm{PG}(N, q) \setminus \mathcal{S}$ *is collinear with two points in* $\mathcal{S}$.

Saturating sets are considered in [5–10, 12, 14, 15], see also the references therein. In the literature, saturating sets are also called "saturated sets" [5, 15], "spanning sets", "dense sets", and "1-saturating sets" [6–8, 12].

Let $s(N, q)$ be *the smallest size of a saturating set* in $\mathrm{PG}(N, q)$.

If q-ary positions of a column of an $r \times n$ parity check matrix of an $[n, n-r]_q 2$ code are treated as homogeneous coordinates of a point in $\mathrm{PG}(r-1, q)$ then this parity check matrix defines a saturating set of size n in $\mathrm{PG}(r-1, q)$ [5–7]. So, there is *the one-to-one correspondence between* $[n, n-r]_q 2$ *codes and saturating sets in* $\mathrm{PG}(r-1, q)$. Therefore,

$$\ell_q(r, 2) = s(r-1, q), \text{ in particular, } \ell_q(3, 2) = s(2, q).$$

In [1, 2], by probabilistic methods the following upper bound is obtained in the geometrical language.

$$s(2, q) \le 2\sqrt{(q+1)\ln(q+1)} + 2 \sim 2\sqrt{q \ln q}. \tag{2}$$

Also, in [1, 2] one can find the previous results and the references on this topic.

In [14], the following bound is proved for the projective plane $\mathrm{PG}(2, q)$.

$$s(2, q) \le (\sqrt{3} + o(1))\sqrt{q \ln q}. \tag{3}$$

The proof of (3) is given in [14] by two approaches: probabilistic and algorithmic. In both the approaches, starting with some stage of the proof, it is assumed (by the context) that q *is large enough.* As the result of the algorithmic proof of [14], the following form of the bound can be derived.

$$s(2, q) \le \left\lceil \sqrt{3q \ln q}\right\rceil + \left\lceil \frac{1}{2}\sqrt{q}\right\rceil \le \sqrt{3q \ln q} + \frac{1}{2}\sqrt{q} + 2, \quad q \text{ large enough.} \tag{4}$$

Note that the first steps of the algorithmic proof in [14] do not need q large enough; this allows us to use these steps in Sect. 2.

Throughout the paper we denote

$$\Upsilon(q) = \sqrt{3\ln q + \ln\ln q} + \sqrt{\frac{1}{3\ln q}} + \frac{3}{\sqrt{q}}\,. \tag{5}$$

Our new results are collected in Theorem 4 based on Theorems 7 and 11.

Theorem 4. *Let q be an* arbitrary *prime power. Let the value of q be* not necessarily large. *Let r be* odd. *For the length function $\ell_q(r,2)$ and for the smallest size $s(r-1,q)$ of a saturating set in the projective space* $\mathrm{PG}(r-1,q)$ *the following upper bounds hold.*

(i) $$\ell_q(3,2) = s(2,q) \le \Upsilon(q)\cdot q^{(3-2)/2} = \Upsilon(q)\sqrt{q}. \tag{6}$$

(ii) $$\ell_q(r,2) = s(r-1,q) \le \Upsilon(q)\cdot q^{(r-2)/2} + 2q^{(r-5)/2}, \quad r = 2t+1 \ge 5, \tag{7}$$
where $r \ne 9, 13,\ t = 2,3,5,$ *and* $t \ge 7,\ q \ge 19.$

$$\ell_q(r,2) = s(r-1,q) \le \Upsilon(q)\cdot q^{(r-2)/2} + 2q^{(r-5)/2} + q^{(r-7)/2} + q^{(r-9)/2}, \tag{8}$$
where $r = 9, 13.$

These upper bounds are smaller (i.e. better) than the previously known ones, see Sect. 4.

The paper is organized as follows. In Sect. 2, a new upper bound on the length function $\ell_q(3,2)$ is obtained. In Sect. 3, upper bounds on the length function $\ell_q(r,2)$, $r \ge 5$ odd, are considered on the base of the results of Sect. 2. Finally, in Sect. 4 we compare the obtained new bounds with the previously known ones.

2 An Upper Bound on the Length Function $\ell_q(3,2)$

Assume that in $\mathrm{PG}(2,q)$ a saturating set is constructed by a step-by-step algorithm adding one new point to the set in every step.

Let $i > 0$ be an integer. Denote by $\mathcal{S}_i$ the running set obtained after the i-th step of the algorithm. A point P of $\mathrm{PG}(2,q) \setminus \mathcal{S}_i$ is covered by $\mathcal{S}_i$ if P lies on a t-secant of $\mathcal{S}_i$ with $t \ge 2$. Let $\mathcal{R}_i$ be the subset of $\mathrm{PG}(2,q) \setminus \mathcal{S}_i$ consisting of points not covered by $\mathcal{S}_i$.

In [14] the following ingenious greedy algorithm is proposed. One takes the line ℓ skew to $\mathcal{S}_i$ such that the cardinality of intersection $|\mathcal{R}_i \cap \ell|$ is the minimal among all skew lines. Then one adds to $\mathcal{S}_i$ the point on ℓ providing the greatest number of new covered points (in comparison with other points of ℓ). As a result we obtain the set $\mathcal{S}_{i+1}$ and the corresponding set $\mathcal{R}_{i+1}$.

In [14, Proposition 3.3, Proof], the following inequality is proved without requirement that q is large enough:

$$|\mathcal{R}_{i+1}| \le |\mathcal{R}_i| \cdot \left(1 - \frac{i(q-1)}{q(q+1)}\right). \tag{9}$$

The running set $\mathcal{S}_2$ contains two points; we consider the line through them. All points on this line are covered by $\mathcal{S}_2$. So, always $\mathcal{R}_2 = (q^2+q+1)-(q+1) = q^2$ where q^2+q+1 and $q+1$ are the number of points in $\mathrm{PG}(2,q)$ and in the line, respectively. Starting from $\mathcal{R}_2 = q^2$ and iteratively applying the relation (9), we obtain for some k the following:

$$|\mathcal{R}_{k+1}| \leq q^2 f_q(k),$$

where

$$f_q(k) = \prod_{i=2}^{k} \left(1 - \frac{i(q-1)}{q(q+1)}\right).$$

Now we consider a *truncated iterative process*. We will stop the iterative process when $|\mathcal{R}_{k+1}| \leq \xi$ where $\xi \geq 1$ is some value that we may *assign arbitrary* to improve estimations.

By [14, Lemma 2.1] after the end of the iterative process we can add at most $\lceil |\mathcal{R}_{k+1}|/2 \rceil$ points to the running subset $\mathcal{S}_{k+1}$ in order to get the final saturating set $\mathcal{S}$. Therefore, the size s of the obtained saturating set $\mathcal{S}$ is

$$s \leq k + 1 + \left\lceil \frac{\xi}{2} \right\rceil \text{ under condition } q^2 f_q(k) \leq \xi. \tag{10}$$

Using the inequality $1 - x \leq e^{-x}$, we obtain that

$$f_q(k) < e^{-\sum\limits_{i=2}^{k} i(q-1)/(q^2+q)} = e^{-(k^2+k-2)(q-1)/(2q^2+2q)},$$

which implies

$$f_q(k) < e^{-(k^2+k-2)(q-1)/(2q^2+2q)} < e^{-k^2/(2q+2)}, \tag{11}$$

provided that

$$\frac{(k^2+k-2)(q-1)}{q} > k^2$$

or, equivalently,

$$\frac{k^2}{k-2} < q - 1,$$

$$k < q - 4. \tag{12}$$

Lemma 5. *Let $\xi \geq 1$ be a fixed value independent of k. The value*

$$k \geq \left\lceil \sqrt{2(q+1)} \sqrt{\ln \frac{q^2}{\xi}} \right\rceil \tag{13}$$

satisfies inequality $q^2 f_q(k) \leq \xi$.

Proof. By (11), to provide $q^2 f_q(k) \leq \xi$ it is sufficient to find k such that

$$e^{-k^2/(2q+2)} < \frac{\xi}{q^2}.$$

□

Theorem 6. *Let q be an arbitrary prime power. In the projective plane $\mathrm{PG}(2,q)$ it holds that*

$$s(2,q) \leq \sqrt{2(q+1)}\sqrt{\ln\frac{q^2}{\xi}} + \frac{\xi}{2} + 3, \quad \xi \geq 1, \tag{14}$$

where ξ is an arbitrarily chosen value.

Proof. We substitute the value k from (13) to (10). The summand "+3" takes into account that the size of a saturating set is an integer. □

In order to get a "good" estimation of $s(2,q)$, we are trying to reduce the right part of (14). For it, let us consider the function of ξ of the form

$$\phi(\xi) = \sqrt{2(q+1)}\sqrt{\ln\frac{q^2}{\xi}} + \frac{\xi}{2} + 3.$$

Its derivative by ξ is

$$\phi'(\xi) = \frac{1}{2} - \frac{1}{\xi}\sqrt{\frac{q+1}{2\ln\frac{q^2}{\xi}}}.$$

It is easy to check that $\phi'(1) < 0$, $\phi'(q) > 0$, and $\phi'(\xi)$ is an increasing function of ξ. This means that for some value $\xi_0 > 1$ it holds that $\phi'(\xi_0) = 0$. Moreover, for $\xi < \xi_0$, the derivative $\phi'(\xi) < 0$ and $\phi(\xi)$ decreases, while for $\xi > \xi_0$, the derivative is positive and $\phi(\xi)$ increases. So, in the point $\xi = \xi_0$ we have the minimum of $\phi(\xi)$. Now we will find a value of ξ such that $\phi'(\xi)$ is close to 0 and, in addition, the expression of the results is relatively simple.

Put $\phi'(\xi) = 0$. Then it is easy to see that

$$\xi^2 = \frac{q+1}{\ln q - \frac{1}{2}\ln\xi}. \tag{15}$$

We find ξ in the form $\xi = \sqrt{\frac{q+1}{c\ln q}}$. By (15),

$$c = 1 - \frac{\ln(q+1)}{4\ln q} + \frac{\ln c + \ln\ln q}{4\ln q}.$$

We choose $c \approx 1 - \frac{\ln(q+1)}{4\ln q} \approx \frac{3}{4}$ and put $\xi = \sqrt{\frac{4q}{3\ln q}}$. The value

$$\phi'\left(\sqrt{\frac{4q}{3\ln q}}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{3(q+1)\ln q}{q\left(3\ln q + \ln\ln q + \ln\frac{3}{4}\right)}}$$

is close to zero for growing q. Also, see below, the expression of the results for such ξ is quite simple.

So, the choice $\xi = \sqrt{\frac{4q}{3\ln q}}$ in (14) seems to be convenient.

Theorem 7. *Let q be an* arbitrary *prime power.*
(i) *In* $\mathrm{PG}(2,q)$*, there is a saturating set of size* $\leq \Upsilon(q)\sqrt{q}$.
(ii) *There exists an* $[n, n-3]_q 2$ *code with* $n \leq \Upsilon(q)\sqrt{q}$.

Proof. **(i)** We substitute $\xi = \sqrt{\frac{4q}{3\ln q}}$ in (14) and obtain

$$s(2,q) \leq \sqrt{(q+1)\left(3\ln q + \ln\ln q + \ln\frac{3}{4}\right)} + \sqrt{\frac{q}{3\ln q}} + 3.$$

It can be shown (e.g. by considering the corresponding derivatives) that

$$\sqrt{(q+1)\left(3\ln q + \ln\ln q + \ln\frac{3}{4}\right)} + \sqrt{\frac{q}{3\ln q}} + 3 < \Upsilon(q)\sqrt{q} \quad \text{for} \quad q \geq 43.$$

Also, the necessary condition (12) holds as $\Upsilon(q)\sqrt{q} < q - 4$.
So, we have proved that a saturating set of size $\leq \Upsilon(q)\sqrt{q}$ exists in $\mathrm{PG}(2,q)$ for $q \geq 43$.
Now note that in [7, Tab. 1], the smallest known (up to September 2010) sizes of saturating sets in $\mathrm{PG}(2,q)$, $q \leq 1217$, are given. All these sizes (including the region $q < 43$) are smaller than $\Upsilon(q)\sqrt{q}$.
The assertion **(i)** is proved.

(ii) The one-to-one correspondence between saturating sets and covering codes, see Introduction, implies the existence of an $[n, n-3]_q 2$ code with $n \leq \Upsilon(q)\sqrt{q}$.

□

Theorem 7 immediately implies the estimation (6) of Theorem 4(i).

Remark 8. Let $\xi = 1$. From (14) we have

$$s(2,q) \leq 2\sqrt{(q+1)\ln q} + 3, \tag{16}$$

that practically coincides with bound (2) from [1, 2].

Let $\xi = \sqrt{q}$. From (14) we obtain the estimation

$$s(2,q) \leq \sqrt{3(q+1)\ln q} + \frac{1}{2}\sqrt{q} + 3 \tag{17}$$

which practically coincides with bound (4) of [14].
However, the value $\xi = \sqrt{\frac{4q}{3\ln q}}$ gives the estimation (6) that is smaller (i.e. better) than (16) and (17), see Sect. 4.

Remark 9. In fact, the estimations (2) from [1, 2], (3) and (4) of [14], and the new estimation (6), proved in this section, hold in an arbitrary finite plane of order q, not necessarily Desarguesian. But in a non-Desarguesian plane we have not the one-to-one correspondence between $[n, n-3]_q 2$ codes and saturating sets. It is why we consider here only the Desarguesian plane $\mathrm{PG}(2,q)$.

3 Upper Bounds on the Length Function $\ell_q(r,2)$, $r \geq 5$ odd

For upper bounds on the length function $\ell_q(r,2)$, $r \geq 5$ odd, an important tool is the inductive construction of [5, 7] providing the following code parameters.

Proposition 10. [5, Ex. 6] [7, Th. 4.4] *Let an $[n_q, n_q-3]_q2$ code exist. Then the following holds.*

(i) *Under conditions $n_q < q$ and $q+1 \geq 2n_q$, there is an infinite family of $[n, n-r]_q2$ codes with the parameters*

$$n = n_q q^{(r-3)/2} + 2q^{(r-5)/2},\ r = 2t-1 \geq 5,\ r \neq 9, 13,\ t = 3, 4, 6,\ \text{and } t \geq 8. \tag{18}$$

(ii) *Under condition $n_q < q$ there is an infinite family of $[n, n-r]_q2$ codes with*

$$n = n_q q^{(r-3)/2} + 2q^{(r-5)/2} + q^{(r-7)/2} + q^{(r-9)/2},\ r = 9, 13. \tag{19}$$

Theorem 11. *Let q be an arbitrary prime power. Then there exists an infinite family of $[n, n-r]_q2$ codes with the parameters*

$$n = \Upsilon(q) \cdot q^{(r-2)/2} + 2q^{(r-5)/2}, \quad r = 2t+1 \geq 5,\ r \neq 9, 13, \tag{20}$$

where $t = 2, 3, 5$, and $t \geq 7$, $q \geq 19$.

Also there exists an infinite family of $[n, n-r]_q2$ codes with the parameters

$$n = \Upsilon(q) \cdot q^{(r-2)/2} + 2q^{(r-5)/2} + q^{(r-7)/2} + q^{(r-9)/2},\ r = 9, 13. \tag{21}$$

Proof. Since $\Upsilon(q)\sqrt{q} < q$, we may put that the starting $[n_q, n_q-3]_q2$ code of Proposition 10 is the $[n, n-3]_q2$ code, $n \leq \Upsilon(q)\sqrt{q}$, of Theorem 7. It is easy to check directly that the condition $q+1 \geq 2\Upsilon(q)\sqrt{q}$ holds for $q \geq 79$. Now, similarly to the proof of Theorem 7, we use the smallest known sizes of saturating sets in $PG(2,q)$ from [7, Tab. 1]. For $q < 79$, these sizes are smaller than $\Upsilon(q)\sqrt{q}$ and, moreover, for $19 \leq q < 79$ they provide the condition $q+1 \geq 2n_q$ for Proposition 10. Now the relations (20) and (21) follow from (18) and (19), respectively. □

Theorem 11 immediately implies the estimations (7) and (8) of Theorem 4(ii).

4 Comparison with the Previously Known Results

Surveys on the results on non-binary covering codes in [7, 10] show that the inductive approach of Proposition 10 is the main tool to obtain upper bounds on the length function $\ell_q(r,2)$, $r \geq 5$ odd. Proposition 10 uses the length function $\ell_q(3,2)$ as the base for inductive estimations. Therefore upper bounds on $\ell_q(3,2)$, smaller than the known ones, provide bounds on $\ell_q(2t+1,2)$, $2t+1 \geq 5$, that are less than the corresponding known results. So, in the beginning we should compare the new bound on $\ell_q(3,2)$, see (6), with the best corresponding known bound, see (4).

First of all we should emphasize that the new bound (6) holds for all q, *not necessary large*, whereas the known bound (4) is proved only for q large enough.

Then we consider the difference $\Delta(q)$ between the bounds (4) and (6) where

$$\Delta(q) = \sqrt{3q\ln q} + \frac{1}{2}\sqrt{q} + 2 - \Upsilon(q)\sqrt{q}.$$

It can be shown (e.g. by considering the derivatives) that $\Delta(q) > 0$ for $q \geq 337$ and, moreover, $\Delta(q)$ and $\frac{\Delta(q)}{\sqrt{q}}$ are increasing functions of q. For illustration, see Fig. 1 where the top curve shows $\Delta(q)$ while the bottom one $\sqrt{q/7}$ is given for comparison.

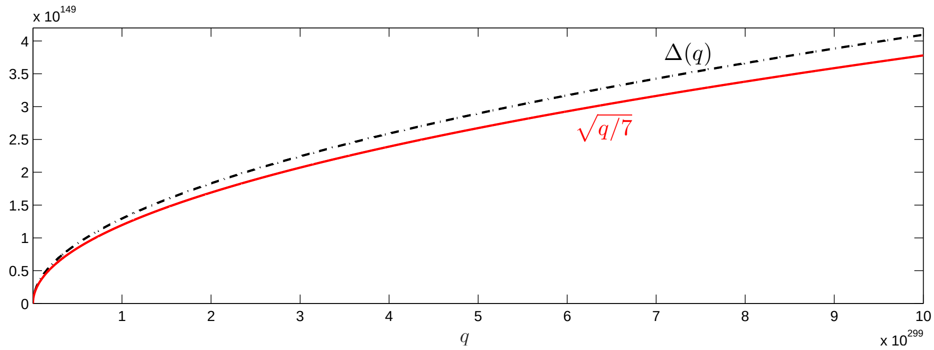

Fig. 1. The difference $\Delta(q)$ (*top dashed-dotted curve*) vs $\sqrt{q/7}$ (*bottom solid curve*)

Note also that

$$\lim_{q\to\infty} \frac{\Delta(q)}{\sqrt{q}} = \lim_{q\to\infty} \left(\sqrt{3\ln q} + \frac{1}{2} - \sqrt{3\ln q + \ln\ln q} - \frac{1}{\sqrt{3\ln q}} - \frac{1}{\sqrt{q}}\right) = \frac{1}{2}.$$

Finally, if one uses Proposition 10 to estimate $\ell_q(r,2)$, $r \geq 5$ odd, then the difference between new and known results will be of order $\Delta(q)q^{(r-3)/2}$. It means that our improvements for $r = 3$ directly expand to odd $r \geq 5$.

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