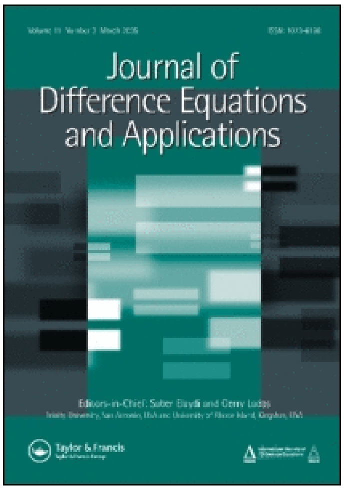

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A relaxation scheme for computation of the joint spectral radius of matrix sets

Victor Kozyakin[a]
[a] Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, Russia

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A relaxation scheme for computation of the joint spectral radius of matrix sets

Victor Kozyakin*

Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoj Karetny lane 19, Moscow 127994 GSP-4, Russia

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The problem of computation of the joint (generalized) spectral radius of matrix sets has been discussed in a number of publications. In this paper, an iteration procedure is considered that allows to build numerically Barabanov norms for the irreducible matrix sets and simultaneously to compute the joint spectral radius of these sets.

Keywords: infinite matrix products; generalized spectral radius; joint spectral radius; extremal norms; Barabanov norms; irreducibility; numerical algorithms

AMS Subject Classification: 15A18; 15A60; 65F15

1. Introduction

Let $\mathscr{A} = \{A_1, \ldots, A_r\}$ be a set of real $m \times m$ matrices. As usual, for $n \geq 1$ denote by $\mathscr{A}^n$ the set of all n-products of matrices from $\mathscr{A}$; $\mathscr{A}^0 = I$. For each $n \geq 1$, define the quantity

$$\rho(\mathscr{A}^n) = \max_{A \in \mathscr{A}^n} \rho(A) = \max_{A_{i_j} \in \mathscr{A}} \rho(A_{i_n} \cdots A_{i_2} A_{i_1}),$$

where maximum is taken over all possible products of n matrices from the set $\mathscr{A}$, and $\rho(\cdot)$ denotes the spectral radius of a matrix, i.e. the maximal magnitude of its eigenvalues. Then the limit

$$\bar{\rho}(\mathscr{A}) = \limsup_{n \to \infty} (\rho(\mathscr{A}^n))^{1/n}$$

is called *the generalized spectral radius* of the matrix set $\mathscr{A}$ [9,11].

Similarly, given a norm $\|\cdot\|$ in $\mathbb{R}^m$, the limit

$$\hat{\rho}(\mathscr{A}) = \limsup_{n \to \infty} \|\mathscr{A}^n\|^{1/n},$$

where

$$\|\mathscr{A}^n\| = \max_{A \in \mathscr{A}^n} \|A\| = \max_{A_{i_j} \in \mathscr{A}} \|A_{i_n} \cdots A_{i_2} A_{i_1}\|$$

*Email: kozyakin@iitp.ru

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is called *the joint spectral radius* of the matrix set $\mathscr{A}$ [29]. Clearly, the value of $\hat{\rho}n(\mathscr{A})$ does not depend on the choice of the norm $\|\cdot\|$.

For bounded matrix sets $\mathscr{A}$, the quantities $\bar{\rho}(\mathscr{A})$ and $\hat{\rho}(\mathscr{A})$ coincide with each other [5], where the values of $\bar{\rho}_n(\mathscr{A})$ and $\hat{\rho}_n(\mathscr{A})$ form lower and upper bounds, respectively, for the joint/generalized spectral radius:

$$\bar{\rho}_n(\mathscr{A}) \leq \bar{\rho}(\mathscr{A}) = \hat{\rho}(\mathscr{A}) \leq \hat{\rho}_n(\mathscr{A}), \quad \forall\ n \geq 0.$$

This last formula may serve as a basis for *a posteriori* estimating the accuracy of computation of $\rho(\mathscr{A})$. The first algorithms of a kind in the context of control theory problems have been suggested in [6], for linear inclusions in [2] and for problems of wavelet theory in [8–10]. Later, the computational efficiency of these algorithms was essentially improved in [13,22]. Unfortunately, the common feature of all such algorithms is that they do not provide any bounds for the number of computational steps required to get the desired accuracy of the approximation of $\rho(\mathscr{A})$.

Some works suggest different formulas to compute $\rho(\mathscr{A})$. So, in [7] it is shown that

$$\rho(\mathscr{A}) = \limsup_{n\to\infty} \max_{A_{i_j}\in\mathscr{A}} \left|\mathrm{tr}(A_{i_n}\cdots A_{i_2}A_{i_1})\right|^{1/n},$$

where, as usual, $\mathrm{tr}(\cdot)$ denotes the trace of a matrix.

In [12,29], it was proved that the spectral radius of the matrix set $\mathscr{A}$ can be defined by the equality

$$\rho(\mathscr{A}) = \inf_{\|\cdot\|}\|\mathscr{A}\|, \tag{1}$$

where infimum is taken over all norms in $\mathbb{R}^d$. For irreducible matrix sets,[1] the infimum in (1) is attained and for such matrix sets there are norms $\|\cdot\|$ in $\mathbb{R}^d$, called *extremal norms*, for which

$$\|\mathscr{A}\| \leq \rho(\mathscr{A}). \tag{2}$$

In the analysis of the joint spectral radius, ideas suggested by Barabanov [2–4] play an important role. These ideas have got further development in a variety of publications among which we would like to distinguish [31].

Theorem 1.1. (N.E. Barabanov). *Let the matrix set $\mathscr{A} = \{A_1, \ldots, A_r\}$ be irreducible. Then the quantity ρ is the joint (generalized) spectral radius of the set A iff there is a norm $\|\cdot\|$ in $\mathbb{R}^m$ such that*

$$\rho\|x\| \equiv \max_i\|A_i x\|. \tag{3}$$

Throughout this paper, a norm satisfying (3) will be called a *Barabanov norm* corresponding to the matrix set $\mathscr{A}$. Note that Barabanov norms are not unique.

Similarly, [27] (Theorem 3.3), [28], the value of ρ equals to $\rho(\mathscr{A})$ if and only if for some central-symmetric convex body[2] S the following equality holds

$$\rho S = \mathrm{conv}\left(\bigcup_{i=1}^{r} A_i S\right), \tag{4}$$

where conv($\cdot$) denotes the convex hull of a set. As is noted in [27], relation (4) was proved by A.N. Dranishnikov and S.V. Konyagin, so it is natural to call the central-symmetric set S the *Dranishnikov-Konyagin-Protasov set*. The set S can be treated as the unit ball of some norm $\|\cdot\|$ in $\mathbb{R}^d$ (recently this norm is usually called the *Protasov norm*). Barabanov norms and Protasov norms are the extremal norms, i.e. they satisfy the inequality (2). In [24,25,32], it is shown that Barabanov and Protasov norms are dual to each other.

Remark that formulas (2)–(4) define the joint or generalized spectral radius for a matrix set in an apparently computationally non-constructive manner. In spite of that, such formulas underlie quite a number of theoretical constructions (see e.g. [1,16,18,23,31,32]) and algorithms [26] for computation of $\rho(\mathscr{A})$.

Different approaches for constructing Barabanov norms to analyse properties of the joint (generalized) spectral radius are discussed, e.g. in [14,15] and [30] (Section 6.6).

In [17], the so-called max-relaxation algorithm was proposed for computation of the joint spectral radius of matrix sets. In this paper, an alternative iteration procedure, a linear relaxation procedure, is introduced that allows to build numerically Barabanov norms for the irreducible matrix sets and simultaneously to compute the joint spectral radius of these sets.

This article is organized as follows: In Section 1, we give basic definitions and present the motivation of the work. In Section 2, the iteration procedures are introduced. This procedure are called the linear relaxation procedure since in it the next approximation to the Barabanov norm is constructed as the linear combination of the current approximation and some auxiliary norm. Section 3 is devoted to the proof of convergence of the iteration procedure. In Section 4, we briefly describe the so-called max-relaxation iteration scheme for computation of the joint spectral radius. Finally, in the concluding Section 5 we present the results of numerical tests and discuss some shortcomings of the proposed approach.

2. Linear relaxation iteration scheme

Let $\mathscr{A} = \{A_1, \ldots, A_r\}$ be an irreducible set of real $m \times m$ matrices, $\|\cdot\|_0$ be a norm in $\mathbb{R}^m$, and $e \neq 0$ be an arbitrary element from $\mathbb{R}^m$ satisfying $\|e\|_0 = 1$.

Let λ^- and λ^+ be fixed but otherwise arbitrary numbers satisfying the condition

$$0 < \lambda^- \leq \lambda^+ < 1.$$

These numbers play the role of boundaries for parameters of the linear relaxation scheme below. Define recursively the sequence of the norms $\|\cdot\|_n$, $n = 1, 2, \ldots,$, according to the following rules:

LR$_1$: if the norm $\|\cdot\|_n$ has been already defined compute the quantities

$$\rho_n^+ = \max_{x \neq 0} \frac{\max_i \|A_i x\|_n}{\|x\|_n}, \quad \rho_n^- = \min_{x \neq 0} \frac{\max_i \|A_i x\|_n}{\|x\|_n}, \quad \gamma_n = \max_i \|A_i e\|_n; \tag{5}$$

LR$_2$: choose an arbitrary number $\lambda_n \in [\lambda^-, \lambda^+]$ and define the norm $\|\cdot\|_{n+1}$:

$$\|x\|_{n+1} = \lambda_n \|x\|_n + (1 - \lambda_n)\gamma_n^{-1} \max_i \|A_i x\|_n. \tag{6}$$

The iteration procedure (5), (6) will be referred to as the linear relaxation procedure (*the LR-procedure*) since in it the next approximation $\|x\|_{n+1}$ to the Barabanov norm is constructed as the linear combination of the current approximation $\|x\|_n$ and some auxiliary norm.

As we will see in Section 3.1, $\rho_n^- \le \rho \le \rho_n^+$ for any $n = 0, 1, \ldots,$ and so the quantities $\{\rho_n^-\}$ form lower bounds for the joint spectral radius ρ of the matrix set $\mathscr{A}$, while the quantities $\{\rho_n^+\}$ form upper bounds for ρ.

Remark that the norm (6) is correctly defined for any choice of γ_n due to irreducibility of the matrix set $\mathscr{A} = \{A_1, \ldots, A_r\}$ for any $x \neq 0$ the vectors $A_1x, \ldots, A_rx$ do not vanish simultaneously, and then $\rho_n^- > 0$ as well as $\gamma_n \ge \rho_n^- \|e\|_n > 0$.

Before we start proving that the LR-procedure converges to some Barabanov norm and that the quantities $\rho_n^\pm$ converge to the joint spectral radius ρ of the matrix set $\mathscr{A}$ make two remarks.

Remark 1. The norms $\|\cdot\|_n$ satisfy the normalization conditions $\|e\|_n \equiv 1$, $n = 1, 2, \ldots,$ which can be derived by the induction from (6). Then by (5)

$$\gamma_n = \frac{\max_i \|A_i \mathrm{e}\|_n}{\|e\|_n}$$

and, therefore,

$$\gamma_n \in \left[\rho_n^-, \rho_n^+]\right], \quad n = 0, 1, \ldots. \tag{7}$$

Remark 2. Instead of the iteration procedure (5), (6), one can consider the following, formally more general, procedure in which the quantities γ_n are chosen arbitrarily if only they satisfy inclusions (7) and the obtained norms are normalized forcibly:

LR_1': provided that the norm $\|\cdot\|_n$ has been already found compute the quantities

$$\rho_n^+ = \max_{x \neq 0} \frac{\max_i \|A_i x\|_n}{\|x\|_n}, \quad \rho_n^- = \min_{x \neq 0} \frac{\max_i \|A_i x\|_n}{\|x\|_n}; \tag{8}$$

LR_2': choose arbitrary numbers $\lambda_n \in [\lambda^-, \lambda^+]$, $\gamma_n \in [\rho_n^-, \rho_n^+]$ and build first the auxiliary norm $\|\cdot\|_{n+1}^\circ$:

$$\|x\|_{n+1}^\circ = \lambda_n \|x\|_n + (1 - \lambda_n)\gamma_n^{-1} \max_i \|A_i x\|_n,$$

and then define the norm $\|\cdot\|_{n+1}$ in such a way that the normalization condition $\|e\|_{n+1} = 1$ be satisfied*:*

$$\|x\|_{n+1} = \frac{\|x\|_{n+1}^\circ}{\|\mathrm{e}\|_{n+1}^\circ}. \tag{9}$$

In fact, if to write down formulas for recalculation of the norms $\|x\|_{n+1}$ via $\|x\|_n$ and to represent them in the form similar to (6):

$$\|x\|_{n+1} = \lambda_n' \|x\|_n + (1 - \lambda_n')(\gamma_n')^{-1} \max_i \|A_i x\|_n,$$

then one can find that the corresponding quantities λ_n' are uniformly separated from zero and unity, while the quantity γ_n' is equal to the quantity γ_n defined by (5). The corresponding calculations are not complicated but cumbersome and are omitted.

So, consideration of the iteration procedures of the form (8), (9) gives nothing new, and such procedures are not studied in what follows.

3. Proof of the main result

Clearly, to prove that the iteration procedure (5), (6) converge to some Barabanov norm $\|\cdot\|^*$ (and that the quantities $\rho_n^{\pm}$ converge to the joint spectral radius ρ of the matrix set $\mathscr{A}$), it suffices to prove the Assertions A1, A2 and A3:

A1: the sequences $\{\rho_n^+\}$ and $\{\rho_n^-\}$ are convegent;
A2: the limits of the sequences $\{\rho_n^+\}$ and $\{\rho_n^-\}$ coincide:

$$\rho = \lim_{n\to\infty} \rho_n^+ = \lim_{n\to\infty} \rho_n^-;$$

A3: the norms $\|\cdot\|_n$ converge pointwise to a limit $\|\cdot\|^*$.

Properties of the iteration procedure (5), (6) needed to prove Assertions A1, A2 and A3 are established below.

3.1 Relations between $\rho_n^{\pm}$ and ρ

Lemma 3.1. *Let α, β be numbers such that in some norm $\|\cdot\|$ the inequalities*

$$\alpha\|x\| \le \max_{A_i \in \mathscr{A}} \|A_i x\| \le \beta\|x\|,$$

hold. Then $\alpha \le \rho \le \beta$, where ρ is the joint spectral radius of the matrix set $\mathscr{A}$.

Proof. Let $\|\cdot\|^*$ be some Barabanov norm for the matrix set $\mathscr{A}$. Since all norms in $\mathbb{R}^m$ are equivalent, there are constants $\sigma^- > 0$ and $\sigma^+ < \infty$ such that

$$\sigma^- \|x\|^* \le \|x\| \le \sigma^+ \|x\|^*. \tag{10}$$

Consider for each $k = 1, 2, \ldots,$ the functions

$$\Delta_k(x) = \max_{1 \le i_1, i_2, \ldots, i_k \le r} \|A_{i_k} \ldots A_{i_2} A_{i_1} x\|.$$

Then, as it is easy to see,

$$\alpha^k\|x\| \leq \Delta_k(x) \leq \beta^k\|x\|. \tag{11}$$

Similarly, consider for each $k = 1, 2, \ldots,$ the functions

$$\Delta_k^*(x) = \max_{1\leq i_1,i_2,\ldots,i_k\leq r} \|A_{i_k}\ldots A_{i_2}A_{i_1}x\|^*.$$

For these functions, by definition of Barabanov norms, the following identity hold

$$\Delta_k^*(x) \equiv \rho^k\|x\|^*, \tag{12}$$

which is stronger than (11).

Now, note that (10) and the definition of the functions $\Delta_k(x)$ and $\Delta_k^*(x)$ imply

$$\sigma^-\Delta_k^*(x) \leq \Delta_k(x) \leq \sigma^+\Delta_k^*(x).$$

Then, by (11) and (12),

$$\frac{\sigma^-}{\sigma^+}\alpha^k \leq \rho^k \leq \frac{\sigma^+}{\sigma^-}\beta^k, \quad \forall k,$$

from which the required estimates $\alpha \leq \rho \leq \beta$ follow. □

So, Lemma 3.1 and definition (5) of $\rho_n^\pm$ imply that the quantities $\{\rho_n^-\}$ form the family of lower bounds for the joint spectral radius ρ of the matrix set $\mathscr{A}$, while the quantities $\{\rho_n^+\}$ form the family of upper bounds for ρ. This allows to estimate *a posteriori* errors of computation of the joint spectral radius with the help of the iteration procedure (5), (6).

3.2 *Convergence of the sequence of norms* $\{\|\cdot\|_\mathbf{n}\}$

Given a pair of norms $\|\cdot\|'$ and $\|\cdot\|''$ in $\mathbb{R}^m$ define the quantities

$$\mathrm{e}^-(\|\cdot\|', \|\cdot\|'') = \min_{x\neq 0}\frac{\|x\|'}{\|x\|''}, \quad \mathrm{e}^+(\|\cdot\|', \|\cdot\|'') = \max_{x\neq 0}\frac{\|x\|'}{\|x\|''}. \tag{13}$$

Since all norms in $\mathbb{R}^m$ are equivalent to each other, the quantities $\mathrm{e}^-(\|\cdot\|', \|\cdot\|'')$ and $\mathrm{e}^+(\|\cdot\|', \|\cdot\|'')$ are correctly defined and

$$0 < \mathrm{e}^-(\|\cdot\|', \|\cdot\|'') \leq \mathrm{e}^+(\|\cdot\|', \|\cdot\|'') < \infty.$$

Therefore, the quantity

$$\mathrm{ecc}(\|\cdot\|', \|\cdot\|'') = \frac{\mathrm{e}^+(\|\cdot\|', \|\cdot\|'')}{\mathrm{e}^-(\|\cdot\|', \|\cdot\|'')} \geq 1, \tag{14}$$

which is called *the eccentricity* of the norm $\|\cdot\|'$ with respect to the norm $\|\cdot\|''$ (see, e.g. [32]), is also correctly defined.

Let us start proving convergence of the sequence of the norms $\|\cdot\|_n$.

LEMMA 3.2. *Let* $\|\cdot\|^*$ *be a Barabanov norm for the matrix set* $\mathscr{A}$. *Then the sequence of the numbers* $\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*)$ *is non-increasing.*

Proof. Denote by ρ the joint spectral radius of the matrix set $\mathscr{A}$. Then by definitions of the function $\mathrm{e}^+(\cdot)$ and of the Barabanov norm $\|\cdot\|^*$ from the relation (5), (6), we obtain:

$$\begin{aligned}\|x\|_{n+1} &= \lambda_n\|x\|_n + (1-\lambda_n)\gamma_n^{-1}\max_i\|A_ix\|_n\\ &\le \mathrm{e}^+(\|\cdot\|_n, \|\cdot\|^*)\left(\lambda_n\|x\|^* + (1-\lambda_n)\gamma_n^{-1}\max_i\|A_ix\|^*\right)\\ &= \mathrm{e}^+(\|\cdot\|_n, \|\cdot\|^*)\big(\lambda_n\|x\|^* + (1-\lambda_n)\gamma_n^{-1}\rho\|x\|^*\big),\end{aligned}$$

from which

$$\mathrm{e}^+(\|\cdot\|_{n+1}, \|\cdot\|^*) \le \mathrm{e}^+(\|\cdot\|_n, \|\cdot\|^*)\big(\lambda_n + (1-\lambda_n)\gamma_n^{-1}\rho\big). \tag{15}$$

Similarly, by definitions of the function $\mathrm{e}^-(\cdot)$ and of the Barabanov norm $\|\cdot\|^*$ from the relation (5), (6) we obtain:

$$\begin{aligned}\|x\|_{n+1} &= \lambda_n\|x\|_n + (1-\lambda_n)\gamma_n^{-1}\max_i\|A_ix\|_n\\ &\ge \mathrm{e}^-(\|\cdot\|_n, \|\cdot\|^*)\left(\lambda_n\|x\|^* + (1-\lambda_n)\gamma_n^{-1}\max_i\|A_ix\|^*\right)\\ &= \mathrm{e}^-(\|\cdot\|_n, \|\cdot\|^*)\big(\lambda_n\|x\|^* + (1-\lambda_n)\gamma_n^{-1}\rho\|x\|^*\big),\end{aligned}$$

from which

$$\mathrm{e}^-(\|\cdot\|_{n+1}, \|\cdot\|^*) \ge \mathrm{e}^-(\|\cdot\|_n, \|\cdot\|^*)\big(\lambda_n + (1-\lambda_n)\gamma_n^{-1}\rho\big). \tag{16}$$

By dividing termwise inequality (15) on (16), we get

$$\mathrm{ecc}(\|\cdot\|_{n+1}, \|\cdot\|^*) = \frac{\mathrm{e}^+(\|\cdot\|_{n+1}, \|\cdot\|^*)}{\mathrm{e}^-(\|\cdot\|_{n+1}, \|\cdot\|^*)} \le \frac{\mathrm{e}^+(\|\cdot\|_n, \|\cdot\|^*)}{\mathrm{e}^-(\|\cdot\|_n, \|\cdot\|^*)} = \mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*).$$

Hence, the sequence $\{\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*)\}$ is non-increasing. □

Denote by $N_{\mathrm{loc}}(\mathbb{R}^m)$ the topological space of norms in $\mathbb{R}^m$ with the topology of uniform convergence on bounded subsets of $\mathbb{R}^m$.

COROLLARY 3.3. *The sequence of norms* $\{\|\cdot\|_n\}$ *is compact in* $N_{\mathrm{loc}}(\mathbb{R}^m)$.

Proof. For each n and any $x \ne 0$ by definition (13) of the functions $\mathrm{e}^+(\cdot)$ and $\mathrm{e}^-(\cdot)$, the following relations hold

$$\mathrm{e}^-(\|\cdot\|_n, \|\cdot\|^*) \le \frac{\|x\|_n}{\|x\|^*} \le \mathrm{e}^+(\|\cdot\|_n, \|\cdot\|^*),$$

and then

$$\mathrm{e}^{-}(\|\cdot\|_n, \|\cdot\|^*) \le \frac{\|e\|_n}{\|e\|^*} \le \mathrm{e}^{+}(\|\cdot\|_n, \|\cdot\|^*),$$

from which

$$\frac{1}{\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*)} \frac{\|x\|^*}{\|e\|^*} \|e\|_n \le \|x\|_n \le \mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*) \frac{\|x\|^*}{\|e\|^*} \|e\|_n.$$

Since here the norms $\|\cdot\|_n$ by Remark 1 satisfy the normalization condition $\|e\|_n \equiv 1$, and by Lemma 3.2 $\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*) \le \mathrm{ecc}(\|\cdot\|_0, \|\cdot\|^*)$, we obtain

$$\frac{1}{\mathrm{ecc}(\|\cdot\|_0, \|\cdot\|^*)} \frac{\|x\|^*}{\|e\|^*} \le \|x\|_n \le \mathrm{ecc}(\|\cdot\|_0, \|\cdot\|^*) \frac{\|x\|^*}{\|e\|^*}.$$

Therefore, the norms $\|\cdot\|_n$, $n \ge 1$ are equicontinuous and uniformly bounded on each bounded subset of $\mathbb{R}^m$. Moreover, their values are also uniformly separated from zero on each bounded subset of $\mathbb{R}^m$ separated from zero. From here by the Arzela–Ascoli theorem, the statement of the corollary follows. □

Corollary 3.4. *If at least one of subsequences of norms from* $\{\|\cdot\|_n\}$ *converges in* $N_{\mathrm{loc}}(\mathbb{R}^m)$ *to some Barabanov norm then the whole sequence* $\{\|\cdot\|_n\}$ *also converges in* $N_{\mathrm{loc}}(\mathbb{R}^m)$ *to the same Barabanov norm.*

Proof. Let $\{\|\cdot\|_{n_k}\}$ be a subsequence of $\{\|\cdot\|_n\}$ which converges in $N_{\mathrm{loc}}(\mathbb{R}^m)$ to some Barabanov norm $\|\cdot\|^*$. Then by definition of the eccentricity of one norm with respect to another

$$\mathrm{ecc}(\|\cdot\|_{n_k}, \|\cdot\|^*) \to 1 \quad \text{as } k \to \infty.$$

Here by Lemma 3.2, the eccentricities $\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*)$ are non-increasing in n, and then the following stronger relation holds

$$\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*) \to 1 \quad \text{as } n \to \infty. \tag{17}$$

Now, note that by the definition (13), (14) of the eccentricity of one norm with respect to another

$$\frac{1}{\mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*)} \le \frac{\|x\|_n}{\|x\|^*} \le \mathrm{ecc}(\|\cdot\|_n, \|\cdot\|^*),$$

from which by (17), it follows that the sequence of norms $\{\|\cdot\|_n\}$ converges in space $N_{\mathrm{loc}}(\mathbb{R}^m)$ to the norm $\|\cdot\|^*$. □

Lemma 3.5. *Assertion A3 is a corollary of Assertions A1 and A2.*

Proof. By Corollary 3.3, the sequence of norms $\{\|\cdot\|_n\}$ has a subsequence $\{\|\cdot\|_{n_k}\}$ that converges in space $N_{\mathrm{loc}}(\mathbb{R}^m)$ to some norm $\|\cdot\|^*$. Then, passing to the limit in (5) as

$n = n_k \to \infty$, we get by Assertions A1 and A2:

$$\rho = \frac{\max_i \|A_i x\|^*}{\|x\|^*}, \quad \forall\ x \neq 0,$$

which means that $\|\cdot\|^*$ is a Barabanov norm for the matrix set $\mathscr{A}$. This and Corollary 3.4 then imply that the sequence $\{\|\cdot\|_n\}$ converges in space $N_{\mathrm{loc}}(\mathbb{R}^m)$ to the Barabanov norm $\|\cdot\|^*$. Assertion A3 is proved.

In view of Lemma 3.5 to prove that the iteration procedure (5), (6) is convergent, it suffices to verify only that Assertions A1 and A2 hold.

3.3 Convergence of the sequences $\{\rho_n^{\pm}\}$

In the same way as in Section 4, from Lemma 3.1 and definition (5) of $\rho_n^{\pm}$ it follows that quantities $\{\rho_n^-\}$ form the family of lower bounds for the joint spectral radius ρ of the matrix set $\mathscr{A}$, while the quantities $\{\rho_n^+\}$ form the family of upper bounds for ρ. This allows to estimate *a posteriori* errors of computation of the joint spectral radius with the help of the iteration procedure (5), (6).

To prove that the sequences $\{\rho_n^{\pm}\}$ are convergent, let us obtain first some auxiliary estimates for $\max_i \|A_i x\|_{n+1}$. By definition,

$$\max_i \|A_i x\|_{n+1} = \max_i \left\{ \lambda_n \|A_i x\|_n + (1-\lambda_n)\gamma_n^{-1} \max_j \|A_j A_i x\|_n \right\}. \tag{18}$$

Here for each i the summand $(1-\lambda_n)\gamma_n^{-1} \max_j \|A_j A_i x\|_n$ in the right-hand part is estimated, by the definition (5) of the quantities $\rho_n^{\pm}$, as follows:

$$\rho_n^-(1-\lambda_n)\gamma_n^{-1}\|A_i x\|_n \leq (1-\lambda_n)\gamma_n^{-1}\max_j \|A_j A_i x\|_n \leq \rho_n^+(1-\lambda_n)\gamma_n^{-1}\|A_i x\|_n.$$

Therefore,

$$\begin{aligned}
&\max_i \left\{ \lambda_n \|A_i x\|_n + \rho_n^-(1-\lambda_n)\gamma_n^{-1}\|A_i x\|_n \right\} \\
&\quad \leq \max_i \left\{ \lambda_n \|A_i x\|_n + (1-\lambda_n)\gamma_n^{-1} \max_j \|A_j A_i x\|_n \right\} \\
&\quad \leq \max_i \left\{ \lambda_n \|A_i x\|_n + \rho_n^+(1-\lambda_n)\gamma_n^{-1}\|A_i x\|_n \right\}.
\end{aligned} \tag{19}$$

Here by the definition (5), (6) of the quantities ρ_n^- and of the norm $\|x\|_{n+1}$, we have

$$\begin{aligned}
&\max_i \left\{ \lambda_n \|A_i x\|_n + \rho_n^-(1-\lambda_n)\gamma_n^{-1}\|A_i x\|_n \right\} \\
&\quad = \left(\lambda_n + \rho_n^-(1-\lambda_n)\gamma_n^{-1}\right)\max_i \|A_i x\|_n \\
&\quad = \lambda_n \max_i \|A_i x\|_n + \rho_n^-(1-\lambda_n)\gamma_n^{-1}\max_i \|A_i x\|_n \\
&\quad \geq \rho_n^- \lambda_n \|x\|_n + \rho_n^-(1-\lambda_n)\gamma_n^{-1}\max_i \|A_i x\|_n \\
&\quad = \rho_n^- \|x\|_{n+1}.
\end{aligned} \tag{20}$$

Similarly, by the definition (5), (6) of the quantities ρ_n^+ and of the norm $\|x\|_{n+1}$, we have

$$\begin{aligned}
&\max_i \big\{ \lambda_n \|A_i x\|_n + \rho_n^+ (1-\lambda_n)\gamma_n^{-1} \|A_i x\|_n \big\} \\
&\quad = \big(\lambda_n + \rho_n^+ (1-\lambda_n)\gamma_n^{-1} \big) \max_i \|A_i x\|_n \\
&\quad = \lambda_n \max_i \|A_i x\|_n + \rho_n^+ (1-\lambda_n)\gamma_n^{-1} \max_i \|A_i x\|_n \\
&\quad \le \rho_n^+ \lambda_n \|x\|_n + \rho_n^+ (1-\lambda_n)\gamma_n^{-1} \max_i \|A_i x\|_n \\
&\quad = \rho_n^+ \|x\|_{n+1}.
\end{aligned} \tag{21}$$

Estimates (18)–(21) imply

$$\rho_n^- \|x\|_{n+1} \le \max_i \|A_i x\|_{n+1} \le \rho_n^+ \|x\|_{n+1},$$

from which

$$\rho_n^- \le \frac{\max_i \|A_i x\|_{n+1}}{\|x\|_{n+1}} \le \rho_n^+, \quad \forall\, x \neq 0,$$

and then

$$\rho_n^- \le \rho_{n+1}^- \le \rho_{n+1}^+ \le \rho_n^+.$$

So, the following lemma is proved.

LEMMA 3.6. *The sequence $\{\rho_n^-\}$ is bounded from above by each member of the sequence $\{\rho_n^+\}$ and is non-decreasing. The sequence $\{\rho_n^+\}$ is bounded from below by each member of the sequence $\{\rho_n^-\}$ and is non-increasing.*

In view of Lemma 3.6, there are the limits

$$\rho^- = \lim_{n\to\infty} \rho_n^-, \quad \rho^+ = \lim_{n\to\infty} \rho_n^+$$

which means that Assertion A1 holds. Hence, to prove that the iteration procedure (5), (6) an convergent, it remains only to justify Assertion A2: $\rho^- = \rho^+$.

To prove that $\rho^- = \rho^+$ below it will be supposed the contrary, which will lead us to a contradiction.

3.4 Transition to a new sequence of norms

To simplify further reasoning we will switch over to a new sequence of norms for which the quantities $\rho_n^\pm$ are independent of n.

As was established in Corollary 3.3, the sequence of the norms $\|\cdot\|_n$ is compact in space $N_{\mathrm{loc}}(\mathbb{R}^m)$. Consequently, there is a subsequence of indices $\{n_k\}$ such that the norms $\|\cdot\|_{n_k}$ converge to some norm $\|\cdot\|_0^\bullet$ satisfying the normalization condition $\|e\|_0^\bullet = 1$, while the quantities λ_{n_k} and γ_{n_k} converge to some numbers μ_0 and η_0, respectively. Then, passing to the limit in (5), by Lemma 3.6 we obtain:

$$\rho^+ = \max_{x\neq 0} \frac{\max_i \|A_i x\|_0^\bullet}{\|x\|_0^\bullet}, \quad \rho^- = \min_{x\neq 0} \frac{\max_i \|A_i x\|_0^\bullet}{\|x\|_0^\bullet}, \quad \eta_0 = \frac{\max_i \|A_i \mathrm{e}\|_0^\bullet}{\|\mathrm{e}\|_0^\bullet}.$$

Now by induction, the following statement can be easily proved.

LEMMA 3.7. *For each* $n = 0, 1, 2, \ldots,$ *the sequence of the norms* $\|\cdot\|_{n_k+n}$ *converges to some norm* $\|\cdot\|_n^\bullet$ *satisfying* $\|e\|_n^\bullet = 1$, *and the sequences of the quantities* λ_{n_k+n} *and* γ_{n_k+n} *converge to some numbers* $\mu_n \in [\lambda^-, \lambda^+]$ *and* η_n, *respectively. Moreover, for each* $n = 0, 1, 2, \ldots,$ *we have the equalities*

$$\max_{x\neq 0} \frac{\max_i \|A_i x\|_n^\bullet}{\|x\|_n^\bullet} = \rho^+, \quad \min_{x\neq 0} \frac{\max_i \|A_i x\|_n^\bullet}{\|x\|_n^\bullet} = \rho^-, \quad \frac{\max_i \|A_i \mathrm{e}\|_n^\bullet}{\|\mathrm{e}\|_n^\bullet} = \eta_n, \tag{22}$$

and the recurrent relations

$$\|x\|_{n+1}^\bullet = \mu_n \|x\|_n^\bullet + (1-\mu_n)\eta_n^{-1} \max_i \|A_i x\|_n^\bullet. \tag{23}$$

Note that the norm (23) and (6) are correctly defined since, by irreducibility of the matrix set $\mathscr{A} = \{A_1, \ldots, A_r\}$, for any $x \neq 0$ the vectors $A_1 x, \ldots, A_r x$ do not vanish simultaneously, and then $\rho^- > 0$ as well as $\eta_n \geq \rho^- > 0$.

3.5 *Sets* ω_n and Ω_n

Define for each $n = 0, 1, 2, \ldots,$ the sets

$$\begin{aligned} \omega_n &= \left\{ x \in \mathbb{R}^m : \rho^- \|x\|_n^\bullet = \max_i \|A_i x\|_n^\bullet \right\}, \\ \Omega_n &= \left\{ x \in \mathbb{R}^m : \rho^+ \|x\|_n^\bullet = \max_i \|A_i x\|_n^\bullet \right\}. \end{aligned} \tag{24}$$

By (22), ω_n and Ω_n are the sets on which the value

$$\frac{\max_i \|A_i x\|_n^\bullet}{\|x\|_n^\bullet}$$

attains its minimum and maximum, respectively.

LEMMA 3.8. *The following relations hold:*

$$\begin{aligned} \|x\|_{n+1}^\bullet &= \left(\mu_n + (1-\mu_n)\eta_n^{-1}\rho^-\right)\|x\|_n^\bullet \quad \textit{for } x \in \omega_n, \\ \|x\|_{n+1}^\bullet &= \left(\mu_n + (1-\mu_n)\eta_n^{-1}\rho^+\right)\|x\|_n^\bullet \quad \textit{for } x \in \Omega_n. \end{aligned}$$

Proof. The statement of the lemma is obvious for $x = 0$ therefore in what follows it will be supposed that $x \in \omega_n$, $x \neq 0$. In this case, (24) and the inequalities $\rho^- \leq \rho^+$ imply

$\max_i\|A_ix\|_n^\bullet = \rho^-\|x\|_n^\bullet$. From here by definition (23) of the norm $\|\cdot\|_{n+1}^\bullet$, we obtain

$$\|x\|_{n+1}^\bullet = \mu_n\|x\|_n^\bullet + (1-\mu_n)\eta_n^{-1}\max_i\|A_ix\|_n^\bullet = \left(\mu_n + (1-\mu_n)\eta_n^{-1}\rho^-\right)\|x\|_n^\bullet.$$

For $x \in \omega_n$, the required equality is proved. For $x \in \Omega_n$, the required equality can be proved similarly. □

LEMMA 3.9. *For each $n = 0, 1, 2, \ldots$, the inclusions $\omega_{n+1} \subseteq \omega_n$, $\Omega_{n+1} \subseteq \Omega_n$ hold.*

Proof. Let $x \in \omega_{n+1}$. If $x = 0$ then clearly $x \in \omega_n$. Therefore, in what follows it suffices to suppose that $x \neq 0$. In this case, by definition of the set ω_{n+1},

$$\max_i\|A_ix\|_{n+1}^\bullet = \rho^-\|x\|_{n+1}^\bullet = \rho^-\left(\mu_n\|x\|_n^\bullet + (1-\mu_n)\eta_n^{-1}\max_i\|A_ix\|_n^\bullet\right). \tag{25}$$

On the other hand, by substituting $\|\cdot\|_n^\bullet$ for the norm $\|\cdot\|_n$ in (18)–(20), and ρ^-, μ_n and η_n for the parameters ρ_n^-, λ_n and γ_n, respectively, we obtain the following estimate for $\max_i\|A_ix\|_{n+1}^\bullet$:

$$\max_i\|A_ix\|_{n+1}^\bullet \geq \mu_n\max_i\|A_ix\|_n^\bullet + (1-\mu_n)\eta_n^{-1}\rho^-\max_i\|A_ix\|_n^\bullet. \tag{26}$$

Since by Lemma 3.7 $\mu_n \geq \lambda^- > 0$, from (25) and (26) it follows that $\rho^-\|x\|_n^\bullet \geq \max_i\|A_ix\|_n^\bullet$ or, what is the same,

$$\rho^- \geq \frac{\max_i\|A_ix\|_n^\bullet}{\|x\|_n^\bullet}.$$

This last inequality by definition of the number ρ^- holds only for the elements $x \in \omega_n$. So, the inclusion $\omega_{n+1} \subseteq \omega_n$ is proved.

Proof of the inclusion $\Omega_{n+1} \subseteq \Omega_n$ can be provided similarly, nevertheless for the sake of completeness it can also be proved.

Let $x \in \Omega_{n+1}$. If $x = 0$ then clearly $x \in \Omega_n$. So, consider further the case when $x \neq 0$. In this case, by definition of the set Ω_{n+1},

$$\max_i\|A_ix\|_{n+1}^\bullet = \rho^+\|x\|_{n+1}^\bullet = \rho^+\left(\mu_n\|x\|_n^\bullet + (1-\mu_n)\eta_n^{-1}\max_i\|A_ix\|_n^\bullet\right). \tag{27}$$

On the other hand, by substituting $\|\cdot\|_n^\bullet$ for the norm $\|\cdot\|_n$ in (18), (19) and (21), and ρ^-, μ_n and η_n for the parameters ρ_n^-, λ_n and γ_n, respectively, we obtain the following estimate for $\max_i\|A_ix\|_{n+1}^\bullet$:

$$\max_i\|A_ix\|_{n+1}^\bullet \leq \mu_n\max_i\|A_ix\|_n^\bullet + (1-\mu_n)\eta_n^{-1}\rho^+\max_i\|A_ix\|_n^\bullet. \tag{28}$$

Since by Lemma 3.7 $\mu_n \geq \lambda^- > 0$, we see that (27) and (28) imply $\rho^+\|x\|_n^\bullet \leq \max_i\|A_ix\|_n^\bullet$ or, what is the same,

$$\rho^+ \leq \frac{\max_i\|A_ix\|_n^\bullet}{\|x\|_n^\bullet}.$$

By definition of the number ρ^-, the last inequality holds only for the elements $x \in \Omega_n$. Thus, the inclusion $\Omega_{n+1} \subseteq \Omega_n$ is also proved. $\square$

COROLLARY 3.10. *$\omega = \cap_{n\geq 0}\omega_n \neq 0$ and $\Omega = \cap_{n\geq 0}\Omega_n \neq 0$.*

Proof. By Lemma 3.9, $\{\omega_n\}$ is a family of embedded closed non-zero conic sets. Then the intersection ω of these sets is also a closed non-zero conic set. The same is valid for the sets $\{\Omega_n\}$. $\square$

3.6 Completion of the proof of Assertion A2

Choose non-zero vectors $g \in \cap_{n\geq 0}\omega_n$, $h \in \cap_{n\geq 0}\Omega_n$ which exist by Corollary 3.10. Then by Lemma 3.9 for each $n \geq 0$, the following equalities hold:

$$\|g\|^\bullet_{n+1} = \left(\mu_n + (1-\mu_n)\eta_n^{-1}\rho^-\right)\|g\|^\bullet_n, \quad \|h\|^\bullet_{n+1} = \left(\mu_n + (1-\mu_n)\eta_n^{-1}\rho^+\right)\|h\|^\bullet_n.$$

From here

$$\|g\|^\bullet_n = \xi_n^-\|g\|^\bullet_0, \quad \|h\|^\bullet_n = \xi_n^+\|h\|^\bullet_0, \quad n \geq 0,$$

where

$$\xi_n^- = \prod_{k=0}^{n}\left\{\mu_k + (1-\mu_k)\eta_k^{-1}\rho^-\right\}, \quad \xi_n^+ = \prod_{k=0}^{n}\left\{\mu_k + (1-\mu_k)\eta_k^{-1}\rho^+\right\}.$$

The eccentricities of the norms $\|\cdot\|^\bullet_n$ are uniformly bounded with respect to some Barabanov norm $\|\cdot\|^*$ (this fact can be proved by verbatim repetition of the analogous proof for the norms $\|\cdot\|_n$). Therefore the norms $\|\cdot\|^\bullet_n$ form a family, uniformly bounded and equicontinuous with respect to the Barabanov norm $\|\cdot\|^*$:

$$\exists\delta^\pm \in (0,\infty): \quad \delta^-\|x\|^* \leq \|x\|^\bullet_n \leq \delta^+\|x\|^*, \quad n = 0,1,2,\ldots.$$

Then the sequences $\{\|g\|^\bullet_n\}$ and $\{\|h\|^\bullet_n\}$ are uniformly bounded and uniformly separated from zero, and the same holds for the sequences $\{\xi_n^-\}$ and $\{\xi_n^+\}$. Let us show that the latter can be valid only under the condition $\rho^- = \rho^+$.

Note first that the inclusions $\eta_k \in [\rho^-, \rho^+]$, valid by (22) for all k, imply

$$\mu_k + (1-\mu_k)\eta_k^{-1}\rho^- \leq 1, \quad k \geq 0, \tag{29}$$

$$\mu_k + (1-\mu_k)\eta_k^{-1}\rho^+ \geq 1, \quad k \geq 0. \tag{30}$$

If we additionally suppose that $\rho^- < \rho^+$ then the inclusions $\mu_n \in [\lambda^-, \lambda^+]$ and $\eta_k \in [\rho^-, \rho^+]$, valid for all k, will imply stronger estimates:

$$\mu_k + (1-\mu_k)\eta_k^{-1}\rho^- \leq \lambda^+ + (1-\lambda^+)\frac{2\rho^-}{\rho^- + \rho^+} < 1 \quad \text{if} \quad \eta_k \in \left[\frac{\rho^- + \rho^+}{2}, \rho^+\right] \tag{31}$$

and

$$\mu_k + (1-\mu_k)\eta_k^{-1}\rho^+ \geq \lambda^- + (1-\lambda^-)\frac{2\rho^+}{\rho^- + \rho^+} > 1 \quad \text{if} \quad \eta_k \in \left[\rho^-, \frac{\rho^- + \rho^+}{2}\right]. \tag{32}$$

Now, note that under the condition $\rho^- < \rho^+$ infinitely many of numbers η_k get into the intervals $[\rho^-, (\rho^- + \rho^+)/2]$ or $[(\rho^- + \rho^+)/2, \rho^+]$. Therefore, either for infinitely many indices k estimate (29) is valid while for the rest of them the estimates (6) holds or for infinitely many indices k estimates (30) is valid while for the rest of them estimate (6) holds. Then in the first case $\xi_n^- \to 0$, while in the second case $\xi_n^+ \to \infty$.

Thus, in any case the assumption $\rho^- < \rho^+$ leads to the conclusion that the sequences $\{\xi_n^-\}$ and $\{\xi_n^+\}$ cannot be uniformly bounded and uniformly separated from zero simultaneously.

So, the proof of the equality $\rho^- = \rho^+$ is completed, and hence the iteration procedure (5), (6) are convergent.

4. Max-relaxation iteration scheme

In [17], for the same purposes, the so-called max-relaxation procedure was introduced. We describe it shortly. Let $\gamma(t,s)$, $t,s > 0$ be a continuous function satisfying

$$\gamma(t,t) = t, \quad \min\{t,s\} < \gamma(t,s) < \max\{t,s\} \quad \text{for } t \neq s.$$

In [17], such a function is called *an averaging function*. Examples for averaging functions are:

$$\gamma(t,s) = \frac{t+s}{2}, \quad \gamma(t,s) = \sqrt{ts}, \quad \gamma(t,s) = \frac{2ts}{t+s}.$$

Given some averaging function $\gamma(\cdot,\cdot)$, construct recursively the norms $\|\cdot\|_n$ and $\|\cdot\|_n^\circ$, $n = 1, 2, \ldots$, in accordance with the following rules:

MR$_1$: if the norm $\|\cdot\|_n$ has been already defined compute the quantities

$$\rho_n^+ = \max_{x\neq 0}\frac{\max_i \|A_i x\|_n}{\|x\|_n}, \quad \rho_n^- = \min_{x\neq 0}\frac{\max_i \|A_i x\|_n}{\|x\|_n}, \quad \gamma_n = \gamma(\rho_n^-, \rho_n^+); \tag{33}$$

MR$_2$: define the norms $\|\cdot\|_{n+1}$ and $\|\cdot\|_{n+1}^\circ$:

$$\|x\|_{n+1} = \max\left\{\|x\|_n, \gamma_n^{-1}\max_i \|A_i x\|_n\right\}, \tag{34}$$

$$\|x\|_{n+1}^\circ = \|x\|_{n+1}/\|e\|_{n+1}. \tag{35}$$

Max-relaxation procedures (33)–(35) (*the MR-procedure*) possesses the same convergence properties as the LR-procedure [17].

5. Examples and concluding remarks

Several dozen numerical tests with 2×2 matrices were carried out with the help of MATLAB. Two of them, quite typical, are presented below. In the LR-procedure the relaxation parameter λ_n was chosen to be identically equal to 0.3, while the averaging function in the MR-procedure was taken as follows: $\lambda(t,s) = (t+s)/2$.

Example 5.1. Consider the family $\mathscr{A} = \{A_1, A_2\}$ of 2×2 matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The functions $\Phi_i(\varphi), H_i(\varphi), R_n(\varphi), R_n^*(\varphi)$ were chosen to be piecewise linear with 3000 nodes uniformly distributed over the interval $[-\pi, \pi]$. Twenty-one steps of the LR-procedure and 22 steps of the MR-procedure were needed to compute the joint spectral radius $\rho(\mathscr{A})$ with the absolute accuracy 10^{-3}. The computed value of the joint spectral radius is $\rho(\mathscr{A}) = 1.389$. The computed unit sphere of the Barabanov norm $\|\cdot\|^*$ is plotted in Figure 1 on the left.

Example 5.2. Consider the family $\mathscr{A} = \{A_1, A_2\}$ of 2×2 matrices

$$A_1 = \begin{pmatrix} 15/17 & -16/17 \\ 4/17 & 15/17 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{pmatrix}.$$

Here the functions $\Phi_i(\varphi), H_i(\varphi), R_n(\varphi), R_n^*(\varphi)$ were also chosen to be piecewise linear with 3000 nodes uniformly distributed over the interval $[-\pi, \pi]$. Thirty-one steps of the LR-procedure and 25 steps of the MR-procedure were needed to compute the joint spectral radius $\rho(\mathscr{A})$ with the absolute accuracy 10^{-3}. The computed value of the joint spectral radius is $\rho(\mathscr{A}) = 1.192$. The computed unit sphere of the Barabanov norm $\|\cdot\|^*$ is plotted in Figure 1 on the right.

As is seen from these examples, the computational 'quality' of the above iteration procedures is approximately the same. At the same time, similar steps in their proofs

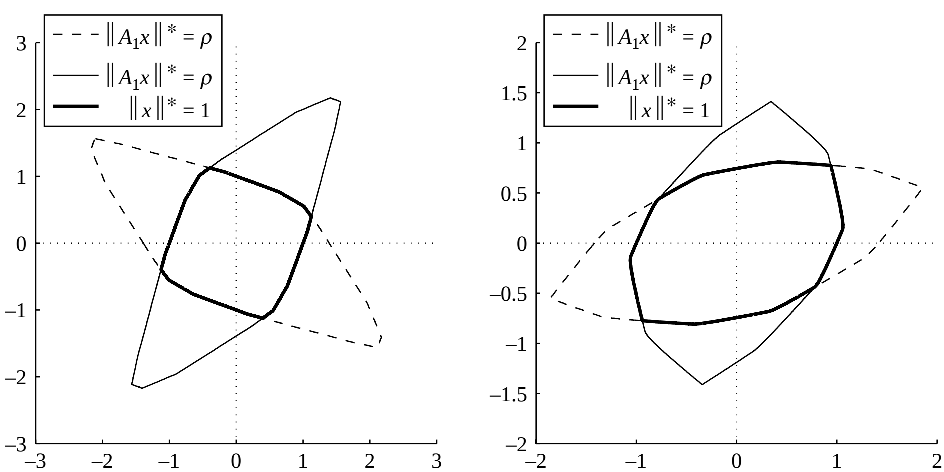

Figure 1. Examples of computation of Barabanov norms for a pair of 2×2 matrices.

require different efforts and potentially may have different theoretical extensions, and now we are unable to predict which of these two algorithms might be more useful in the future.

In conclusion, note that the above algorithms allow us to calculate the joint spectral radius of a finite matrix family with any required accuracy and to evaluate *a posteriori* the computational error. At the same time, the question about the accuracy of approximation of the Barabanov norm $\|\cdot\|^*$ by the norms $\|\cdot\|_n$ is open. It seems that the difficulty in answering this question is caused by the fact that, in general, the Barabanov norms for a matrix family are determined ambiguously, namely to overcome this difficulty we preferred to consider relaxation algorithms instead of direct ones. Moreover, if to set $\lambda_n \equiv 0$ in (6) then, as numerical tests demonstrate, the obtained direct computational analogue of the LR-procedure may turn out to be non-convergent.

The question about the rate of convergence of the sequences $\{\rho_n^+\}$ and $\{\rho_n^-\}$ to the joint spectral radius is also open.

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Notes

1. A matrix set $\mathscr{A}$ is called *irreducible*, if the matrices from $\mathscr{A}$ have no common invariant subspaces except $\{0\}$ and $\mathbb{R}^m$. In [19–21] such a matrix set was called quasi-controllable.
2. The set is called body if it contains at least one interior point.

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