Conformal model of hypercolumns in V1 cortex and Möbius group. Application to the visual stability problem

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1 Introduction

We present a conformal spherical model of hypercolumns of primary visual cortex V1, which is a modification of the Bressloff- Cowan Riemannian spherical model and is closely related to the Sarti-Citti-Petit symplectic model of V1 cortex. Application to visual stability problem will be considered.

D. Hubel and T. Wiesel put forward the idea that the visual cortex should be viewed as a fiber bundle over the retina R. Fiber of the bundle corresponds to different internal parameters (orientation, spatial frequency, ocular dominance, direction of motion, curvature, etc.) that affect the excitation of visual neurons. N.V. Swindale [?] estimated the dimension of the fibers (= the number of internal parameters) as 6-7 or 9-10.
In 1989 , W. Hoffman [?] stated that the primary visual cortex is a contact bundle.

Following the idea by Hubel and Wiesel, J.Petitot [?] proposed a contact model of V1 cortex as the contact bundle $\pi : F \to R$ of orientations (directions) over the retina R (identified with the Euclidean plane $R = \mathbb{R}^2$). The manifold F has coordinates (x, y, θ) where $(x, y) \in \mathbb{R}^2$ and the orientation θ is the angle between the tangent line to a contour in retina and the axis $0x$. The manifold F is identified with the bundle of (oriented) orthonormal frames and with the group $SE(2) = SO(2) \cdot \mathbb{R}^2$ of (unimodular) Euclidean isometries.
The basic assumption is that simple neurons are parametrized by points of $F = SE(2)$. More precisely, the simple neuron, associated to a frame $f \in F$, is working as the mother Gabor filter in the Euclidean coordinates defined by the frame f.
Note that in this model, "points" of retina correspond to pinwheels, that is, singular columns of cortex, which contains simple neurons of any orientation. Recall that all simple neurons of a regular column act as (almost) identical

Gabor filters with (almost) the same receptive field D and they fire only if the contour on the retia, which cross D, has an appropriate orientation θ.

Recently, this model (with an appropriate sub-Riemannian metric) had been successfully applied by B. Franceschiello, A. Mashtakov, G. Citti and A. Sarti for explanation of some optical illusions.

The contact model had been extended by Sarti, Citti and Petitot [**?**] to a symplectic model, with two-dimensional fiber, associated with the orientation θ and the scaling σ. In this model, simple cells are parametrized by conformal frames or points of the conformal group $Sim(E^2) = \mathbb{R}^+ \cdot SE(2)$. Again, the simple neuron, associated with a frame f acts as Gabor filter, written w.r.t. coordinates associated with f.

P. Bressloff and J. Cowan [**?**] proposed a Riemannian spherical model of a hypercolumn H, associated with the orientation θ and spatial frequency p. They assume that a hypercolumn H is associated with two pinwheels S, N , which correspond to minimum and maximum of the spatial frequency. Simple neurons are parametrized by θ and normalised spatial frequency $\sigma \in [-\pi/2, \pi/2]$. More precisely, this means that the simple neuron $n(\theta, \sigma)$ is fired if a stimulus has the orientation θ and the normalised spatial frequency σ. The exception are simple neurons from singular columns, which corresponds to South and North poles S, N and have spatial frequency $\sigma = -\pi/2$ and , respectively, $\pi/2$. They contain simple neurons of any orientation and the longitude coordinate θ is not defined for them.

We present a modification of this model, based on the assumption that a hypercolumn H is a conformal sphere. Simple neurons of H are working as the mother Gabor filter with respect to conformal coordinates, obtained from some standard coordinates by transformations from the Möbius group $G = SL(2.\mathbb{C})$.
This corresponds to the Cartan approach to conformal geometry, bases on the construction of so-called Cartan connection. In the case of conformal sphere, the Cartan connection is the principal bundle $G = Sl(2, \mathbb{C}) \to S^2 = G/Sim(E^2)$ with the Maurer-Cartan form $\mu : TG \to \mathfrak{sl}(2, \mathbb{C})$ (which identifies tangent spaces T_gG with the Lie algebra $\mathfrak{sl}(2.\mathbb{C})$). Moreover, points of the sphere (which correspond to columns of the hypercolumn H) ara parametrised by the stability subgroups. Remark that a hypercolumn in the conformal model can be considered as the Tits model of the conformal sphere (where points are defined as stability subgroups).

We show that in a neighborhood of each pinwheel, the conformal model reduces to the symplectic model of Sarti, Citti and Petitot.
Application of this model to the problem of visual stability is considered. The visual stability problem consists in explanation how we perceive stable objects as stable in spite of change their retina images caused by eyes rotation.

1.1 Riemannian spinor model of conformal sphere

To describe our conformal model of hypercolumns, which is a conformal modification of the model by Bressloff andf Cowanm, we recall Riemann spinor model of conformal sphere as Riemann sphere $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with two distinguished points $S = 0$ and $N = \infty$ (which correspond to two pinwheels of the hypercolumn) and complex coordinate $z \in S^2 \setminus N$ and $w = \frac{1}{z} \in S^2 \setminus S$. The group $G = SL(2, \mathbb{C})$ acts on S^2 as conformal group by fractional-linear transformations

$$z \mapsto Az = \frac{az + b}{cz + d}, \; a, b, c, d \in \mathbb{C}, \det A = 1$$

Remark that this group is acts non-effectively, and the effectively acts its quotient $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\{\pm \mathrm{Id}\}$, which is isomorphic to the Lorentz group $SO^0(1, 3)$.

Denote by

$$G = G^- \cdot G^0 \cdot G^+ = \begin{pmatrix} 1 & 0 \\ \mathbb{C} & 0 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}, a \in \mathbb{C}^*$$

the Gauss decomposition. Then the stability subgroups G_S, G_N of points $S = 0$ and $N = \infty$ are $B_\mp = G_0 \cdot G^\mp \simeq Sim(E^2) = CO_2 \cdot \mathbb{R}^2$. As a homogeneous manifold, the sphere is $S^2 = G/B_\mp = SL_2(\mathbb{C})/Sim(E^2)$.

1.2 Conformal spherical model of hypercolumns

We present a conformal modification of the Bressloff-Cowan model. We assume that a hypercolumn associated with two pinwheels N, S is a conformal sphere with spherical coordinates θ, σ. Simple neurons are parametrized by the conformal Möbius group $G \simeq SL(2, \mathbb{C})$, hence depends of 6 parameters.

More precisely, each simple neuron acts as the mother Gabot filtder w.r.t. conformal coordinated, obtained from the standard coordinates by a conformal transformation from G.
We show that in a small neighborhood D_S of the South pole , responsible for perception of low frequency stimuli, the model reduces to the symplectic model by Sarti, Citti, Petitot.
Similarly, a small neighborhood D_N of the north pole N, responsible for perception of higher frequency stimuli, is identify with another copy of the symplectic model. The identification is realised by the stereographic projections from North and , respectively , South pole.

1.3 Relation with symplectic Sarti-Citti-Petitot model

Using stereographic projections, we show that Sarti-Citti-Petitot model is an approximation of the conformal spherical model in neighborhood of the pinwheels N, S.
Let $\sigma_N : S^2 \to T_S S^2 = E^2$ be the stereographic projection from the North pole N to the tangent space at the South pole. The transitive action of $G_N \simeq \mathrm{Sim}(E^2)$ on $S^2 \setminus N$ corresponds to the the action of $G_N = B_+ = G^0 \cdot G^+$ as the homothety group on $T_S S^2 = E^2$.

More precisely, the subgroup G^+ acts on $T_S S^2 = E^2$ by parallel translations,
the group $SO_2 = \{\mathrm{diag}(e^{i\alpha}, e^{-i\alpha})\}$ acts by rotations
the group $\mathbb{R}^+ = \{\mathrm{diag}(\lambda, \lambda^{-1})\}$ acts by homotheties.
The subgroup $G^+ \subset G_S$ acts trivially on $T_S S^2 = E^2$. We conclude:
Simple neurons in a neighbourhood of the South pole depends only on 4-parameters and are parametrized by the points of the group $G_N = Sim(E^2)$ of similarities according to the Sarti-Citti-Petitot model.

1.4 Principle of invariancy

We will state the following obvious general principle of invariancy:
Let G be a group of transformations of a space V and $\mathcal{O} = Gx$ an orbit.
Principle of invariancy The information, which observers, distributed along the orbit $\mathcal{O}$, send to some center is invariant w.r.t. the group G.

Application. The information about low spatial frequency stimulus, coded in simple cells neat South pinwheel and parametrized by the group

G_N, is invariant w.r.t. G_N.
Similarly, the information about high spatial frequency stimulus is invariant w.r.t. the group G_S.
The information about local stimulus, coded in simple cells of a hypercolumn and parametrized by the conformal group $G = SL_2(\mathbb{C})$, is invariant with respect to the conformal group G.
In particular, if we assume that the remapping of the retina image after each saccade is carried out by a conformal transformation, then the simple neurons of a hypercolumns contain information, which is sufficient to identify retina images before and after saccade.

In the next section, we justify the conjecture that the remapping after each saccades is described by a conformal transformation of the retina image.

1.5 The cental projection

Let $M \subset E^3$ be a surface , whose points are sources of diffuse reflected light. We assume that all the light rays emitted from a point $A \in M$ carry the same energy density $E(A)$. The retina image of the surface is described by the central projection of the surface to the eye sphere with respect to the nodal point F of the eye.
The **central projection** of a surface $M \subset E^3$ onto a sphere $S^2 \subset E^3$ with center $F \in S^2$ is defined by

$$\varphi : M \ni A \to \bar{A} = \ell_{AF} \cap S^2 \subset S^2$$

where $\ell_{AF} \cap S^2$ is the second point of intersection of the ray ℓ_{AF}, which goes through the point F, with the sphere.
We may assume that the central projection is a (local) diffeomorphism and that the energy density $I(\bar{A})$ at a point $\bar{A}$ is proportional to $E(A)$. So the input function $I : S^2 \to \mathbb{R}$ contains information about illumination of points of the surface M.
We are assuming that the point F belongs to the sphere. It is not completely true for the case of the eye, where the point F, called the **node point** or **optical center**, is located inside the eye ball, but very close to the eye sphere.

Consider the retina image of a an object , e.g. a plane Π, described by the central projection with respect to the node point F on the eye sphere (the boundary of eyeball). With respect to the retina coordinated (fixed w.r.t. to eye sphere S^2) the rotation R_α of the sphere D^2 on an angle α w.r.t. some axis (say $0z$) corresponds to rotation $R_\alpha^{-1} = R_{-\alpha}$ of the space E^3 which transforms the plane Π onto the plane $\Pi' = \Pi_{R_{-\alpha}n}$. The problem of remapping is to identify planes Π, Π' such that their retina images $\pi_F(|Pi), \pi_F(\Pi')$ will be related by a simple way.

An important idea about remapping had been proposed by art historian Ernst Gombrich, see [**?**], [**?**]. He supposes that for remapping, the brain control information about new position of only 3-4 salient points of the scene. It is sufficient to reconstruct the new retina image of all scene, using redundancy in the scene and previous experience with the given type of environment.
He claimed that **"Only a few (3-4) salient stimuli are contained in the trans-saccadic visual memory and update."**
We will propose a realisation of this idea , based on assumption that remapping of retina images after saccades are described by conformal transformations. First of all, we recall some basic facts about conformal geometry of sphere.

1.6 Möbius projective model of conformal sphere

If an inertial coordinate system is fixed, the Minkowski space-time $M^{1,3}$ is identified with the vector space

$$M^{1,3} = \mathbb{R}^{1,3} = V = \mathbb{R}e_0 + E^3 \ni X = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3 = (x^0, \vec{x})$$

with the Lorentz scalar product $g(X,Y) = -x^0 y^0 + \vec{x} \cdot \vec{y}$.
The light cone at 0 is the set $V_0 = \{X \in V,\ g(X,X) = 0\}$ of isotropic vectors.
Up to scalung, there are three orbits of the (connected) Lorentz group $G = SO^0(V) = SO^0_{1,3}$:
$V_T = Ge_0 = G/SO_3$ - Lobachevski space,
$V_S = Ge_1 = G/SO_{1,2}$- De Sitter space ,
$V_0 = Gp = G/SE(2)$ - isotropic (light) cone, where $SE(2) = SO_2 \cdot \mathbb{R}^2$.
Projectivisation of these orbits gives three G-orbits in the projective space $P^3 = PV$:
The ball $B^3 = PV_T \simeq V_T$, the conformal sphere (projective quadric) $S^2 = Q = PV_0 = G/Sim(E^2) = G/(\mathbb{R}^+ \cdot SE(2))$;
and the exterior of the ball $PV_S \simeq V_S$, The action of G in the projective quadric Q is conformal.

1.7 Projective duality between points and planes in PV

We have the following correspondence between projective points and planes in the projectividation PV of the Minkowski space, define by the Minkowski metric:

$$\begin{array}{ccccc} V_T \ni n & \Leftrightarrow & \Pi_n = Pn^{\perp} & \Pi_n \cap Q = \emptyset \\ PV_0 = Q \ni [p] = F & \Leftrightarrow & \Pi_A = p^{\perp} & \Pi_F \cap Q = F \\ PV_S \ni m & \Leftrightarrow & \Pi_m = P_m^{\perp} & \Pi_m \cap Q = S^1. \end{array}$$

Euclidean interpretation
Let $n = e_0 \in V_T$, (or any other vector from V_T) and $E^3_{e_0} = e_0 + e_0^{\perp}$ the Euclidean hypeplane. Then $Q = PV_0$ is identified with $S^2(e_0) := Q \cap E^3_{e_0}$, and projective planes $\Pi_v = Pv^{\perp}$ are identified with the Euclidean planes $v^{\perp} \cap E^3_{e_0}$.

Lemma The stability group $G_F = Sim(E^2)$ of a point $F \in Q$ acts transitivity on V_T, hence, on the set of projective planes $\{\Pi_n,\, n \in V_T\}$ which do not intersect the quadric Q.

1.8 Conjecture: remapping is defined by a conformal transformation

Due to Lemma, there is a Lorentz transformation $L \in SO(V)_F$ (which fix the point F) and transforms the plane $\Pi = \Pi_n$ into any other plane $\Pi' = \Pi_{n'}$ (which does not intersect Q).
If the brain identify the planes Π, Pi' using this Lorentz transformation, the retina images $\pi_F(\Pi), \pi_F(\Pi') \subset S^2$ before and after saccade are related by the conformal transformation $L|S^2$.
Such conformal transformation (and the Lorentz transformation L) is determined by the images of three points of the sphere $S^2 = Q$, which is consistent with Etcetera Principle by Gombrich.
Then the problem of stability reduces to solution of the classical problem of conformal geometry - description of a curve on the conformal sphere up to a conformal transformation (the conformal generalisation of the Frenet theory). It was solved by Fialkov, Sulanke, Sharp, Shelechov and others. Recently V. Lychagin and N. Konovenko [?] give a new elegant solution of this problem in terms of differential invariants.

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