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# On the Dimension of the Subspace of Liouvillian Solutions of a Fuchsian System

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**Abstract**—The paper deals with the problem of determining the dimension of the subspace of Liouvillian solutions of a Fuchsian system of linear differential equations in the case where this question can be answered directly in terms of the matrix of coefficients of the system.

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## 1. INTRODUCTION

On the Riemannian sphere $\overline{\mathbb{C}}$, we consider a *Fuchsian* system of $p$ linear differential equations

$$\frac{dy}{dz}=\biggl(\sum_{i=1}^{n}\frac{B_i}{z-a_i}\biggr)y, \qquad y(z)\in\mathbb{C}^p, \tag{1.1}$$

where $B_i$ are constant $p\times p$ matrices. For definiteness, we can set $\sum_{i=1}^{n}B_i=0$, i.e., we can assume that the system has $n$ singular points $a_1,\dots,a_n$ and the infinite point is not a singular point.

The *Piccard–Vessiot extension* of the field $\mathbb{C}(z)$ of rational functions corresponding to system (1.1) is the differential field $F=\mathbb{C}(z)\langle Y\rangle$ obtained by supplementing $\mathbb{C}(z)$ with all elements $y_{ij}$ of a fundamental matrix $Y(z)$ of system (1.1). We note that (by the Cauchy theorem) the functions $y_{ij}$ can be treated as elements of the field of germs of meromorphic functions at a nonsingular point $z_0$ of this system. System (1.1) is said to be *solvable in generalized quadratures* if the elements of the matrix $Y$ can be expressed in elementary or algebraic functions and their primitives or, more formally, if the field $F$ is contained in the extension of the field $\mathbb{C}(z)$ obtained by successively adding exponentials, integrals, and algebraic functions:

$$\mathbb{C}(z)=F_1\subseteq\dots\subseteq F_m, \qquad F\subseteq F_m,$$

where the field $F_{i+1}=F_i\langle x_i\rangle$ is generated over the field $F_i$ by an element $x_i$ which is an exponential or an integral of an element of the field $F_i$ or an algebraic element over the field $F_i$. Such an extension $\mathbb{C}(z)\subseteq F_m$ is called a *generalized Liouville extension* and the solvability in generalized quadratures thus means that the Piccard–Vessiot extension $F$ is contained in a generalized Liouville extension of the field of rational functions.

An individual solution of system (1.1) is said to be *Liouvillian* if all of its components are contained in a generalized Liouville extension of the field of rational functions. The set $\mathcal{L}$ of all Liouvillian solutions is a subspace in the space of solutions of the system and $0\le\dim\mathcal{L}\le p$ (the solvability of a system in generalized quadratures means that $\dim\mathcal{L}=p$). We note that the quantity $\dim\mathcal{L}$ can take the values $0,1,\dots,p-2$, and $p$, because the fact that there are $p-1$ linearly independent Liouvillian solutions implies that there exists another solution which is linearly independent of them. This can easily be verified in the case of a scalar linear differential equation

$$Lu=u^{(p)}+b_1(z)u^{(p-1)}+\dots+b_p(z)u=0, \qquad b_i\in\mathbb{C}(z), \tag{1.2}$$

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of order $p$. Indeed, if $u_1, \dots, u_{p-1}$ are linearly independent Liouvillian solutions of this equation, then one can consider a linear differential operator $L_1$ of order $p-1$ with coefficients from a generalized Liouville extension $K = \mathbb{C}(z)\langle u_1, \dots, u_{p-1}\rangle$ of the field of rational functions

$$L_1 u = W(u, u_1, \dots, u_{p-1}),$$

where $W(u, u_1, \dots, u_{p-1})$ is the Wronskian of the unknown function $u$ and the functions $u_1, \dots, u_{p-1}$. Since the functions $u_1, \dots, u_{p-1}$ form a fundamental system of solutions of the equation $L_1 u = 0$, then $L = \widetilde{L} \circ L_1$, where $\widetilde{L}$ is a first-order linear differential operator with coefficients from the field $K$. Therefore, determining the last basic solution $u_p$ of Eq. (1.2) reduces to solving the inhomogeneous linear differential equation $L_1 u = f$, where $f$ is a nonzero solution of the equation $\widetilde{L} f = 0$. Because $f$ is necessarily Liouvillian, the method of variation of constants shows that each solution of the equation $L_1 u = f$ is also Liouvillian and linearly independent of $u_1, \dots, u_{p-1}$ (because $f \neq 0$). The case of a system of linear differential equations can be reduced to the case of a scalar equation by using the Deligne lemma on cyclic vectors (see the proof of Lemma 2).

By analogy with classical Galois theory, the solvability or unsolvability of a linear system in quadratures is related to the properties of its differential Galois group. The *differential Galois group* $\mathbf{G} = \mathrm{Gal}(F/\mathbb{C}(z))$ of system (1.1) (the Piccard–Vessiot extensions $\mathbb{C}(z) \subseteq F$) is the group of differential automorphisms of the field $F$ (the automorphisms commuting with the operation of differentiation) which leave the elements of the field $\mathbb{C}(z)$ fixed:

$$\mathbf{G} = \left\{ \sigma : F \to F \;\middle|\; \sigma \circ \frac{d}{dz} = \frac{d}{dz} \circ \sigma,\ \sigma(f) = f,\ f \in \mathbb{C}(z) \right\}.$$

It follows from the definition that the image $\sigma(Y)$ of the fundamental matrix $Y$ of system (1.1) under the action of an arbitrary element $\sigma$ of the Galois group is again a fundamental matrix of this system and, therefore, $\sigma(Y) = Y(z)C$, $C \in \mathrm{GL}(p, \mathbb{C})$. Since each element of the differential Galois group is uniquely determined by its action on the fundamental matrix, the group $\mathbf{G}$ can be considered as a subgroup of the matrix group $\mathrm{GL}(p, \mathbb{C})$. Moreover, this subgroup $\mathbf{G} \subseteq \mathrm{GL}(p, \mathbb{C})$ is algebraic, i.e., closed in the Zariski topology of the space $\mathrm{GL}(p, \mathbb{C})$ (see [1, Theorem 5.5]).

The differential Galois group $\mathbf{G}$ can be represented as a union of finitely many nonintersecting connected sets simultaneously closed and open (in the Zariski topology), and a set containing a unit matrix is called a *connected component of unity*. The connected component of the unity $\mathbf{G}^0 \subseteq \mathbf{G}$ is a normal subgroup of a finite index (see [1, Lemma 4.5]). According to the Piccard–Vessiot theorem, the solvability of system (1.1) in generalized quadratures is equivalent to the solvability of the group $\mathbf{G}^0$ (see [1, Theorem 5.12], [2, Chap. 3, Theorem 5.1]). Thus, the Piccard–Vessiot theory (which was later completed by Kolchin, who considered other types of solvability and their dependence on the properties of the differential Galois group) explains why linear differential equations can be solvable or unsolvable in quadratures. But the theory cannot answer this question posed for each specific system, because the dependence of the differential Galois group of a system on its coefficients remains unknown. Therefore, it is interesting to study the cases where the problem of solvability of a system in generalized quadratures (or, more generally, the problem of the existence of Liouvillian solutions) can be answered directly in terms of the coefficients of the system.

It was shown in [3] that, in the case where the eigenvalues of the residue-matrices $B_i$ of Fuchsian system (1.1) are sufficiently small, the solvability of such a system in generalized quadratures is equivalent to the triangularity of all matrices $B_i$ (in a certain common basis). Following a simple and elegant remark of Professor Y. Haraoka, we here specify that this solvability criterion also holds in the case where the *pairwise differences* of eigenvalues of each matrix $B_i$ are sufficiently small. We use this to propose the following answer to a more general question about the dimension of the subspace $\mathcal{L}$ of Liouvillian solutions of system (1.1).

**Theorem 1.** *Suppose that* $\dim \mathcal{L} = k > 0$ *and the eigenvalues* $\beta_i^j$ *of each residue-matrix* $B_i$ *satisfy the condition*

$$\operatorname{Re} \beta_i^j > -\frac{1}{nk}, \qquad j = 1, \dots, p, \tag{1.3}$$

*and all differences satisfy the condition* $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$. *Then all matrices* $B_i$ (*in a certain basis*) *have the form*

$$B_i = \begin{pmatrix} B_i' & * \\ 0 & * \end{pmatrix}, \qquad i = 1, \dots, n,$$

*where* $B_i'$ *are upper-triangular* $k \times k$ *blocks.*

**Theorem 2.** *The assertion of Theorem* 1 *also holds if conditions* (1.3) *are replaced by the conditions*

$$|\operatorname{Re} \beta_i^j - \operatorname{Re} \beta_i^l| < \frac{1}{nk}, \qquad j, l = 1, \dots, p.$$

## 2. ON THE SOLVABILITY OF A FUCHSIAN SYSTEM IN GENERALIZED QUADRATURES

Along with the differential Galois group $\mathbf{G}$, we consider the *monodromy group* $\mathbf{M}$ of system (1.1) generated by the monodromy matrices $M_1, \dots, M_n$ of the system that correspond to the analytic continuation of the fundamental matrix $Y$ around the singular points $a_1, \dots, a_n$, respectively. Each matrix $M_i$ is defined as follows: since the operations of analytic continuation and differentiation commute, the matrix $Y$ extended from a neighborhood of a nonsingular point $z_0$ along a simple loop $\gamma_i$ surrounding only one singular point $a_i$ is again a fundamental matrix $YM_i$ (generally a different one). The analytic continuation also preserves the elements of the field $\mathbb{C}(z)$, because they are singly-valued functions. Therefore, we have $\mathbf{M} \subseteq \mathbf{G}$. Moreover, the differential Galois group of a Fuchsian system coincides with the closure of its monodromy group in the Zariski topology, namely, $\mathbf{G} = \overline{\mathbf{M}}$ (see [2, Chap. 6, Corollary 1.3]). Therefore, system (1.1) is solvable in generalized quadratures if and only if the group $\mathbf{G}^0 \cap \mathbf{M}$ is solvable.

Recall that, in the geometric interpretation, Fuchsian system (1.1) determines the logarithmic (Fuchsian) connection $\nabla$ in a holomorphically trivial vector bundle $E$ of rank $p$ over the Riemannian sphere (see [4, Lectures 1–7], [5, Chap. 17] for details). The *exponents* $\beta_i^1, \dots, \beta_i^p$ of the system (connection $\nabla$) at a singular point $a_i$ are eigenvalues of the matrix $B_i$. If the monodromy group $\mathbf{M}$ is reducible (i.e., the monodromy matrices $M_1, \dots, M_n$ form a general invariant subspace of dimension $k$), then the bundle $E$ has a subbundle $E' \subset E$ (of rank $k$) invariant under the connection $\nabla$. In this case, the exponents of the logarithmic connection $\nabla'$ (the restriction of the connection $\nabla$ to the subbundle $E'$) at a singular point $a_i$ coincide with arbitrarily many $k$ exponents of the $p$ exponents $\beta_i^1, \dots, \beta_i^p$. The degree of the holomorphic vector bundle equipped with the logarithmic connection can be calculated as the sum of exponents of the connection over all of its singular points (the degree is an integer).

The following criterion for solvability of Fuchsian system (1.1) in quadratures holds.

**Theorem 3** ([3], [6]). *Suppose that the exponents* $\beta_i^1, \dots, \beta_i^p$ *at each singular point* $a_i$ *of system* (1.1) *satisfy the inequalities*

$$\operatorname{Re} \beta_i^j > -\frac{1}{n(p-1)}, \qquad j = 1, \dots, p,$$

*and all the differences satisfy the condition* $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$. *Then the solvability of this system in generalized quadratures is equivalent to the triangularity of all matrices* $B_i$ (*in a certain common basis*).

To show that the conditions of Theorem 3 can be replaced by somewhat more general conditions, we assume that there are numbers $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ whose sum is zero and the exponents of system (1.1) at each singular point $a_i$ satisfy the inequalities

$$\operatorname{Re} \beta_i^j > \lambda_i - \frac{1}{n(p-1)}, \qquad j = 1, \dots, p.$$

The transformation $\widetilde{y}=(z-a_1)^{-\lambda_1}\cdots(z-a_n)^{-\lambda_n}y$ results in the Fuchsian system

$$\frac{d\widetilde{y}}{dz}=\biggl(\sum_{i=1}^{n}\frac{B_i-\lambda_i I}{z-a_i}\biggr)\widetilde{y}$$

with the same singular points $a_1,\dots,a_n$ (the infinite point is not a singular point, because $\sum_{i=1}^n\lambda_i=0$). In this case, the exponents of the obtained system satisfy the conditions of Theorem 3. Therefore, Theorem 3, which is a criterion of solvability in generalized quadratures, can be applied both to the transformed system and and to initial system. In particular, if the pairwise differences of the exponents at each singular point $a_i$ of system (1.1) satisfy the inequalities

$$|\operatorname{Re}\beta_i^j-\operatorname{Re}\beta_i^l|<\frac{1}{n(p-1)},\qquad j,l=1,\dots,p,$$

then one can take the numbers

$$\lambda_i=\frac{1}{p}\sum_{j=1}^{p}\operatorname{Re}\beta_i^j,\qquad i=1,\dots,n,$$

for the numbers $\lambda_1,\dots,\lambda_n$. Indeed, in this case, we have $\sum_{i=1}^n\lambda_i=0$ due to the relation $\sum_{i=1}^n\operatorname{tr}B_i=0$, and for each singular point $a_i$, we have

$$\operatorname{Re}\beta_i^j-\lambda_i=\frac{1}{p}((\operatorname{Re}\beta_i^j-\operatorname{Re}\beta_i^1)+\dots+(\operatorname{Re}\beta_i^j-\operatorname{Re}\beta_i^p))>-\frac{1}{n(p-1)},\qquad j=1,\dots,p.$$

Thus, we arrive at the following assertion.

**Theorem 4.** *Suppose that the pairwise differences of the exponents of system* (1.1) *at each singular point $a_i$ satisfy the inequalities*

$$|\operatorname{Re}\beta_i^j-\operatorname{Re}\beta_i^l|<\frac{1}{n(p-1)},\qquad j,l=1,\dots,p,$$

*and do not belong to the set $\mathbb{Q}\setminus\mathbb{Z}$. Then the solvability of this system in generalized quadratures is equivalent to the triangularity of all matrices $B_i$* (*in a certain common basis*).

## 3. PROOF OF THE THEOREM ON THE DIMENSION OF THE SPACE $\mathcal{L}$

**Lemma 1.** *Let $y$ be a Liouvillian solution of system* (1.1). *Then $\sigma(y)$ is also a Liouvillian solution of this system for each element $\sigma$ of its monodromy group* $\mathbf{M}$.

**Proof.** By definition, all components of the solution $y$ (as elements of the field of germs of meromorphic functions at a nonsingular point $z_0$) are contained in a generalized Liouville extension $\mathbb{C}(z)\subseteq F_m$ of the field of rational functions. Then all components of the solution $\sigma(y)$ are contained in the extension $\mathbb{C}(z)\subseteq\sigma(F_m)$ which is also a generalized Liouville extension. Indeed, if

$$\mathbb{C}(z)=F_1\subseteq\dots\subseteq F_m$$

is a chain of elementary extensions, where $F_{i+1}=F_i\langle x_i\rangle$ and the element $x_i$ is an exponential or an integral of an element of the field $F_i$ or an algebraic element over the field $F_i$, then

$$\mathbb{C}(z)=\sigma(F_1)\subseteq\dots\subseteq\sigma(F_m)$$

is also a chain of elementary extensions, where $\sigma(F_{i+1})=\sigma(F_i)\langle\sigma(x_i)\rangle$ and the element $\sigma(x_i)$ is respectively an exponential of an integral of an element of the field $\sigma(F_i)$ or an algebraic element over the field $\sigma(F_i)$. The lemma is proved. □

It follows from Lemma 1 that the subspace $\mathcal{L}$ of Liouvillian solutions of system (1.1) is invariant under the action of the monodromy group. Thus, the monodromy matrices of the system have a common invariant subspace of dimension $k$. Therefore, the holomorphically trivial vector bundle $E$ has a subbundle $E' \subset E$ of rank $k$ which is invariant under the connection $\nabla$. The degree of any subbundle of a trivial bundle is nonpositive, and if this degree is zero, then the subbundle itself is also holomorphically trivial (see [5, Corollary 17.25]). We show that the subbundle $E'$ is therefore holomorphically trivial. As was explained at the beginning of the preceding section,

$$\deg E' = \sum_{i=1}^{n} \sum_{j=1}^{k} \operatorname{Re} \widetilde{\beta}_i^j,$$

where $\widetilde{\beta}_i^1, \dots, \widetilde{\beta}_i^k$ are exponents of the logarithmic connection $\nabla'$ (the restriction of the connection $\nabla$ to the subbundle $E'$) at a singular point $a_i$ (they coincide with arbitrarily many $k$ exponents of the $p$ exponents $\beta_i^1, \dots, \beta_i^p$ of the connection $\nabla$). Under the conditions of Theorem 2, we determine the quantities

$$\lambda_i = \frac{1}{p} \sum_{j=1}^{p} \operatorname{Re} \beta_i^j, \qquad i = 1, \dots, n,$$

whose sum is zero and, as in the proof of Theorem 4, obtain

$$\operatorname{Re} \widetilde{\beta}_i^j - \lambda_i > -\frac{1}{nk}.$$

Therefore,

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \operatorname{Re} \widetilde{\beta}_i^j = \sum_{i=1}^{n} \biggl( \sum_{j=1}^{k} \operatorname{Re} \widetilde{\beta}_i^j - k\lambda_i \biggr) = \sum_{i=1}^{n} \sum_{j=1}^{k} (\operatorname{Re} \widetilde{\beta}_i^j - \lambda_i) > -1,$$

which implies that $\deg E' = 0$. (Obviously, under the conditions of Theorem 1, it is not required to introduce the quantities $\lambda_i$.)

By [6, Lemma 1], it follows from the holomorphic triviality of the subbundle $E'$ that the matrices $B_i$ can simultaneously be reduced to the upper-triangular block form

$$C B_i C^{-1} = \begin{pmatrix} B_i' & * \\ 0 & * \end{pmatrix}, \qquad i = 1, \dots, n, \quad C \in \mathrm{GL}(p, \mathbb{C}),$$

where $B_i'$ are $k \times k$ blocks. Now the assertions of Theorems 1 and 2 are direct consequences of the following lemma and Theorems 3 and 4.

**Lemma 2.** *The following Fuchsian system of $k$ equations*

$$\frac{dy'}{dz} = \biggl( \sum_{i=1}^{n} \frac{B_i'}{z - a_i} \biggr) y', \qquad y'(z) \in \mathbb{C}^k, \tag{3.1}$$

*is solvable in generalized quadratures.*

**Proof.** Let $Y$ be a fundamental matrix of system (1.1) whose first $k$ columns are Liouvillian solutions of this system. The monodromy matrices $M_i$ corresponding to the fundamental matrix $Y$ have the upper-triangular block form

$$M_i = \begin{pmatrix} M_i' & * \\ 0 & * \end{pmatrix}, \qquad i = 1, \dots, n,$$

where the $k \times k$ blocks $M_i'$ generate the monodromy group $\mathbf{M}'$ of system (3.1) (the connection $\nabla'$ in the holomorphically trivial vector bundle $E'$). To prove the lemma, it suffices to show that the group $\mathbf{M}'$ has a normal subgroup of a finite index.

By the Deligne lemma [7, Lemma II.1.3] on cyclic vectors (which was analytically proved in [8]), there exists a matrix function $\Gamma(z)$, whose elements are rational functions, such that the elements $u_1, \dots, u_p$ in the first row of the matrix $\Gamma Y$ form a basis in the space of solutions of a scalar Fuchsian equation of order $p$. This equation can have extra singular points in addition to $a_1, \dots, a_n$, but the monodromy is trivial at them. We note that the functions $u_1, \dots, u_k$ are Liouvillian by construction and the set $(u_1, \dots, u_k)$ is transformed into the set $(u_1, \dots, u_k)M_i'$ by the analytic continuation around a singular point $a_i$, $i = 1, \dots, n$. Thus, the set of functions $u_1, \dots, u_k$ permits constructing the scalar Fuchsian equation

$$\frac{W(u, u_1, \dots, u_k)}{W(u_1, \dots, u_k)} = 0$$

of order $k$ for the unknown $u$ (where the $W$ are the corresponding Wronskians), and this set is a basis in the space of solutions of this equation. The equation thus constructed turns out to be solvable in generalized quadratures, and the group $\mathbf{M}'$ is its monodromy group. Therefore, the group $\mathbf{M}'$ has a normal subgroup of a finite index. The lemma is proved. □

It is easy to note that Fuchsian system (1.1) with residue-matrix of the form

$$B_i = \begin{pmatrix} B_i' & * \\ 0 & * \end{pmatrix}, \qquad i = 1, \dots, n, \tag{3.2}$$

where $B_i'$ are upper-triangular $k \times k$ blocks, has at least $k$ linearly independent Liouvillian solutions independently of the value of the exponents, because, in this case, Fuchsian system (3.1) of $k$ equations with triangular residue-matrix $B_i'$ is solvable in generalized quadratures. Therefore, we have the following direct consequences of Theorems 1 and 2.

**Corollary 1.** *Suppose that the eigenvalues $\beta_i^j$ of each residue-matrix $B_i$ of the Fuchsian system* (1.1) *satisfy the condition*

$$\operatorname{Re} \beta_i^j > -\frac{1}{n(p-1)}, \qquad j = 1, \dots, p, \tag{3.3}$$

*and all the differences satisfy the condition $\beta_i^j - \beta_i^l \notin \mathbb{Q} \setminus \mathbb{Z}$. Then $\dim \mathcal{L} = k$ if and only if all matrices $B_i$ can simultaneously be reduced to the form* (3.2)*, where $B_i'$ are upper-triangular $k \times k$ blocks and cannot be reduced to the same form with upper-triangular blocks $B_i'$ of a greater size.*

**Corollary 2.** *The assertion of Corollary* 1 *also holds if conditions* (3.3) *are replaced by the conditions*

$$|\operatorname{Re} \beta_i^j - \operatorname{Re} \beta_i^l| < \frac{1}{n(p-1)}, \qquad j, l = 1, \dots, p.$$

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