Probability free methods for universal portfolio selection

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The model of stock market

- Market vector $\mathbf{x} = (x_1, \ldots, x_N)$ represents price relatives for a given trading period: $x_j = c_j / c_{j-1}$, where c_j is closing price and c_{j-1} is opening price of the asset j.
- Investor can rebalance his wealth $S_t = S_{t-1} \cdot (\mathbf{p} \cdot \mathbf{x})$ in each round t according to a portfolio vector $\mathbf{p} = (p_1, \ldots, p_N)$, where $p_j \geq 0$ for all j, and $\sum\limits_{j=1}^{N} b_j = 1$. In what follows $S_0 = 1$.
- $\mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1})$ – causal portfolio at time moment t; $S_T = \prod_{t=1}^{T} (\mathbf{p}_t(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}) \cdot \mathbf{x}_t)$ – wealth (capital) achieved in T trading periods; $W_T = \frac{1}{T} \ln S_T$ – growth rate; constant rebalanced portfolio $\mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_{t-1}) = \mathbf{p}$ for all t.

Data model

Assumptions for data model:

- $\mathbf{X}_1, \mathbf{X}_2, \dots \sim$ i.i.d. (a.s. and in expectation performance bounds)
- $\mathbf{X}_1, \mathbf{X}_2, \dots$ is a stationary and ergodic process (a.s. and in expectation performance bounds for portfolio algorithms)
- $\mathbf{x}_1, \mathbf{x}_2, \dots$ – no stochastic assumptions – black box (worst case performance bounds)

Probability free algorithm for portfolio selection does not use any probability distribution on price relatives.

We show that nonstochastic setting is a part of game-theoretic Prediction with Expert Advice (PEA) approach.

Universal portfolio

Reference classes of portfolios:

- Class of constant rebalanced portfolios (CRP): $S_T(\mathbf{p}) = \prod_{t=1}^{T}(\mathbf{p}\cdot\mathbf{x}_t)$ - capital.
- Class of switching portfolios $\mathbf{E} = \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T}$, where $\mathbf{e}_i$ is a unit vector: $S_T(\mathbf{E}) = \prod_{j=1}^{T}(\mathbf{e}_{i_t}\cdot\mathbf{x}_t)$.
- Class of all causal portfolio strategies $\mathbf{p}(\mathbf{x}_1^t)$: $S_T(\mathbf{p}) = \prod_{t=1}^{T}(\mathbf{p}(\mathbf{x}_1^{t-1})\cdot\mathbf{x}_t)$.

A portfolio strategy $\mathbf{p}^*$ is called **universal** with respect to a reference class $\mathscr{A}$ of portfolio strategies if

$$\liminf_{T\to\infty}\frac{1}{T}\ln\frac{S_T(\mathbf{p}^*)}{S_T(\mathbf{p})} \geq 0 \text{ (almost surely for stochastic model)}$$

for each $\mathbf{p}\in\mathscr{A}$.

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Example of a constant rebalanced portfolio

A constant rebalanced portfolio (CRP) is an investment strategy which keeps the same distribution of wealth among a set of stocks from day to day.

Consider sequence of market vectors

$$\mathbf{x}_t = (1, \frac{1}{2}), (1, 2), (1, \frac{1}{2}), (1, 2), (1, \frac{1}{2}), (1, 2), \dots$$

Buy and Hold strategy earns no profit.
The portfolio strategy $\mathbf{p}_t = \mathbf{p} = (\frac{1}{2}, \frac{1}{2})$ earns for $2T$ rounds the gain

$$S_T = 1 \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2\right) \cdots = \left(\frac{9}{8}\right)^T \to \infty$$

for $T \to \infty$.

Example of probability depended portfolio algorithm

Log-optimal constant rebalanced portfolio: i.i.d. case

$\mathbf{X}_1, \mathbf{X}_2, \ldots$ be i.i.d random market vectors (with bounded logarithm)

Let $\mathbf{p}^*$ be such that $E(\ln(\mathbf{p}^* \cdot \mathbf{X})) = \sup_{\mathbf{p}} E(\ln(\mathbf{p} \cdot \mathbf{X}))$.

$S_T(\mathbf{p}^*) = \prod_{t=1}^{T}(\mathbf{p}^* \cdot \mathbf{X_t})$ – the wealth of $\mathbf{p}^*$ after T rounds.

Asymptotic optimality of the log-optimal constant rebalanced portfolio in i.i.d case:

$$\frac{1}{T} \ln S_T(\mathbf{p}^*) \to E_{X \sim \mathbf{P}} \ln(\mathbf{p}^* \cdot \mathbf{X}) \text{ and}$$

$$\liminf_{T \to \infty} \frac{1}{T} \ln \frac{S_T(\mathbf{p}^*)}{S_T(\mathbf{p})} \geq 0 \text{ almost surely,}$$

where $S_T(\mathbf{p}) = \prod_{t=1}^{T}(\mathbf{p}(\mathbf{X_1}, \ldots, \mathbf{X_{i-1}}) \cdot \mathbf{X_i})$ be the wealth after T rounds using any causal portfolio $\mathbf{p}(\mathbf{X_1}, \ldots, \mathbf{X_{i-1}})$.

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Example of probability-free portfolio (Cover, Morvai, Udina)

The log-optimal portfolio according to the empirical distribution of price relatives observed in the past

$\mathbf{b}(\mathbf{x}_1,\ldots,\mathbf{x}_{t-1}) = \mathrm{argmax}_{\mathbf{p}} \frac{1}{t-1}\sum_{i=1}^{t-1} \ln(\mathbf{p}\cdot\mathbf{x}_i)$,

where $\mathbf{x}_1,\ldots,\mathbf{x}_{t-1}$ are observed market vectors.

Theorem

Let $\mathbf{X}_1,\mathbf{X}_2,\ldots$ be a stationary and ergodic sequence of market vectors such that $E|\ln X_{i,t}| < \infty$ for all i and t. Then

$\lim_{T\to\infty} \frac{1}{T} \ln \prod_{t=1}^{T} (\mathbf{b}(\mathbf{X}_1,\ldots,\mathbf{X}_{t-1})\cdot\mathbf{X}_t) = E(\ln(\mathbf{p}^*\cdot\mathbf{X}))$ *a.s.,*

where $\mathbf{p}^*$ is an optimal constant rebalanced portfolio, i.e.,

$E(\ln(\mathbf{p}^*\cdot\mathbf{X})) = \sup_{\mathbf{p}\in\Gamma_N} E(\ln(\mathbf{p}\cdot\mathbf{X}))$.

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Machine learning framework

Prediction with Expert Advice

- No assumptions on a source of price relatives
- Game between Market and Trader
- Worst case performance bounds
- We consider additive quantities and losses instead of rewards

Hedge algorithm by Freund and Shapire (1997)

Define initial weights $\mathbf{p}_1 = \frac{1}{N}\mathbf{e}$, where $\mathbf{e} = (1, \ldots, 1)$, $\eta > 0$.

FOR $t = 1, \ldots T$

Learner presents a prediction $\mathbf{p}^*_t = \frac{\mathbf{p}_t}{\|\mathbf{p}_t\|_1}$.

Experts reveal their losses $\mathbf{l}_t = (l_{1,t}, \ldots, l_{N,t})$.

Learner suffers the mixloss $m_t = -\frac{1}{\eta} \ln \sum_{i=1}^N p^*_{i,t} e^{-\eta l_{i,t}}$.

Learner modifies weights $p_{i,t+1} = p_{i,t} e^{-\eta l_{i,t}}$.

ENDFOR

$L^i_T = \sum_{t=1}^T l_{i,t}$ – cumulative loss of any expert i, $M_T = \sum_{t=1}^T m_t$.

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Analysis of Hedge

$L_T^i = \sum_{t=1}^T l_{i,t}$ – cumulative loss of the expert i

$W_t = \sum_{i=1}^N p_{i,t} = \sum_{i=1}^N p_{i,1} e^{-\eta L_t^i}$

$m_t = -\frac{1}{\eta} \ln \sum_{i=1}^N p_{i,t}^* e^{-\eta l_{i,t}} = -\frac{1}{\eta} \ln \frac{W_{t+1}}{W_t}$.

$M_T = \sum_{t=1}^T m_t = -\frac{1}{\eta} \sum_{t=1}^T \ln \frac{W_{t+1}}{W_t} = -\frac{1}{\eta} \ln W_T = -\frac{1}{\eta} \sum_{i=1}^N p_{i,1} e^{-\eta L_T^i}$.

Lemma. $M_T = -\frac{1}{\eta} \ln \sum_{i=1}^N p_{i,1} e^{-\eta L_T^i}$.

Theorem

Performance of Hedge: $M_T \leq \min_i L_T^i + \frac{\ln N}{\eta}$ *for each* i.

Analogy with portfolio theory

Define initial weights $\mathbf{p}_1 = \frac{1}{N}\mathbf{e}$, where $\mathbf{e} = (1, \ldots, 1)$, $\eta = 1$.

FOR $t = 1, \ldots T$

Trader presents a prediction $\mathbf{p}_t^* = \frac{\mathbf{p}_t}{\|\mathbf{p}_t\|_1}$.

Market reveals their losses $\mathbf{l}_t = (l_{1,t}, \ldots, l_{N,t}) = (-\ln x_{1,t}, \ldots, -\ln x_{N,t})$.

Trader suffers the mixloss $m_t = -\ln \sum_{i=1}^N p_{i,t}^* e^{-\eta l_{i,t}} = -\ln(\mathbf{p}_t^* \cdot \mathbf{x}_i)$.

Trader modifies weights $p_{i,t+1} = p_{i,t} e^{-\eta l_{i,t}} = p_{i,t} x_{i,t}$.

ENDFOR

$L_T^i = \sum_{i=1}^T l_t^i = -\sum_{t=1}^T \ln x_{i,t} = -\ln \prod_{t=1}^T x_{i,t} = -\ln S(\mathbf{e}_i)$.

$M_T = -\sum_{t=1}^T \ln(\mathbf{p}_t^* \cdot x_t) = -\ln S_T^* = -\ln \frac{1}{N} \sum_{i=1}^N S_T(\mathbf{e}_i)$

Theorem

$S_T^* = \prod_{t=1}^T (\mathbf{p}_t^* \cdot \mathbf{x}_t) \geq \frac{1}{N} \max_{1 \leq i \leq N} S_T(\mathbf{e}_i)$ *for each i.*

Switching portfolio algorithm (Fixed Share)

Define initial portfolio $\mathbf{p}_1 = \frac{1}{N}\mathbf{e}$, where $\mathbf{e} = (1, \dots, 1)$, $0 < \alpha < 1$.

FOR $t = 1, \dots T$

Trader predicts a portfolio $\mathbf{p}_t$.

Market reveals market vector $\mathbf{x}_t = (x_{1,t}, \dots, x_{N,t})$.

Trader modifies portfolio weights $\mathbf{p}^m_t$: $p^m_{i,t} = \frac{p_{i,t} x_{i,t}}{(\mathbf{p}_t \cdot \mathbf{x}_t)}$.

Trader defines portfolio for the next step $\mathbf{p}_{t+1} = \frac{\alpha}{N}\mathbf{e} + (1 - \alpha)\mathbf{p}^m_t$, where $\mathbf{e} = (1, 1, \dots, 1)$.

ENDFOR

Performance of swithing portfolio algorithm

Let $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T}$ be a sequence of portfolios, where $\mathbf{e}_i$ is a unit vector, whose ith coordinate is 1 and other ones are 0. (T is a fixed horizon).

$s(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T}) = |\{s : i_{s-1} \ne i_s\}|$ – complexity of this portfolio strategy and

$S_T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T}) = \prod_{t=1}^{T} x_{i_t,t}$ is its cumulative wealth.

$S_T^* = \prod_{t=1}^{T}(\mathbf{p}_t \cdot \mathbf{x}_t)$ is wealth of the switching portfolio algorithm.

Theorem

For a suitable variable parameter $\alpha = \alpha_t$, for any T and k and for any sequence of market vectors given on-line,
$S_T^* \ge (TN)^{-(k+1)} \max_{s(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T}) \le k} S_T(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_T})$ – guarantee for wealth achieved by switching portfolio algorithm.

Method of multiplicative updates by Helmbold et al. 1998

$F(\mathbf{p}_t) = \eta \ln(\mathbf{p}_t \cdot \mathbf{x}_t) - D(\mathbf{p}_t \| \mathbf{p}_{t-1}) \to \max$
under constaints $\|\mathbf{p}_t\|_1 = 1$, where $D(\mathbf{p}\|\mathbf{q})$ is relative entropy.

$\mathbf{p}^* = \{\mathbf{p}_t^*\}$, where $\mathbf{p}_t^*$ is a maximizer of the first order Taylor approximation of $F(\mathbf{p}_t)$ around $\mathbf{p}_{t-1}$.

Theorem

For a suitable variable parameter η, for any T and for any sequence of market vectors given on-line,

$\ln S_T(\mathbf{p}^*) \geq \max_{\mathbf{p}} \ln S_T(\mathbf{p}) - \frac{\sqrt{2T\ln N}}{2r}$, *where $x_{i,t} \geq r > 0$ for all i and t.*

Also, $S_T(\mathbf{p}^*) \geq e^{-\frac{\sqrt{2T\ln N}}{2r}} \max_{\mathbf{p}} S_T(\mathbf{p})$.

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Aggregating algorithm AA (Vovk - 1991)

$\lambda(\omega,\gamma)$ – loss function ($\lambda(\omega,\gamma) = (\omega-\gamma)^2$ – example).

$dP_1(\theta)$ – initial distribution, $\int dP_1(\theta) = 1$, $0 < \eta < 1$.

FOR $t = 1, \dots T$

Expert θ reveals a forecast ξ_t^θ.

Learner reveals a forecast γ_t satisfying mixability condition:
$\lambda(\omega,\gamma_t) \le -\frac{1}{\eta} \ln \int e^{-\eta\lambda(\omega,\xi_t^\theta)} dP_t^*(\theta)$ for all ω, where
$dP_t^*(\mathbf{p}) = \frac{dP_t(\mathbf{p})}{\int dP_t(\mathbf{p})}$.

Nature reveals an outcome $\omega_t \in \Omega$.

Expert $\theta \in \Theta$ suffers loss $\lambda(\omega_t, \xi_t^\theta)$.

Learner suffers loss $\lambda(\omega_t, \gamma_t)$.

Learner modifies weights $dP_{t+1}(\theta) = e^{-\eta\lambda(\omega_t,\xi_t^\theta)} dP_t(\theta)$.

ENDFOR

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Performance of AA

$L_T(\theta) = \sum_{t=1}^{T} \lambda(\omega_t, \xi_t^{\theta})$ – cumulative loss of the expert θ.

$L_T = \sum_{t=1}^{T} \lambda(\omega_t, \gamma_t)$ – cumulative loss of AA.

Since $\lambda(\omega, \gamma_t) \le -\frac{1}{\eta} \ln \int e^{-\eta\lambda(\omega, \xi_t^{\theta})} dP_t^*(\theta)$ for all ω, this is true for $\omega = \omega_t$, and we have $\lambda(\omega_t, \gamma_t) \le m_t$ and then $L_T \le M_T$.

Lemma.

(1) $M_T = -\frac{1}{\eta} \ln \int e^{-\eta L_T(\theta)} dP_1(\theta)$ (as for Hedge).

(2) $L_T \le -\frac{1}{\eta} \ln \int e^{-\eta L_T(\theta)} dP_1(\theta)$.

(3) $L_T \le \min_\theta L_T(\theta) + \frac{\ln N}{\eta}$ if there are N experts θ.

In general case we have only (2).

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Universal portfolio algorithm by Thomas Cover (1991-1997)

$\lambda(\mathbf{x},\mathbf{p}) = -\ln(\mathbf{p}\cdot\mathbf{x})$ – loss function.

$dP_1(\theta)$ – initial distribution, $\eta = 1$.

FOR $t = 1,\dots T$

Expert $\mathbf{p}$ reveals a forecast $\mathbf{p} = (p_1,\dots,p_N)$ – portfolio.

Trader reveals a forecast $\mathbf{p}_t^* = \int \mathbf{p}\, dP_t^*(\mathbf{p})$, where

$dP_t^*(\mathbf{p}) = \frac{dP_t(\mathbf{p})}{\int dP_t(\mathbf{p})}$.

Market reveals a market vector $\mathbf{x}_t = (x_{1,t},\dots,x_{N,t})$.

Expert $\mathbf{p}$ multiplies its wealth by $(\mathbf{p}\cdot\mathbf{x}_t)$.

Trader multiplies its wealth by $(\mathbf{p}_t^*\cdot\mathbf{x}_t)$.

Trader modifies experts weights $dP_{t+1}(\mathbf{p}) = (\mathbf{p}\cdot\mathbf{x}_t)dP_t(\mathbf{p})$.

ENDFOR

Duality: loss = $-\ln(\text{wealth})$

$L_T(\mathbf{p}) = -\sum_{t=1}^{T} \ln(\mathbf{p}\cdot\mathbf{x}_t)$ and $S_T(\mathbf{p}) = \prod_{t=1}^{T}(\mathbf{p}\cdot\mathbf{x}_t)$ are cumulative loss and wealth of the constant rebalanced portfolio **p**.

$L_T^* = -\sum_{t=1}^{T} \ln(\mathbf{p}_t^*\cdot\mathbf{x}_t)$ and $S_T^* = \prod_{t=1}^{T}(\mathbf{p}_t\cdot\mathbf{x}_t)$ are cumulative loss and cumulative wealth of universal portfolio.

Mixability condition holds as equality
$\lambda(\mathbf{x},\mathbf{p}_t) = -\ln\left(\mathbf{x}\cdot\int \mathbf{p}\,dP_{t-1}^*(\mathbf{p})\right) = -\ln\int(\mathbf{x}\cdot\mathbf{p})dP_{t-1}^*(\mathbf{p})$ and then
$\lambda(\mathbf{x}_t,\mathbf{p}_t) = m_t$ for all t, then $L_T^* = M_T$.

Since $L_T^* = M_T$, by Lemma
$L_T^* = -\ln\int e^{-L_T(\mathbf{p})}dP_1(\mathbf{p})$ and $S_T^* = \int S_T(\mathbf{p})dP_1(\mathbf{p})$.

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Performance of the universal portfolio

Theorem

The portfolio strategy

$$\mathbf{p}_t^* = \frac{\int \mathbf{p} \prod_{s=1}^{t} (\mathbf{p} \cdot \mathbf{x}_s) dP_1(\mathbf{p})}{\int \prod_{s=1}^{t} (\mathbf{p} \cdot \mathbf{x}_s) dP_1(\mathbf{p})} \qquad .$$

is universal for the class of all constant rebalanced portfolios:

$$S_T^* \geq c T^{-\frac{N-1}{2}} \max_{\mathbf{p}} S_T(\mathbf{p})$$

for all T, where $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is an arbitrary sequence of market vectors, $P_1(\mathbf{p})$ is the Dirichlet distribution on the simplex.

$\frac{1}{T} \ln \frac{S_T^*}{S_T(\mathbf{p})} \geq -\frac{N-1}{2T} \ln T + O\left(\frac{1}{T}\right)$ – in terms of growth rates.

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Details of the proof

Dirichlet distribution with parameters $(1/2, \ldots, 1/2)$ in the simplex of all portfolios:

$$dP_1(\mathbf{p}) = \frac{\gamma(N/2)}{[\gamma(1/2)]^N} \prod_{j=1}^{N} p_j^{-1/2} d\mathbf{p}, \text{ where } \gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx.$$

Recalling that $S_T^* = \int S_T(\mathbf{p}) dP_1(\mathbf{p})$, we should prove that

$$\frac{\sup_{\mathbf{p}} S_T(\mathbf{p})}{\int S_T(\mathbf{p}) dP_1(\mathbf{p})} = \frac{\sup_{\mathbf{p}} \prod_{t=1}^{T} (\mathbf{p} \cdot \mathbf{x}_t)}{\int \prod_{t=1}^{T} (\mathbf{p} \cdot \mathbf{x}_t) dP_1(\mathbf{p})} \leq \varepsilon^{-1} T^{\frac{N-1}{2}}$$

for all **x**.

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Stationary markets

$\mathbf{X}_1, \mathbf{X}_2, \ldots$ – stationary and ergodic process.

Log-optimal causal portfolio strategy $\mathbf{b}^*(\cdot)$:

$$E\left(\ln(\mathbf{b}^*(\mathbf{X}_1^{t-1}) \cdot \mathbf{X}_t) | \mathbf{X}_1^{t-1}\right) = \sup_{\mathbf{b}(\cdot)} E\left(\ln(\mathbf{b}(\mathbf{X}_1^{t-1}) \cdot \mathbf{X}_t) | \mathbf{X}_1^{t-1}\right).$$

Algoet and Cover (1988) proved asymptotic optimality property of this strategy:

$\liminf_{T\to\infty} \frac{1}{T} \ln \frac{S_T(\mathbf{b}^*)}{S_T(\mathbf{b})} \geq 0$ a.s. and

$\lim_{T\to\infty} \frac{1}{T} \ln S_T(\mathbf{b}^*) = W^*$ a.s.,

where $W^* = E\left(\sup_{\mathbf{b}(\cdot)} E\left(\ln(\mathbf{b}(\mathbf{X}_{-\infty}^{-1}) \cdot \mathbf{X}_0) | \mathbf{X}_{-\infty}^{-1}\right)\right)$ is the **maximal possible growth rate** of any investment strategy (with respect to the process).

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Györfi and Schäfer Kernel (histogram) universal strategy

$$\mathbf{h}^{k,l}(\mathbf{x}_1^{t-1}) = \mathrm{argmax}_{\mathbf{b}\in\Gamma_N} \prod_{k\le i\le l:\mathbf{x}_{i-k}^{i-1}\sim\mathbf{x}_{t-k}^{t-1}} (\mathbf{b}\cdot\mathbf{x}_i),$$

where $\mathbf{x}_{i-k}^{i-1} \sim \mathbf{x}_{t-k}^{t-1}$ means that they are in the same element of some partition. This portfolio vector is optimal for those past trading periods whose preceding k trading periods have identical discretized market vectors to the present one.

Universal portfolio strategy $S_T(B) = \sum_{k,l} q_{k,l} S_T(h_{k,l})$.

Theorem

For each stationary ergodic sequence of market vectors $\lim_{T\to\infty} \frac{1}{T} \ln S_T(\mathbf{B}) = W^*$, *where* $W^* = E\left(\sup_{\mathbf{b}(\cdot)} E\left(\ln(\mathbf{b}(\mathbf{X}_{-\infty}^{-1})\cdot\mathbf{X}_0)|\mathbf{X}_{-\infty}^{-1}\right)\right)$.

Universal well calibrated portfolio by Cover and Gluss

$A_\varepsilon = \{\mathbf{a}_1, \dots, \mathbf{a}_M\}$ is an ε-net in the set of all market vectors.

$P(A_\varepsilon)$ – set of all probability distributions (vectors) on the set A_ε and $\mathscr{P}_\varepsilon = \{\mathbf{s}_1, \dots, \mathbf{s}_N\}$ is an ε-net in $P(A_\varepsilon)$.

We also discretize the set of all histories $\dots, \mathbf{z}_{-1}, \mathbf{x}_{-1}, \mathbf{z}_0, \mathbf{x}_0, \mathbf{z}_1, \mathbf{x}_1, \dots, \mathbf{z}_t, \mathbf{x}_t, \dots$.

Using the Blackwell approachability theorem a randomized algorithm can be constructed which with probability one generates a sequence of probability vectors $\mathbf{c}_1, \mathbf{c}_2, \dots \in P(A_\varepsilon)$ such that for any i the probability $\mathbf{c}_i$ is close to empirical distribution of market vectors $\mathbf{x}_i$ (calibration property).

At any step t, using artificial probability $\mathbf{c}_t$, define

$$\mathbf{p}_t^* = \mathrm{argmax}_{\mathbf{p}} E_{\mathbf{X} \sim \mathbf{c}_t}(\ln(\mathbf{p} \cdot \mathbf{X})). \tag{1}$$

Performance of randomized portfolio strategy

Theorem

The randomized portfolio strategy $\mathbf{p}^* = \{\mathbf{p}_t^*\}$ *is universal:*

$$\liminf_{T\to\infty} \frac{1}{T} \ln \frac{S_T(\mathbf{p}^*)}{S(\mathbf{p})} \geq 0 \qquad (2)$$

for almost all trajectories $\mathbf{c}_1, \mathbf{c}_2, \ldots,$ *where* $S_T(\mathbf{p}^*) = \prod_{t=1}^{T}(\mathbf{p}_t^* \cdot \mathbf{x}_t)$ *and* $S_T(\mathbf{p}) = \prod_{t=1}^{T}(\mathbf{p}(\sigma_t) \cdot \mathbf{x}_t)$ *is the wealth achieved by an arbitrary Lipschitz continuous causal portfolio* $\mathbf{p}(\sigma_t)$, *and* $\sigma_t = \ldots, \mathbf{z}_{-1}, \mathbf{x}_{-1}, \mathbf{z}_0, \mathbf{x}_0, \mathbf{z}_1, \mathbf{x}_1, \ldots, \mathbf{x}_{t-1}, \mathbf{z}_t$ *is the history at any round* t.

Rate of convergence

Poor rate of convergence $O\left(\frac{1}{(\ln T)^{\frac{1}{N+1}-\nu}}\right)$ in the theorem, where ν is an arbitrary small positive real number, and N is the number of assets.
More precise, for any $\delta > 0$, with probability $1-\delta$,

$$\frac{1}{T}\ln S_T^* \geq \frac{1}{T}\ln S_T - \left(\ln\left(\frac{T}{\ln\frac{1}{\delta}}\right)\right)^{-\frac{1}{N+1}+\nu}$$

for all T, where $S_T = \prod_{t=1}^{T}(\mathbf{b}(\sigma_t)\cdot\mathbf{x}_t)$ is the wealth achieved by an arbitrary Lipschitz continuous portfolio $\mathbf{b}(\cdot)$.

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