Towards the convergence of generalized power series solutions of algebraic ODEs

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Abstract. The aim of this work is to provide another proof of the sufficient condition of the convergence of a generalized power series (with complex power exponents) formally satisfying an algebraic (polynomial) ordinary differential equation. This proof is based on the implicit mapping theorem for Banach spaces rather than on the majorant method used in our previous proof. We also discuss some examples of a such type formal solutions of Painlevé equations.

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1. Introduction

Let us consider an ordinary differential equation (ODE)

$$F(z, u, \delta u, \ldots, \delta^m u) = 0 \tag{1.1}$$

of order m with respect to the unknown u, where $F(z, u_0, u_1, \ldots, u_m) \not\equiv 0$ is a polynomial of $m+2$ variables, $\delta = z\frac{d}{dz}$.

In the paper we study *generalized* power series solutions of (1.1) of the form

$$\varphi = \sum_{n=0}^{\infty} c_n z^{s_n}, \qquad c_n \in \mathbb{C}, \qquad s_n \in \mathbb{C}, \tag{1.2}$$

with the power exponents satisfying conditions

$$0 \leqslant \operatorname{Re} s_0 \leqslant \operatorname{Re} s_1 \leqslant \ldots, \qquad \lim_{n\to\infty} \operatorname{Re} s_n = +\infty$$

(the latter, in particular, implies that a set of exponents having a fixed real part is finite).

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Note that substituting the series (1.2) into the equation (1.1) makes sense, as only a finite number of terms in φ contribute to any term of the form cz^s in the expansion of $F(z,\Phi) = F(z,\varphi,\delta\varphi,\dots,\delta^m\varphi)$ in powers of z. Indeed, $\delta^j\varphi = \sum_{n=0}^{\infty} c_n s_n^j z^{s_n}$, and an equation $s = s_{n_0} + s_{n_1} + \dots + s_{n_l}$ has a finite number of solutions $(s_{n_0}, s_{n_1},\dots,s_{n_l})$, since $0 \leqslant \operatorname{Re} s_n \to +\infty$. Furthermore, for any integer N an inequality $\operatorname{Re}(s_{n_0}+s_{n_1}+\dots+s_{n_l}) \leqslant N$ has also a finite number of solutions, so that powers of z in the expansion of $F(z,\Phi)$ can be ordered by the increasing of real parts. Thus, one may correctly define the notion of a formal solution of (1.1) in the form of a generalized power series. In particular, the Painlevé III, V, VI equations are known to have such formal solutions (see [1] – [9]). Their convergence in sectorial domains near zero is also proved in some of those papers. Here we are interested in convergence for an equation of the general form (1.1).

There is the following sufficient condition [10] of the convergence of a generalized power series solution of (1.1).

Theorem 1.1. *Let the generalized power series* (1.2) *formally satisfy the equation* (1.1)*,* $\frac{\partial F}{\partial u_m}(z,\Phi) \neq 0$*, and for each* $i = 0,1,\dots,m$ *one have*

$$\frac{\partial F}{\partial u_i}(z,\Phi) = A_i z^{\lambda} + B_i z^{\lambda_i} + \dots, \qquad \operatorname{Re}\lambda_i > \operatorname{Re}\lambda, \quad A_m \neq 0. \tag{1.3}$$

Then for any sector S of sufficiently small radius with the vertex at the origin and of the opening less than 2π, the series φ converges uniformly in S.

In this paper we propose a shorter proof of Theorem 1.1 based on the implicit mapping theorem, whereas in the original proof in [10] we used the majorant method. In the case of integer powers $s_n = n \in \mathbb{Z}_+$, Theorem 1.1 was obtained by Malgrange [11], and in the case of real powers $s_n \in \mathbb{R}$ this theorem was formulated in a somewhat different form in [12, Th. 3.4].

2. Auxiliary lemmas

The proof of Theorem 1.1 is preceded by some auxiliary lemmas which have been proved in [10].

Lemma 2.1. *Under the assumptions of Theorem* 1.1*, there exists an integer* $\mu' \geqslant 0$ *such that for any integer* $\mu \geqslant \mu'$ *satisfying* $\operatorname{Re}(s_{\mu+1} - s_\mu) > 0$*, a transformation*

$$u = \sum_{n=0}^{\mu} c_n z^{s_n} + z^{s_\mu} v \tag{2.1}$$

reduces the equation (1.1) *to an equation of the form*

$$L(\delta)v + N(z,v,\delta v,\dots,\delta^m v) = 0, \tag{2.2}$$

where

- *L is a polynomial of degree m,*
- *$L(s) \neq 0$ for any s with* $\operatorname{Re} s > 0$*, and*

– *N is a finite linear combination of monomials of the form*

$$z^{\beta}v^{q_0}(\delta v)^{q_1}\dots(\delta^m v)^{q_m}, \qquad \beta\in\mathbb{C},\ \operatorname{Re}\beta>0,\ q_i\in\mathbb{Z}_+.$$

As follows from the form of the transformation (2.1), the reduced equation (2.2) has a generalized power series solution $\psi=\sum_{n=\mu+1}^{\infty}c_n z^{s_n-s_\mu}$. The second auxiliary lemma describes a structure of the set of power exponents $s_n-s_\mu\in\mathbb{C}$ of this series.

Let us define an additive semi-group Γ generated by a (finite) set of power exponents of the variable z containing in $N(z,v,\delta v,\dots,\delta^m v)$, and let $r_1,\dots,r_l$ be generators of this semi-group, that is,

$$\Gamma=\{m_1r_1+\ldots+m_lr_l\mid m_i\in\mathbb{Z}_+,\ \sum_{i=1}^{l}m_i>0\}, \qquad \operatorname{Re}r_i>0.$$

Lemma 2.2. *All the numbers s_n-s_μ, $n\geqslant\mu+1$, belong to the semi-group Γ.*

We may assume that the generators $r_1,\dots,r_l$ of Γ are linearly independent over $\mathbb{Z}$. This is provided by the following lemma.

Lemma 2.3. *There are complex numbers $\rho_1,\dots,\rho_\tau$ linearly independent over $\mathbb{Z}$, such that all $\operatorname{Re}\rho_i>0$, and an additive semi-group Γ' generated by them contains the above semi-group Γ generated by $r_1,\dots,r_l$.*

3. Proof of Theorem 1.1

For the simplicity of exposition we assume that the semi-group Γ is generated by two numbers:

$$\Gamma=\{m_1r_1+m_2r_2\mid m_1,m_2\in\mathbb{Z}_+,\ m_1+m_2>0\}, \qquad \operatorname{Re}r_1,\ \operatorname{Re}r_2>0.$$

In the case of an arbitrary number l of generators all constructions are analogous, only multivariate Taylor series in l rather than in two variables are involved.

We should establish the convergence of the generalized power series

$$\psi=\sum_{n=\mu+1}^{\infty}c_n z^{s_n-s_\mu},$$

which satisfies the equality

$$L(\delta)\psi+N(z,\psi,\delta\psi,\dots,\delta^m\psi)=0. \tag{3.1}$$

According to Lemma 2.2, all the exponents s_n-s_μ belong to the semi-group Γ:

$$s_n-s_\mu=m_1r_1+m_2r_2, \qquad (m_1,m_2)\in M\subseteq\mathbb{Z}_+^2\setminus\{0\},$$

for some set M such that the map $n\mapsto(m_1,m_2)$ is a bijection from $\mathbb{N}\setminus\{1,\dots,\mu\}$ to M. Hence,

$$\psi=\sum_{(m_1,m_2)\in M}c_{m_1,m_2}z^{m_1r_1+m_2r_2}=\sum_{(m_1,m_2)\in\mathbb{Z}_+^2\setminus\{0\}}c_{m_1,m_2}z^{m_1r_1+m_2r_2}$$

(in the last series one puts $c_{m_1,m_2}=0$, if $(m_1,m_2)\not\in M$).

Now we define a natural linear map $\sigma:\mathbb{C}[[z^\Gamma]]\to\mathbb{C}[[z_1,z_2]]_*$ from the $\mathbb{C}$-algebra of generalized power series with exponents in Γ to the $\mathbb{C}$-algebra of Taylor series in two variables without a constant term,

$$\sigma:\sum_{\gamma=m_1r_1+m_2r_2\in\Gamma}a_\gamma z^\gamma\mapsto\sum_{\gamma=m_1r_1+m_2r_2\in\Gamma}a_\gamma z_1^{m_1}z_2^{m_2}.$$

As follows from the linear independence of the generators r_1,r_2 over $\mathbb{Z}$,

$$\sigma(\eta_1\eta_2)=\sigma(\eta_1)\sigma(\eta_2)\qquad\forall\eta_1,\eta_2\in\mathbb{C}[[z^\Gamma]],$$

hence σ is an isomorphism. The differentiation $\delta:\mathbb{C}[[z^\Gamma]]\to\mathbb{C}[[z^\Gamma]]$ naturally induces a linear bijective map Δ of $\mathbb{C}[[z_1,z_2]]_*$ to itself,

$$\Delta:\sum_{\gamma\in\Gamma}a_\gamma z_1^{m_1}z_2^{m_2}\mapsto\sum_{\gamma\in\Gamma}\gamma\,a_\gamma z_1^{m_1}z_2^{m_2},$$

which clearly satisfies $\Delta\circ\sigma=\sigma\circ\delta$, so that the following commutative diagramme holds:

$$\begin{array}{ccc}\mathbb{C}[[z^\Gamma]] & \stackrel{\delta}{\longrightarrow} & \mathbb{C}[[z^\Gamma]]\\ \downarrow\sigma & & \downarrow\sigma\\ \mathbb{C}[[z_1,z_2]]_* & \stackrel{\Delta}{\longrightarrow} & \mathbb{C}[[z_1,z_2]]_*\end{array}$$

Thus we have the representation

$$\tilde\psi=\sigma(\psi)=\sum_{\gamma\in\Gamma}c_\gamma z_1^{m_1}z_2^{m_2}$$

of the formal solution ψ of (2.2) by a multivariate Taylor series, where $c_\gamma=c_{m_1,m_2}$ for every $\gamma=m_1r_1+m_2r_2$. Now we apply the map σ to the both sides of the equality (3.1) and obtain a relation for $\tilde\psi$:

$$L(\Delta)\tilde\psi+\widetilde{N}(z_1,z_2,\tilde\psi,\Delta\tilde\psi,\ldots,\Delta^m\tilde\psi)=0,\tag{3.2}$$

where $\widetilde{N}(z_1,z_2,u_0,\ldots,u_m)$ is a polynomial such that $\widetilde{N}(0,0,u_0,\ldots,u_m)\equiv0$.

We conclude the proof of Theorem 1.1 establishing the convergence of the bivariate Taylor series $\tilde\psi$, which represents the generalized power series ψ and satisfies the relation (3.2). We use the dilatation method based on the implicit mapping theorem for Banach spaces. This was originally used by Malgrange [11] for proving Theorem 1.1 in the case of integer powers $s_n=n\in\mathbb{Z}_+$.

Let us define the following Banach spaces H^j of (formal) Taylor series in two variables without a constant term:

$$H^j=\Big\{\eta=\sum_{\gamma\in\Gamma}a_\gamma z_1^{m_1}z_2^{m_2}\mid\sum_{\gamma\in\Gamma}|\gamma|^j\,|a_\gamma|<+\infty\Big\},\qquad j=0,1,\ldots,m,$$

with the norm

$$\|\eta\|_j=\sum_{\gamma\in\Gamma}|\gamma|^j\,|a_\gamma|=\|\Delta^j\eta\|_0.$$

(The completeness of each H^j is checked in a way similar to that how one checks the completeness of the space l_2; see, for example, [13, Ch. 6, §4].) One clearly has $H^m \subset H^{m-1} \ldots \subset H^0$ and

$$\Delta : H^j \to H^{j-1}, \qquad j = 1, \ldots, m,$$

are continuous linear mappings.

We recall below the implicit mapping theorem for Banach spaces (see [13, Th. 10.2.1]).

Let $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ be Banach spaces, A an open subset of the direct product $\mathcal{E} \times \mathcal{F}$, and $h : A \to \mathcal{G}$ a continuously differentiable mapping. Consider a point $(x_0, y_0) \in A$ such that $h(x_0, y_0) = 0$ and $\frac{\partial h}{\partial y}(x_0, y_0)$ is a bijective linear mapping from $\mathcal{F}$ to $\mathcal{G}$.

Then there are a neighbourhood $U_0 \subset \mathcal{E}$ of the point x_0 and a unique continuous mapping $g : U_0 \to \mathcal{F}$ such that $g(x_0) = y_0$, $(x, g(x)) \in A$, and $h(x, g(x)) = 0$ for any $x \in U_0$.

We will apply this theorem to the Banach spaces $\mathbb{C}$, H^m, H^0, and to the mapping $h : \mathbb{C} \times H^m \to H^0$ defined by

$$h : (\lambda, \eta) \mapsto L(\Delta)\eta + \widetilde{N}(\lambda z_1, \lambda z_2, \eta, \Delta\eta, \ldots, \Delta^m \eta),$$

with L and $\widetilde{N}$ coming from (3.2). This mapping is continuously differentiable, moreover $h(0,0) = 0$, and $\frac{\partial h}{\partial \eta}(0,0) = L(\Delta)$ is a bijective linear mapping from H^m to H^0. Indeed,

$$L(\Delta) : a_\gamma z_1^{m_1} z_2^{m_2} \mapsto a_\gamma L(\gamma) z_1^{m_1} z_2^{m_2} \ (= 0 \Longleftrightarrow a_\gamma = 0),$$

therefore $\ker L(\Delta) = \{0\}$ (recall that $L(\gamma) \neq 0$ for any γ with $\operatorname{Re}\gamma > 0$). In the same time, if $\sum_{\gamma \in \Gamma} a_\gamma z_1^{m_1} z_2^{m_2} \in H^0$, then $\sum_{\gamma \in \Gamma} (a_\gamma / L(\gamma))\, z_1^{m_1} z_2^{m_2} \in H^m$, that is, the image of $L(\Delta)$ coincides with H^0.

Hence, by the implicit mapping theorem, there are a real number $\rho > 0$ and $\eta_\rho \in H^m$ such that

$$L(\Delta)\eta_\rho + \widetilde{N}(\rho z_1, \rho z_2, \eta_\rho, \Delta\eta_\rho \ldots, \Delta^m \eta_\rho) = 0.$$

Making the change of variables $(z_1, z_2) \mapsto (\frac{z_1}{\rho}, \frac{z_2}{\rho})$, which induces an automorphism $\eta(z_1, z_2) \mapsto \eta(\frac{z_1}{\rho}, \frac{z_2}{\rho})$ of $\mathbb{C}[[z_1, z_2]]_*$ commuting with Δ, one can easily see that the above relation implies that the power series $\eta_\rho(\frac{z_1}{\rho}, \frac{z_2}{\rho})$ satisfies the same equality (3.2) as $\tilde{\psi} = \sum_{\gamma \in \Gamma} c_\gamma z_1^{m_1} z_2^{m_2}$ does. Hence, these two series coincide (the coefficients of a series satisfying (3.2) are determined uniquely by this equality) and $\tilde{\psi}$ has a non-zero radius of convergence. This implies (substitute $z_1 = z^{r_1}$, $z_2 = z^{r_2}$ remembering that $\operatorname{Re} r_1, \operatorname{Re} r_2 > 0$) the convergence of the series

$$\sum_{\gamma \in \Gamma} c_\gamma z^\gamma = \sum_{n=\mu+1}^{\infty} c_n z^{s_n - s_\mu}$$

for any z from a sector S of sufficiently small radius with the vertex at the origin and of the opening less than 2π, whence Theorem 1.1 follows.

4. Examples

As was mentioned in Introduction, for the Painlevé III, V, VI equations, their generalized power series solutions of the form (1.2) converge in some sectorial domains near the origin. This is proved in each case mainly by using a kind of majorant series. Here we give examples of such formal solutions and illustrate how Theorem 1.1 can be applied to prove their convergence.

Let us consider the Painlevé III equation with the parameters $a = b = 0$, $c = d = 1$:

$$\frac{d^2u}{dz^2} = \frac{1}{u}\left(\frac{du}{dz}\right)^2 - \frac{1}{z}\frac{du}{dz} + u^3 + \frac{1}{u}.$$

Rewritten in the form (1.1), this becomes

$$u\,\delta^2 u - (\delta u)^2 - z^2u^4 - z^2 = 0 \quad \text{or} \quad F(z, u, \delta u, \delta^2 u) = 0, \tag{4.1}$$

where

$$F(z, u_0, u_1, u_2) = u_0u_2 - u_1^2 - z^2(u_0^4 + 1).$$

As known [2], [5], the equation (4.1) has a two-parametric family of formal solutions

$$\varphi = c_r z^r + \sum_{s\in \mathrm{K}} c_s z^s, \tag{4.2}$$

where $c_r \neq 0$ is an arbitrary complex number and r is any complex number with $-1 \leqslant \mathrm{Re}\, r \leqslant 1$. The other coefficients c_s are determined uniquely by c_r, and the set K of power exponents is of the form

$$\mathrm{K} = \{r + m_1(1-r) + m_2(1+r) \mid m_1, m_2 \in \mathbb{Z}_+, m_1 + m_2 > 0\}.$$

Denote $r = \rho + i\sigma$. There are two essentially different types of formal solutions in the family above.

1) Solutions with $\rho \in (-1, 1)$. For any solution of such type there is only a finite number of exponents $s = r + m_1(1-r) + m_2(1+r)$ with a fixed real part $\mathrm{Re}\, s = \rho + m_1(1-\rho) + m_2(1+\rho)$, since $1-\rho$ and $1+\rho$ are positive. Therefore, such solutions are of the form (1.2), and we will apply Theorem 1.1 to prove their convergence.

2) Solutions with $\rho = \pm 1$. For any solution of such type there are infinitely many exponents $s = r + m_1(1-r) + m_2(1+r)$ with a fixed real part $\mathrm{Re}\, s = \rho + m_1(1-\rho) + m_2(1+\rho)$, since the latter depends only on one of the two indexes m_1, m_2. Therefore, such solutions are not generalized power series of the form (1.2), and Theorem 1.1 cannot be used for studying their convergence. In fact, such series diverge along some rays coming to the origin, which will be explained below.

Let us consider a formal solution φ of the **first type**. To prove the convergence of φ in sectors of small radius, it is sufficient to find the partial derivatives $\frac{\partial F}{\partial u_0}$, $\frac{\partial F}{\partial u_1}$, $\frac{\partial F}{\partial u_2}$ along φ and verify the assumption (1.3). Note that in this case $\mathrm{Re}\, s > \rho$

for any $s \in \mathrm{K}$. One has

$$\frac{\partial F}{\partial u_0} = u_2 - 4z^2 u_0^3, \qquad \frac{\partial F}{\partial u_1} = -2u_1, \qquad \frac{\partial F}{\partial u_2} = u_0.$$

Hence,

$$\begin{aligned}
\frac{\partial F}{\partial u_2}(z,\Phi) &= \varphi = c_r z^r + \sum_{s\in\mathrm{K}} c_s z^s, \quad c_r \neq 0, \quad \operatorname{Re} s > \rho \ \ \forall s \in \mathrm{K},\\
\frac{\partial F}{\partial u_1}(z,\Phi) &= -2\,\delta\varphi = -2r\,c_r z^r - \sum_{s\in\mathrm{K}} 2s\,c_s z^s, \ \ \operatorname{Re} s > \rho \ \ \forall s \in \mathrm{K},\\
\frac{\partial F}{\partial u_0}(z,\Phi) &= \delta^2\varphi - 4z^2\varphi^3.
\end{aligned}$$

To prove the convergence of φ, it is remaining to find first terms of the expansion of $\frac{\partial F}{\partial u_0}(z,\Phi)$. Since

$$\begin{aligned}
\delta^2\varphi &= r^2 c_r z^r + \sum_{s\in\mathrm{K}} s^2 c_s z^s, \qquad \operatorname{Re} s > \rho \ \ \forall s \in \mathrm{K},\\
z^2\varphi^3 &= c_r^3 z^{3r+2} + \ldots, \qquad\qquad 3\rho + 2 > \rho \ (\text{as } -1 < \rho < 1),
\end{aligned}$$

one finally has

$$\frac{\partial F}{\partial u_0}(z,\Phi) = r^2 c_r z^r + B_0 z^{\lambda_0} + \ldots, \qquad \operatorname{Re}\lambda_0 > \rho,$$

whence the convergence follows.

Now we consider a formal solution φ of the **second type**. Let $\rho = -1$. Then φ can be written in the form

$$\begin{aligned}
\varphi &= \sum_{(m_1,m_2)\in\mathbb{Z}_+^2} c_{m_1,m_2} z^{r+m_1(1-r)+m_2(1+r)} =\\
&= \sum_{m_1=0}^{\infty} z^{2m_1-1} \sum_{m_2=0}^{\infty} c_{m_1,m_2} z^{i\sigma(1-m_1+m_2)} = \sum_{l=0}^{\infty} z^{2l-1} y_l(z), \qquad (4.3)
\end{aligned}$$

where

$$y_l(z) = z^{i\sigma(1-l)} \sum_{m=0}^{\infty} c_{l,m} z^{i\sigma m}, \qquad l = 0, 1, \ldots .$$

Applying the technique of the Newton–Bruno polygon (see [12]), one can check that the first term $z^{-1}y_0(z)$ of the series (4.3) is a solution of the truncated equation

$$u\,\delta^2 u - (\delta u)^2 - z^2 u^4 = 0. \qquad (4.4)$$

A general solution of (4.4) has the form

$$u = z^{-1} \frac{4c_1^2}{(c_2/z^{c_1}) - 4c_1^2(z^{c_1}/c_2)},$$

where $c_1, c_2 \in \mathbb{C}$, $c_2 \neq 0$, are arbitrary constants. This solution coincides with the first term $z^{-1}y_0(z)$ of the series (4.3) only if $\operatorname{Re} c_1 = 0$. Let $c_1 = i\mu$, $\mu \in \mathbb{R} \setminus \{0\}$. Then

$$u = z^{-1}\frac{-4\mu^2}{(c_2/z^{i\mu}) + 4\mu^2(z^{i\mu}/c_2)},$$

which has expansions

$$z^{-1}\left(-\frac{4\mu^2}{c_2}\right) z^{i\mu} \sum_{m=0}^{\infty} (-1)^m \left(\frac{2\mu}{c_2}\right)^{2m} z^{2i\mu m}, \qquad |z^{i\mu}| < |c_2/2\mu|, \tag{4.5}$$

$$z^{-1}(-c_2)z^{-i\mu} \sum_{m=0}^{\infty} (-1)^m \left(\frac{c_2}{2\mu}\right)^{2m} z^{-2i\mu m}, \qquad |z^{i\mu}| > |c_2/2\mu|. \tag{4.6}$$

These expansions coincide with

$$z^{-1}y_0(z) = z^{-1}z^{i\sigma} \sum_{m=0}^{\infty} c_{0,m} z^{i\sigma m}, \qquad c_{0,0} = c_r \neq 0,$$

if one puts $\mu = \sigma$, $c_2 = -4\sigma^2/c_{0,0}$ for (4.5), and $\mu = -\sigma$, $c_2 = -c_{0,0}$ for (4.6).

Thus, the first term $z^{-1}y_0(z)$ of the formal solution (4.3) converges to the function

$$u = z^{-1}\frac{4\sigma^2}{c_{0,0}z^{i\sigma} + (4\sigma^2/c_{0,0})z^{-i\sigma}} \tag{4.7}$$

in sectors contained in the domain $\{|z^{i\sigma}| < |2\sigma/c_{0,0}|\}$ with the boundary ray

$$\{|z^{i\sigma}| = |2\sigma/c_{0,0}|\} = \{\arg z = (-1/\sigma)\ln|2\sigma/c_{0,0}|\}.$$

The poles of the function (4.7) accumulate to the origin along this ray. For example, if $c_{0,0} = 2\sigma$, then

$$u = z^{-1}\frac{2\sigma}{z^{i\sigma} + z^{-i\sigma}} = z^{-1}\frac{\sigma}{\cos(\sigma \ln z)},$$

whose poles $z_k = e^{(\pi+2\pi k)/2\sigma}$ accumulate to the origin along the positive real axis. In this case the series $z^{-1}y_0(z)$ cannot converge in a whole sector containing this ray.

Concerning the formal solution φ, in general the points where it diverges do not coincide with the poles of the solution (4.7) of the truncated equation, but we guess that they are asymptotically distributed near those poles (something similar holds for the pole distribution of some Painlevé VI transcendents near its critical point, see [14]). This means that the formal solution φ (of the second type) converges not in any sector of sufficiently small radius and opening less than 2π, but convergence depends on the bisecting direction of a sector. Such solutions are "of measure null", as they form a (real) three-parametric subfamily in the (real) four-parametric family (4.2) of formal solutions of (4.1), whereas "most" solutions (4.2) converge in any sector near the origin (which could correspond to the accumulation of poles along spirals around the origin).

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We would like to thank the referee for pointing us a minor error in the statement of Lemma 1 of [10]. Indeed, in the statement of this lemma we have written "for any integer $\mu \geqslant \mu'$" and missed an additional assumption "$\mathrm{Re}(s_{\mu+1} - s_\mu) > 0$" which was used in its proof. Taking this into consideration, in Lemma 2.1 we have written a more precise "for any integer $\mu \geqslant \mu'$ satisfying $\mathrm{Re}(s_{\mu+1} - s_\mu) > 0$".

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