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Smallness of the formal exponents of an irregular linear differential system, with an application to solvability by quadratures

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We prove that the formal exponents of a linear differential system with non-resonant irregular singular points whose coefficient matrix is small, are also small enough. This implies that such a system is solvable by quadratures if, and only if its coefficient matrix is conjugated to a triangular one (via a constant conjugating matrix), which generalizes the corresponding theorem by Ilyashenko–Khovanskii for Fuchsian systems.

Key words and phrases: irregular singular point, formal exponents, solvability by quadratures.

1. Introduction

We consider a linear differential system

$$z\frac{dy}{dz} = B(z)\,y, \qquad y(z)\in\mathbb{C}^p, \tag{1}$$

of p equations near a non-resonant irregular singular point $z=0$ of Poincaré rank $r>0$. This means that the coefficient matrix B has the Laurent expansion

$$B(z) = \frac{1}{z^r}\Big(B_0 + B_1 z + \ldots\Big) \tag{2}$$

and the eigenvalues $\alpha_1,\ldots,\alpha_p$ of the matrix B_0 are pairwise distinct. Such a system has a formal fundamental matrix $\widehat{Y}$ of the form

$$\widehat{Y}(z) = \widehat{F}(z) z^{\Lambda} e^{Q(1/z)},$$

where $\Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_p)$ is a diagonal matrix, $\widehat{F}$ is a matrix formal Taylor series, and Q is a polynomial (they will be described in more details further).

The elements $\lambda_1,\ldots,\lambda_p$ of the matrix Λ are called the *formal exponents* of the system (1) at the (non-resonant) irregular singular point $z=0$. Their dependence on the coefficient matrix is not so evident as the similar dependence of proper exponents in the Fuchsian case (*i.e.*, in the case of Poincaré rank $r=0$): the latter coincide with the eigenvalues of the residue matrix B_0. In particular, a Fuchsian system having a small residue matrix has small exponents. Here we are interested in the following question: *when can one expect the smallness of the formal exponents at an irregular singular point?* The answer would have some application to the solvability of linear differential systems by quadratures.

To give an answer to the addressed question, let us recall the procedure of a formal transformation of the system (1), (2) to a diagonal form (from §11 in [1]).

2. Transformation to the non-resonant formal normal form

Under a transformation $y = T(z)\tilde{y}$, the system is changed as follows:

$$z\frac{d\tilde{y}}{dz} = A(z)\,\tilde{y}, \qquad A(z) = T^{-1}B(z)T - zT^{-1}\frac{dT}{dz},$$

and a new coefficient matrix A has the Laurent expansion

$$A(z) = \frac{1}{z^r}\Bigl(A_0 + A_1 z + \ldots\Bigr).$$

One may always assume that $T(z) = I + T_1 z + \ldots$ and $B_0 = A_0 = \operatorname{diag}(\alpha_1, \ldots, \alpha_p)$. Then gathering the coefficients at each power z^k from the relation

$$T(z)\,z^r A(z) - z^r B(z)\,T(z) + z^{r+1}\frac{dT}{dz} = 0,$$

we obtain

$$T_k B_0 - B_0 T_k + A_k - B_k + \sum_{l=1}^{k-1}(T_l A_{k-l} - B_{k-l}T_l) + (k-r)T_{k-r} = 0$$

(the last summand equals zero for $k \leqslant r$). There are two sets of unknowns in this system of matrix equations: $A_k = (A_k^{ij})$ and $T_k = (T_k^{ij})$. Requiring all the A_k's to be diagonal and assuming that $A_1, \ldots, A_{k-1}$ and $T_1, \ldots, T_{k-1}$ are already found, one firstly obtains

$$A_k^{ii} = B_k^{ii} - H_k^{ii},$$

where $H_k = \sum_{l=1}^{k-1}(T_l A_{k-l} - B_{k-l}T_l) + (k-r)T_{k-r}$, and then

$$T_k^{ij} = \frac{1}{\alpha_j - \alpha_i}\bigl(B_k^{ij} - H_k^{ij}\bigr),\ i \neq j, \qquad T_k^{ii} = 0.$$

Thus one can see that $\Lambda = A_r$ and

$$\widehat{F}(z) = e^{A_{r+1}z + A_{r+2}\frac{z^2}{2} + \ldots}\,T(z), \quad Q(1/z) = -\frac{A_0}{rz^r} - \frac{A_1}{(r-1)z^{r-1}} - \ldots - \frac{A_{r-1}}{z}.$$

Therefore,

$$\lambda_i = A_r^{ii} = B_r^{ii} - H_r^{ii}.$$

This implies the estimate

$$|\lambda_i| \leqslant \|A_r\| \leqslant \|B_r\| + \|H_r\|$$

(we will use, for example, the matrix 1-norm $\|\cdot\|_1$ here).

3. On the smallness of the formal exponents

Now we prove the smallness of the formal exponents λ_i's in the case of small coefficients $B_1, \ldots, B_r$. For this, we should prove the smallness of $H_1, \ldots, H_r$. Denoting $\rho = \min_{i \neq j} |\alpha_j - \alpha_i| > 0$ we will have

$$|T_k^{ij}| \leqslant \frac{1}{\rho}\bigl(|B_k^{ij}| + |H_k^{ij}|\bigr).$$

This implies the norm estimate

$$\|T_k\| \leqslant \frac{1}{\rho}\bigl(\|B_k\| + \|H_k\|\bigr),$$

hence

$$\|H_k\| \leqslant \sum_{l=1}^{k-1} \|T_l\|\bigl(\|A_{k-l}\| + \|B_{k-l}\|\bigr) \leqslant \sum_{l=1}^{k-1} \frac{1}{\rho}\bigl(\|B_l\| + \|H_l\|\bigr)\bigl(2\|B_{k-l}\| + \|H_{k-l}\|\bigr).$$

Assume that all $\|B_k\| < \varepsilon$, $k = 1, \ldots, r$. Then

$$\begin{aligned}\|H_k\| &\leqslant \sum_{l=1}^{k-1} \frac{1}{\rho}\bigl(2\varepsilon^2 + \varepsilon\bigl(2\|H_l\| + \|H_{k-l}\|\bigr) + \|H_l\|\,\|H_{k-l}\|\bigr) = \\ &= \frac{1}{\rho}\Bigl(2(k-1)\varepsilon^2 + 3\varepsilon \sum_{l=1}^{k-1} \|H_l\| + \sum_{l=1}^{k-1} \|H_l\|\,\|H_{k-l}\|\Bigr).\end{aligned}$$

Thus having

$$H_1 = 0, \qquad \|H_2\| \leqslant 2\frac{\varepsilon^2}{\rho},$$

one obtains by induction

$$\|H_r\| \leqslant \frac{\varepsilon^2}{\rho}\, P_{r-2}(\varepsilon/\rho) \quad \text{and} \quad |\lambda_i| \leqslant \varepsilon P_{r-1}(\varepsilon/\rho),$$

where P_{r-2} and P_{r-1} are polynomials of degree $r-2$ and $r-1$ respectively. Indeed, if

$$\|H_k\| \leqslant \frac{\varepsilon^2}{\rho}\, P_{k-2}(\varepsilon/\rho) \quad \text{for} \quad k = 2, 3, \ldots, r-1$$

(where P_{k-2} is a polynomial of degree $k-2$) then

$$\begin{aligned}\|H_r\| &\leqslant \frac{1}{\rho}\Bigl(2(r-1)\varepsilon^2 + 3\varepsilon \sum_{l=1}^{r-1} \frac{\varepsilon^2}{\rho}\, P_{l-2}(\varepsilon/\rho) + \sum_{l=1}^{r-1} \frac{\varepsilon^4}{\rho^2}\, P_{l-2}(\varepsilon/\rho) P_{r-l-2}(\varepsilon/\rho)\Bigr) = \\ &= \frac{\varepsilon^2}{\rho}\Bigl(2(r-1) + \sum_{l=1}^{r-1} 3\frac{\varepsilon}{\rho}\, P_{l-2}(\varepsilon/\rho) + \sum_{l=1}^{r-1} \Bigl(\frac{\varepsilon}{\rho}\Bigr)^2 P_{l-2}(\varepsilon/\rho) P_{r-l-2}(\varepsilon/\rho)\Bigr) = \\ &= \frac{\varepsilon^2}{\rho}\, P_{r-2}(\varepsilon/\rho).\end{aligned}$$

This leads to the following assertion.

Proposition 1. *Consider a (Zariski open) subset* $W \subset \mathrm{Mat}(p, \mathbb{C})$ *of* $p \times p$*-matrices having pairwise distinct eigenvalues. For any open disk* $D \Subset W$ *and any* $\varepsilon > 0$*, there exists* $\delta = \delta(D, \varepsilon) > 0$ *such that every system* (1), (2) *with*

$$B_0 \in D, \quad \|B_1\| < \delta, \quad \ldots, \quad \|B_r\| < \delta,$$

has formal exponents satisfying the condition $|\lambda_i| < \varepsilon$.

Remark 1. Note that if the Poincaré rank $r = 1$, then $|\lambda_i| \leqslant \|B_1\|$ thus one has no necessity to restrict the leading term B_0 on some disk D and may take any $B_0 \in W$. There is also no necessity for restricting B_0 in the case $B_1 = \ldots = B_r = 0$: then all the formal exponents equal zero.

4. Application to solvability by quadratures

Now we consider a system

$$\frac{dy}{dz} = B(z)\, y, \qquad y(z) \in \mathbb{C}^p, \tag{3}$$

defined on the whole Riemann sphere $\overline{\mathbb{C}}$. Assume that it has non-resonant irregular singular points $a_1, \ldots, a_n$ of Poincaré rank $r_1, \ldots, r_n$ respectively (for determinance we assume here that the system has no Fuchsian singular points). Then the coefficient matrix B has the form

$$B(z) = \sum_{i=1}^{n} \Bigl(\frac{B_0^i}{(z - a_i)^{r_i + 1}} + \ldots + \frac{B_{r_i}^i}{z - a_i} \Bigr), \qquad \sum_{i=1}^{n} B_{r_i}^i = 0 \tag{4}$$

(if ∞ is a non-singular point).

One says that a solution y of the system (3) is *Liouvillian* if there is a tower of elementary extensions

$$\mathbb{C}(z) = F_0 \subset F_1 \subset \ldots \subset F_m$$

of the field $\mathbb{C}(z)$ of rational functions such that all components of y belong to F_m. Here each $F_{i+1} = F_i\langle x_i \rangle$, where x_i is either an integral or an exponential of integral of some element in F_i, or algebraic over F_i. The system is said to be solvable in the Liouvillian sense (or, *by quadratures*), if all its solutions are Liouvillian.

The question of the solvability of a linear differential equation or system by quadratures is studied usually by purely algebraic methods, since an answer depends on properties of the differential Galois group of the system. However in some cases, with the use of analytic methods one can leave aside the Galois group and obtain an answer in terms of the coefficients of the system, which essentially simplifies the study. As is explained below, one of such cases is a system with a small coefficient matrix.

For the system (3) whose formal exponents are sufficiently small, the solvability by quadratures is equivalent to the existence of a constant matrix $C \in \mathrm{GL}(p, \mathbb{C})$ such that $CB(z)C^{-1}$ is triangular (see [2]). Thus Proposition 1 leads to the following observation. Let $\mathcal{S}_{r_1,\ldots,r_n}^{a_1,\ldots,a_n}$ be a set of systems (3), (4):

$$\mathcal{S}_{r_1,\ldots,r_n}^{a_1,\ldots,a_n} = \Bigl\{ (B_0^1, \ldots, B_{r_1}^1), \ldots, (B_0^n, \ldots, B_{r_n}^n) \mid B_0^1, \ldots, B_0^n \in W \Bigr\}$$

(recall that $W \subset \mathrm{Mat}(p, \mathbb{C})$ is a set of $p \times p$-matrices having pairwise distinct eigenvalues).

Theorem 1. *For any open disk* $D \Subset W$ *there exists* $\varepsilon = \varepsilon(D, p, n) > 0$ *such that in a subset*

$$\Bigl\{ (B_0^i, \ldots, B_{r_i}^i)_{i=1}^n \in \mathcal{S}_{r_1,\ldots,r_n}^{a_1,\ldots,a_n} \mid B_0^i \in D, \|B_1^i\| < \varepsilon, \ldots, \|B_{r_i}^i\| < \varepsilon \Bigr\}$$

of systems with sufficiently small non-leading coefficients, solvable by quadratures systems are determined by the following algebraic condition: their matrices $B_0^i, B_1^i, \dots, B_{r_i}^i$ $(i = 1, \dots, n)$ *are simultaneously reduced to a triangular form.*

This means, roughly speaking, that a system with a sufficiently small coefficient matrix is solvable by quadratures if and only if it is triangular. Thus we have a generalization of the result by Ilyashenko–Khovanskii which claims that a Fuchsian system with sufficiently small residue matrices is solvable by quadratures if and only if it is triangular (see Ch. 6 in [3]).

Taking into consideration Remark 1 we also obtain two evident corollaries.

Corollary 1 (systems with singular points of Poincaré rank 1). *There exists* $\varepsilon = \varepsilon(p, n) > 0$ *such that in a subset*

$$\left\{(B_0^i, B_1^i)_{i=1}^n \in \mathcal{S}^{a_1,\dots,a_n}_{1,\dots,1} \mid \|B_1^i\| < \varepsilon\right\}$$

of systems with sufficiently small non-leading coefficients, solvable by quadratures systems are determined by the following algebraic condition: their matrices B_0^i, B_1^i $(i = 1, \dots, n)$ *are simultaneously reduced to a triangular form.*

Corollary 2. *Non-resonant system* (3) *with the coefficient matrix* B *of the form*

$$B(z) = \sum_{i=1}^{n} \frac{B_0^i}{(z - a_i)^{r_i+1}}, \qquad r_i > 0,$$

is solvable by quadratures if and only if all the matrices $B_0^1, \dots, B_0^n$ *are simultaneously reduced to a triangular form.*

5. Conclusions

Let us note that the result formulated at the end of the previous section, looking like a theoretical one, is not far from to be applied for a practical realization. For example, the required smallness of the formal exponents of the system under consideration is concrete: their absolute value should be not greater than $1/n(p-1)$ (see [2]). The practical calculation of the formal exponents can definitely be implemented, as follows from the formulae for the elements A_r^{ii} from Section 2 (see [4] for more general questions in this context). At last, we guess that the simultaneous triangularizability of a set of matrices is equivalent to the nilpotence of the Lie algebra generated by them, which is also can be checked.

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Малость формальных показателей иррегулярной системы линейных дифференциальных уравнений, с приложением к проблеме разрешимости в квадратурах

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Доказывается, что формальные показатели системы линейных дифференциальных уравнений с нерезонансными иррегулярными особыми точками, матрица коэффициентов которой мала, также достаточно малы. Из этого следует, что такая система разрешима в квадратурах, если и только если ее матрица коэффициентов приводится к треугольному виду сопряжением на постоянную матрицу. Это обобщает соответствующую теорему Ильяшенко–Хованского для фуксовых систем.

Ключевые слова: иррегулярная особая точка, формальные показатели, разрешимость в квадратурах.