On Almost Complete Subsets of a Conic in PG$(2,q)$, Completeness of Normal Rational Curves and Extendability of Reed-Solomon Codes

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Abstract

A subset $\mathcal{S}$ of a conic $\mathcal{C}$ in the projective plane PG$(2,q)$ is called *almost complete* (AC-subset for short) if it can be extended to a larger arc in PG$(2,q)$ only by the points of $\mathcal{C}\setminus\mathcal{S}$ and by the nucleus of $\mathcal{C}$ when q is even. New upper bounds on the smallest size $t(q)$ of an AC-subset are obtained, in particular,

$$t(q) < \sqrt{q(3\ln q+\ln\ln q+\ln 3)}+\sqrt{\frac{q}{3\ln q}}+4\sim\sqrt{3q\ln q};$$
$$t(q) < 1.835\sqrt{q\ln q}.$$

The new bounds are used to increase regions of pairs (N,q) for which it is proved that every normal rational curve in PG(N,q) is a complete $(q+1)$-arc or, equivalently, that no $[q+1,N+1,q-N+1]_q$ generalized doubly-extended Reed-Solomon code can be extended to a $[q+2,N+1,q-N+2]_q$ MDS code.

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1 Introduction

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the Galois field $\mathbb{F}_q$ of order q. An n-arc in $\mathrm{PG}(N,q)$ with $n > N+1$ is a set of n points such that no $N+1$ points belong to the same hyperplane of $\mathrm{PG}(N,q)$. An n-arc of $\mathrm{PG}(N,q)$ is complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(N,q)$. In $\mathrm{PG}(N,q)$ with $2 \leq N \leq q-2$, a normal rational curve is any $(q+1)$-arc projectively equivalent to the arc $\{(1,t,t^2,\ldots,t^N) : t \in \mathbb{F}_q\} \cup \{(0,\ldots,0,1)\}$. For an introduction to projective geometries over finite fields see [1–3].

Let an $[n,k,d]_q$ code be a q-ary linear code of length n, dimension k, and minimum distance d. If $d = n-k+1$, it is a maximum distance separable (MDS) code. The code dual to an $[n,k,n-k+1]_q$ MDS code is an $[n,n-k,k+1]_q$ MDS code.

Points (in the homogeneous coordinates) of an n-arc in $\mathrm{PG}(N,q)$ treated as columns define a generator matrix of an $[n,N+1,n-N]_q$ MDS code. If an n-arc in $\mathrm{PG}(N,q)$ is complete then the corresponding $[n,N+1,n-N]_q$ MDS code cannot be extended to an $[n+1,N+1,n-N+1]_q$ MDS code. For properties of linear MDS codes and their equivalence to arcs see e.g. [1–14].

The j-th column of a generator matrix of a $[q+1,N+1,q-N+1]_q$ generalized doubly-extended Reed-Solomon (GDRS) code has the form $(v_j, v_j\alpha_j, v_j\alpha_j^2, \ldots, v_j\alpha_j^N)^T$, where $j = 1,2,\ldots,q$; $\alpha_1,\ldots,\alpha_q$ are distinct elements of $\mathbb{F}_q$; $v_1,\ldots,v_q$ are nonzero (not necessarily distinct) elements of $\mathbb{F}_q$. Also, this matrix contains one more column $(0,\ldots,0,v)^T$ with $v \neq 0$. The code, dual to a GDRS code, is a GDRS code too.

Points (in the homogeneous coordinates) of a normal rational curve in $\mathrm{PG}(N,q)$ treated as columns define a generator matrix of a $[q+1,N+1,q-N+1]_q$ GDRS code. Proposition 1.1 is well known.

Proposition 1.1. *Let N and q be fixed integers with $2 \leq N \leq q-2$. Moreover, let q be a prime power. The following statements are equivalent:*
- *Every normal rational curve in $\mathrm{PG}(N,q)$ is a complete $(q+1)$-arc;*
- *No $[q+1,N+1,q-N+1]_q$ GDRS code can be extended to a $[q+2,N+1,q-N+2]_q$ MDS code.*

Due to Proposition 1.1, all results given below on completeness of normal rational curves can be reformulated in coding theory language for extendability of GDRS codes.

The completeness of normal rational curves and related problems are considered in numerous works starting from Segre's paper [15] of 1955; see for example [1–20], where surveys and references can be found. In particular, the following conjecture, connected with the famous Segre's three problems, is well known.

Conjecture 1.2. *Let* $2 \leq N \leq q-2$. *Every normal rational curve in* $\mathrm{PG}(N,q)$ *is a complete* $(q+1)$*-arc except for the cases* q *even and* $N \in \{2, q-2\}$ *when one point can be added to the curve.*

Remark 1.3. As a comment to Conjecture 1.2 for q even, note the following. If $N=2$, the point which can be added to a normal rational curve is unique. But if $N=q-2$, there are many points in $\mathrm{PG}(q-2,q)$ which extend a normal rational curve to a $(q+2)$-arc, see [13, Theorem 3.10] for the geometrical characterization of these points.

Remark 1.4. If $k \geq q$ then an $[n,k,n-k+1]_q$ MDS code has length $n \leq k+1$, see e.g. [10, 11]. For $2 \leq N \leq q-2$, the well known *MDS conjecture* assumes that an $[n, N+1, n-N]_q$ MDS code (or equivalently an n-arc in $\mathrm{PG}(N,q)$) has length $n \leq q+1$ except for the cases q even and $N \in \{2, q-2\}$ when $n \leq q+2$. The MDS conjecture considers all MDS codes (or all arcs) whereas Conjecture 1.2 says only something about normal rational curves (or GDRS codes). If the MDS conjecture holds for some pair (N,q) then Conjecture 1.2 holds too, but in general the reverse is not true.

For many pairs (N,q) Conjecture 1.2 is proved, see [1–11, 14–20] and the references therein; but in general, *completeness of normal rational curves is an open problem.* The main known results are given in Table 1, where p and $p_0(h)$ are *prime.* For rows 1–6 of Table 1, in fact, the MDS conjecture is proved. In [5], see row 7 of Table 1, it is proved that a subset of size $3(N-1)-6$ of a normal rational curve in $\mathrm{PG}(N,q)$, q odd, cannot be extended to an arc of size $q+2$. This means that $3N-3 \leq q+1$ (otherwise the curve could not contain a such subset). So, $N \leq \frac{q+4}{3}$. The regions of N in rows 10–11 cover the ones in rows 6–8; we included rows 6–8 in Table 1 as the methods used for them are useful for further research.

For the problem of completeness of normal rational curves we use tools connected with almost complete subsets of a conic in the projective plane $\mathrm{PG}(2,q)$.

An n-arc in $\mathrm{PG}(2,q)$ is a set of n points no three of which are collinear. A point P of $\mathrm{PG}(2,q)$ is covered by an arc $\mathcal{K} \subset \mathrm{PG}(2,q)$ if P lies on a bisecant of $\mathcal{K}$. Throughout the paper, $\mathcal{C} = \{(1,t,t^2) : t \in \mathbb{F}_q\} \cup \{(0,0,1)\}$ is a fixed conic in $\mathrm{PG}(2,q)$. Any point subset of $\mathcal{C}$ is an arc. For even q, denote by $\mathcal{O}$ the nucleus of $\mathcal{C}$. Let

$$\mathcal{M}_q := \begin{cases} \mathrm{PG}(2,q) \setminus \mathcal{C} & \text{if } q \text{ odd} \\ \mathrm{PG}(2,q) \setminus (\mathcal{C} \cup \{\mathcal{O}\}) & \text{if } q \text{ even} \end{cases}.$$

Definition 1.5. **(i)** In $\mathrm{PG}(2,q)$, an *almost complete subset* of the conic $\mathcal{C}$ (*AC-subset*, for short) is a proper subset of $\mathcal{C}$ covering all the points of $\mathcal{M}_q$. An n-AC-subset is an AC-subset of size n.

(ii) An AC-subset is *minimal* if it does not contain a smaller AC-subset.

Table 1: Pairs (N,q) for which it is proved that every normal rational curve in $\mathrm{PG}(N,q)$ is a complete $(q+1)$-arc

no.	q	N	Reference
1	$q=p^{2h+1}$, $p\geq 3$, $h\geq 1$	$q-\frac{1}{4}\sqrt{pq}+\frac{29}{16}p-3<N\leq q-3$	[2, Table 3.4]
2	$q=p^h$, $p\geq 5$	$q-\frac{1}{2}\sqrt{q}+1<N\leq q-3$	[2, Table 3.4]
3	$q=p^h\geq 23^2$; $p\geq 3$; $q\neq 5^5,3^6$; h even for $q=3$	$q-\frac{1}{2}\sqrt{q}-1<N\leq q-3$	[2, Table 3.4]
4	$q=2^h$, $h>2$	$q-\frac{1}{2}\sqrt{q}-\frac{11}{4}<N\leq q-5$	[2, Table 3.4]
5	$q=p$	$2\leq N\leq p-1$	[4,6,16,17]
6	$q=p^2$	$2\leq N\leq 2\sqrt{q}-3$	[4,6,16,17]
7	q odd	$N\leq\frac{q+4}{3}$	[5, Theorem 1.4]
8	all q	$3\leq N\leq q+2-6\sqrt{q\ln q}$	[19, Theorem 3.3]
9	$q=p^{2h+1}$; $p\geq p_0(h)$; $p_0(h)$ is the smallest $\widehat{p}$ satisfying $\sqrt{\widehat{p}}>24\sqrt{(2h+1)\ln\widehat{p}}$ $+\frac{29}{4\widehat{p}^{\,h-0.5}}-\frac{20}{\widehat{p}^{\,h+0.5}}$	$2\leq N\leq q-2$	[19, Theorem 3.5]
10	q odd	$2\leq N\leq q-2-\sqrt{7(q+1)\ln q}$	[18, Theorem 9.2]
11	q even	$3\leq N\leq q-1-\sqrt{7(q+1)\ln q}$	[18, Theorem 9.2]

Note that an AC-subset $\mathcal{S}$ is an arc that can be extended to a larger arc in $\mathrm{PG}(2,q)$ only by the points of $\mathcal{C}\setminus\mathcal{S}$ and by the nucleus $\mathcal{O}$ when q is even. The term "almost completeness" was introduced in [18, p. 94] for objects in the affine plane $\mathrm{AG}(2,q)$.

Denote by $t(q)$ *the smallest size of an AC-subset in* $\mathrm{PG}(2,q)$.

In this work we provide new *upper bounds* on $t(q)$. This is an *open problem*. It is addressed, for example, in [19–21]. In [21], by probabilistic methods, it is proved that

$$t(q)<6\sqrt{q\ln q}. \tag{1.1}$$

In [19, Theorem 3.1], using the results and approaches of [20], the following connection between $t(q)$ and the completeness of normal rational curves is proved:

under the condition

$$3\leq N\leq q+2-t(q), \tag{1.2}$$

every normal rational curve in $\mathrm{PG}(N,q)$ *is a complete* $(q+1)$*-arc.*

The *aims of this paper* are as follows: obtain new upper bounds on the smallest size of an AC-subset of a conic in $\mathrm{PG}(2,q)$; using the bounds, extend regions of pairs (N,q) for which it is proved that every normal rational curve in $\mathrm{PG}(N,q)$ is a complete $(q+1)$-arc.

The paper is organized as follows. In Section 2 the main results of this paper are formulated. In Section 3, we consider an estimate of the number of new covered points in one step of a step-by-step algorithm constructing AC-subsets. In Section 4, implicit and explicit upper bounds on $t(q)$, based on the results of Section 3, are obtained. In Section 5, computer assisted bounds on $t(q)$ are studied. In Section 6, new bounds on $t(q)$ are applied to the problem of completeness of normal rational curves. Finally, in Appendix tables of the smallest known sizes $\overline{t}(q)$ of AC-subsets in $\mathrm{PG}(2,q)$ are given.

2 The main results

We introduce the following set of prime powers.

$$Q_1 := \{8 \le q \le 139129,\ q = p^m,\ p \text{ prime},\ m \ge 2\}. \tag{2.1}$$

Throughout the paper we denote

$$\Phi(q) = \sqrt{q(3\ln q + \ln\ln q + \ln 3)} + \sqrt{\frac{q}{3\ln q}} + 4 \sim \sqrt{3}\sqrt{q\ln q}; \tag{2.2}$$

$$\Theta(q) = \begin{cases} 1.62\sqrt{q\ln q} & \text{for} \quad 8 \le q \le 17041 \\ 1.635\sqrt{q\ln q} & \text{for} \quad 17041 < q \le 33013 \\ 1.674\sqrt{q\ln q} & \text{for} \quad q \in Q_1 \\ \min\{1.835\sqrt{q\ln q}, \Phi(q)\} & \text{for} \quad \text{all } q \ge 5 \end{cases}, \tag{2.3}$$

where

$$\min\{1.835\sqrt{q\ln q}, \Phi(q)\} = \begin{cases} 1.835\sqrt{q\ln q} & \text{for} \quad q < 12755807 \\ \Phi(q) & \text{for} \quad 12755807 \le q \end{cases}.$$

The main result of this paper is Theorem 2.1 based on Theorems 4.10, 4.12, and 5.1.

Theorem 2.1. *The following upper bound on the smallest size $t(q)$ of an AC-subset of the conic $\mathcal{C}$ in $\mathrm{PG}(2,q)$ holds:*

$$t(q) < \Theta(q). \tag{2.4}$$

Similarly to [19], we use upper bounds on $t(q)$ to prove the completeness of the normal rational curves as arcs in projective spaces. From Theorem 2.1 and [19, Theorems 3.1,3.5] we obtained Corollaries 2.2 and 2.3; see Section 6.

Corollary 2.2. *Let*

$$3 \le N \le q + 2 - \Theta(q). \tag{2.5}$$

Then every normal rational curve in $\mathrm{PG}(N,q)$ is a complete $(q+1)$-arc.

Corollary 2.3. *Let $h \geq 1$ be a fixed integer. Let $p_0(1) = 757$, $p_0(2) = 1399$, $p_0(3) = 2129$, $p_0(4) = 2887$, $p_0(5) = 3623$. Also, for $h \geq 6$ let $p_0(h)$ be the smallest odd prime $\widehat{p}$ satisfying*

$$\sqrt{\widehat{p}} > 4c\sqrt{(2h+1)\ln \widehat{p}} + \frac{29}{4\widehat{p}^{\,h-0.5}} - \frac{20}{\widehat{p}^{\,h+0.5}}\ , \tag{2.6}$$

where $c = 1.62$ for $6 \leq h \leq 19$, $c = 1.635$ for $20 \leq h \leq 28$, $c = 1.835$ for $h \geq 29$.

Then for every odd prime $p \geq p_0(h)$ in $\mathrm{PG}(N,q)$ with $q = p^{2h+1}$, $2 \leq N \leq q-2$, every normal rational curve is a complete $(q+1)$-arc.

Remark 2.4. In (2.6), the term $\frac{29}{4\widehat{p}^{\,h-0.5}} - \frac{20}{\widehat{p}^{\,h+0.5}}$ quickly decreases when h grows. Therefore, practically, use of inequality $\widehat{p} > 16c^2(2h+1)\ln \widehat{p}$ gives the same result as for (2.6). In particular, we have checked this for $h \leq 16$.

In Section 4 we consider also implicit upper bounds on $t(q)$.

All bounds on $t(q)$ obtained in this paper are better than the bound of (1.1).

Corollaries 2.2 and 2.3 extend regions of pairs (N,q) for which it is proved that every normal rational curve in $\mathrm{PG}(N,q)$ is a complete $(q+1)$-arc.

Corollary 2.2 improves the results of [18, Theorem 9.2], cf. (2.5) and rows 10–11 of Table 1; in (2.5) the region on N values is greater by $\sim 0.8\sqrt{q \ln q}$.

Corollary 2.3 gives essentially smaller values $p_0(h)$ than [19, Theorem 3.5]. By Corollary 2.3, we have $\{p_0(1), p_0(2), \ldots, p_0(16)\} = \{757, 1399, 2129, 2887, 3623, 4621, 5417, 6247, 7079, 7919, 8779, 9629, 10499, 11383, 12253, 13147\}$. For comparison, [19, Theorem 3.5], see row 9 of Table 1, provides $\{p_0(1), p_0(2), \ldots, p_0(16)\} = \{16831, 29663, 43037, 56747, 70769, 85009, 99431, 114031, 128767, 143651, 158647, 173741, 188953, 204251, 219629, 235091\}$.

3 The number of new covered points in one step of a step-by-step algorithm constructing AC-subsets

Assume that an AC-subset is constructed by a step-by-step algorithm (Algorithm, for short) adding a new point to the subset on every step. As an example, we mention the greedy algorithm that on every step adds to the subset a point providing the maximal possible (for the given step) number of new covered points.

Let $w > 0$ be a fixed integer. Consider the $(w+1)$st step of Algorithm. This step starts from a w-subset $\mathcal{K}_w \subset \mathcal{C}$ constructed in the previous w steps. Let $\mathcal{U}(\mathcal{K}_w)$ be the subset of points of $\mathcal{M}_q$ not covered by the subset $\mathcal{K}_w$.

Let the subset $\mathcal{K}_w$ consist of w points $A_1, A_2, \ldots, A_w$. Let $A_{w+1} \in \mathcal{C} \setminus \mathcal{K}_w$ be the point that will be included into the subset in the $(w+1)$st step. Denote by $\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\})$ the subset of points of $\mathcal{M}_q$ not covered by the new subset $\mathcal{K}_w \cup \{A_{w+1}\}$.

Let $\overline{AB}$ be the line through points A and B. The point A_{w+1} defines a bundle $\mathcal{B}(A_{w+1}) = \{\overline{A_1A_{w+1}}, \overline{A_2A_{w+1}}, \ldots, \overline{A_wA_{w+1}}\}$ of w tangents (unisecants) to $\mathcal{K}_w$ which are bisecants of $\mathcal{C}$. In order to obtain the next subset $\mathcal{K}_{w+1}$, we may include to $\mathcal{K}_w$ any of $q+1-w$ points of $\mathcal{C}\setminus\mathcal{K}_w$. So, there exist $q+1-w$ distinct points A_{w+1} and $q+1-w$ distinct bundles. Introduce the set of $w(q+1-w)$ lines

$$\mathcal{B}_{w+1}^{\cup}=\bigcup_{A_{w+1}\in\mathcal{C}\setminus\mathcal{K}_w}\mathcal{B}(A_{w+1}).$$

Let $P_{w+1}^{\cup}$ be the point multiset consisting of all points of $\mathcal{B}_{w+1}^{\cup}$. A point that is the intersection of m lines of $\mathcal{B}_{w+1}^{\cup}$ has multiplicity m in $P_{w+1}^{\cup}$.

Let $\Delta(A_{w+1})$ be the number of the *new covered* points in the $(w+1)$st step. Denote by $\mathcal{N}(A_{w+1})$ the set of *new* points *covered* by $\mathcal{K}_w\cup\{A_{w+1}\}$. By definition,

$$\begin{aligned}&\mathcal{N}(A_{w+1})=\mathcal{U}(\mathcal{K}_w)\setminus\mathcal{U}(\mathcal{K}_w\cup\{A_{w+1}\}),\\&\Delta(A_{w+1})=\#\mathcal{N}(A_{w+1})=\#\mathcal{U}(\mathcal{K}_w)-\#\mathcal{U}(\mathcal{K}_w\cup\{A_{w+1}\}).\end{aligned}$$

Introduce the point multiset

$$\mathcal{N}_{w+1}^{\cup}=\bigcup_{A_{w+1}\in\mathcal{C}\setminus\mathcal{K}_w}\mathcal{N}(A_{w+1})\subset P_{w+1}^{\cup}.$$

By the definitions above,

$$\#\mathcal{N}_{w+1}^{\cup}=\sum_{A_{w+1}\in\mathcal{C}\setminus\mathcal{K}_w}\Delta(A_{w+1}).$$

Let $P\in\mathcal{U}(\mathcal{K}_w)\subset\mathcal{M}_q$ be a point not covered by $\mathcal{K}_w$. Every point of $\mathcal{M}_q$ lies at most on two tangents of $\mathcal{C}$. The rest of lines through this point and the points of $\mathcal{C}$ are bisecants. Therefore, among the w lines connecting P with $\mathcal{K}_w$ there are at least $w-2$ bisecants of $\mathcal{C}$. None of those bisecants is a bisecant of $\mathcal{K}_w$ otherwise the point P would be covered. Hence, all bisecants of $\mathcal{C}$ through P and $\mathcal{K}_w$ belong to $\mathcal{B}_{w+1}^{\cup}$. It means that *every point of $\mathcal{U}(\mathcal{K}_w)$ is included in $\mathcal{N}_{w+1}^{\cup}$ at least $w-2$ times.* So,

$$\#\mathcal{N}_{w+1}^{\cup}\geq(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w).\tag{3.1}$$

Remark 3.1. For even q, every point of $\mathcal{M}_q$ lies on one tangent of $\mathcal{C}$. Therefore for even q, in relation (3.1) we may change $w-2$ by $w-1$. Also, for odd q, an internal point does not belong to any tangent of a conic whereas each of the $\frac{1}{2}q(q+1)$ external points lies on two distinct tangents. Hence for odd q, in (3.1) we may change $(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w)$ by $(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w)+2\max\{0,\#\mathcal{U}(\mathcal{K}_w)-\frac{1}{2}q(q+1)\}$. These changes could slightly improve estimates below. However, for simplicity of presentation, we save relation (3.1) as it is.

By the above, the average number, say $\Delta^{\text{aver}}_{w+1}$, of new covered points in a bundle in the $(w+1)$st step is as follows

$$\Delta^{\text{aver}}_{w+1} = \frac{\sum\limits_{A_{w+1}\in\mathcal{C}\setminus\mathcal{K}_w} \Delta(A_{w+1})}{q+1-w} \geq \frac{(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w)}{q+1-w}.$$

Clearly,

$$\max_{A_{w+1}\in\mathcal{C}\setminus\mathcal{K}_w} \Delta(A_{w+1}) \geq \left\lceil \Delta^{\text{aver}}_{w+1} \right\rceil .$$

So, we have proved the following lemma.

Lemma 3.2. *For an arbitrary step-by-step algorithm, there exists a point A_{w+1} providing*

$$\Delta(A_{w+1}) \geq \left\lceil \frac{(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w)}{q+1-w} \right\rceil . \tag{3.2}$$

Note that the greedy algorithm always finds the point A_{w+1} with property (3.2).

4 Upper bounds on the smallest size of an AC-subset based on properties of step-by-step algorithms

We denote

$$t^*(q) = \frac{t(q)}{\sqrt{q\ln q}} .$$

Let $t(q) < f(q)$. Then $t^*(q) < f(q)/\sqrt{q\ln q}$. The upper bounds on $t^*(q)$ are more convenient for graphical representation than bounds on $t(q)$. If $f(q)$ is called "Bound L", say, then we call $f(q)/\sqrt{q\ln q}$ "Bound L*".

4.1 Implicit bound A

By Section 3,

$$\#\mathcal{U}(\mathcal{K}_w\cup\{A_{w+1}\}) = \#\mathcal{U}(\mathcal{K}_w) - \Delta(A_{w+1}) \leq \#\mathcal{U}(\mathcal{K}_w) - \left\lceil \frac{(w-2)\cdot\#\mathcal{U}(\mathcal{K}_w)}{q+1-w} \right\rceil . \tag{4.1}$$

Define U_w as an upper bound on $\#\mathcal{U}(\mathcal{K}_w)$:

$$\#\mathcal{U}(\mathcal{K}_w) = U_w - \delta \leq U_w; \quad \delta \geq 0. \tag{4.2}$$

By (4.1), (4.2),

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \le U_w - \delta - \left\lceil \frac{(w-2)(U_w-\delta)}{q+1-w} \right\rceil =$$
$$U_w - \left\lceil \frac{(w-2)U_w + (q+3-2w)\delta}{q+1-w} \right\rceil .$$

From now on, we suppose

$$q+3 > 2w. \tag{4.3}$$

Under condition (4.3), it holds that

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = \#\mathcal{U}(\mathcal{K}_w) - \Delta(A_{w+1}) \le U_w - \left\lceil \frac{(w-2)U_w}{q+1-w} \right\rceil . \tag{4.4}$$

Assume that there exists a w_0-subset $\mathcal{K}_{w_0} \subset \mathcal{C} \subset PG(2,q)$ that does not cover at most U_{w_0} points of $\mathcal{M}_q$. Then, starting from values w_0 and U_{w_0}, one can iteratively apply the relation (4.4) and obtain eventually $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = 0$ for some w, say w_{fin}. Clearly, w_{fin} depends on w_0 and U_{w_0}, i.e. we have a function $w_{\text{fin}}(w_0, U_{w_0})$. The size k of the obtained AC-subset is as follows:

$$k = w_{\text{fin}}(w_0, U_{w_0}) + 1 \text{ under condition } \#\mathcal{U}(\mathcal{K}_{w_{\text{fin}}(w_0,U_{w_0})} \cup \{A_{w_{\text{fin}}(w_0,U_{w_0})+1}\}) = 0.$$

From the above we have the following theorem.

Theorem 4.1. **(*implicit bound* $A(w_0, U_{w_0})$)** *Let the values w_0, U_{w_0}, and $w_{fin}(w_0, U_{w_0})$ be defined and calculated as above. Let also $w_{fin}(w_0, U_{w_0}) < \frac{q+3}{2}$. Then it holds that*

$$t(q) \le w_{fin}(w_0, U_{w_0}) + 1.$$

It is easily seen that, for any q, there exists a 5-subset $\mathcal{K}_5 \subset \mathcal{C} \subset PG(2,q)$ that does not cover $\#\mathcal{U}(\mathcal{K}_5) = \#\mathcal{M}_q - (10q-25) \le U_5 = (q-5)^2$ points of $\mathcal{M}_q$. The corresponding implicit bound $A^*(5,(q-5)^2)$ (i.e. the value $(w_{\text{fin}}(5,(q-5)^2)+1)/\sqrt{q \ln q}\,$) is shown by the third blue curve on Figs. 1 and 2.

Observation 4.2. *In the region $7 \le q \le 55711$ the implicit bound $A^*(5,(q-5)^2)$ tends to increase with the maximal value $A^*(5,(q-5)^2) \sim 1.8341$ for $q = 55711$. In the region $55711 < q \le 14000029$ the bound $A^*(5,(q-5)^2)$ tends to decrease with the minimal value $A^*(5,(q-5)^2) \sim 1.8180$ for $q = 13995829$, see Fig.* 2.

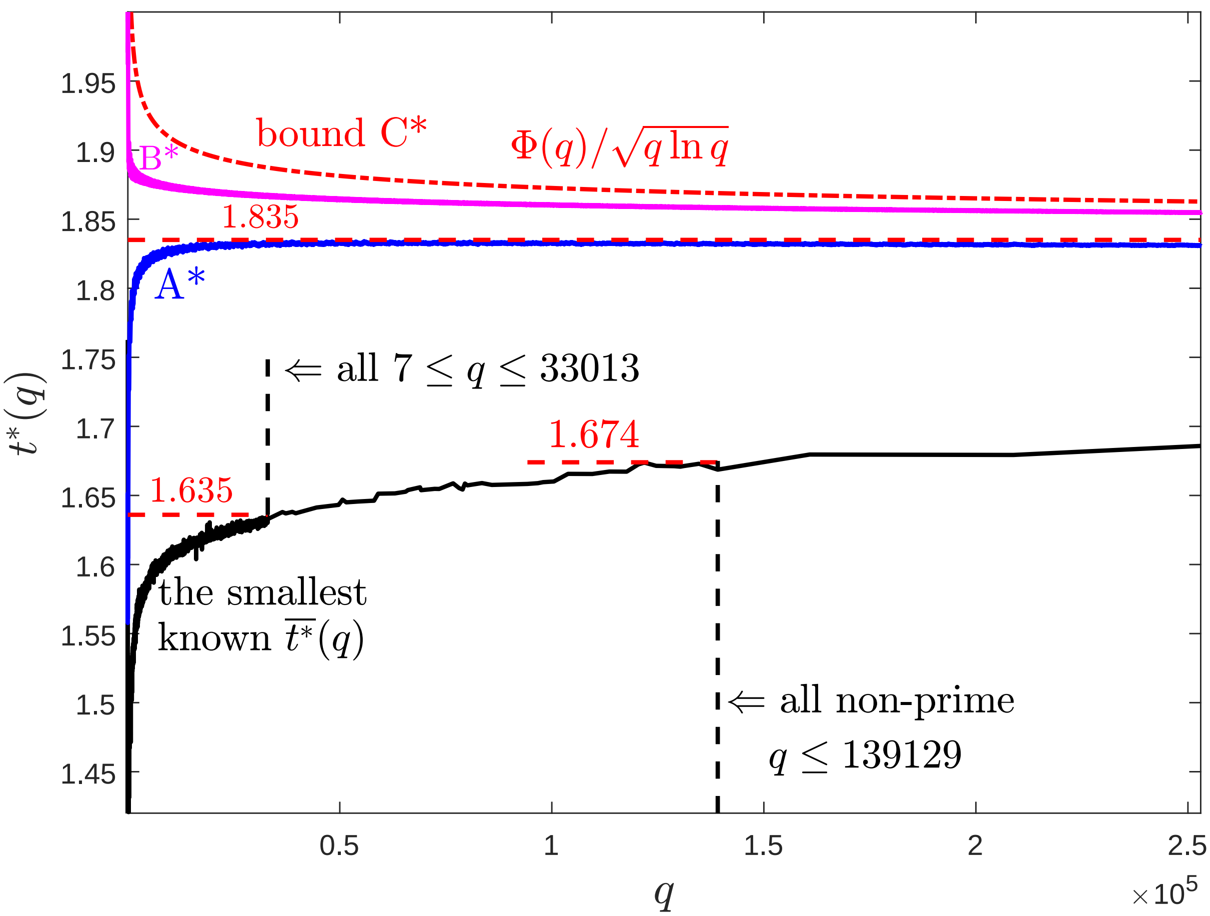

Figure 1: **Upper bounds on sizes of AC-subsets divided by $\sqrt{q \ln q}$, $q \le 253009$:** bound C* equal to $\Phi(q)/\sqrt{q \ln q}$ (*top dashed-dotted red curve*); implicit bound B* (*the 2-nd magenta curve*); bound (4.21) (*dashed red line* $y = 1.835$); implicit bound $A^*(5, (q-5)^2)$ (*the 3-rd blue curve*); bound (5.6) (*dashed red line* $y = 1.635$); bound (5.7) (*dashed red line* $y = 1.674$); the smallest known sizes of AC-subsets divided by $\sqrt{q \ln q}$, i.e. values $\overline{t^*}(q)$ (*bottom black curve*). Vertical dashed lines $x = 33013$ and $x = 139129$ mark regions of complete computer search, respectively, for all prime powers q and all non-prime q's

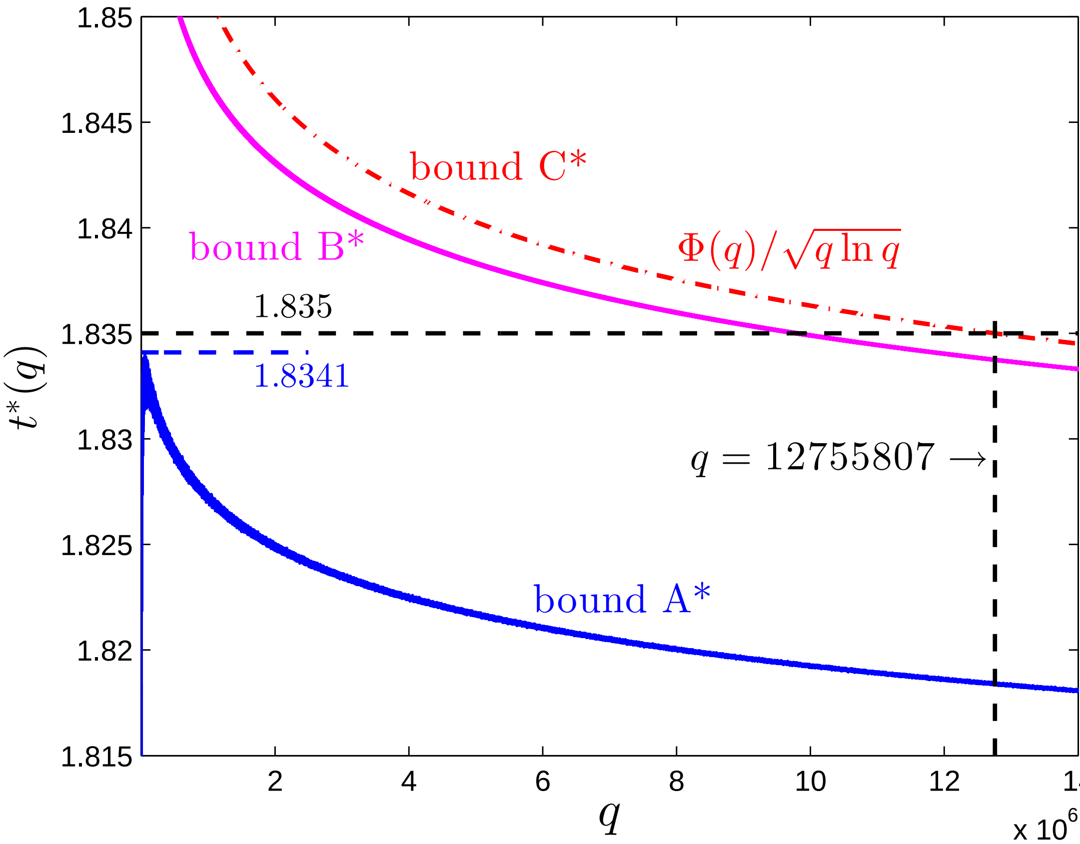

Figure 2: **Upper bounds on sizes of AC-subsets divided by $\sqrt{q \ln q}$,** $q \leq 14000029$**:** bound C* equal to $\Phi(q)/\sqrt{q \ln q}$ (*top dashed-dotted red curve*); implicit bound B* (*the 2-nd magenta curve*); bound (4.21) (*dashed red line* $y = 1.835$); implicit bound $A^*(5,(q-5)^2)$ (*the 3-rd blue curve*)

4.2 A truncated iterative process

From (4.1) we have that

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = \#\mathcal{U}(\mathcal{K}_w) - \Delta(A_{w+1}) \leq U_w \left(1 - \frac{w-2}{q+1-w}\right). \tag{4.5}$$

Clearly, $\#\mathcal{U}(\mathcal{K}_1) \leq U_1 = q^2$. Using (4.5) iteratively, we obtain

$$\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) = U_{w+1} \leq q^2 f_q(w), \tag{4.6}$$

where

$$f_q(w) = \prod_{i=1}^{w} \left(1 - \frac{i-2}{q+1-i}\right). \tag{4.7}$$

From now on, we will stop the iterative process when $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi$ where $\xi \geq 1$ is some value that we may assign to improve estimates. Note that if some point $P \in \mathcal{M}_q$ is not covered by $\mathcal{K}_w \cup \{A_{w+1}\}$, one always can find a point $A_{w+2} \in \mathcal{C} \setminus (\mathcal{K}_w \cup \{A_{w+1}\})$ such that P is covered by $\mathcal{K}_w \cup \{A_{w+1}, A_{w+2}\}$. It means that after the end of the iterative process we can add at most ξ points of $\mathcal{C}$ to the running subset in order to get a k-AC-subset with size k satisfying

$$w + 1 \leq k \leq w + 1 + \xi \text{ under condition } \#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi. \tag{4.8}$$

Theorem 4.3. *Let $\xi \geq 1$ be a fixed value independent of w. Let $w < \frac{q+3}{2}$ satisfy*

$$f_q(w) = \prod_{i=1}^{w} \left(1 - \frac{i-2}{q+1-i}\right) \leq \frac{\xi}{q^2}. \tag{4.9}$$

Then it holds that

$$t(q) \leq w + 1 + \xi. \tag{4.10}$$

Proof. By (4.6), to provide the inequality $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi$ it is sufficient to find w such that $q^2 f_q(w) \leq \xi$. Now (4.10) follows from (4.8). □

Clearly, we should choose ξ such that $w + 1 + \xi$ is small under condition $\#\mathcal{U}(\mathcal{K}_w \cup \{A_{w+1}\}) \leq \xi$.

In order to get more simple forms of upper bounds on $t(q)$ we will find an upper bound on $f_q(w)$ of (4.7). To this end we use the Taylor series $e^{-\alpha} = 1 - \alpha + \frac{\alpha^2}{2} - \frac{\alpha^3}{6} + \ldots$, whence

$$1 - \alpha < e^{-\alpha} \text{ for } \alpha \neq 0. \tag{4.11}$$

4.3 Implicit bound B

Lemma 4.4. *It holds that*

$$f_q(w) = \prod_{i=1}^{w} \left(1 - \frac{i-2}{q+1-i}\right) < e^{-S}, \tag{4.12}$$

where

$$-w + (q-1)\ln\frac{q+1}{q+1-w} < S < -w + (q-1)\ln\frac{q}{q-w}. \tag{4.13}$$

Proof. By (4.11),

$$\prod_{i=1}^{w} \left(1 - \frac{i-2}{q+1-i}\right) < e^{-S}, \quad S = \sum_{i=1}^{w} \frac{i-2}{q+1-i}.$$

Also,

$$S = \sum_{i=1}^{w} \frac{i-2}{q+1-i} = \sum_{u=-1}^{w-2} \frac{u}{q-1-u} = -w + \sum_{u=-1}^{w-2} \left(\frac{u}{q-1-u} + 1\right) =$$
$$-w + (q-1)\sum_{u=-1}^{w-2} \frac{1}{q-1-u} = -w + (q-1)\sum_{t=q+1-w}^{q} \frac{1}{t}.$$

It is well known that

$$\ln(q+1) < \sum_{t=1}^{q} \frac{1}{t} < 1 + \ln q.$$

Therefore,

$$\ln(q+1) - \ln(q+1-w) < \sum_{t=q+1-w}^{q} \frac{1}{t} = \sum_{t=1}^{q} \frac{1}{t} - \sum_{t=1}^{q-w} \frac{1}{t} < \ln q - \ln(q-w). \quad \square$$

Corollary 4.5. *Let $\xi \geq 1$ be a fixed value independent of w. Let $w < \frac{q+3}{2}$ satisfy*

$$w - (q-1)\ln\frac{q+1}{q+1-w} \leq \ln\frac{\xi}{q^2}.$$

Then it holds that

$$t(q) \leq w + 1 + \xi.$$

Proof. We substitute (4.12) and (4.13) in (4.9). $\square$

Corollary 4.6. ***(implicit bound B)*** *Let* $w < \frac{q+3}{2}$ *satisfy*

$$w - (q-1)\ln\frac{q+1}{q+1-w} \le \ln\frac{1}{q\sqrt{3q\ln q}}.$$

Then it holds that

$$t(q) \le w + 1 + \sqrt{\frac{q}{3\ln q}}.$$

Proof. In the assertions of Corollary 4.5, we use $\xi = \sqrt{\frac{q}{3\ln q}}$. □

The implicit bound B* is shown by the second magenta curve on Figs. 1 and 2.

4.4 Explicit bounds

By (4.7) and (4.11), we have

$$f_q(w) < \prod_{i=1}^{w}\left(1 - \frac{i-2}{q}\right) < \prod_{i=1}^{w} e^{-(i-2)/q} = e^{-(w^2-3w)/2q} < e^{-(w-2)^2/2q}. \tag{4.14}$$

Lemma 4.7. *Let* $\xi \ge 1$ *be a fixed value independent of* w*. The value*

$$\frac{q+3}{2} > w \ge \sqrt{2q}\sqrt{\ln\frac{q^2}{\xi}} + 3 \tag{4.15}$$

satisfies inequality (4.9)*.*

Proof. By (4.14), to provide (4.9) it is sufficient to find w such that

$$e^{-(w-2)^2/2q} < \frac{\xi}{q^2}.$$

As w should be an integer, in (4.15) one is added. Inequality $w < \frac{q+3}{2}$ is obvious. □

Theorem 4.8. *In* $PG(2,q)$ *it holds that*

$$t(q) \le \sqrt{2q}\sqrt{\ln\frac{q^2}{\xi}} + \xi + 4, \quad \xi \ge 1, \tag{4.16}$$

where ξ *is an arbitrarily chosen value.*

Proof. The assertion follows from (4.10) and (4.15). □

Remark 4.9. We consider the function of ξ of the form

$$\phi(\xi) = \sqrt{2q}\sqrt{\ln\frac{q^2}{\xi}} + \xi + 4.$$

Its derivative by ξ is

$$\phi'(\xi) = 1 - \frac{1}{\xi}\sqrt{\frac{q}{2\ln\frac{q^2}{\xi}}}.$$

Put $\phi'(\xi) = 0$. Then

$$\xi^2 = \frac{q}{4\ln q - 2\ln\xi}. \tag{4.17}$$

We find ξ in the form $\xi = \sqrt{\frac{q}{c\ln q}}$. By (4.17), $c = 3 + \frac{\ln c + \ln\ln q}{\ln q}$. For simplicity, we choose $c = 3$. Then $\xi = \sqrt{\frac{q}{3\ln q}}$ and the value

$$\phi'\left(\sqrt{\frac{q}{3\ln q}}\right) = 1 - \sqrt{\frac{3\ln q}{3\ln q + \ln\ln q + \ln 3}}$$

is close to zero for growing q. Also, it is easy to check the following: $\phi'(1) < 0$ if $q \geq 9$, $\phi'(\xi)$ is an increasing function, $0 < \phi'\left(\sqrt{\frac{q}{3\ln q}}\right) < \phi'(\sqrt{q}) = 1 - \sqrt{\frac{1}{3\ln q}}$.

So, the choice $\xi = \sqrt{\frac{q}{3\ln q}}$ in (4.16) seems to be convenient.

Theorem 4.10. ***(Bound C)*** *The following upper bound on the smallest size $t(q)$ of an AC-subset in* PG$(2,q)$ *holds.*

$$t(q) < \Phi(q) = \sqrt{q(3\ln q + \ln\ln q + \ln 3)} + \sqrt{\frac{q}{3\ln q}} + 4 \sim \sqrt{3q\ln q}. \tag{4.18}$$

Proof. We substitute $\xi = \sqrt{\frac{q}{3\ln q}}$ in (4.16). □

The bound C* (i.e. the value $\Phi(q)/\sqrt{q\ln q}$) is shown by the top dashed-dotted red curve on Figs. 1 and 2.

Remark 4.11. If in (4.16) we take $\xi = 1$ and $\xi = \sqrt{q}$, we obtain bounds (4.19) and (4.20).

$$t(q) < 2\sqrt{q\ln q} + 5. \tag{4.19}$$

$$t(q) < \sqrt{3q\ln q} + \sqrt{q} + 4. \tag{4.20}$$

It can be shown that bounds (4.19) and (4.20) are worse than (4.18).

If we put, see Remark 4.9, $c = 3 + \frac{1 + \ln\ln q}{\ln q}$, $\xi = \sqrt{\frac{q}{3\ln q + \ln\ln q + 1}}$, we improve bound (4.18). But, the improvement is unessential whereas the bound takes a lengthy form.

Theorem 4.12. *The following upper bound on the smallest size* $t(q)$ *of an AC-subset in* $\mathrm{PG}(2,q)$ *holds.*

$$t(q) < 1.835\sqrt{q \ln q}. \tag{4.21}$$

Proof. For $q \le 12755807$ we checked by computer that the implicit bound $A(5,(q-5)^2) < 1.8341\sqrt{q\ln q}$; so in this region the assertion is provided by the bound $A(5,(q-5)^2)$, see Observation 4.2 and Fig. 2. It is easy to see that $\Phi(q)/\sqrt{q\ln q}$ is a decreasing function of q. Moreover, $\Phi(q)/\sqrt{q\ln q} < 1.835$ for $q = 12755807$. So, for $q > 12755807$ the assertion is provided by the bound C. □

The bound (4.21) is presented by the dashed red line $y = 1.835$ in Figs. 1 and 2.

5 Computer assisted results on $t(q)$ and $t^*(q)$

Let $\bar{t}(q)$ be the smallest *known* size of an AC-subset in $\mathrm{PG}(2,q)$. Let $\overline{t^*}(q) = \bar{t}(q)/\sqrt{q\ln q}$. We denote the following sets of values of q: $Q_2 := \{5 \le q \le 33013,\ q \text{ prime power}\}$; $Q_3 := \{5 \le q \le 32,\ q \text{ prime power}\}$; $Q_4 := Q_1 \cup \{160801, 208849, 253009\}$. Let Q_1 be as in (2.1).

For the set Q_3 we obtained by computer search the smallest sizes $t(q)$ of AC-subsets of $\mathcal{C}$ in $\mathrm{PG}(2,q)$, see Table 2. The algorithm, used in the search, fixes a conic, computes all the non-equivalent point subsets of the conic of a certain size (6 in our complete cases) and extends each of them trying to obtain a minimal AC-subset. Each time an example is found only smaller examples are looked for. Minimality is checked explicitly: once we have found an AC-subset we test that deleting from it a point in all possible ways no almost complete subset is obtained. All computations are performed using the system for symbol calculations MAGMA [22].

Table 2: The smallest sizes $t(q)$ of AC-subsets of $\mathcal{C}$ in $\mathrm{PG}(2,q)$, $q \in Q_3$

q	$t(q)$	q	$t(q)$	q	$t(q)$	q	$t(q)$	q	$t(q)$	q	$t(q)$	q	$t(q)$	q	$t(q)$
5	5	7	6	8	6	9	6	11	8	13	8	16	9	17	10
19	11	23	12	25	12	27	13	29	13	31	14	32	15		

For the sets Q_2 and Q_4 we obtained small AC-subsets of $\mathcal{C}$ in $\mathrm{PG}(2,q)$ by computer search[3]. For it we used step-by-step randomized greedy algorithms similar to those

[3]The computer search for $q \in Q_2 \cup Q_4$ has been carried out using computing resources of the federal collective usage center Complex for Simulation and Data Processing for Mega-science Facilities at NRC "Kurchatov Institute", http://ckp.nrcki.ru/.

from [23], see also the references therein. Recall that at each step a randomized greedy algorithm maximizes some objective function f, but some steps are executed in a random manner. Also, if one and the same maximum of f can be obtained in different ways, the choice is made at random. As the value of the objective function, the number of points lying on bisecants of the running subset is considered.

As far as the authors know, sizes of AC-subsets, obtained by the mentioned computer search, are the smallest known. The corresponding values of $\overline{t^*}(q)$ are shown by the bottom black curve in Fig. 1. Recall that,

$$t^*(q) = \frac{t(q)}{\sqrt{q \ln q}} .$$

The values $\overline{t}(q)$ and $\overline{t^*}(q)$ for $q \in Q_4$ and prime $q \in Q_2$ are given in Tables 3 and 4, respectively, see Appendix. As values of $\overline{t^*}(q)$ are not integers, in Tables 3 and 4 we give rounded values of $\overline{t^*}(q)$, moreover we round up. This explains the entry "$\overline{t^*}(q) <$" in the top of columns.

In Table 4, the values $\overline{t^*}(q)$ are written for not all q's. The rules for entries $\overline{t^*}(q)$ are as follows. Assume that the following holds: $q' < q''$; the values $\overline{t^*}(q')$ and $\overline{t^*}(q'')$ are written in Table 4; no value $\overline{t^*}(q)$ is written in the table if $q' < q < q''$. Then $\overline{t^*}(q') \leq \overline{t^*}(q'')$ and $\overline{t^*}(q) \leq \overline{t^*}(q')$ with $q' < q < q''$.

For example, one may take $q' = 19$ and $q'' = 307$. We see that no value $\overline{t^*}(q)$ is written in Table 4 if $19 < q < 307$. We have $\overline{t^*}(19) \approx 1.471 < \overline{t^*}(307) \approx 1.479$ and $\overline{t^*}(q) \leq 1.471$ with $19 < q < 307$.

So, in Table 4, the blank on place $\overline{t^*}(q)$ means that $\overline{t^*}(q) \leq \overline{t^*}(q')$ under the conditions that $q' < q$, value $\overline{t^*}(q')$ is written in the table, and no value $\overline{t^*}(q^\bullet)$ is written if $q' < q^\bullet < q$.

By computer search for the sets Q_2 and Q_4, see Tables 3 and 4, we have Theorem 5.1.

Theorem 5.1. *The following upper bounds on the smallest size $t(q)$ of an AC-subset of the conic $\mathcal{C}$ in* $\mathrm{PG}(2, q)$ *hold:*

$$t(q) < 1.525\sqrt{q \ln q}, \qquad 8 \leq q \leq 887,\ q \text{ prime power},\ q \neq 11; \tag{5.1}$$

$$t(q) < 1.548\sqrt{q \ln q}, \qquad 887 < q \leq 1553,\ q \text{ prime power}; \tag{5.2}$$

$$t(q) < 1.572\sqrt{q \ln q}, \qquad 1553 < q \leq 2351,\ q \text{ prime power},\ q = 11; \tag{5.3}$$

$$t(q) < 1.585\sqrt{q \ln q}, \qquad 2351 < q \leq 4027,\ q \text{ prime power}; \tag{5.4}$$

$$t(q) < 1.620\sqrt{q \ln q}, \qquad 4027 < q \leq 17041,\ q \text{ prime power}; \tag{5.5}$$

$$t(q) < 1.635\sqrt{q \ln q}, \qquad 17041 < q \leq 33013,\ q \text{ prime power},\ q = 7; \tag{5.6}$$

$$t(q) < 1.674\sqrt{q \ln q}, \qquad q = p^m,\ p \text{ prime},\ m \geq 2,\ q \in Q_1; \tag{5.7}$$

$$t(q) < 1.686\sqrt{q \ln q}, \qquad q = 160801, 208849, 253009.$$

The bounds (5.6), (5.7) are presented by dashed red lines $y = 1.635$, $y = 1.674$ in Fig. 1.

6 New bounds on $t(q)$ and completeness of normal rational curves

Proof of Corollary 2.2. We substitute the new bounds of Theorem 2.1 in relation (1.2) taken from [19, Theorem 3.1]. □

Proof of Corollary 2.3. We act analogously to the proof of [19, Theorem 3.5], changing in it $6\sqrt{q \ln q}$ by $c\sqrt{q \ln q}$. As the result we obtain inequality (2.6).

By (2.6), for $c = 1.835$, $h \geq 29$, we have $p_0(h) \geq 33079 > 33013$; but for $c = 1.835$, $h \leq 28$, it holds that $p_0(h) \leq 31840 < 33013$. So, by (4.21) and (5.1)–(5.6), we may take $c = 1.835$ for $h \geq 29$ and $c = 1.635$ for $h \leq 28$.

Again we use (2.6). For $c = 1.635$, $h \geq 20$, we have $p_0(h) \geq 17091 > 17041$; but for $c = 1.635$, $h \leq 19$, it holds that $p_0(h) \leq 16164 < 17041$. So, by (5.1)–(5.6), we may take $c = 1.635$ for $20 \leq h \leq 28$ and $c = 1.62$ for $h \leq 19$.

Now for $h = 1, \dots, 5$ we found $p_0(h)$ as a solution of (2.6) taking c on the base Theorem 5.1. For the given h, at the beginning we obtain $p_0(h)$ with $c = 1.62$. Then we decrease c using (5.1)–(5.4) and get a smaller $p_0(h)$. For $c = 1.62$ we obtain $p_0(1) = 877$, $p_0(2) = 1543$, $p_0(3) = 2273$, $p_0(4) = 3037$, $p_0(5) = 3821$. So, we may put $c = 1.525$ for $h = 1$, $c = 1.548$ for $h = 2$, $c = 1.572$ for $h = 3$, and $c = 1.585$ for $h = 4, 5$, see (5.1), (5.2), (5.3), and (5.4), respectively. Solutions of inequality (2.6) for these (c, h) are the values $p_0(1), \dots, p_0(5)$ written in the assertion of the corollary. □

Remark 6.1. We can also improve the result of [19, Theorem 3.4]. If in the proof of [19, Theorem 3.4] one uses the new bound $t(q) < 1.835\sqrt{q \ln q}$ instead of (1.1) then the following assertion can be proved: *for prime* $p \geq 76207$ *every normal rational curve in* $\mathrm{PG}(N, p)$ *with* $2 \leq N \leq p - 2$ *is a complete* $(q + 1)$*-arc.*

For comparison note that in [19, Theorem 3.4] the value $p > 1007215$ is pointed out.

Of course, due to the results of [4, 16, 17], see row 5 of Table 1, we know that for *all primes* p normal rational curves in $\mathrm{PG}(N, p)$ are complete.

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Appendix. Tables of the smallest known sizes $\overline{t}(q)$ of AC-subsets in $\mathrm{PG}(2,q)$

Table 3. The smallest known sizes $\overline{t}(q)$ of AC-subsets in $\mathrm{PG}(2,q)$ and values $\overline{t^*}(q)$, q non-prime, $q \in \{8 \le q \le 139129,\ q = p^m,\ p \text{ prime},\ m \ge 2\} \cup \{160801, 208849, 253009\}$

q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ <
8	2^3	6	1.48	9	3^2	6	1.35	16	2^4	9	1.36
25	5^2	12	1.34	27	3^3	13	1.38	32	2^5	15	1.43
49	7^2	18	1.31	64	2^6	22	1.35	81	3^4	25	1.33
121	11^2	33	1.37	125	5^3	35	1.43	128	2^7	35	1.41
169	13^2	41	1.40	243	3^5	53	1.46	256	2^8	55	1.46
289	17^2	58	1.44	343	7^3	66	1.48	361	19^2	66	1.44
512	2^9	84	1.49	529	23^2	85	1.48	625	5^4	96	1.52
729	3^6	102	1.48	841	29^2	114	1.52	961	31^2	122	1.51
1024	2^{10}	127	1.51	1331	11^3	150	1.54	1369	37^2	152	1.53
1681	41^2	173	1.55	1849	43^2	182	1.55	2048	2^{11}	194	1.56
2187	3^7	203	1.57	2197	13^3	203	1.57	2209	47^2	203	1.56
2401	7^4	214	1.57	2809	53^2	235	1.58	3125	5^5	250	1.58
3481	59^2	267	1.59	3721	61^2	277	1.59	4096	2^{12}	292	1.59
4489	67^2	309	1.60	4913	17^3	325	1.60	5041	71^2	330	1.60
5329	73^2	341	1.60	6241	79^2	373	1.60	6561	3^8	383	1.60
6859	19^3	393	1.60	6889	83^2	394	1.60	7921	89^2	426	1.60
8192	2^{13}	435	1.61	9409	97^2	472	1.61	10201	101^2	493	1.61
10609	103^2	503	1.61	11449	107^2	526	1.61	11881	109^2	538	1.62
12167	23^3	545	1.62	12769	113^2	561	1.62	14641	11^4	607	1.62
15625	5^6	629	1.62	16129	127^2	634	1.61	16384	2^{14}	645	1.62
16807	7^5	655	1.62	17161	131^2	663	1.63	18769	137^2	700	1.63
19321	139^2	712	1.631	19683	3^9	717	1.626	22201	149^2	767	1.628
22801	151^2	778	1.627	24389	29^3	808	1.628	24649	157^2	814	1.631
26569	163^2	849	1.632	27889	167^2	870	1.629	28561	13^4	882	1.630
29791	31^3	904	1.632	29929	173^2	906	1.632	32041	179^2	941	1.633
32761	181^2	952	1.632	32768	2^{15}	953	1.633	36481	191^2	1014	1.639
37249	193^2	1025	1.638	38809	197^2	1049	1.639	39601	199^2	1060	1.638
44521	211^2	1133	1.642	49729	223^2	1205	1.644	50653	37^3	1220	1.647
51529	227^2	1230	1.646	52441	229^2	1242	1.646	54289	233^2	1266	1.646
57121	239^2	1302	1.647	58081	241^2	1314	1.647	59049	3^{10}	1330	1.652
63001	251^2	1378	1.652	65536	2^{16}	1409	1.653	66049	257^2	1416	1.654

Table 3. Continue

q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	p^m	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$
68921	41^3	1451	1.656	69169	263^2	1452	1.654	72361	269^2	1489	1.655
73441	271^2	1501	1.655	76729	277^2	1541	1.659	78125	5^7	1553	1.656
78961	281^2	1561	1.655	79507	43^3	1571	1.659	80089	283^2	1576	1.658
83521	17^4	1614	1.659	85849	293^2	1637	1.658	94249	307^2	1723	1.659
96721	311^2	1748	1.659	97969	313^2	1761	1.660	100489	317^2	1786	1.661
103823	47^3	1824	1.666	109561	331^2	1878	1.666	113569	337^2	1917	1.668
117649	7^6	1954	1.668	120409	347^2	1985	1.673	121801	349^2	1999	1.674
124609	353^2	2021	1.672	128881	359^2	2058	1.672	130321	19^4	2070	1.671
131072	2^{17}	2077	1.672	134689	367^2	2110	1.673	139129	373^2	2142	1.669
160801	401^2	2332	1.680	208849	457^2	2686	1.680	253009	503^2	2991	1.686

Table 4. The smallest known sizes $\overline{t}(q)$ of AC-subsets in PG$(2,q)$ and values $\overline{t^*}(q)$, $13 \le q \le 33013$, q prime

q	$\overline{t}(q)$	$\overline{t^*}(q) <$	q	$\overline{t}(q)$	$\overline{t^*}(q) <$	q	$\overline{t}(q)$	$\overline{t^*}(q) <$	q	$\overline{t}(q)$	$\overline{t^*}(q) <$	q	$\overline{t}(q)$	$\overline{t^*}(q) <$
13	8	1.386	17	10	1.441	19	11	1.471	23	12		29	13	
31	14		37	16		41	16		43	16		47	18	
53	20		59	21		61	22		67	23		71	24	
73	24		79	25		83	26		89	24		97	29	
101	30		103	30		107	31		109	32		113	32	
127	35		131	36		137	37		139	37		149	38	
151	39		157	40		163	41		167	42		173	43	
179	43		181	44		191	45		193	46		197	46	
199	47		211	48		223	50		227	51		229	51	
233	51		239	53		241	53		251	54		257	55	
263	55		269	56		271	57		277	58		281	58	
283	58		293	59		307	62	1.479	311	62		313	62	
317	62		331	64		337	65		347	66		349	66	
353	67		359	68	1.480	367	68		373	69		379	70	
383	70		389	71		397	72		401	73	1.489	409	73	
419	75	1.492	421	74		431	76		433	76		439	76	
443	77		449	78		457	79	1.494	461	79		463	79	
467	80		479	81		487	82	1.494	491	82		499	83	
503	83		509	84		521	85		523	85		541	87	
547	88	1.499	557	89	1.500	563	89		569	90		571	90	
577	91	1.503	587	92	1.504	593	92		599	93		601	93	
607	93		613	94		617	95	1.509	619	95		631	96	
641	97		643	97		647	97		653	98		659	98	
661	99	1.512	673	100		677	100		683	101	1.513	691	101	
701	102		709	103		719	104		727	104		733	105	
739	105		743	106		751	107	1.518	757	107		761	107	
769	108		773	108		787	109		797	111	1.522	809	112	1.522
811	112		821	112		823	113		827	113		829	113	
839	114		853	115		857	116	1.525	859	116		863	116	
877	117		881	117		883	118		887	118		907	120	1.527
911	120		919	120		929	121		937	122		941	122	
947	123		953	123		967	124		971	125	1.530	977	125	
983	125		991	126		997	127	1.531	1009	128	1.533	1013	128	
1019	128		1021	128		1031	129		1033	129		1039	129	
1049	130		1051	131		1061	131		1063	132	1.534	1069	132	
1087	133		1091	133		1093	134		1097	134		1103	134	
1109	135		1117	136	1.537	1123	136		1129	137	1.538	1151	138	
1153	138		1163	139		1171	139		1181	140		1187	140	
1193	141		1201	141		1213	143	1.541	1217	142		1223	143	
1229	144		1231	144		1237	144		1249	145		1259	146	
1277	147		1279	147		1283	147		1289	148		1291	148	
1297	148		1301	149	1.543	1303	149		1307	149		1319	150	
1321	150		1327	150		1361	153	1.544	1367	153		1373	154	1.547

Table 4. Continue 1

q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$
1381	153		1399	155		1409	156		1423	157		1427	157	
1429	157		1433	157		1439	158		1447	158		1451	159	1.548
1453	159		1459	159		1471	160		1481	160		1483	161	1.548
1487	161		1489	161		1493	161		1499	162	1.548	1511	162	
1523	163		1531	163		1543	164		1549	165		1553	165	
1559	166	1.551	1567	166		1571	166		1579	167		1583	167	
1597	167		1601	168		1607	168		1609	169		1613	169	
1619	169		1621	169		1627	170		1637	171	1.554	1657	172	
1663	172		1667	172		1669	172		1693	174		1697	175	1.558
1699	174		1709	175		1721	175		1723	176		1733	177	
1741	177		1747	177		1753	178		1759	178		1777	179	
1783	179		1787	180		1789	180		1801	181		1811	182	1.562
1823	182		1831	183		1847	183		1861	185	1.563	1867	185	
1871	185		1873	185		1877	185		1879	185		1889	186	
1901	187		1907	187		1913	188	1.564	1931	189	1.564	1933	189	
1949	190	1.564	1951	190		1973	191		1979	191		1987	191	
1993	192		1997	192		1999	192		2003	192		2011	193	
2017	193		2027	194		2029	194		2039	194		2053	195	
2063	196		2069	196		2081	197		2083	197		2087	197	
2089	197		2099	198		2111	199	1.566	2113	199		2129	200	1.566
2131	200		2137	200		2141	201	1.569	2143	201		2153	202	1.572
2161	202		2179	203		2203	204		2207	204		2213	205	
2221	205		2237	206		2239	206		2243	206		2251	206	
2267	208	1.572	2269	207		2273	208		2281	208		2287	208	
2293	209		2297	208		2309	209		2311	208		2333	211	
2339	211		2341	211		2347	212		2351	212		2357	213	1.575
2371	212		2377	212		2381	213		2383	212		2389	213	
2393	213		2399	214		2411	214		2417	214		2423	215	
2437	216		2441	217		2447	217		2459	217		2467	218	
2473	219	1.576	2477	219		2503	219		2521	220		2531	220	
2539	221		2543	222		2549	223	1.578	2551	222		2557	223	
2579	223		2591	224		2593	223		2609	224		2617	225	
2621	225		2633	225		2647	227		2657	228		2659	227	
2663	228		2671	229	1.578	2677	229		2683	229		2687	230	1.580
2689	230		2693	230		2699	231	1.582	2707	231		2711	231	
2713	231		2719	231		2729	232		2731	231		2741	231	
2749	231		2753	231		2767	232		2777	233		2789	233	
2791	233		2797	234		2801	234		2803	234		2819	235	
2833	235		2837	236		2843	236		2851	236		2857	237	
2861	237		2879	238		2887	238		2897	238		2903	239	
2909	239		2917	239		2927	240		2939	241		2953	242	
2957	241		2963	241		2969	242		2971	242		2999	244	
3001	243		3011	244		3019	244		3023	245		3037	245	

Table 4. Continue 2

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
3041	245		3049	246		3061	247		3067	248		3079	248	
3083	249	1.583	3089	248		3109	249		3119	249		3121	250	
3137	250		3163	251		3167	251		3169	252		3181	252	
3187	253		3191	253		3203	253		3209	254		3217	255	
3221	254		3229	255		3251	255		3253	255		3257	256	
3259	256		3271	256		3299	258		3301	258		3307	257	
3313	258		3319	259		3323	259		3329	258		3331	258	
3343	260		3347	260		3359	261		3361	260		3371	261	
3373	261		3389	262		3391	262		3407	262		3413	262	
3433	264		3449	264		3457	264		3461	265		3463	264	
3467	265		3469	265		3491	266		3499	266		3511	267	
3517	267		3527	268		3529	268		3533	268		3539	268	
3541	268		3547	269		3557	269		3559	269		3571	270	
3581	270		3583	270		3593	271		3607	271		3613	271	
3617	272		3623	271		3631	273	1.583	3637	273		3643	273	
3659	274		3671	274		3673	274		3677	274		3691	275	
3697	275		3701	275		3709	276		3719	276		3727	277	
3733	277		3739	277		3761	278		3767	279	1.585	3769	279	
3779	279		3793	280		3797	280		3803	280		3821	281	
3823	281		3833	281		3847	282		3851	282		3853	282	
3863	282		3877	283		3881	283		3889	283		3907	284	
3911	284		3917	284		3919	284		3923	284		3929	285	
3931	285		3943	286		3947	286		3967	286		3989	288	
4001	288		4003	288		4007	288		4013	288		4019	289	
4021	288		4027	288		4049	291	1.587	4051	290		4057	290	
4073	291		4079	291		4091	292		4093	292		4099	291	
4111	292		4127	293		4129	293		4133	294		4139	293	
4153	294		4157	294		4159	295		4177	295		4201	297	
4211	297		4217	297		4219	297		4229	298		4231	298	
4241	298		4243	298		4253	298		4259	298		4261	299	
4271	299		4273	300	1.588	4283	300		4289	299		4297	300	
4327	301		4337	302		4339	302		4349	302		4357	303	
4363	303		4373	304	1.588	4391	304		4397	305	1.589	4409	305	
4421	306	1.589	4423	305		4441	306		4447	307		4451	306	
4457	307		4463	307		4481	307		4483	308		4493	308	
4507	309		4513	308		4517	309		4519	309		4523	309	
4547	310		4549	310		4561	311		4567	311		4583	312	
4591	311		4597	312		4603	312		4621	313		4637	314	
4639	314		4643	314		4649	314		4651	314		4657	314	
4663	315		4673	315		4679	315		4691	316		4703	317	1.590
4721	317		4723	318	1.591	4729	317		4733	317		4751	318	
4759	319		4783	320		4787	320		4789	320		4793	320	
4799	320		4801	320		4813	321		4817	321		4831	322	

Table 4. Continue 3

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
4861	323		4871	323		4877	323		4889	323		4903	324	
4909	325	1.592	4919	325		4931	325		4933	325		4937	325	
4943	326		4951	325		4957	326		4967	327		4969	327	
4973	327		4987	327		4993	328		4999	328		5003	328	
5009	328		5011	329	1.593	5021	329		5023	329		5039	328	
5051	330		5059	330		5077	331		5081	331		5087	331	
5099	331		5101	332		5107	332		5113	332		5119	332	
5147	334	1.593	5153	334		5167	334		5171	334		5179	335	
5189	336	1.595	5197	335		5209	335		5227	336		5231	337	
5233	337		5237	337		5261	338		5273	338		5279	338	
5281	338		5297	339		5303	340		5309	339		5323	341	1.596
5333	341		5347	341		5351	342		5381	342		5387	343	
5393	343		5399	343		5407	343		5413	343		5417	343	
5419	344		5431	344		5437	345		5441	345		5443	345	
5449	344		5471	346		5477	346		5479	346		5483	346	
5501	346		5503	347		5507	347		5519	347		5521	348	
5527	348		5531	348		5557	349		5563	349		5569	350	1.597
5573	350		5581	350		5591	351	1.599	5623	352		5639	352	
5641	352		5647	352		5651	352		5653	353		5657	352	
5659	353		5669	354	1.600	5683	354		5689	354		5693	354	
5701	354		5711	355		5717	354		5737	356		5741	356	
5743	356		5749	355		5779	357		5783	357		5791	357	
5801	358		5807	358		5813	358		5821	358		5827	359	
5839	360	1.600	5843	360		5849	360		5851	360		5857	360	
5861	360		5867	360		5869	361		5879	361		5881	361	
5897	362	1.600	5903	362		5923	363	1.601	5927	362		5939	363	
5953	364		5981	364		5987	365		6007	366	1.601	6011	365	
6029	366		6037	365		6043	366		6047	366		6053	366	
6067	368		6073	367		6079	367		6089	367		6091	367	
6101	368		6113	369		6121	368		6131	369		6133	367	
6143	370		6151	369		6163	370		6173	370		6197	371	
6199	371		6203	372		6211	371		6217	372		6221	373	
6229	373		6247	373		6257	373		6263	374		6269	373	
6271	374		6277	375		6287	374		6299	375		6301	374	
6311	375		6317	376		6323	375		6329	375		6337	376	
6343	377		6353	376		6359	376		6361	377		6367	377	
6373	377		6379	378		6389	377		6397	378		6421	379	
6427	379		6449	380		6451	379		6469	381		6473	381	
6481	381		6491	382		6521	382		6529	383		6547	383	
6551	383		6553	383		6563	385	1.604	6569	384		6571	384	
6577	385		6581	384		6599	385		6607	384		6619	385	
6637	386		6653	386		6659	387		6661	386		6673	387	
6679	388		6689	388		6691	387		6701	388		6703	387	

Table 4. Continue 4

q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$
6709	388		6719	389		6733	389		6737	389		6761	390	
6763	390		6779	390		6781	391		6791	391		6793	391	
6803	391		6823	392		6827	391		6829	392		6833	392	
6841	392		6857	392		6863	393		6869	393		6871	393	
6883	394		6899	395		6907	395		6911	395		6917	395	
6947	396		6949	396		6959	397		6961	397		6967	397	
6971	398		6977	398		6983	397		6991	397		6997	397	
7001	398		7013	398		7019	399		7027	398		7039	399	
7043	399		7057	400		7069	400		7079	400		7103	401	
7109	401		7121	402		7127	402		7129	402		7151	403	
7159	403		7177	403		7187	404		7193	404		7207	405	
7211	405		7213	405		7219	405		7229	405		7237	406	
7243	406		7247	405		7253	406		7283	407		7297	408	
7307	407		7309	408		7321	408		7331	409		7333	409	
7349	408		7351	409		7369	410		7393	411		7411	411	
7417	411		7433	412		7451	412		7457	412		7459	412	
7477	413		7481	413		7487	413		7489	413		7499	414	
7507	414		7517	414		7523	415		7529	415		7537	415	
7541	416	1.604	7547	415		7549	416		7559	416		7561	416	
7573	416		7577	416		7583	417		7589	416		7591	417	
7603	417		7607	417		7621	418		7639	419	1.604	7643	418	
7649	419		7669	419		7673	419		7681	420		7687	420	
7691	420		7699	420		7703	420		7717	421		7723	420	
7727	421		7741	422		7753	422		7757	422		7759	422	
7789	422		7793	423		7817	424		7823	424		7829	424	
7841	424		7853	425		7867	426	1.604	7873	426		7877	425	
7879	426		7883	425		7901	426		7907	427		7919	427	
7927	427		7933	427		7937	428		7949	428		7951	428	
7963	428		7993	429		8009	429		8011	428		8017	429	
8039	431		8053	431		8059	431		8069	431		8081	432	
8087	433	1.606	8089	432		8093	432		8101	432		8111	433	
8117	433		8123	433		8147	434		8161	434		8167	434	
8171	434		8179	435		8191	435		8209	435		8219	436	
8221	436		8231	437		8233	436		8237	437		8243	437	
8263	437		8269	437		8273	438		8287	438		8291	437	
8293	438		8297	438		8311	438		8317	440	1.606	8329	439	
8353	440		8363	440		8369	441		8377	441		8387	440	
8389	442		8419	442		8423	442		8429	443		8431	442	
8443	444	1.608	8447	443		8461	444		8467	443		8501	445	
8513	445		8521	445		8527	445		8537	445		8539	444	
8543	446		8563	446		8573	447		8581	447		8597	448	
8599	447		8609	448		8623	448		8627	448		8629	448	
8641	449		8647	450	1.608	8663	449		8669	450		8677	450	

Table 4. Continue 5

q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$
8681	450		8689	451		8693	450		8699	450		8707	451	
8713	450		8719	451		8731	451		8737	451		8741	452	
8747	452		8753	452		8761	452		8779	452		8783	452	
8803	453		8807	453		8819	454		8821	454		8831	455	
8837	454		8839	455		8849	455		8861	455		8863	456	
8867	456		8887	455		8893	456		8923	456		8929	456	
8933	457		8941	458		8951	458		8963	458		8969	458	
8971	458		8999	459		9001	459		9007	458		9011	459	
9013	459		9029	460		9041	461		9043	460		9049	460	
9059	462	1.609	9067	461		9091	462		9103	462		9109	462	
9127	463		9133	464		9137	464		9151	463		9157	464	
9161	464		9173	464		9181	464		9187	465		9199	465	
9203	465		9209	465		9221	466		9227	465		9239	466	
9241	466		9257	467		9277	467		9281	468		9283	467	
9293	468		9311	469		9319	469		9323	469		9337	470	1.609
9341	469		9343	470		9349	469		9371	470		9377	471	
9391	471		9397	470		9403	470		9413	471		9419	471	
9421	471		9431	473	1.611	9433	471		9437	471		9439	472	
9461	473		9463	472		9467	473		9473	473		9479	473	
9491	473		9497	474		9511	474		9521	474		9533	475	
9539	475		9547	475		9551	475		9587	476		9601	475	
9613	477		9619	477		9623	477		9629	478		9631	478	
9643	477		9649	479		9661	478		9677	478		9679	479	
9689	478		9697	479		9719	479		9721	480		9733	481	
9739	480		9743	480		9749	480		9767	481		9769	480	
9781	482		9787	481		9791	482		9803	482		9811	482	
9817	482		9829	483		9833	483		9839	484		9851	484	
9857	483		9859	483		9871	484		9883	484		9887	485	
9901	485		9907	486		9923	485		9929	485		9931	485	
9941	485		9949	487		9967	488	1.611	9973	487		10007	488	
10009	487		10037	489		10039	489		10061	490		10067	490	
10069	489		10079	490		10091	490		10093	490		10099	491	
10103	490		10111	490		10133	492		10139	492		10141	492	
10151	492		10159	492		10163	492		10169	492		10177	493	
10181	493		10193	492		10211	494		10223	494		10243	495	
10247	494		10253	495		10259	495		10267	494		10271	496	
10273	494		10289	496		10301	496		10303	495		10313	496	
10321	496		10331	498	1.612	10333	497		10337	496		10343	497	
10357	497		10369	498		10391	499		10399	498		10427	499	
10429	499		10433	500		10453	501		10457	500		10459	500	
10463	501		10477	500		10487	501		10499	501		10501	503	1.614
10513	502		10529	503		10531	502		10559	503		10567	503	
10589	503		10597	504		10601	504		10607	505		10613	504	

Table 4. Continue 6

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
10627	505		10631	505		10639	504		10651	505		10657	505	
10663	507		10667	505		10687	506		10691	506		10709	507	
10711	507		10723	507		10729	507		10733	508		10739	507	
10753	508		10771	509		10781	509		10789	509		10799	510	
10831	510		10837	511		10847	511		10853	511		10859	511	
10861	511		10867	511		10883	511		10889	512		10891	511	
10903	512		10909	512		10937	513		10939	513		10949	513	
10957	514		10973	514		10979	514		10987	515		10993	515	
11003	515		11027	516		11047	516		11057	515		11059	516	
11069	517		11071	516		11083	517		11087	517		11093	517	
11113	518		11117	518		11119	518		11131	518		11149	519	
11159	519		11161	519		11171	519		11173	518		11177	520	
11197	521		11213	521		11239	522		11243	521		11251	521	
11257	522		11261	521		11273	522		11279	522		11287	522	
11299	523		11311	523		11317	523		11321	522		11329	523	
11351	524		11353	525		11369	525		11383	525		11393	525	
11399	525		11411	525		11423	526		11437	527		11443	527	
11447	526		11467	528		11471	527		11483	528		11489	526	
11491	528		11497	528		11503	527		11519	529		11527	529	
11549	530		11551	528		11579	530		11587	530		11593	531	
11597	530		11617	531		11621	531		11633	532		11657	532	
11677	533		11681	533		11689	533		11699	533		11701	533	
11717	534		11719	534		11731	534		11743	534		11777	536	1.614
11779	536		11783	536		11789	535		11801	536		11807	536	
11813	536		11821	537		11827	537		11831	537		11833	536	
11839	537		11863	537		11867	537		11887	538		11897	538	
11903	538		11909	539		11923	539		11927	539		11933	539	
11939	539		11941	539		11953	540		11959	539		11969	540	
11971	540		11981	540		11987	541		12007	541		12011	541	
12037	542		12041	542		12043	542		12049	542		12071	543	
12073	542		12097	543		12101	544		12107	544		12109	544	
12113	543		12119	544		12143	545		12149	544		12157	545	
12161	545		12163	544		12197	546		12203	546		12211	547	1.614
12227	545		12239	546		12241	546		12251	547		12253	546	
12263	547		12269	547		12277	548		12281	548		12289	548	
12301	549		12323	549		12329	549		12343	550		12347	549	
12373	549		12377	550		12379	550		12391	551		12401	550	
12409	551		12413	551		12421	552		12433	551		12437	552	
12451	552		12457	552		12473	553		12479	553		12487	552	
12491	554	1.614	12497	553		12503	553		12511	554		12517	554	
12527	554		12539	555		12541	555		12547	554		12553	555	
12569	555		12577	555		12583	555		12589	555		12601	556	
12611	556		12613	556		12619	556		12637	557		12641	556	

Table 4. Continue 7

q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$
12647	556		12653	558	1.615	12659	557		12671	557		12689	558	
12697	558		12703	558		12713	559		12721	559		12739	559	
12743	559		12757	559		12763	559		12781	561		12791	561	
12799	560		12809	561		12821	561		12823	562		12829	561	
12841	560		12853	562		12889	563		12893	563		12899	563	
12907	564		12911	564		12917	563		12919	563		12923	564	
12941	565	1.615	12953	564		12959	565		12967	565		12973	565	
12979	565		12983	565		13001	566		13003	565		13007	566	
13009	565		13033	566		13037	566		13043	566		13049	567	
13063	568	1.615	13093	568		13099	568		13103	568		13109	568	
13121	569		13127	567		13147	569		13151	570		13159	569	
13163	570		13171	570		13177	570		13183	570		13187	570	
13217	573	1.618	13219	571		13229	571		13241	572		13249	572	
13259	573		13267	572		13291	574		13297	573		13309	573	
13313	573		13327	574		13331	573		13337	573		13339	573	
13367	574		13381	575		13397	576		13399	576		13411	574	
13417	576		13421	576		13441	577		13451	576		13457	578	
13463	578		13469	576		13477	577		13487	578		13499	578	
13513	579		13523	579		13537	578		13553	579		13567	580	
13577	580		13591	581		13597	580		13613	580		13619	582	
13627	581		13633	582		13649	581		13669	582		13679	582	
13681	582		13687	582		13691	583		13693	583		13697	583	
13709	583		13711	583		13721	582		13723	582		13729	583	
13751	583		13757	584		13759	583		13763	584		13781	586	
13789	584		13799	584		13807	586		13829	586		13831	586	
13841	585		13859	586		13873	587		13877	586		13879	586	
13883	587		13901	588		13903	587		13907	588		13913	588	
13921	587		13931	589		13933	588		13963	590		13967	589	
13997	590		13999	590		14009	590		14011	590		14029	591	
14033	590		14051	591		14057	591		14071	591		14081	592	
14083	592		14087	592		14107	592		14143	594		14149	593	
14153	593		14159	593		14173	595		14177	595		14197	593	
14207	594		14221	595		14243	596		14249	596		14251	595	
14281	596		14293	597		14303	597		14321	597		14323	597	
14327	599		14341	598		14347	598		14369	599		14387	600	
14389	599		14401	600		14407	599		14411	600		14419	599	
14423	600		14431	601		14437	601		14447	600		14449	600	
14461	602		14479	601		14489	603	1.619	14503	601		14519	602	
14533	603		14537	602		14543	603		14549	603		14551	603	
14557	603		14561	604		14563	603		14591	604		14593	603	
14621	605		14627	605		14629	604		14633	605		14639	605	
14653	605		14657	605		14669	606		14683	605		14699	607	
14713	607		14717	607		14723	607		14731	607		14737	607	

Table 4. Continue 8

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
14741	607		14747	608		14753	608		14759	608		14767	608	
14771	607		14779	609		14783	609		14797	608		14813	609	
14821	609		14827	609		14831	610		14843	610		14851	610	
14867	610		14869	610		14879	610		14887	611		14891	611	
14897	611		14923	611		14929	611		14939	613		14947	613	
14951	612		14957	612		14969	612		14983	613		15013	614	
15017	614		15031	614		15053	614		15061	614		15073	615	
15077	614		15083	615		15091	616		15101	617	1.619	15107	616	
15121	616		15131	617		15137	617		15139	616		15149	618	
15161	617		15173	618		15187	618		15193	618		15199	617	
15217	618		15227	619		15233	618		15241	620		15259	620	
15263	620		15269	619		15271	620		15277	619		15287	621	
15289	621		15299	621		15307	621		15313	621		15319	621	
15329	621		15331	622		15349	621		15359	622		15361	621	
15373	622		15377	623		15383	622		15391	623		15401	622	
15413	623		15427	623		15439	623		15443	623		15451	624	
15461	625		15467	624		15473	624		15493	626	1.620	15497	625	
15511	625		15527	626		15541	625		15551	626		15559	626	
15569	626		15581	627		15583	626		15601	628		15607	628	
15619	627		15629	628		15641	629		15643	628		15647	628	
15649	628		15661	629		15667	629		15671	628		15679	628	
15683	630		15727	630		15731	630		15733	630		15737	631	
15739	630		15749	631		15761	631		15767	630		15773	631	
15787	631		15791	631		15797	631		15803	632		15809	633	
15817	632		15823	631		15859	634		15877	634		15881	633	
15887	634		15889	634		15901	635		15907	634		15913	635	
15919	635		15923	635		15937	635		15959	634		15971	636	
15973	635		15991	636		16001	636		16007	637		16033	637	
16057	638		16061	637		16063	637		16067	638		16069	638	
16073	638		16087	639		16091	637		16097	639		16103	639	
16111	639		16127	639		16139	640		16141	640		16183	641	
16187	641		16189	641		16193	640		16217	641		16223	641	
16229	641		16231	642		16249	642		16253	642		16267	642	
16273	642		16301	643		16319	643		16333	644		16339	644	
16349	645	1.620	16361	644		16363	645		16369	645		16381	645	
16411	645		16417	646		16421	645		16427	646		16433	645	
16447	647		16451	646		16453	647		16477	646		16481	647	
16487	648	1.620	16493	646		16519	648		16529	647		16547	648	
16553	649		16561	648		16567	648		16573	649		16603	649	
16607	650		16619	650		16631	650		16633	651		16649	651	
16651	651		16657	651		16661	651		16673	652		16691	650	
16693	651		16699	652		16703	652		16729	653		16741	653	
16747	653		16759	654	1.620	16763	653		16787	654		16811	654	

Table 4. Continue 9

q	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	$\overline{t}(q)$	$\overline{t^*}(q)$ <	q	$\overline{t}(q)$	$\overline{t^*}(q)$ <
16823	654		16829	654		16831	654		16843	655		16871	656	
16879	656		16883	655		16889	655		16901	656		16903	657	
16921	657		16927	657		16931	657		16937	657		16943	656	
16963	657		16979	657		16981	658		16987	658		16993	657	
17011	659		17021	659		17027	659		17029	658		17033	659	
17041	659		17047	661	1.622	17053	660		17077	660		17093	661	
17099	661		17107	660		17117	660		17123	660		17137	661	
17159	662		17167	663		17183	662		17189	664		17191	662	
17203	664		17207	662		17209	662		17231	664		17239	663	
17257	664		17291	664		17293	665		17299	665		17317	665	
17321	666		17327	666		17333	666		17341	666		17351	666	
17359	666		17377	667		17383	667		17387	667		17389	667	
17393	667		17401	668		17417	668		17419	668		17431	667	
17443	668		17449	670	1.623	17467	669		17471	669		17477	668	
17483	669		17489	668		17491	669		17497	666		17509	670	
17519	671		17539	669		17551	671		17569	671		17573	670	
17579	671		17581	670		17597	671		17599	672		17609	672	
17623	672		17627	673		17657	672		17659	673		17669	673	
17681	673		17683	673		17707	674		17713	674		17729	674	
17737	674		17747	674		17749	674		17761	675		17783	676	
17789	675		17791	676		17807	676		17827	676		17837	676	
17839	676		17851	676		17863	677		17881	677		17891	677	
17903	678		17909	678		17911	678		17921	679		17923	678	
17929	677		17939	679		17957	679		17959	680		17971	679	
17977	679		17981	679		17987	680		17989	680		18013	679	
18041	681		18043	682		18047	681		18049	681		18059	682	
18061	681		18077	681		18089	682		18097	683		18119	682	
18121	682		18127	683		18131	683		18133	683		18143	683	
18149	682		18169	684		18181	683		18191	683		18199	683	
18211	685		18217	684		18223	685		18229	685		18233	685	
18251	685		18253	685		18257	685		18269	685		18287	686	
18289	687		18301	686		18307	686		18311	687		18313	686	
18329	687		18341	688		18353	687		18367	688		18371	687	
18379	687		18397	688		18401	689		18413	689		18427	688	
18433	689		18439	689		18443	689		18451	688		18457	690	
18461	690		18481	689		18493	691		18503	691		18517	691	
18521	692		18523	690		18539	691		18541	691		18553	691	
18583	692		18587	693		18593	692		18617	693		18637	692	
18661	693		18671	694		18679	695		18691	696	1.624	18701	694	
18713	696		18719	694		18731	696		18743	695		18749	695	
18757	696		18773	697		18787	697		18793	697		18797	697	
18803	697		18839	699		18859	698		18869	698		18899	698	
18911	699		18913	698		18917	699		18919	699		18947	700	

Table 4. Continue 10

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
18959	700		18973	700		18979	701		19001	702		19009	701	
19013	701		19031	702		19037	702		19051	702		19069	702	
19073	702		19079	703		19081	703		19087	703		19121	704	
19139	704		19141	705		19157	704		19163	703		19181	706	1.624
19183	704		19207	706		19211	705		19213	705		19219	706	
19231	705		19237	705		19249	706		19259	707		19267	706	
19273	707		19289	708		19301	708		19309	707		19319	708	
19333	707		19373	709		19379	709		19381	709		19387	709	
19391	709		19403	709		19417	710		19421	709		19423	710	
19427	710		19429	711		19433	710		19441	709		19447	709	
19457	712	1.625	19463	710		19469	711		19471	710		19477	711	
19483	711		19489	711		19501	712		19507	710		19531	712	
19541	712		19543	713		19553	713		19559	712		19571	713	
19577	713		19583	713		19597	712		19603	714		19609	713	
19661	714		19681	714		19687	715		19697	715		19699	716	
19709	716		19717	716		19727	716		19739	716		19751	717	
19753	717		19759	717		19763	717		19777	716		19793	717	
19801	719	1.625	19813	718		19819	719		19841	718		19843	718	
19853	719		19861	719		19867	719		19889	720		19891	720	
19913	719		19919	721		19927	720		19937	720		19949	721	
19961	721		19963	721		19973	720		19979	721		19991	721	
19993	722		19997	722		20011	722		20021	721		20023	722	
20029	722		20047	723		20051	722		20063	722		20071	723	
20089	723		20101	724		20107	724		20113	724		20117	724	
20123	723		20129	725		20143	723		20147	724		20149	725	
20161	725		20173	726		20177	725		20183	726		20201	725	
20219	726		20231	727		20233	727		20249	727		20261	727	
20269	727		20287	726		20297	728		20323	728		20327	729	
20333	728		20341	729		20347	728		20353	729		20357	728	
20359	728		20369	729		20389	730		20393	729		20399	730	
20407	729		20411	729		20431	731		20441	729		20443	730	
20477	731		20479	731		20483	731		20507	732		20509	732	
20521	733		20533	732		20543	732		20549	732		20551	733	
20563	733		20593	733		20599	734		20611	733		20627	734	
20639	735		20641	734		20663	734		20681	735		20693	736	
20707	736		20717	735		20719	736		20731	736		20743	737	
20747	737		20749	738	1.626	20753	736		20759	737		20771	737	
20773	737		20789	738		20807	737		20809	738		20849	739	
20857	740		20873	740		20879	740		20887	739		20897	739	
20899	740		20903	740		20921	741		20929	740		20939	740	
20947	741		20959	740		20963	743	1.627	20981	741		20983	741	
21001	742		21011	741		21013	742		21017	741		21019	741	
21023	742		21031	742		21059	743		21061	742		21067	743	

Table 4. Continue 11

q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\overline{t}(q)$	$\overline{t^*}(q)$ $<$
21089	745		21101	745		21107	743		21121	745		21139	745	
21143	746		21149	743		21157	745		21163	745		21169	744	
21179	745		21187	746		21191	745		21193	745		21211	746	
21221	746		21227	746		21247	746		21269	747		21277	747	
21283	748		21313	749		21317	748		21319	747		21323	748	
21341	749		21347	748		21377	749		21379	749		21383	748	
21391	748		21397	749		21401	750		21407	750		21419	751	
21433	751		21467	750		21481	750		21487	751		21491	752	
21493	751		21499	752		21503	752		21517	752		21521	753	
21523	752		21529	752		21557	752		21559	753		21563	753	
21569	753		21577	752		21587	753		21589	754		21599	754	
21601	753		21611	754		21613	755		21617	754		21647	754	
21649	755		21661	754		21673	755		21683	756		21701	755	
21713	756		21727	755		21737	756		21739	756		21751	756	
21757	756		21767	758		21773	757		21787	758		21799	757	
21803	758		21817	758		21821	757		21839	759		21841	758	
21851	758		21859	757		21863	760		21871	759		21881	759	
21893	759		21911	759		21929	760		21937	758		21943	759	
21961	760		21977	761		21991	759		21997	760		22003	762	
22013	761		22027	761		22031	762		22037	762		22039	764	1.628
22051	763		22063	763		22067	761		22073	762		22079	763	
22091	763		22093	763		22109	762		22111	763		22123	764	
22129	764		22133	764		22147	763		22153	764		22157	765	
22159	764		22171	764		22189	766		22193	765		22229	765	
22247	766		22259	765		22271	767		22273	766		22277	766	
22279	766		22283	766		22291	767		22303	768		22307	768	
22343	768		22349	768		22367	768		22369	768		22381	770	
22391	769		22397	769		22409	770		22433	770		22441	768	
22447	770		22453	770		22469	769		22481	771		22483	770	
22501	771		22511	771		22531	772		22541	771		22543	772	
22549	773		22567	773		22571	773		22573	773		22613	773	
22619	773		22621	773		22637	773		22639	774		22643	774	
22651	773		22669	775		22679	774		22691	774		22697	775	
22699	774		22709	776		22717	775		22721	775		22727	775	
22739	775		22741	776		22751	775		22769	777		22777	777	
22783	775		22787	777		22807	778		22811	777		22817	777	
22853	778		22859	777		22861	778		22871	778		22877	779	
22901	777		22907	779		22921	779		22937	779		22943	780	
22961	780		22963	780		22973	780		22993	780		23003	781	
23011	782		23017	781		23021	782		23027	781		23029	781	
23039	781		23041	781		23053	782		23057	781		23059	782	
23063	782		23071	782		23081	782		23087	783		23099	782	
23117	783		23131	783		23143	783		23159	783		23167	784	

Table 4. Continue 12

		$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$
q	$\bar{t}(q)$	<	q	$\bar{t}(q)$	<	q	$\bar{t}(q)$	<	q	$\bar{t}(q)$	<	q	$\bar{t}(q)$	<
23173	784		23189	784		23197	785		23201	784		23203	784	
23209	785		23227	785		23251	786		23269	786		23279	785	
23291	786		23293	786		23297	786		23311	787		23321	787	
23327	786		23333	787		23339	787		23357	788		23369	788	
23371	788		23399	789		23417	788		23431	788		23447	789	
23459	789		23473	789		23497	791		23509	790		23531	791	
23537	791		23539	791		23549	792		23557	793	1.629	23561	792	
23563	792		23567	792		23581	791		23593	792		23599	792	
23603	793		23609	793		23623	792		23627	793		23629	793	
23633	792		23663	792		23669	793		23671	793		23677	793	
23687	794		23689	794		23719	794		23741	794		23743	794	
23747	795		23753	795		23761	793		23767	795		23773	795	
23789	796		23801	796		23813	795		23819	796		23827	798	1.629
23831	796		23833	797		23857	796		23869	796		23873	797	
23879	797		23887	797		23893	798		23899	798		23909	798	
23911	797		23917	798		23929	799		23957	798		23971	799	
23977	799		23981	800		23993	800		24001	800		24007	799	
24019	799		24023	800		24029	799		24043	800		24049	800	
24061	800		24071	801		24077	802		24083	801		24091	801	
24097	801		24103	802		24107	802		24109	802		24113	803	
24121	801		24133	802		24137	803		24151	802		24169	803	
24179	803		24181	803		24197	803		24203	803		24223	804	
24229	802		24239	804		24247	804		24251	805		24281	805	
24317	805		24329	806		24337	805		24359	806		24371	807	
24373	807		24379	806		24391	808		24407	808		24413	807	
24419	807		24421	808		24439	808		24443	807		24469	808	
24473	809		24481	808		24499	809		24509	808		24517	810	
24527	809		24533	809		24547	808		24551	811		24571	810	
24593	811		24611	811		24623	811		24631	811		24659	812	
24671	812		24677	812		24683	812		24691	813		24697	813	
24709	813		24733	812		24749	814		24763	814		24767	814	
24781	814		24793	815		24799	814		24809	814		24821	816	
24841	816		24847	815		24851	815		24859	815		24877	815	
24889	816		24907	816		24917	817		24919	817		24923	817	
24943	817		24953	818		24967	818		24971	818		24977	818	
24979	817		24989	817		25013	818		25031	818		25033	820	1.629
25037	818		25057	820		25073	819		25087	821	1.629	25097	818	
25111	821		25117	820		25121	820		25127	821		25147	820	
25153	820		25163	822		25169	820		25171	821		25183	820	
25189	822		25219	821		25229	823		25237	823		25243	824	1.629
25247	822		25253	822		25261	822		25301	823		25303	824	
25307	824		25309	823		25321	825		25339	824		25343	825	
25349	824		25357	825		25367	824		25373	825		25391	825	

Table 4. Continue 13

q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$	q	$\bar{t}(q)$	$\overline{t^*}(q)$ $<$
25409	825		25411	826		25423	827		25439	827		25447	826	
25453	827		25457	826		25463	826		25469	827		25471	826	
25523	829		25537	827		25541	829		25561	828		25577	828	
25579	829		25583	829		25589	828		25601	830		25603	829	
25609	830		25621	830		25633	831	1.630	25639	830		25643	830	
25657	831		25667	830		25673	830		25679	830		25693	831	
25703	831		25717	831		25733	832		25741	834	1.632	25747	831	
25759	832		25763	831		25771	831		25793	833		25799	833	
25801	833		25819	834		25841	833		25847	833		25849	834	
25867	834		25873	833		25889	835		25903	835		25913	836	
25919	836		25931	836		25933	836		25939	835		25943	836	
25951	835		25969	835		25981	835		25997	835		25999	837	
26003	836		26017	836		26021	837		26029	838		26041	837	
26053	837		26083	837		26099	837		26107	838		26111	838	
26113	838		26119	839		26141	839		26153	839		26161	839	
26171	840		26177	841		26183	840		26189	841		26203	840	
26209	841		26227	840		26237	841		26249	841		26251	841	
26261	840		26263	841		26267	841		26293	841		26297	842	
26309	841		26317	843		26321	841		26339	843		26347	843	
26357	841		26371	843		26387	844		26393	843		26399	843	
26407	844		26417	844		26423	844		26431	844		26437	844	
26449	845		26459	845		26479	845		26489	845		26497	846	
26501	846		26513	846		26539	846		26557	846		26561	847	
26573	847		26591	848		26597	847		26627	848		26633	848	
26641	848		26647	847		26669	849		26681	848		26683	848	
26687	849		26693	850		26699	849		26701	848		26711	850	
26713	849		26717	850		26723	850		26729	848		26731	850	
26737	849		26759	851		26777	851		26783	850		26801	851	
26813	851		26821	850		26833	851		26839	852		26849	851	
26861	851		26863	853		26879	852		26881	850		26891	852	
26893	852		26903	853		26921	852		26927	853		26947	853	
26951	853		26953	854		26959	855		26981	854		26987	854	
26993	854		27011	854		27017	856		27031	855		27043	855	
27059	856		27061	854		27067	856		27073	856		27077	855	
27091	856		27103	856		27107	856		27109	856		27127	857	
27143	857		27179	856		27191	856		27197	857		27211	857	
27239	859		27241	859		27253	858		27259	859		27271	860	
27277	858		27281	859		27283	860		27299	859		27329	860	
27337	860		27361	862		27367	861		27397	861		27407	861	
27409	861		27427	861		27431	862		27437	862		27449	862	
27457	862		27479	862		27481	863		27487	863		27509	862	
27527	864		27529	863		27539	864		27541	864		27551	864	
27581	865		27583	864		27611	866		27617	864		27631	866	

Table 4. Continue 14

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
27647	864		27653	865		27673	867		27689	866		27691	866	
27697	867		27701	866		27733	868		27737	867		27739	867	
27743	867		27749	867		27751	867		27763	868		27767	868	
27773	866		27779	869		27791	868		27793	869		27799	868	
27803	869		27809	868		27817	868		27823	868		27827	868	
27847	868		27851	869		27883	870		27893	869		27901	870	
27917	871		27919	871		27941	871		27943	872		27947	870	
27953	872		27961	872		27967	871		27983	872		27997	871	
28001	872		28019	872		28027	873		28031	872		28051	873	
28057	872		28069	874		28081	872		28087	875	1.632	28097	874	
28099	873		28109	874		28111	874		28123	873		28151	876	
28163	875		28181	875		28183	875		28201	876		28211	876	
28219	875		28229	875		28277	876		28279	876		28283	876	
28289	875		28297	876		28307	877		28309	878		28319	877	
28349	876		28351	878		28387	879		28393	877		28403	878	
28409	878		28411	879		28429	878		28433	880		28439	881	
28447	880		28463	879		28477	880		28493	881		28499	881	
28513	880		28517	881		28537	881		28541	881		28547	881	
28549	882		28559	883		28571	882		28573	883		28579	883	
28591	882		28597	882		28603	884	1.632	28607	882		28619	882	
28621	882		28627	883		28631	884		28643	882		28649	884	
28657	883		28661	883		28663	883		28669	883		28687	884	
28697	884		28703	883		28711	885		28723	884		28729	884	
28751	883		28753	885		28759	883		28771	885		28789	886	
28793	885		28807	884		28813	887		28817	885		28837	887	
28843	884		28859	886		28867	886		28871	886		28879	888	
28901	888		28909	887		28921	889		28927	888		28933	887	
28949	888		28961	888		28979	889		29009	888		29017	889	
29021	890		29023	888		29027	890		29033	891		29059	891	
29063	891		29077	889		29101	891		29123	892		29129	892	
29131	890		29137	892		29147	891		29153	891		29167	893	
29173	893		29179	893		29191	892		29201	894		29207	893	
29209	893		29221	893		29231	892		29243	892		29251	894	
29269	893		29287	894		29297	894		29303	894		29311	895	
29327	894		29333	896		29339	895		29347	895		29363	895	
29383	895		29387	896		29389	896		29399	895		29401	896	
29411	895		29423	895		29429	897		29437	896		29443	896	
29453	896		29473	898		29483	898		29501	898		29527	898	
29531	897		29537	899		29567	898		29569	900		29573	900	
29581	899		29587	900		29599	899		29611	899		29629	900	
29633	900		29641	900		29663	900		29669	900		29671	901	
29683	900		29717	901		29723	902		29741	902		29753	902	
29759	902		29761	902		29789	902		29803	903		29819	903	

Table 4. Continue 15

q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <	q	$\bar{t}(q)$	$\overline{t^*}(q)$ <
29833	903		29837	903		29851	904		29863	905		29867	903	
29873	905		29879	904		29881	903		29917	904		29921	905	
29927	904		29947	905		29959	906		29983	906		29989	907	
30011	907		30013	906		30029	907		30047	906		30059	908	
30071	907		30089	908		30091	907		30097	908		30103	909	
30109	909		30113	908		30119	909		30133	908		30137	909	
30139	909		30161	909		30169	910		30181	909		30187	910	
30197	910		30203	910		30211	910		30223	909		30241	910	
30253	911		30259	911		30269	910		30271	911		30293	911	
30307	912		30313	913	1.633	30319	912		30323	911		30341	911	
30347	911		30367	913		30389	912		30391	914		30403	913	
30427	914		30431	914		30449	913		30467	913		30469	915	
30491	914		30493	914		30497	914		30509	915		30517	915	
30529	915		30539	915		30553	915		30557	916		30559	916	
30577	916		30593	915		30631	917		30637	917		30643	917	
30649	916		30661	915		30671	917		30677	918		30689	917	
30697	918		30703	917		30707	918		30713	917		30727	919	
30757	919		30763	919		30773	920		30781	918		30803	921	
30809	919		30817	921		30829	920		30839	921		30841	920	
30851	919		30853	921		30859	922	1.633	30869	920		30871	921	
30881	919		30893	922		30911	922		30931	922		30937	920	
30941	922		30949	921		30971	922		30977	921		30983	923	
31013	925	1.634	31019	923		31033	924		31039	923		31051	925	
31063	925		31069	924		31079	925		31081	925		31091	924	
31121	926		31123	926		31139	925		31147	924		31151	927	
31153	925		31159	926		31177	926		31181	926		31183	927	
31189	928	1.634	31193	927		31219	927		31223	927		31231	927	
31237	928		31247	927		31249	928		31253	927		31259	927	
31267	928		31271	928		31277	928		31307	928		31319	929	
31321	929		31327	929		31333	930		31337	928		31357	928	
31379	929		31387	930		31391	930		31393	930		31397	930	
31469	931		31477	930		31481	931		31489	932		31511	930	
31513	930		31517	931		31531	932		31541	932		31543	932	
31547	933		31567	934		31573	933		31583	933		31601	933	
31607	935	1.634	31627	933		31643	934		31649	934		31657	934	
31663	935		31667	934		31687	936		31699	936		31721	934	
31723	937	1.635	31727	935		31729	935		31741	936		31751	937	
31769	937		31771	936		31793	937		31799	936		31817	936	
31847	937		31849	938		31859	938		31873	938		31883	937	
31891	938		31907	939		31957	939		31963	938		31973	939	
31981	940		31991	940		32003	940		32009	941		32027	941	
32029	939		32051	941		32057	941		32059	940		32063	941	
32069	940		32077	941		32083	941		32089	941		32099	941	

Table 4. Continue 16

		$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$			$\overline{t^*}(q)$
q	$\bar{t}(q)$	$<$	q	$\bar{t}(q)$	$<$	q	$\bar{t}(q)$	$<$	q	$\bar{t}(q)$	$<$	q	$\bar{t}(q)$	$<$
32117	942		32119	941		32141	941		32143	942		32159	943	
32173	943		32183	943		32189	943		32191	941		32203	944	
32213	944		32233	944		32237	943		32251	943		32257	945	
32261	945		32297	945		32299	946		32303	946		32309	946	
32321	945		32323	945		32327	945		32341	944		32353	946	
32359	947		32363	946		32369	947		32371	947		32377	946	
32381	946		32401	947		32411	947		32413	946		32423	946	
32429	947		32441	946		32443	947		32467	947		32479	948	
32491	948		32497	948		32503	948		32507	947		32531	949	
32533	947		32537	947		32561	948		32563	950		32569	950	
32573	949		32579	949		32587	950		32603	950		32609	950	
32611	951		32621	950		32633	951		32647	951		32653	951	
32687	951		32693	951		32707	952		32713	952		32717	952	
32719	951		32749	953		32771	953		32779	953		32783	952	
32789	952		32797	953		32801	952		32803	953		32831	953	
32833	953		32839	954		32843	954		32869	954		32887	954	
32909	956		32911	955		32917	956		32933	955		32939	955	
32941	957	1.635	32957	955		32969	956		32971	956		32983	956	
32987	955		32993	957		32999	957		33013	957				