Tables, bounds and graphics of short linear codes with covering radius 3 and codimension 4 and 5 *

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Abstract. The length function $\ell_q(r,R)$ is the smallest length of a q-ary linear code of codimension r and covering radius R.

In this work, by computer search in wide regions of q, we obtained short $[n,n-4,5]_q3$ quasi-perfect MDS codes and $[n,n-5,5]_q3$ quasi-perfect Almost MDS codes of covering

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radius $R = 3$. The new codes imply the following upper bounds:

$$\ell_q(4,3) < 2.8\sqrt[3]{q\ln q} \quad \text{for} \quad 8 \le q \le 6229;$$
$$\ell_q(5,3) < 3\sqrt[3]{q^2\ln q} \quad \text{for} \quad 5 \le q \le 761.$$

For $r \neq 3t$ and $q \neq (q')^3$, the new bounds have the form

$$\ell_q(r,3) < c\sqrt[3]{\ln q}\cdot q^{(r-3)/3}, \quad c \text{ is a universal constant}, \quad r = 4, 5.$$

As far as it is known to the authors, such bounds have not been previously described in the literature.

In computer search, we use the leximatrix algorithm to obtain parity check matrices of codes. The algorithm is a version of the recursive g-parity check algorithm for greedy codes.

Keywords: Covering codes, saturating sets, the length function, upper bounds, projective spaces.

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1 Introduction

1.1 Covering codes. The length function. Saturating sets in projective spaces

Let F_q be the Galois field with q elements. Let F_q^n be the n-dimensional vector space over F_q. Denote by $[n, n-r]_q$ a q-ary linear code of length n and codimension (redundancy) r, that is, a subspace of F_q^n of dimension $n-r$. The sphere of radius R with center c in F_q^n is the set $\{v : v \in F_q^n, d(v,c) \le R\}$ where $d(v,c)$ is the Hamming distance between the vectors v and c.

Definition 1.1. (i) The covering radius of a linear $[n, n-r]_q$ code is the least integer R such that the space F_q^n is covered by spheres of radius R centered at the codewords.

(ii) A linear $[n, n-r]_q$ code has covering radius R if every column of F_q^r is equal to a linear combination of at most R columns of a parity check matrix of the code, and R is the smallest value with such property.

Definitions 1.1(i) and 1.1(ii) are equivalent. Let an $[n, n-r]_qR$ code be an $[n, n-r]_q$ code of covering radius R. Let an $[n, n-r, d]_qR$ code be an $[n, n-r]_qR$ code of minimum distance d. For an introduction to coverings of vector Hamming spaces over finite fields, see [6, 7].

The covering density μ of an $[n, n-r]_q R$-code is defined as the ratio of the total volume of all q^{n-r} spheres of radius R centered at the codewords to the volume q^n of the space F_q^n. By Definition 1.1(i), we have $\mu \geq 1$. In the other words,

$$\mu = \left(q^{n-r}\sum_{i=0}^{R}(q-1)^i\binom{n}{i}\right)\frac{1}{q^n} = \frac{1}{q^r}\sum_{i=0}^{R}(q-1)^i\binom{n}{i} \geq 1. \tag{1.1}$$

The covering quality of a code is better if its covering density is smaller. For fixed q, r, and R, the covering density of an $[n, n-r]_q R$ code decreases with decreasing n.

Codes investigated from the point view of the covering quality are usually called *covering codes* [7]; see an online bibliography [22], works [6, 8–10, 12–15, 20, 21], and the references therein.

This work is devoted to non-binary covering codes with radius $R = 3$. Note that for relatively small $q > 2$ many results are given in [10, 12, 13] and the references therein.

Definition 1.2. [6, 7] *The length function* $\ell_q(r, R)$ is the smallest length of a q-ary linear code of codimension r and covering radius R.

From (1.1), see also Definition 1.1(ii), one can get an *approximate lower bound on* $\ell_q(r, R)$. In particular, if n is considerable larger than R (this is the natural situation in covering codes investigations) and if q is large enough, we have

$$\mu \approx \frac{1}{q^r}(q-1)^R\binom{n}{R} \approx q^{R-r}\frac{n^R}{R!} \gtrsim 1, \; n \gtrsim \sqrt[R]{R!}\cdot q^{(r-R)/R},$$

and, in a more general form,

$$\ell_q(r, R) \gtrsim cq^{(r-R)/R}, \tag{1.2}$$

where c is independent of q but it is possible that c is dependent of r and R. In [10], see also the references therein including [8, 12], the bound (1.2) is given in another (asymptotic) form and infinite families of covering codes, achieving the bound, are obtained for the following situations:

$$r = tR, \quad \text{arbitrary } q;$$
$$r \neq tR, \quad q = (q')^R;$$
$$R = sR', \quad r = Rt + s, \quad q = (q')^{R'}.$$

Here t, s are integers, q' is a prime power.

In the general case, *for arbitrary* r, R, q, *the problem to achieve the bound* (1.2) *is open.*

In the last decades, upper bounds on $\ell_q(r, R)$ have been intensively investigated, see [6–10, 12–15, 20–22] and the references therein.

The *goal of this work* is to obtain new *upper bounds on the length functions* $\ell_q(4, 3)$ *and* $\ell_q(5, 3)$ *with* $r \neq tR$ *and arbitrary* q, in particular with $q \neq (q')^3$ where q' is a prime power. It is an open problems.

Let $\mathrm{PG}(N, q)$ be the N-dimensional projective space over the field F_q; see [16–18] for an introduction to the projective spaces over finite fields, see also [14, 17, 20, 21] for connections between coding theory and Galois geometries.

Effective methods to obtain upper bounds on $\ell_q(r, R)$ are connected with saturating sets in $\mathrm{PG}(N, q)$.

Definition 1.3. A point set $\mathcal{S} \subseteq \mathrm{PG}(N, q)$ is ρ-saturating if for any point A of $\mathrm{PG}(N, q) \setminus \mathcal{S}$ there exist $\rho + 1$ points in $\mathcal{S}$ generating a subspace of $\mathrm{PG}(N, q)$ containing A, and ρ is the smallest value with such property.

By Definition 1.3, every point A from $\mathrm{PG}(N, q)$ can be written as a linear combination of at most $\rho + 1$ points of a ρ-saturating set, cf. Definition 1.1(ii).

Saturating sets are considered, for instance, in [1–3, 6, 8–12, 14, 15, 19–21, 25]. In the literature, saturating sets are also called "saturated sets", "spanning sets", "dense sets".

Let $s_q(N, \rho)$ be *the smallest size of a* ρ*-saturating set* in $\mathrm{PG}(N, q)$.

If q-ary positions of a column of an $r \times n$ parity check matrix of an $[n, n-r]_q R$ code are treated as the homogeneous coordinates of a point in $\mathrm{PG}(r-1, q)$ then this parity check matrix defines an $(R-1)$-saturating set of size n in $\mathrm{PG}(r-1, q)$ [8–10, 14, 15, 19–21]. So, there is a *one-to-one correspondence between* $[n, n-r]_q R$ *codes and* $(R-1)$*-saturating sets in* $\mathrm{PG}(r-1, q)$. Therefore,

$$\ell_q(r, R) = s_q(r-1, R-1),$$

in particular, $\ell_q(4, 3) = s_q(3, 2)$, $\ell_q(5, 3) = s_q(4, 2)$.

Complete arcs in $\mathrm{PG}(N, q)$ are an important class of saturating sets. An n-arc in $\mathrm{PG}(N, q)$ with $n > N+1$ is a set of n points such that no $N+1$ points belong to the same hyperplane of $\mathrm{PG}(N, q)$. An n-arc of $\mathrm{PG}(N, q)$ is complete if it is not contained in an $(n+1)$-arc of $\mathrm{PG}(N, q)$. A complete arc in $\mathrm{PG}(N, q)$ is an $(N-1)$-saturating set. Points (in the homogeneous coordinates) of a complete n-arc in $\mathrm{PG}(N, q)$, treated as columns, form a parity check matrix of an $[n, n-(N+1), N+2]_q N$ maximum distance separable (MDS) code. If $N = 2, 3$ these codes are quasi-perfect.

Let $s_q^{\mathrm{arc}}(N, N-1)$ be *the smallest size of a complete arc* in $\mathrm{PG}(N, q)$. By above,

$$\ell_q(N+1, N) = s_q(N, N-1) \leq s_q^{\mathrm{arc}}(N, N-1).$$

1.2 Covering codes with radius 3

For arbitrary q, covering $[n, n-r]_q 3$ codes of length close to the lower bound (1.2) are known only for $r = tR = 3t$ [10, 12]. In particular, the following bounds are obtained by algebraic constructions [10, Sect. 5, eq. (5.2)], [12, Th. 12]:

$$\ell_q(r,3) \le 3q^{(r-3)/3} + q^{(r-6)/3},\ r = 3t \ge 6, r \ne 9,\ q \ge 5,\ \text{and } r = 9,\ q = 16, q \ge 23.$$
$$\ell_q(r,3) \le 3q^{(r-3)/3} + 2q^{(r-6)/3} + 1,\ r = 9,\ q = 7, 8, 11, 13, 17, 19.$$
$$\ell_q(r,3) \le 3q^{(r-3)/3} + 2q^{(r-6)/3} + 2,\ r = 9,\ q = 5, 9.$$

If $r = 3t+1$ or $r = 3t+2$, covering codes of length close to the lower bound (1.2) are known only when $q = (q')^3$, where q' is a prime power [8–10, 15]. In particular, the following bounds are obtained by algebraic constructions, see [8, 9], [10, Sect. 5, eqs. (5.3),(5.4)]:

$$\ell_q(r,3) \le \left(4 + \frac{4}{\sqrt[3]{q}}\right) q^{(r-3)/3},\ r = 3t+1 \ge 4,\ q = (q')^3 \ge 64.$$
$$\ell_q(r,3) \le \left(9 - \frac{8}{\sqrt[3]{q}} + \frac{4}{\sqrt[3]{q^2}}\right) q^{(r-3)/3},\ r = 3t+2 \ge 5,\ q = (q')^3 \ge 27.$$

For arbitrary $q \ne (q')^3$, in the literature, computer results are given for $[n, n-4]_q 3$ codes with $q \le 563$ [13, Tab. 1] and $[n, n-5]_q 3$ codes with $q \le 43$ [9, Tab. 1], [13, Tab. 2].

In this work, by computer search, we obtain new results for $[n, n-4, 5]_q 3$ quasi-perfect MDS codes with $q \le 6229$, and $[n, n-5, 5]_q 3$ quasi-perfect Almost MDS codes with $q \le 761$. This gives upper bounds on $\ell_q(4,3)$ and $\ell_q(5,3)$ for a set of values q essentially greater than the one in [9, 13].

The following theorem summarizes the **new results** of this paper, see Sections 3 and 4.

Theorem 1.4. *Let $c_4 = 2.8$ and $c_5 = 3$. For the length function $\ell_q(r,3)$ and for the smallest size $s_q(r-1,2)$ of a 2-saturating set in the projective space* $\mathrm{PG}(r-1,q)$ *the following upper bounds hold:*

(i) $$\ell_q(4,3) = s_q(3,2) \le s_q^{\mathrm{arc}}(3,2) < c_4 \sqrt[3]{\ln q} \cdot q^{(4-3)/3} = c_4 \sqrt[3]{\ln q} \cdot \sqrt[3]{q} \quad (1.3)$$
for $11 \le q \le 6229$ *and* $q = 8$;

$$\ell_q(4,3) = s_q(3,2) < c_4 \sqrt[3]{\ln q} \cdot q^{(4-3)/3} = c_4 \sqrt[3]{\ln q} \cdot \sqrt[3]{q} \quad \textit{for } q = 9. \quad (1.4)$$

(ii) $$\ell_q(5,3) = s_q(4,2) < c_5 \sqrt[3]{\ln q} \cdot q^{(5-3)/3} = c_5 \sqrt[3]{\ln q} \cdot \sqrt[3]{q^2} \quad \textit{for } 5 \le q \le 761. \quad (1.5)$$

Remark 1.5. The assertion (1.3) is provided by $[n, n-4, 5]_q 3$ quasi-perfect MDS codes, see Proposition 3.3(ii). These codes correspond to complete n-arcs in $\mathrm{PG}(3,q)$. Therefore, the assertion (1.3) gives upper bounds on $\ell_q(4,3) = s_q(3,2)$ sa well as on $s_q^{\mathrm{arc}}(3,2)$.

On the other hand, the assertion (1.4) is provided by an $[n, n-4, 4]_q 3$ code, see Proposition 3.3(i). Also, the assertion (1.5) is provided by $[n, n-5, 4]_q 3$ codes for $5 \le q < 37$ and $[n, n-5, 5]_q 3$ codes for $37 \le q \le 761$, see Proposition 4.1. The mentioned codes do not correspond to arcs; therefore the assertions (1.4) and (1.5) give upper bounds only on $\ell_q(4,3) = s_q(3,2)$ and $\ell_q(5,3) = s_q(4,2)$.

We emphasize that, for $r \ne 3t$ and $q \ne (q')^3$, the new bounds of Theorem 1.4 have the form

$$\ell_q(r,3) < c\sqrt[3]{\ln q} \cdot q^{(r-3)/3}, \quad c \text{ is a universal constant}, \quad r = 4, 5.$$

As far as it is known to the authors, such bounds have not been previously described in the literature.

Our results, in particular figures and observations in Sections 3 and 4, allow us to conjecture the following.

Conjecture 1.6. *Let $c_4 = 2.8$ and $c_5 = 3$. For the length function $\ell_q(r,3)$ and for the smallest size $s_q(r-1,2)$ of a 2-saturating set in the projective space* $\mathrm{PG}(r-1,q)$ *the following upper bounds hold:*

- **(i)** $\ell_q(4,3) = s_q(3,2) \le s_q^{\mathrm{arc}}(3,2) < c_4\sqrt[3]{\ln q} \cdot q^{(4-3)/3} = c_4\sqrt[3]{\ln q} \cdot \sqrt[3]{q}$ *for all* $q \ge 11$;
- **(ii)** $\ell_q(5,3) = s_q(4,2) < c_5\sqrt[3]{\ln q} \cdot q^{(5-3)/3} = c_5\sqrt[3]{\ln q} \cdot \sqrt[3]{q^2}$ *for all* $q \ge 5$.

The paper is organized as follows. In Section 2, we describe a leximatrix algorithm to obtain parity check matrices of covering codes. In Sections 3 and 4, upper bounds on the length functions $\ell_q(4,3)$ and $\ell_q(5,3)$ are considered. In Conclusion, the results of this work are briefly analyzed; some tasks for investigation of the leximatrix algorithm are formulated. In Appendix, tables with sizes of codes obtained in this work are given.

2 Leximatrix algorithm to obtain parity check matrices of covering codes

The following is a version of the recursive g-parity check algorithm for greedy codes, see e.g. [5, p. 25], [23], [24, Section 7].

Let $F_q = \{0, 1, \dots, q-1\}$ be the Galois field with q elements.

If q is prime, the elements of F_q are treated as integers modulo q.

If $q = p^m$ with p prime and $m \ge 2$, the elements of F_{p^m} are represented by integers as follows: $F_{p^m} = F_q = \{0, 1 = \alpha^0, 2 = \alpha^1, \dots, u = \alpha^{u-1}, \dots, q-1 = \alpha^{q-2}\}$, where α is a root of a primitive polynomial of F_{p^m}.

For a q-ary code of codimension r, covering radius R, and minimum distance $d = R+2$, we construct a parity check matrix from nonzero columns h_i of the form

$$h_i = (x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)})^T, \; x_u^{(i)} \in F_q,$$

where the first (leftmost) non-zero element is 1. The number of distinct columns is $(q^r - 1)/(q - 1)$. For h_i we put $i = \sum\limits_{u=1}^{r} x_u^{(i)} q^{r-u}$. We order the columns in the list as $h_1, h_2, \ldots, h_{(q^r-1)/(q-1)}$. The columns of the list are candidates to be included in the parity check matrix.

By the above arguments connected with the formula for i and the order of columns, a column h_i is treated as its number i in our list written in the q-ary scale of notation. The considered order of columns is lexicographical.

The first column of the list should be included into the matrix. Then step-by-step, one takes the next column from the list which cannot be represented as a linear combination of at most R columns already chosen. The process ends when no new column may be included into the matrix. The obtained matrix H_n is a parity check matrix of an $[n, n-r, R+2]_q R$ code.

We call a ***leximatrix*** the obtained parity check matrix. We call a ***leximatrix code*** the corresponding code.

It is important to note that **for prime q, length n of a leximatrix code and the form of the leximatrix H_n depend on q and R only**. No other factors affect code length and structure. Actually, assume that after some step a current matrix is obtained. At the next step we should remove from our current list all columns that are linear combination of R or less columns of the current matrix. For prime q and the given R, the result of removing is unequivocal; hence, the next column is taken uniquely.

For non-prime q, the length n of a leximatrix code depends on q and on the form of the primitive polynomial of the field. In this work, we use primitive polynomials that are created by the program system MAGMA [4] by default, see Table A. In any case, the choice of the polynomial changes the leximatrix code length unessentially.

By the leximatrix algorithm, if $R = 1$, we obtain the q-ary Hamming code. If $R = 2$, we obtain a quasi-perfect $[n, n-r, 4]_q 2$ code; for $r = 3$ such code is an MDS code and corresponds to a complete arc in PG$(2, q)$. If $R = 3$, we obtain a quasi-perfect $[n, n-r, 5]_q 3$ code; for $r = 4$ such code is an MDS code and corresponds to a complete arc in PG$(3, q)$; for $r = 5$ it is an Almost MDS code.

Let $n_q^{\mathrm{L}}(r, R)$ be **length of the q-ary leximatrix code of codimension r and covering radius R**. It is assumed that for a non-prime field F_q, one uses the primitive polynomial created by the program system MAGMA [4] by default; in particular, for non-prime $q \leq 5329$, the polynomial from Table A should be taken.

Future, we represent length of an $[n_q^L(r, R), n_q^L(r, R) - r, R+2]_q R$ leximatrix code in

Table A. Primitive polynomials used for leximatrix $[n,n-r,5]_q3$ quasi-perfect codes with non-prime q

$q=p^m$	primitive polynomial	$q=p^m$	primitive polynomial	$q=p^m$	primitive polynomial
$4=2^2$	x^2+x+1	$8=2^3$	x^3+x+1	$9=3^2$	x^2+2x+2
$16=2^4$	x^4+x^3+1	$25=5^2$	x^2+x+2	$27=3^3$	x^3+2x^2+x+1
$32=2^5$	x^5+x^3+1	$49=7^2$	x^2+x+3	$64=2^6$	$x^6+x^4+x^3+1$
$81=3^4$	x^4+x+2	$121=11^2$	x^2+4x+2	$125=5^3$	x^3+3x+2
$128=2^7$	x^7+x+1	$169=13^2$	x^2+x+2	$243=3^5$	x^5+2x+1
$256=2^8$	$x^8+x^4+x^3+$ x^2+1	$289=17^2$	x^2+x+3	$343=7^3$	x^3+3x+2
$361=19^2$	x^2+x+2	$512=2^9$	x^9+x^4+1	$529=23^2$	x^2+2x+5
$625=5^4$	x^4+x^2+2x+2	$729=3^6$	x^6+x+2	$841=29^2$	$x^2+24x+2$
$961=31^2$	$x^2+29x+3$	$1024=2^{10}$	$x^{10}+x^6+x^5+$ x^3+x^2+x+1	$1331=11^3$	x^3+2x+9
$1369=37^2$	$x^2+33x+2$	$1681=41^2$	$x^2+38x+6$	$1849=43^2$	x^2+x+3
$2048=2^{11}$	$x^{11}+x^2+1$	$2187=3^7$	x^7+x^2+2x+1	$2197=13^3$	x^3+x^2+7
$2209=47^2$	x^2+x+13	$2401=7^4$	x^4+5x^2+4x+3	$2809=53^2$	$x^2+49x+2$
$3125=5^5$	x^5+4x+2	$3481=59^2$	$x^2+58x+2$	$3721=61^2$	$x^2+60x+2$
$4096=2^{12}$	$x^{12}+x^8+x^2+$ $x+1$	$4489=67^2$	$x^2+63x+2$	$4913=17^3$	x^3+x+14
$5041=71^2$	$x^2+69x+7$	$5329=73^2$	$x^2+70x+5$		

the form

$$n_q^{\mathrm{L}}(r,R)=c_q^{\mathrm{L}}(r,R)\,\sqrt[R]{\ln q}\cdot q^{(r-R)/r}, \tag{2.1}$$

where $c_q^{\mathrm{L}}(r,R)$ is a coefficient entirely given by r,R,q (if q is prime) or by r,R,q, and the primitive polynomial of F_q (if q is non-prime).

Remark 2.1. In the literature on the projective geometry, the columns are considered as points in the homogenous coordinates; the algorithm, described above, is called an "algorithm with fixed order of points" (FOP) [2,3].

3 Upper bounds on the length functions $\ell_q(4,3)$

The following properties of the leximatrix algorithm are useful for implementation.

Proposition 3.1. *Let q be a prime. Then the v-th column of the leximatrix of an $[n,n-4,5]_q3$ code is the same for all $q\ge q_0(v)$ where $q_0(v)$ is large enough.*

Proof. Let $H_j = [h^{(1)}, h^{(2)}, \dots, h^{(j)}]$ be the matrix obtained in the j-th step of the leximatrix algorithm. Here $h^{(v)}$ is a column of the matrix. A column from the list, not included in H_j, is covered by H_j if it can be represented as a linear combination of at most 3 columns of H_j. Suppose that $h^{(j)} = h_s$, where h_s is the s-th column in the lexicographical list of candidates. A column $Q = h_u \notin H_j$ is the next chosen column, if and only if all the columns h_m with $m \in [s+1, u-1]$ are covered by H_j. This means that, for any $m \in [s+1, u-1]$, at least one of the determinants $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m)$, with $h^{(v_1)}, h^{(v_2)}, h^{(v_3)} \in H_j$, is equal to zero modulo q. This can happen only in two cases:

- $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m) = 0$, we say that h_m is "absolutely" covered by H_j;
- $\det(h^{(v_1)}, h^{(v_2)}, h^{(v_3)}, h_m) = B \neq 0$, but $B \equiv 0 \bmod q$.

For q large enough, q does not divide any of the possible values of B and then, at least for j relatively small, the columns covered are just the absolutely covered columns. Therefore, when q is large enough the leximatrices share a certain number of columns. □

The values of $q_0(v)$ can be found with the help of calculations based on the proof of Proposition 3.1. Also, we can directly consider leximatrices for a convenient region of q.

Example 3.2. Values of $q_0(v)$, $v \le 20$, together with columns $(x_1^{(v)}, x_2^{(v)}, x_3^{(v)}, x_4^{(v)})^T$, are given in Table B. So, for all prime $q \ge 233$ (resp. $q \ge 1321$) the first 14 (resp. 20) columns of a parity check matrix of an $[n, n-4, 5]_q 3$ MDS leximatrix code are as in Table B.

Table B. The first 20 columns of parity check matrices of $[n, n-4, 5]$ leximatrix MDS codes, q prime

v	$x_1^{(v)}$	$x_2^{(v)}$	$x_3^{(v)}$	$x_4^{(v)}$	$q_0(v)$	v	$x_1^{(v)}$	$x_2^{(v)}$	$x_3^{(v)}$	$x_4^{(v)}$	$q_0(v)$
1	0	0	0	1	2	11	1	7	11	8	67
2	0	0	1	0	2	12	1	8	6	13	109
3	0	1	0	0	2	13	1	9	13	16	199
4	1	0	0	0	2	14	1	10	12	22	233
5	1	1	1	1	2	15	1	11	7	29	269
6	1	2	3	4	5	16	1	12	22	15	769
7	1	3	2	5	11	17	1	13	16	20	769
8	1	4	5	3	29	18	1	14	17	7	1283
9	1	5	4	2	41	19	1	15	21	10	1283
10	1	6	8	9	41	20	1	16	9	38	1321

Proposition 3.3. (i) *There exists a* $[7, 7-4, 4]_9 3$ *code, length* $n = 7$ *of which satisfies* $n < 2.8\sqrt[3]{q \ln q}$.

(ii) *There exist* $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4,5]_q3$ *quasi-perfect MDS leximatrix codes of length* $n_q^{\mathrm{L}}(4,3) < 2.8\sqrt[3]{q\ln q}$ *for* $11 \le q \le 6229$ *and* $q = 8$.

Proof. **(i)** The existence of the code is noted in [13, Tab. 1], see also the references therein.

(ii) The needed codes are obtained by computer search, using the leximatrix algorithm, Proposition 3.1, and Example 3.2.

□

Proposition 3.3 implies assertions of Theorem 1.4(i).

Lengths of $[n_q^L(4,3), n_q^L(4,3)-4,5]_q3$ leximatrix quasi-perfect MDS codes are collected in Table 1 (see Appendix) and presented in Figure 1 by the bottom solid black curve. The bound $2.8\sqrt[3]{q\ln q}$ is shown in Figure 1 by the top dashed red curve.

We denote by $\delta_q(4,3)$ the difference between the bound $2.8\sqrt[3]{q\ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of the leximatrix code. Let $\delta_q^{\%}(4,3)$ be the corresponding percent difference. Thus,

$$\delta_q(4,3) = 2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3);$$
$$\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}}100\%.$$

The difference $\delta_q(4,3)$ and the percent difference $\delta_q^{\%}(4,3)$ are presented in Figures 2 and 3, respectively.

By (2.1), we represent length of an $[n_q^L(4,3), n_q^L(4,3)-4,5]_q3$ leximatrix code in the form

$$n_q^{\mathrm{L}}(4,3) = c_q^{\mathrm{L}}(4,3)\sqrt[3]{q\ln q}, \tag{3.1}$$

where $c_q^{\mathrm{L}}(4,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field F_q (if q is non-prime). The coefficients $c_q^{\mathrm{L}}(4,3) = \frac{n_q^{\mathrm{L}}(4,3)}{\sqrt[3]{q\ln q}}$ are shown in Figure 4.

Observation 3.4. **(i)** *The difference* $\delta_q(4,3)$ *tends to increase when* q *grows, see Figures* 1 *and* 2.

(ii) *The percent difference* $\delta_q^{\%}(4,3)$ *oscillates around the horizontal line* $y = 6\%$. *For growing* q, *the oscillation amplitude decreases, see Figure* 3.

(iii) *Coefficients* $c_q^L(4,3)$ *oscillate around the horizontal line* $y = 2.64$ *with a small amplitude. For growing* q, *the oscillation amplitude decreases, see Figure* 4.

Observation 3.4 gives rise to Conjecture 1.6(i) for $[n, n-4]_q3$ codes.

Note that Observations 3.4(ii) and 3.4(iii) are connected with each other. Actually,

$$\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}}100\% = \left(1 - \frac{c_q^{\mathrm{L}}(4,3)}{2.8}\right)100\%.$$

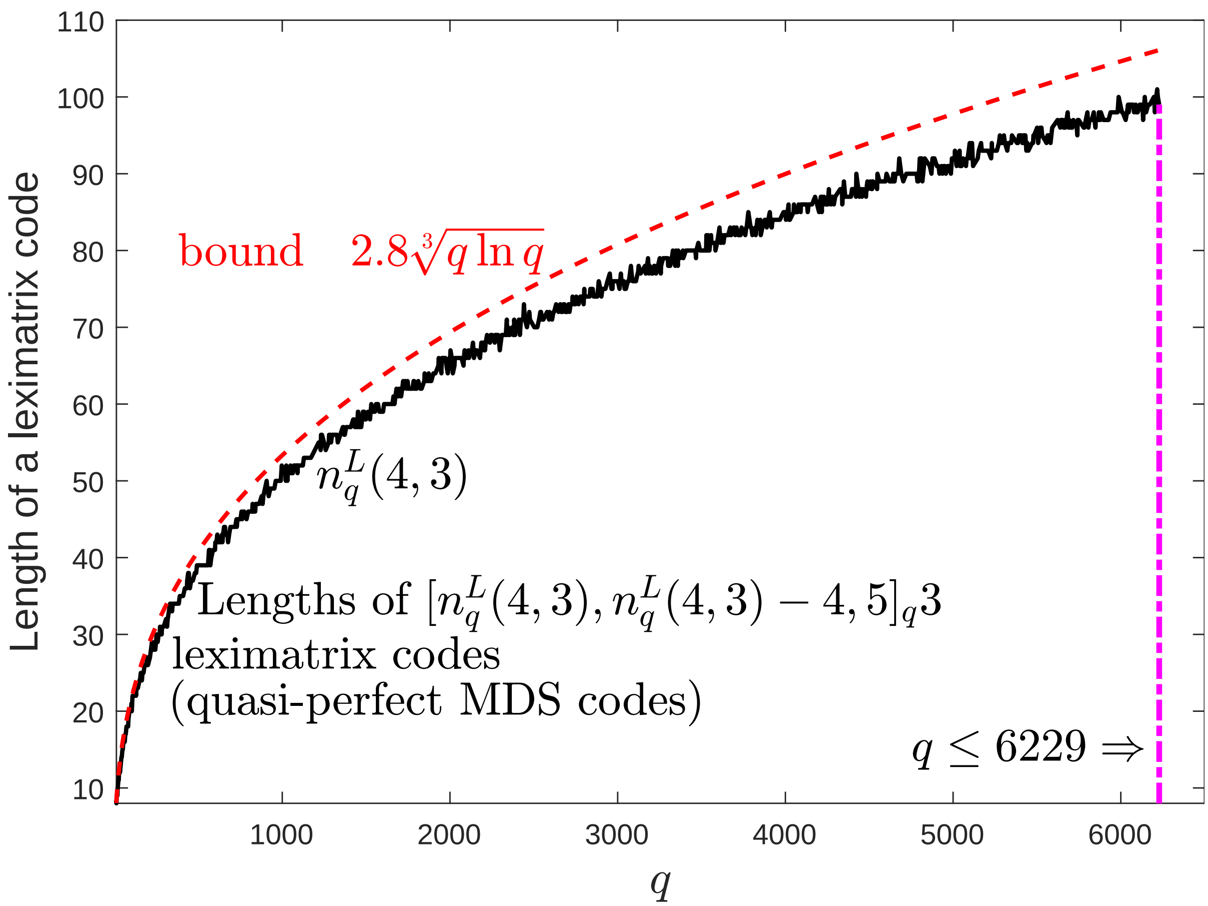

Figure 1: Lengths $n_q^L(4,3)$ of $[n_q^L(4,3), n_q^L(4,3)-4, 5]_q3$ leximatrix codes (quasi-perfect MDS codes) (*bottom solid black curve*) vs bound $2.8\sqrt[3]{q\ln q}$ (*top dashed red curve*); $11 \leq q \leq 6229$. *Vertical magenta line* marks region $q \leq 6229$

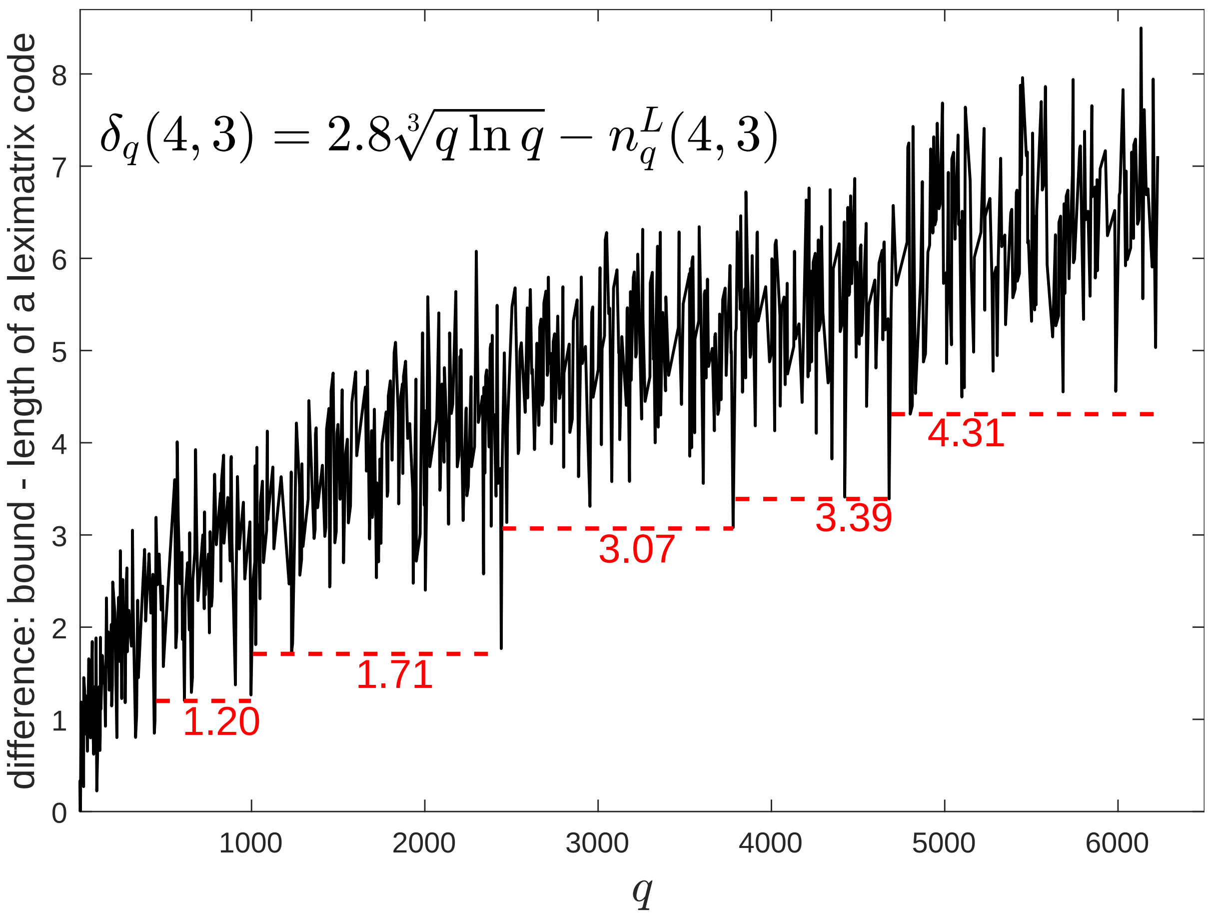

Figure 2: Difference $\delta_q(4,3)$ between bound $2.8\sqrt[3]{q \ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4, 5]_q 3$ leximatrix codes; $11 \le q \le 6229$

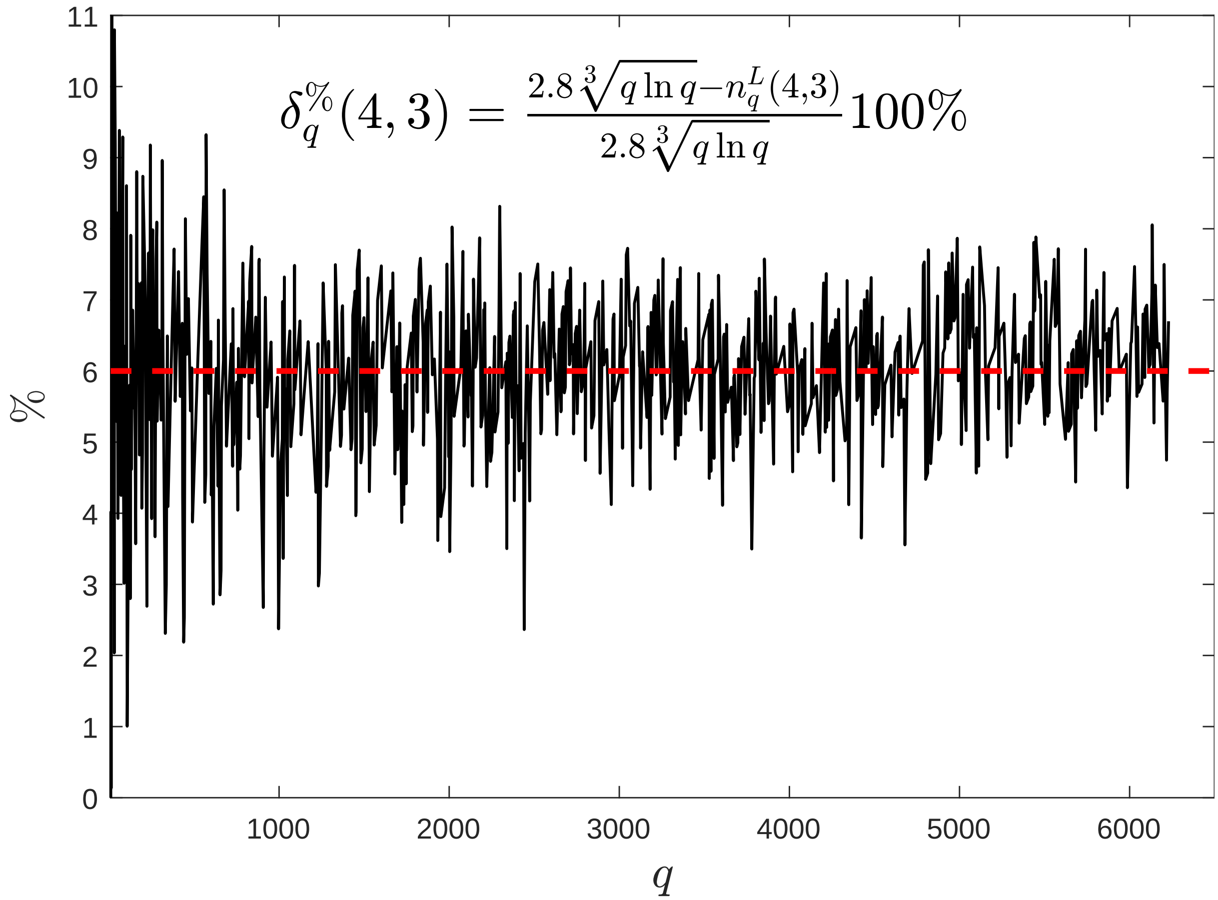

Figure 3: Percent difference $\delta_q^{\%}(4,3) = \frac{2.8\sqrt[3]{q\ln q} - n_q^{\mathrm{L}}(4,3)}{2.8\sqrt[3]{q\ln q}}100\%$ between bound $2.8\sqrt[3]{q\ln q}$ and length $n_q^{\mathrm{L}}(4,3)$ of $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4,5]_q3$ leximatrix code; $11 \le q \le 6229$

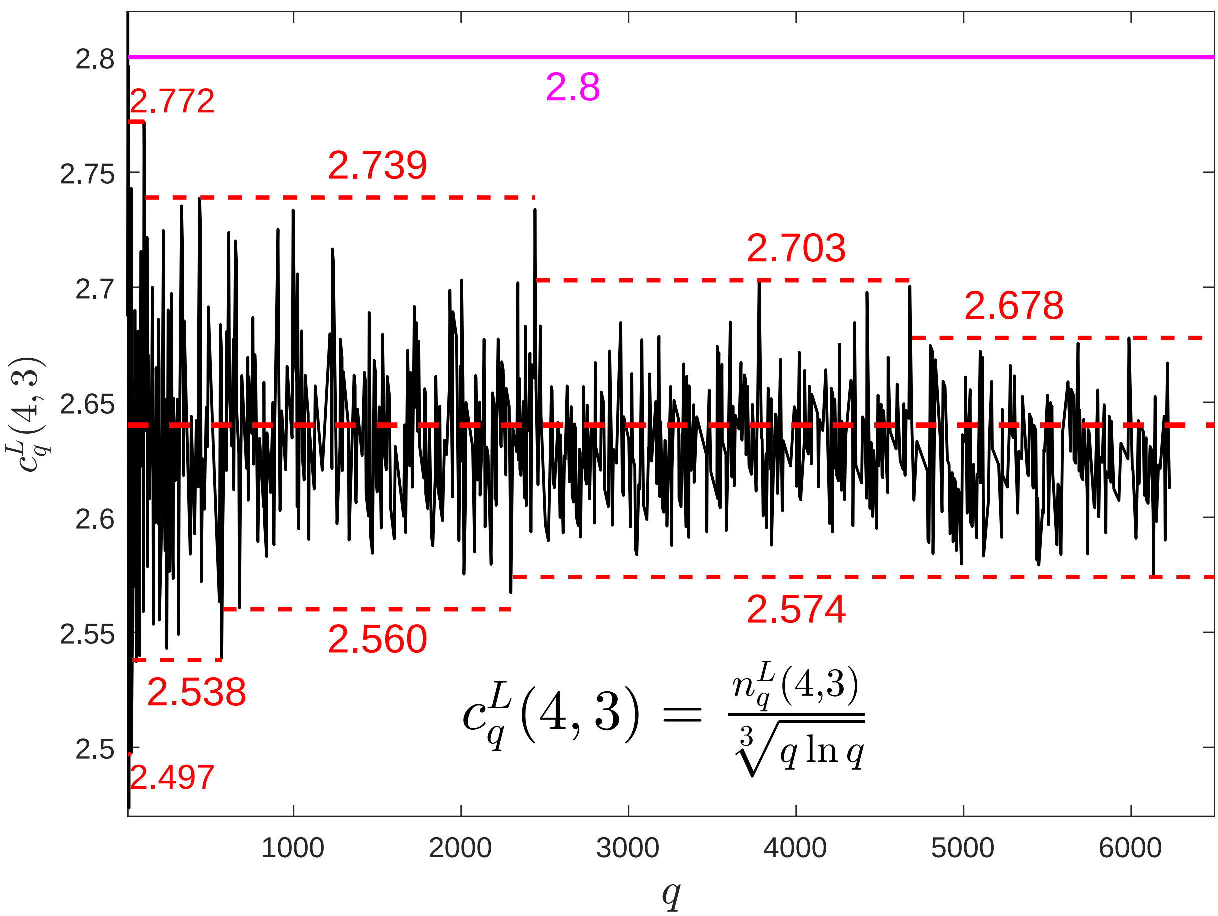

Figure 4: Coefficients $c_q^{\mathrm{L}}(4,3) = n_q^{\mathrm{L}}(4,3)/\sqrt[3]{q \ln q}$ for $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3) - 4, 5]_q 3$ leximatrix codes (quasi-perfect MDS codes); $11 \le q \le 6229$

Remark 3.5. It is interesting that the oscillation of the coefficients $c_q^{\mathrm{L}}(4,3)$ around a horizontal line, in principle, is similar to the oscillation of the values $h^{\mathrm{L}}(q)$ around a horizontal line in [2, Fig. 12, Observation 3.7], [3, Fig. 5, Observation 3.7].

In the papers [2,3], small complete $t_2^L(2,q)$-arcs in the projective plane $\mathrm{PG}(2,q)$ are constructed by computer search using algorithm with fixed order of points (FOP). These arcs correspond to $[t_2^L(2,q), t_2^L(2,q)-3,4]_q2$ quasi-perfect MDS codes while the algorithm FOP is analogous to the leximatrix algorithm of Section 2. Moreover, the value $h^{\mathrm{L}}(q)$ is defined in [2,3] as $h^{\mathrm{L}}(q) = t_2^L(2,q)/\sqrt{3q\ln q}$. So, see (3.1), the coefficients $c_q^{\mathrm{L}}(4,3)$ and the values $h^{\mathrm{L}}(q)$ have the similar nature. It is possible that the oscillations mentioned have similar reasons too. However, in the present time the **enigma of the oscillations** is incomprehensible,

4 Upper bounds on the length functions $\ell_q(5,3)$

Proposition 4.1. **(i)** *There exist* $[n,n-5,4]_q3$ *codes with* $n < 3\sqrt[3]{q^2\ln q}$ *for* $5 \le q < 37$.

(ii) *There exist* $[n_q^L(5,3), n_q^L(5,3)-5,5]_q3$ *Almost MDS leximatrix codes with* $n_q^L(5,3) < 3\sqrt[3]{q^2\ln q}$ *for* $37 \le q \le 761$.

Proof. **(i)** The existence of the codes is noted in [9, Tab. 1], [13, Tab. 2], see also the references therein.

(ii) The needed codes are obtained by computer search, using the leximatrix algorithm. □

Proposition 4.1 implies assertions of Theorem 1.4(ii).

Lengths of $[n_q^L(5,3), n_q^L(5,3)-5,5]_q3$ leximatrix Almost MDS codes are collected in Table 2 (see Appendix) and presented in Figure 5 by the bottom solid black curve. The bound $3\sqrt[3]{q^2\ln q}$ is shown in Figure 5 by the top dashed red curve.

We denote by $\delta_q(5,3)$ the difference between the bound $3\sqrt[3]{q^2\ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of the leximatrix code. Let $\delta_q^{\%}(5,3)$ be the corresponding percent difference. Thus,

$$\delta_q(5,3) = 3\sqrt[3]{q^2\ln q} - n_q^{\mathrm{L}}(5,3);$$
$$\delta_q^{\%}(5,3) = \frac{3\sqrt[3]{q^2\ln q} - n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2\ln q}}100\%.$$

The difference $\delta_q(5,3)$ and the percent difference $\delta_q^{\%}(5,3)$ are presented in Figures 6 and 7, respectively.

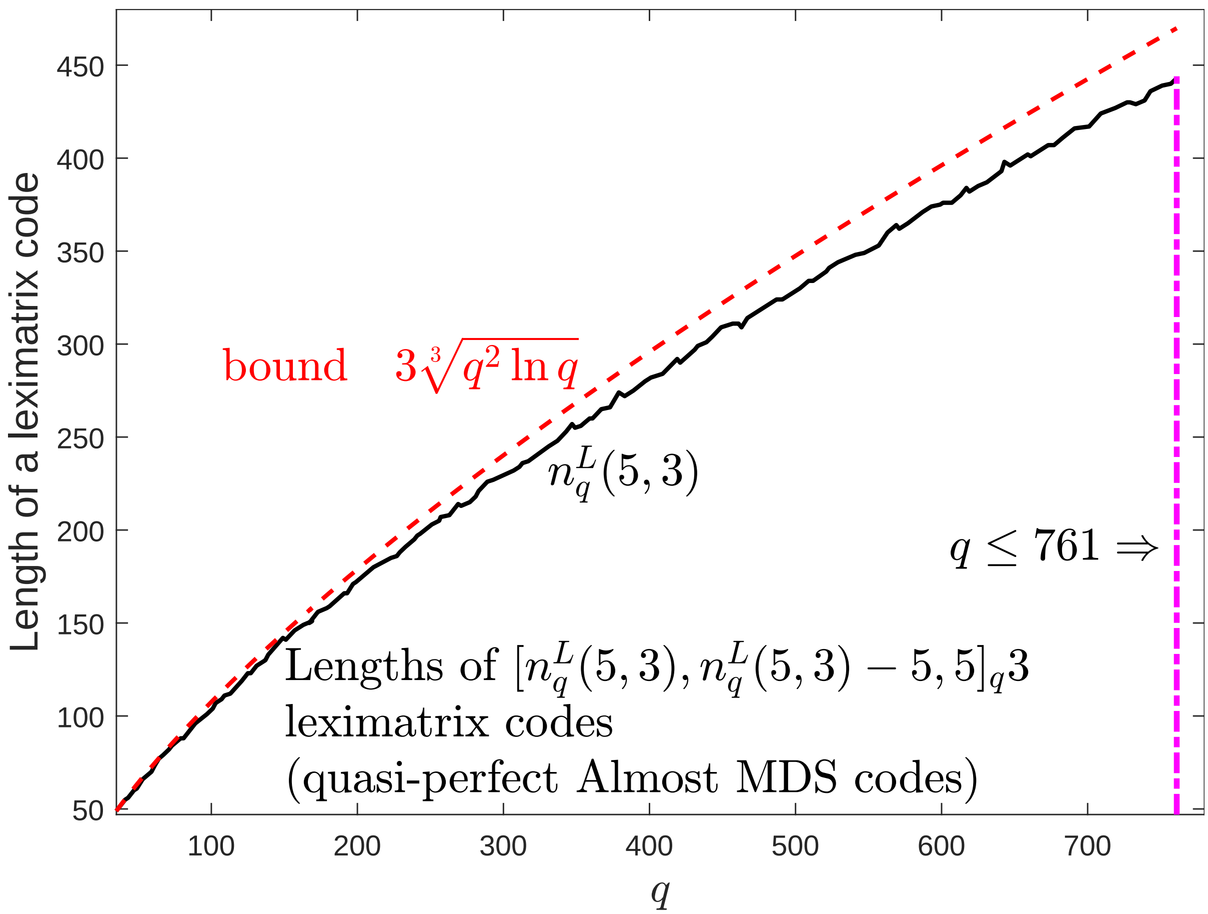

Figure 5: Lengths $n_q^{\mathrm{L}}(5,3)$ of $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5, 5]_q3$ leximatrix codes (quasi-perfect Almost MDS codes) (*bottom solid black curve*) vs bound $3\sqrt[3]{q^2 \ln q}$ (*top dashed red curve*); $37 \leq q \leq 761$. *Vertical magenta line* marks region $q \leq 761$

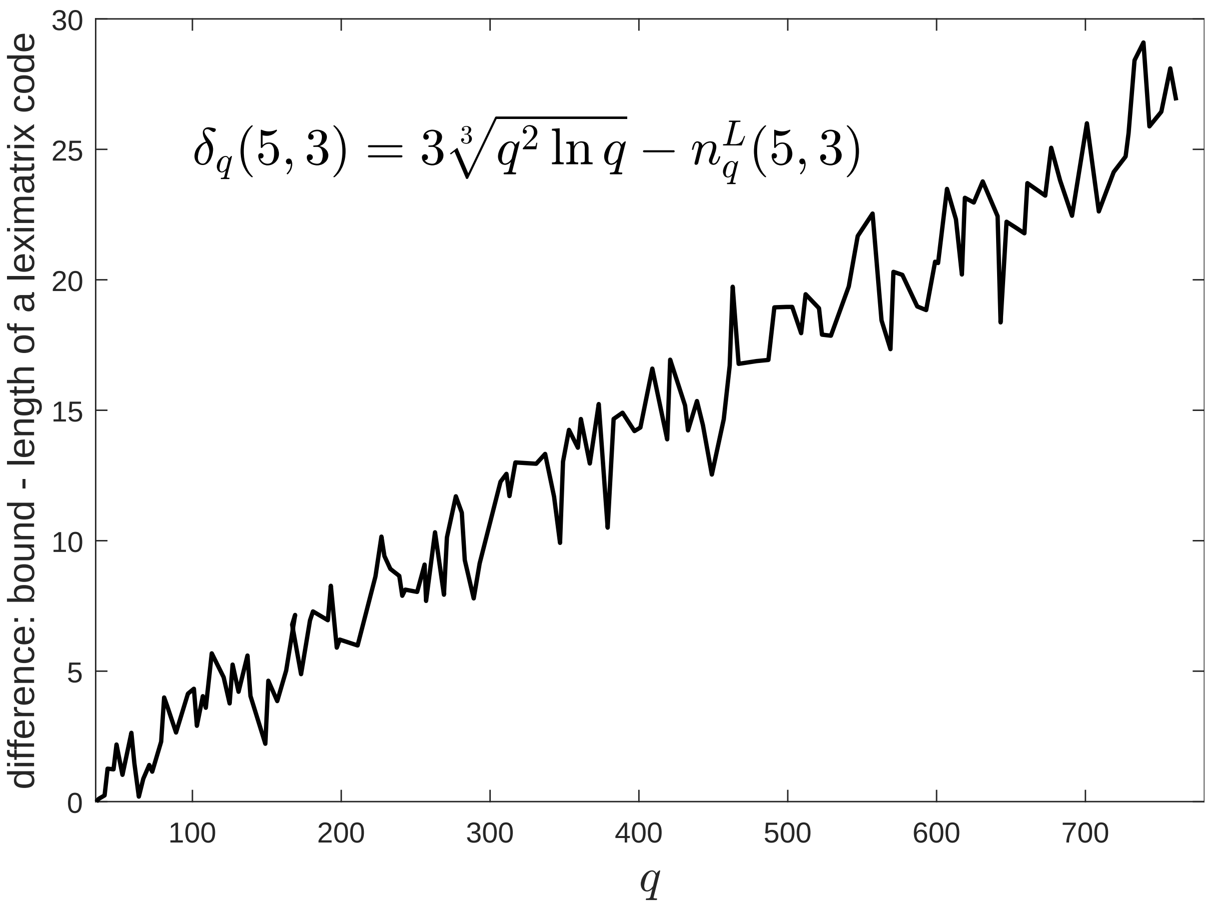

Figure 6: Difference $\delta_q(5,3)$ between bound $3\sqrt[3]{q^2 \ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5, 5]_q 3$ leximatrix code; $37 \le q \le 761$

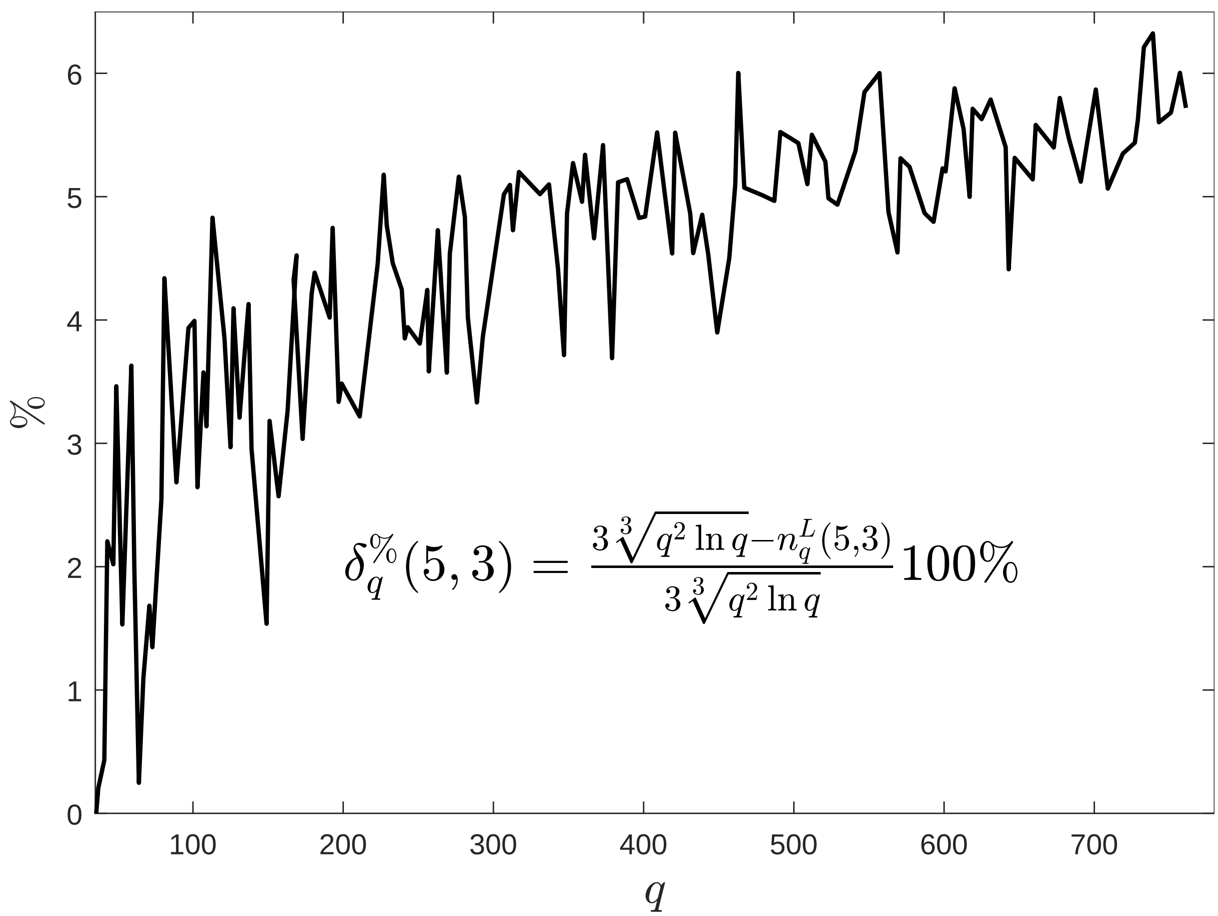

Figure 7: Percent difference $\delta_q^{\%}(5,3) = \frac{3\sqrt[3]{q^2 \ln q} - n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2 \ln q}} 100\%$ between bound $3\sqrt[3]{q^2 \ln q}$ and length $n_q^{\mathrm{L}}(5,3)$ of $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3) - 5, 5]_q 3$ leximatrix code; $37 \leq q \leq 761$

By (2.1), we represent length of an $[n_q^L(5,3), n_q^L(5,3)-5, 5]_q 3$ leximatrix code in the form

$$n_q^{\mathrm{L}}(5,3) = c_q^{\mathrm{L}}(5,3)\sqrt[3]{q^2 \ln q}, \tag{4.1}$$

where $c_q^{\mathrm{L}}(5,3)$ is a coefficient entirely given by q (if q is prime) or by q and the primitive polynomial of the field F_q (if q is non-prime). The coefficients $c_q^{\mathrm{L}}(5,3) = \frac{n_q^{\mathrm{L}}(5,3)}{\sqrt[3]{q^2 \ln q}}$ are shown in Figure 8.

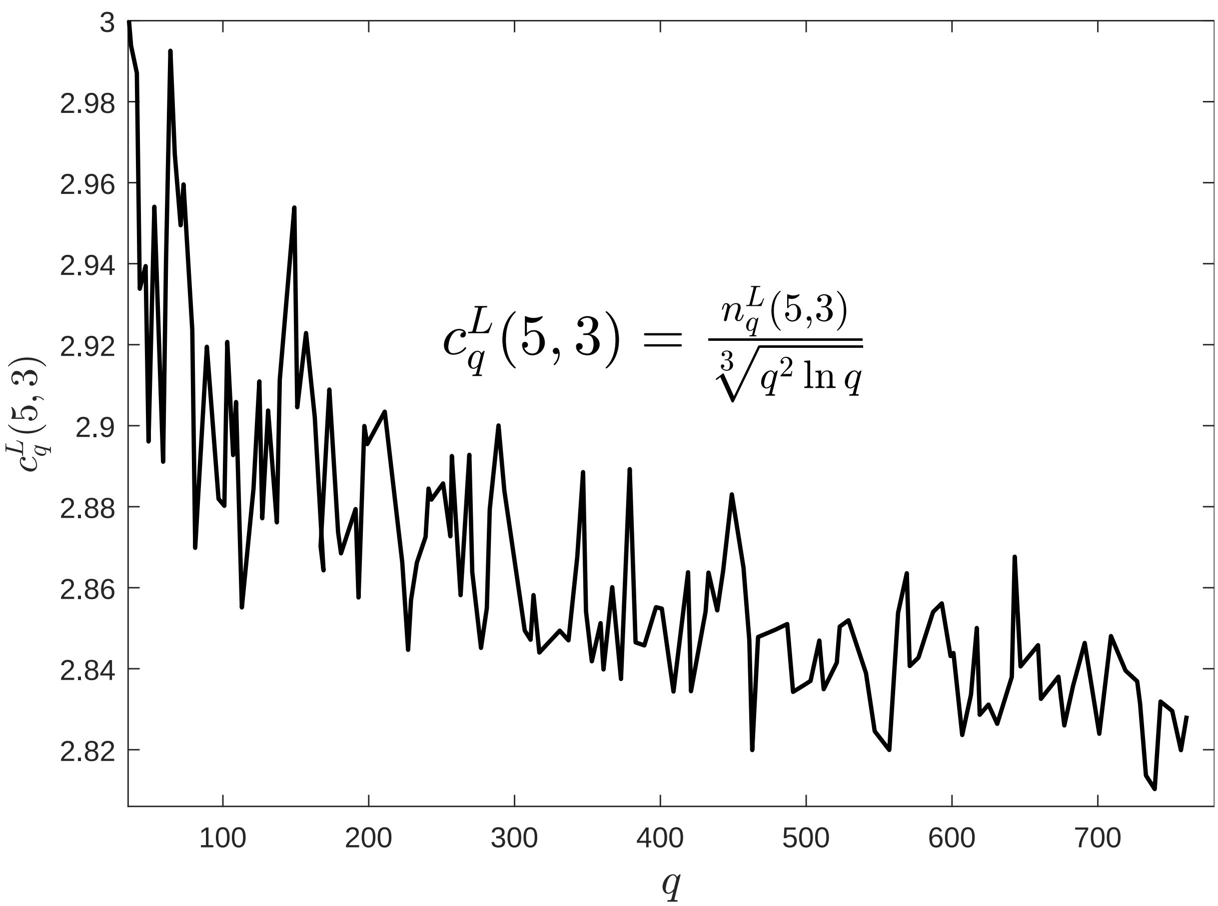

Figure 8: Coefficients $c_q^{\mathrm{L}}(5,3) = n_q^{\mathrm{L}}(5,3)/\sqrt[3]{q^2 \ln q}$ for $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5, 5]_q 3$ leximatrix codes (quasi-perfect Almost MDS codes); $37 \le q \le 761$

Observation 4.2. **(i)** *The difference* $\delta_q(5,3)$ *tends to increase when* q *grows, see Figures* 5 *and* 6.

(ii) *The percent difference* $\delta_q^{\%}(5,3)$ *tends to increase when* q *grows, see Figure* 7.

(iii) *Coefficients* $c_q^{\mathrm{L}}(5,3)$ *tend to decrease when* q *grows, see Figure* 8.

Observation 4.2 gives rise to Conjecture 1.6(ii) for $[n,n-5]_q3$ codes.

Note that Observations 4.2(ii) and 4.2(iii) directly follow each from other. Actually,

$$\delta_q^{\%}(5,3)=\frac{3\sqrt[3]{q^2\ln q}-n_q^{\mathrm{L}}(5,3)}{3\sqrt[3]{q^2\ln q}}100\%=\left(1-\frac{c_q^{\mathrm{L}}(5,3)}{3}\right)100\%.$$

5 Conclusion

The length function $\ell_q(r,R)$ is the smallest length of a q-ary linear code of covering radius R and codimension r. In this work, we consider upper bounds on the length functions $\ell_q(4,3)$ and $\ell_q(5,3)$. For $r=3t$ and $q=(q')^3$ upper bounds on $\ell_q(r,3)$ close to a lower bound are known in literature.

In this work, by computer search in wide regions of q, we obtained short $[n,n-4,5]_q3$ quasi-perfect MDS codes and $[n,n-5,5]_q3$ quasi-perfect Almost MDS codes with covering radius $R=3$. For $r\neq 3t$ and values of $q\neq (q')^3$, the new codes imply upper bounds of the form

$$\ell_q(r,3)<c\sqrt[3]{\ln q}\cdot q^{(r-3)/3},\quad c \text{ is a universal constant},\quad r=4,5.$$

As far as it is known to the authors, such bounds have not been previously described in the literature.

In computer search, we use the step-by-step leximatrix algorithm to obtain parity check matrices of codes. The algorithm is a version of the recursive g-parity check algorithm for greedy codes.

In future, it would be useful to investigate and understand properties of the leximatrix algorithm and structure of leximatrices. In particular, the following is of great interest:

• Initial part of the parity check matrices that is the same for all matrices with greater prime q, see Proposition 3.1 and Example 3.2.

• The working mechanism and its quantitative estimates for the leximatrix algorithm; see, for instance, the work [1] where the working mechanism of a greedy algorithm for complete arcs in the projective plane $\mathrm{PG}(2,q)$ is studied.

• The oscillation of the coefficients $c_q^{\mathrm{L}}(4,3)$ around a horizontal line and its likenesses with the oscillation of the values $h^{\mathrm{L}}(q)$ around a horizontal line in [2, Fig. 12, Observation 3.7], [3, Fig. 5, Observation 3.7], see Figure 4 and Remark 3.5.

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Appendix

Table 1. Lengths $n_q^{\mathrm{L}}(4,3)$ of $[n_q^{\mathrm{L}}(4,3), n_q^{\mathrm{L}}(4,3)-4, 5]_q 3$ leximatrix codes (quasi-perfect MDS codes), $2 \le q \le 6229$

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
2	5	3	5	4	5	5	6	7	8	8	7
9	9	11	8	13	9	16	9	17	9	19	10
23	11	25	11	27	12	29	12	31	13	32	12
37	13	41	14	43	14	47	15	49	15	53	16
59	16	61	16	64	17	67	17	71	18	73	18
79	18	81	18	83	19	89	20	97	20	101	21
103	20	107	22	109	22	113	22	121	22	125	23
127	23	128	22	131	23	137	23	139	23	149	24
151	24	157	25	163	24	167	25	169	25	173	25
179	26	181	26	191	26	193	27	197	27	199	26
211	27	223	29	227	28	229	28	233	28	239	29
241	29	243	28	251	30	256	29	257	29	263	30
269	30	271	31	277	30	281	30	283	31	289	31
293	31	307	32	311	32	313	31	317	32	331	34
337	34	343	33	347	34	349	34	353	34	359	34
361	34	367	34	373	34	379	34	383	34	389	35
397	35	401	35	409	35	419	36	421	36	431	36
433	37	439	38	443	38	449	36	457	37	461	37
463	37	467	37	479	38	487	38	491	39	499	39
503	39	509	39	512	39	521	39	523	39	529	39
541	39	547	39	557	39	563	41	569	41	571	39
577	40	587	41	593	41	599	41	601	42	607	42
613	43	617	42	619	42	625	42	631	42	641	43
643	42	647	43	653	44	659	44	661	43	673	43
677	42	683	43	691	44	701	44	709	44	719	44
727	45	729	44	733	45	739	45	743	45	751	45
757	46	761	45	769	46	773	46	787	45	797	46
809	46	811	46	821	46	823	47	827	46	829	46
839	46	841	47	853	47	857	47	859	47	863	47
877	48	881	47	883	47	887	48	907	50	911	49
919	48	929	49	937	49	941	49	947	49	953	49
961	50	967	50	971	50	977	50	983	50	991	50
997	52	1009	51	1013	51	1019	51	1021	50	1024	52
1031	50	1033	51	1039	51	1049	52	1051	51	1061	51

Table 1. Continue 1

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
1063	51	1069	52	1087	52	1091	51	1093	52	1097	52
1103	52	1109	52	1117	52	1123	52	1129	53	1151	53
1153	53	1163	53	1171	53	1217	55	1223	55	1229	54
1231	56	1237	56	1249	55	1259	54	1277	55	1279	56
1283	56	1289	56	1291	55	1297	56	1301	56	1303	56
1307	56	1319	56	1321	56	1327	56	1331	55	1361	57
1367	57	1369	56	1373	56	1381	57	1399	57	1409	57
1423	58	1427	58	1429	58	1433	57	1439	57	1447	57
1451	59	1453	59	1459	57	1471	57	1481	59	1483	59
1487	59	1489	59	1493	58	1499	58	1511	59	1523	58
1531	60	1543	59	1549	59	1553	59	1559	60	1567	60
1571	60	1579	59	1583	59	1597	59	1601	59	1607	60
1609	60	1613	60	1619	60	1621	60	1627	60	1637	60
1657	60	1663	61	1667	61	1669	60	1681	62	1693	61
1697	62	1699	62	1709	61	1721	63	1723	62	1733	63
1741	62	1747	63	1753	62	1759	62	1777	62	1783	63
1787	63	1789	62	1801	62	1811	63	1823	62	1831	62
1847	63	1849	64	1861	63	1867	63	1871	63	1873	64
1877	63	1879	63	1889	63	1901	64	1907	64	1913	64
1931	65	1933	66	1949	64	1951	66	1973	66	1979	65
1987	64	1993	65	1997	66	1999	65	2003	67	2011	66
2017	64	2027	65	2029	66	2039	66	2048	66	2053	66
2063	66	2069	66	2081	65	2083	66	2087	67	2089	67
2099	66	2111	67	2113	66	2129	67	2131	67	2137	68
2141	67	2143	66	2153	67	2161	67	2179	66	2187	68
2197	68	2203	67	2207	68	2209	67	2213	68	2221	69
2237	68	2239	68	2243	69	2251	69	2267	68	2269	69
2273	69	2281	69	2287	69	2293	68	2297	67	2309	69
2311	69	2333	69	2339	71	2341	69	2347	70	2351	69
2357	69	2371	70	2377	69	2381	69	2383	71	2389	69
2393	70	2399	70	2401	70	2411	71	2417	69	2423	71
2437	71	2441	73	2447	71	2459	70	2467	71	2473	72
2477	71	2503	70	2521	70	2531	71	2539	72	2543	72
2549	71	2551	71	2557	71	2579	72	2591	71	2593	72
2609	71	2617	72	2621	72	2633	73	2647	72	2657	73
2659	73	2663	72	2671	72	2677	73	2683	73	2687	72
2689	72	2693	72	2699	72	2707	73	2711	73	2713	72
2719	73	2729	73	2731	74	2741	73	2749	73	2753	74

Table 1. Continue 2

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
2767	73	2777	74	2789	74	2791	74	2797	73	2801	75
2803	74	2809	74	2819	74	2833	74	2837	75	2843	75
2851	75	2857	74	2861	74	2879	74	2887	76	2897	75
2903	74	2909	75	2917	75	2927	75	2939	76	2953	77
2957	76	2963	75	2969	75	2971	76	2999	76	3001	76
3011	75	3019	77	3023	76	3037	76	3041	75	3049	75
3061	76	3067	76	3079	78	3083	77	3089	76	3109	76
3119	77	3121	77	3125	78	3137	77	3163	78	3167	77
3169	77	3181	79	3187	77	3191	78	3203	77	3209	77
3217	78	3221	78	3229	77	3251	79	3253	78	3257	77
3259	78	3271	79	3299	79	3301	78	3307	78	3313	78
3319	79	3323	79	3329	80	3331	79	3343	78	3347	80
3359	78	3361	80	3371	79	3373	79	3389	80	3391	79
3407	80	3413	80	3433	80	3449	80	3457	80	3461	80
3463	80	3467	79	3469	80	3481	81	3491	80	3499	80
3511	80	3517	80	3527	80	3529	82	3533	80	3539	82
3541	80	3547	80	3557	82	3559	81	3571	81	3581	81
3583	80	3593	81	3607	83	3613	81	3617	81	3623	82
3631	81	3637	82	3643	82	3659	82	3671	83	3673	82
3677	82	3691	83	3697	83	3701	82	3709	83	3719	82
3721	82	3727	82	3733	82	3739	83	3761	82	3767	83
3769	83	3779	85	3793	83	3797	83	3803	82	3821	83
3823	82	3833	84	3847	83	3851	84	3853	82	3863	83
3877	84	3881	84	3889	83	3907	85	3911	84	3917	83
3919	83	3923	84	3929	84	3931	84	3943	84	3947	84
3967	84	3989	85	4001	85	4003	84	4007	85	4013	85
4019	86	4021	84	4027	84	4049	85	4051	86	4057	85
4073	85	4079	86	4091	85	4093	86	4096	86	4099	86
4111	86	4127	86	4129	86	4133	85	4139	86	4153	86
4157	86	4159	86	4177	87	4201	85	4211	87	4217	85
4219	87	4229	86	4231	87	4241	86	4243	86	4253	86
4259	88	4261	87	4271	86	4273	87	4283	87	4289	86
4297	87	4327	88	4337	88	4339	86	4349	89	4357	87
4363	87	4373	87	4391	87	4397	88	4409	88	4421	87
4423	90	4441	87	4447	88	4451	88	4457	87	4463	88
4481	87	4483	88	4489	89	4493	88	4507	89	4513	88
4517	88	4519	89	4523	89	4547	88	4549	90	4561	89
4567	89	4583	89	4591	89	4597	89	4603	90	4621	89

Table 1. Continue 3

q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$	q	$n_q^{\mathrm{L}}(4,3)$
4637	89	4639	89	4643	90	4649	89	4651	89	4657	90
4663	90	4673	90	4679	92	4691	90	4703	89	4721	90
4723	90	4729	90	4733	90	4751	90	4759	90	4783	90
4787	89	4789	89	4793	89	4799	91	4801	92	4813	92
4817	89	4831	92	4861	91	4871	90	4877	92	4889	92
4903	91	4909	91	4913	91	4919	90	4931	91	4933	91
4937	90	4943	91	4951	91	4957	90	4967	91	4969	91
4973	91	4987	90	4993	92	4999	92	5003	92	5009	92
5011	93	5021	91	5023	92	5039	93	5041	91	5051	91
5059	92	5077	91	5081	92	5087	92	5099	94	5101	92
5107	93	5113	94	5119	91	5147	92	5153	93	5167	94
5171	93	5179	93	5189	93	5197	93	5209	93	5227	92
5231	94	5233	93	5237	93	5261	93	5273	94	5279	95
5281	94	5297	94	5303	95	5309	94	5323	93	5329	94
5333	94	5347	94	5351	95	5381	94	5387	94	5393	95
5399	95	5407	95	5413	94	5417	94	5419	95	5431	95
5437	93	5441	94	5443	94	5449	93	5471	94	5477	94
5479	95	5483	95	5501	96	5503	95	5507	94	5519	96
5521	95	5527	96	5531	95	5557	94	5563	95	5569	95
5573	95	5581	94	5591	96	5623	97	5639	96	5641	97
5647	97	5651	97	5653	97	5657	97	5659	96	5669	96
5683	98	5689	96	5693	97	5701	96	5711	96	5717	97
5737	96	5741	95	5743	97	5749	97	5779	96	5783	96
5791	97	5801	98	5807	96	5813	97	5821	97	5827	97
5839	98	5843	97	5849	96	5851	97	5857	97	5861	97
5867	97	5869	98	5879	97	5881	98	5897	97	5903	97
5923	97	5927	97	5939	98	5953	98	5981	98	5987	100
6007	98	6011	98	6029	97	6037	98	6043	99	6047	98
6053	99	6067	99	6073	99	6079	98	6089	99	6091	98
6101	98	6113	99	6121	99	6131	98	6133	97	6143	100
6151	98	6163	99	6173	99	6197	100	6199	100	6203	98
6211	100	6217	101	6221	100	6229	99				

Table 2. Lengths of $[n_q^{\mathrm{L}}(5,3), n_q^{\mathrm{L}}(5,3)-5,5]_q3$ leximatrix codes (quasi-perfect Almost MDS codes), $3 \le q \le 761$

q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$	q	$n_q^{\mathrm{L}}(5,3)$
3	11	4	10	5	11	7	16	8	17	9	19	11	22
13	24	16	28	17	28	19	31	23	36	25	37	27	40
29	43	31	46	32	46	37	51	41	55	43	56	47	60
49	61	53	66	59	70	61	73	64	77	67	79	71	82
73	84	79	88	81	88	83	90	89	96	97	101	101	104
103	107	107	109	109	111	113	112	121	119	125	123	127	123
128	124	131	127	137	130	139	133	149	142	151	141	157	146
163	149	169	151	167	150	173	156	179	158	181	159	191	166
193	166	197	171	199	172	211	180	223	185	227	186	229	188
233	191	239	195	241	197	243	198	251	203	256	205	257	207
263	208	269	214	271	213	277	215	281	218	283	221	289	226
293	227	307	232	311	234	313	236	317	237	331	245	337	248
343	253	347	257	349	255	353	256	359	260	361	260	367	265
373	266	379	274	383	272	389	275	397	280	401	282	409	284
419	292	421	290	431	297	433	299	439	301	443	304	449	309
457	311	461	311	463	309	467	314	479	320	487	324	491	324
499	328	503	330	509	334	512	334	521	339	523	341	529	344
541	348	547	349	557	353	563	360	569	364	571	362	577	365
587	371	593	374	599	375	601	376	607	376	613	380	617	384
619	382	625	385	631	387	641	393	643	398	647	396	653	399
659	402	661	401	673	407	677	407	683	411	691	416	701	417
709	424	719	427	727	430	729	430	733	429	739	431	743	436
751	439	757	440	761	443								