DECOMPOSABLE $(4,7)$ SOLUTIONS IN ELEVEN-DIMENSIONAL SUPERGRAVITY

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ABSTRACT. Consider an oriented four-dimensional Lorentzian manifold $(\widetilde{M}^{3,1},\widetilde{g})$ and an oriented seven-dimensional Riemannian manifold (M^7,g). We describe a class of decomposable eleven-dimensional supergravity backgrounds on the product manifold $(\mathcal{M}^{10,1}=\widetilde{M}^{3,1}\times M^7, g_{\mathcal{M}}=\widetilde{g}+g)$, endowed with a flux form given in terms of the volume form on $\widetilde{M}^{3,1}$ and a closed 4-form F^4 on M^7. We show that the Maxwell equation for such a flux form can be read in terms of the co-closed 3-form $\phi=\star_7 F^4$. Moreover, the supergravity equation reduces to the condition that $(\widetilde{M}^{3,1},\widetilde{g})$ is an Einstein manifold with negative Einstein constant and (M^7,g,F) is a Riemannian manifold which satisfies the Einstein equation with a stress-energy tensor associated to the 3-form ϕ. Whenever this 3-form is generic, the Maxwell equation induces a weak G_2-structure on M^7 and then we obtain decomposable supergravity backgrounds given by the product of a weak G_2-manifold (M^7,ϕ,g) with a Lorentzian Einstein manifold $(\widetilde{M}^{3,1},\widetilde{g})$. We classify homogeneous 7-manifolds $M^7=G/H$ of a compact Lie group G and indicate the cosets which admit an invariant or non-invariant G_2-structure, or even no G_2-structure. Then we construct examples of compact homogeneous Riemannian 7-manifolds endowed with non-generic invariant 3-forms which satisfy the Maxwell equation, but the construction of decomposable homogeneous supergravity backgrounds of this type remains an open problem.

1. Introduction

Ten-dimensional supersymmetric string theories and their eleven-dimensional unified analogue, called M-theory, are some of the most promising approaches to a consistent model for the unification of fundamental forces of nature. Indeed, supergravity theories merge the theory of general relativity with supersymmetry and are crucial for understanding the dynamics of massless fields in string theories, since they determine the appropriate backgrounds in which strings propagate (see [BBS07] for a comprehensive survey). Nowadays there are several known consistent supergravity theories in different dimensions. For example, in dimension ten there are at least 5 different types of string theories, namely Type I, Type IIA and IIB and some heterotic $\mathrm{E}_8\times\mathrm{E}_8$ and SO_{32} theories. In dimension eleven physicists are concerned with the (weak) coupling limits of these theories via T-duality and other kinds of dualities that yield a unique eleven-dimensional M-theory.

The eleven-dimensional supergravity theory has as bosonic fields some Lorentzian metric $g_{\mathcal{M}}$ and a 3-form potential A with 4-form field strength $\mathcal{F}=\mathrm{d}\,A$, the so-called *flux form*, satisfying the supergravity field equations (with zero gravitino):

$$\left\{\begin{array}{rcll} \mathrm{d}\,\mathcal{F} & = & 0, & \textit{Closure} \quad (\mathscr{C}),\\ \mathrm{d}\star\mathcal{F} & = & (1/2)\mathcal{F}\wedge\mathcal{F}, & \textit{Maxwell} \quad (\mathscr{M}),\\ \mathrm{Ric}^{g_{\mathcal{M}}}(X,Y) & = & (1/2)\langle X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}\rangle-(1/6)g_{\mathcal{M}}(X,Y)\|\mathcal{F}\|^2, & \textit{Einstein} \quad (\mathscr{E}).\end{array}\right.$$

Here, $\mathrm{d}\equiv\mathrm{d}^{g_{\mathcal{M}}}$ is the exterior derivative of differential forms on the Lorentzian manifold $(\mathcal{M}^{10,1},g_{\mathcal{M}})$, $\mathrm{Ric}^{g_{\mathcal{M}}}$ is the Ricci tensor of the Levi-Civita connection on $\mathcal{M}$, and

$$\langle X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}\rangle=\frac{1}{3!}g_{\mathcal{M}}(X\lrcorner\mathcal{F}, Y\lrcorner\mathcal{F}),\qquad \|\mathcal{F}\|^2=\frac{1}{4!}g_{\mathcal{M}}(\mathcal{F},\mathcal{F}).$$

The second equation is referred to as the *Maxwell-like equation* and the third one as the *supergravity Einstein equation.* Note that usually one asks from $\mathcal{M}^{10,1}$ to be also spin, but in this work we are not interested in the supersymmetries of the model, so we do not pay much attention to this condition.

Classification of supergravity backgrounds, i.e. Lorentzian manifolds $(\mathcal{M}^{10,1}, g_{\mathcal{M}}, \mathcal{F}^4)$ solving the above system, can be considered in several different contexts. For example, besides the construction of Killing superalgebras (see [FO'F01, FO'FP03]), there are also methods based on the theory of G-structures (see for example [DfNP86, BhJ03, GPR05, MaC05, Wit10]). In this paper we are concerned with eleven-dimensional oriented Lorentzian manifolds $\mathcal{M} \equiv \mathcal{M}^{10,1} := \widetilde{M}^{3,1} \times M^7$ given by a product of a four-dimensional oriented Lorentzian manifold $(\widetilde{M} \equiv \widetilde{M}^{3,1}, \widetilde{g})$ and a seven-dimensional (compact) oriented Riemannian manifold $(M \equiv M^7, g)$ and analyse the supergravity equations from a purely geometric perspective. In particular, we consider the following type of flux forms on $\mathcal{M}$

$$\mathcal{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} + F^4, \qquad (*)$$

where F^4 is a closed 4-form on M and $f \in \mathbb{R}$ is assumed to be a constant. Solutions of eleven-dimensional supergravity for such 4-forms and with respect to the product metric $g_{\mathcal{M}} = \widetilde{g} + g$, will be called $(4,7)$-*decomposable supergravity backgrounds.*

For this specific Ansatz the core observation (see Proposition 2.2) is that the Maxwell equation $(\mathscr{M})$ is equivalent to the equation

$$\mathrm{d} \star_7 F^4 = f \cdot F^4,$$

which by setting $\phi := \star_7 F^4$ can be rewritten as

$$\mathrm{d}\, \phi = f \star_7 \phi. \qquad (**)$$

Moreover, the closure condition $(\mathscr{C})$ of $\mathcal{F}$ can be rephrased as $\mathrm{d} \star_7 \phi = 0$. For brevity, 3-forms on M^7 satisfying the last two conditions for some constant $f \in \mathbb{R}$, will be referred to as *special 3-forms.* In these terms one has that the specific flux form $\mathcal{F}$ is a solution of the closure condition $(\mathscr{C})$ and the supergravity Maxwell equation $(\mathscr{M})$ if and only if the associated 3-form $\phi := \star_7 F^4$ on M^7 is special.

Turning now to the corresponding supergravity Einstein equation $(\mathscr{E})$, we conclude that the four-dimensional Lorentzian manifold $(\widetilde{M}, \widetilde{g})$ must be Einstein with negative Einstein constant $\Lambda := -\frac{1}{6}\left(2f^2 + \|\phi\|^2\right)$ (Proposition 2.8). Moreover, we see that the Ricci tensor of (M, g) must satisfy the equation

$$\mathrm{Ric}^g(X,Y) = \frac{1}{6} g(X,Y)\left(f^2 + 2\|\phi\|^2_M\right) + q_\phi(X,Y), \qquad (***)$$

where $q_\phi(X,Y)$ is the symmetric bilinear form defined by $q_\phi(X,Y) := -\frac{1}{2}\langle X \lrcorner \phi, Y \lrcorner \phi\rangle_M$. We then proceed with a description of some special situations arising by focussing on $(***)$. In particular, we examine the following basic classes of special 3-forms on (M,g):

- the trivial 3-form, i.e. $\phi = 0$ (and hence $F = 0$) but with $f \neq 0$,
- non-zero harmonic 3-forms, i.e. $\phi \neq 0$, $f = 0$,
- non-harmonic 3-forms, i.e. $\phi \neq 0$, $f \neq 0$.

For these three cases we analyse the supergravity equations and describe solutions. In particular, for the more general third case the construction of $(4,7)$-decomposable supergravity backgrounds relies on the theory of G_2-structures (see also [AW01, BDSf02, BhJ03, AF03, HM05, Df11] for the role of G_2-geometries in M-theory). Here, we show that whenever $\phi := \star_7 F^4$ is a co-closed *generic* 3-form on M^7 satisfying equation $(**)$ for $f \neq 0$, i.e. a *generic special 3-form with* $f \neq 0$, which is equivalent to say that ϕ induces a weak G_2-structure on M, then the pair

$$(\mathcal{M} = \widetilde{M} \times M,\ g_{\mathcal{M}} = \widetilde{g} + g),$$

where g is the Einstein metric induced by ϕ, provides $(4,7)$-decomposable supergravity solutions. In particular, we obtain that

THEOREM A. *Assume that the product* $(\mathcal{M} = \widetilde{M} \times M, g_{\mathcal{M}} = \widetilde{g} + g)$ *is endowed with the 4-form* $\mathcal{F}^4 := f \cdot \mathrm{vol}_{\widetilde{M}} + F^4$, *for some constant* $0 \neq f \in \mathbb{R}$ *and some closed 4-form* $F^4 \in \Omega^4_{\mathrm{cl}}(M)$ *on* M, *such that* $\phi := \star_7 F^4$ *is a generic 3-form on* M. *Then* $(\mathcal{M}, g_{\mathcal{M}}, \mathcal{F}^4)$ *gives rise to a* $(4,7)$-*decomposable supergravity background if and only if* $(M, g, \phi := \star_7 F^4)$ *is a weak* G_2-*manifold and* $(\widetilde{M}, \widetilde{g})$ *is Einstein with negative Einstein constant. In particular,* f *takes the values* $f = \pm 2$.

Weak G_2-structures are spin 7-manifolds (M, g, ϕ) endowed with a generic 3-form ϕ satisfying the differential equation $\mathrm{d}\,\phi = \lambda \star_7 \phi$, for some non-zero constant λ. Such G_2-structures are extremely interesting in theoretical and mathematical physics, since they are manifolds admitting non-trivial solutions of the Killing spinor equation (see [FKMS97]). We should emphasize that our approach to Theorem A does *not* take into account the theory of Killing superalgebras, i.e. we reach Theorem A by solving only the zero gravitino supergravity equations, independently of the supersymmetries that preserves the corresponding model $\mathcal{M}$. Moreover, our Ansatz serves well the purpose of finding obstructions to the existence of $(4,7)$-decomposable supergravity backgrounds. For example, whenever $\phi = \star_7 F^4$ is a *generic special 3-form with* $f = 0$, which means that it induces a *parallel* G_2-*structure* on M, we obtain the following non-existence result.

COROLLARY A. *If* $f = 0$ *and* $\phi := \star_7 F^4$ *is a generic 3-form on* M^7, *where* $F^4 \in \Omega^4_{\mathrm{cl}}(M^7)$, *then the closure condition* $(\mathscr{C})$ *and the Maxwell equation* $(\mathscr{M})$ *for our Ansatz* $(*)$, *imply that* ϕ *is* ∇^g-*parallel, i.e.* ϕ *induces a parallel* G_2-*structures and hence* (M, g) *is Ricci flat. In this case the eleven-dimensional Lorentzian manifold* $(\mathcal{M} = \widetilde{M} \times M, g_{\mathcal{M}} = \widetilde{g} + g, \mathcal{F}^4)$ *does not give rise to a* $(4,7)$-*decomposable supergravity background.*

The rest of the article is devoted to the homogeneous case, where the calculations related to the supergravity equations become more attractive, since the tensor fields $g_{\mathcal{M}}$ and $\mathcal{F}^4$ are invariant under the action of a Lie group. In this case we obtain a series of examples serving Theorem A, and these are based on the the classification of compact homogeneous weak G_2-manifolds and homogeneous Lorentz Einstein 4-manifolds, given in [FKMS97] and [Km01, FeR06], respectively. Then we examine the supergravity equations for invariant *non-generic 3-forms* $\phi := \star_7 F^4$. To this end, we classify all almost effective seven-dimensional homogeneous manifolds $M^7 = G/H$ of a compact Lie group G (see Table 2 and Theorem 4.4). This extends the classification of simply-connected homogeneous 7-manifolds $M^7 = G/H$ of a semisimple compact group G, which was used for classifying homogeneous Einstein 7-manifolds, see [CRW84, Nk04]. In combination with the classification of compact homogeneous 7-manifolds admitting invariant G_2-structures given in [LM10, Rd10], we obtain the complete list of all compact (almost) effective homogeneous 7-manifolds which admit a G_2-structure but no invariant G_2-structure (and hence no invariant spin structure, see Theorem 4.6). We then describe all invariant special 3-forms ϕ (i.e. solutions of Maxwell equation) on the non-spin manifold $\mathbb{C}P^2 \times \mathrm{S}^3 = \mathrm{SU}_3 / \mathrm{U}_2 \times \mathrm{SU}_2$. We also discuss the case of the Lie group $\mathrm{S}^3 \times \mathrm{T}^4 = \mathrm{SU}_2 \times \mathrm{T}^4$. In both cases we show that there are invariant special 3-forms which are not generic.

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2. 11D supergravity backgrounds of the form $\mathcal{M}^{10,1} = \widetilde{M}^{3,1} \times M^7$

We begin by fixing some conventions, relevant to our subsequent computations.

Conventions. Consider an n-dimensional pseudo-Riemannian manifold (N, h) of signature (p, q). At any point $x \in N$, the tangent space $V := T_x N = \mathbb{R}^{p,q}$ $(n = p + q)$ is a pseudo-Euclidean vector space endowed with a non-degenerate inner product of signature

$$(p, q) = (n - q, q) = (+ \cdots +, - \cdots -).$$

When the signature is $(n, 0)$ (resp. $(n - 1, 1)$), then we say that (N, h) is a *Riemannian* (resp. *Lorentzian*) manifold. We shall denote by $\mathfrak{so}(V)$ the Lie algebra of skew-symmetric endomorphisms of V; for any $u, v \in V$ let $w \wedge u$ the skew-symmetric endomorphism on V, given by $(u \wedge v)(z) = h(v, z)u - h(u, z)v$. Hence, here we take the convention $\omega_1 \wedge \omega_2 := \omega_1 \otimes \omega_2 - \omega_1 \otimes \omega_2$ for any two elements $\omega_1, \omega_2 \in \bigwedge^1 T_x^* N$. The metric tensor h induces a metric in $\bigwedge^\bullet TN$ and its dual, namely

$$\langle \phi, \psi \rangle := \det(\langle \phi_i, \psi_j \rangle) = \frac{1}{k!} h(\phi, \psi),$$

for any decomposable k-vector $\phi = \phi_1 \wedge \ldots \wedge \phi_k$ and $\psi = \psi_1 \wedge \ldots \wedge \psi_k$. We choose a volume form $\mathrm{vol}^{(n)}$ normalised as $\langle \mathrm{vol}^{(n)}, \mathrm{vol}^{(n)} \rangle = (-1)^q$. Equivalently, if $\{e_1, \ldots, e_p, e_{p+1}, \ldots e_{p+q}\}$ is a pseudo-orthonormal frame with

$$h(e_i, e_j) = \delta_{ij}, \quad h(e_k, e_\ell) = -\delta_{k\ell}, \quad h(e_i, e_k) = 0, \quad \text{for} \;\; 1 \leq i, j \leq p, \;\; p + 1 \leq k, \ell \leq p + q,$$

then $\mathrm{vol}^{(n)}(e_1, e_2, \ldots, e_n) = 1$. The Hodge star operator is defined by $\phi \wedge \star\psi = \langle \phi, \psi \rangle \, \mathrm{vol}^{(n)}$ for any k-form ϕ and ψ. In particular, for any $\phi \in \bigwedge^k T_x^* N$ we have the identities

$$\star 1 = \mathrm{vol}^{(n)}, \quad \star \mathrm{vol}^{(n)} = (-1)^q, \quad \star\star\phi = (-1)^{k(n-k)+q}\phi,$$

and hence $\phi \wedge \psi = (-1)^{k(n-k)+q} \langle \phi, \star\psi \rangle \, \mathrm{vol}^{(n)}$, for any $\phi \in \bigwedge^k T_x^* N$ and $\psi \in \bigwedge^{n-k} T_x^* N$.

2.1. **Supergravity backgrounds of the form $\mathcal{M}^{10,1} = \widetilde{M}^{3,1} \times M^7$.** Let us consider an eleven-dimensional Lorentzian manifold $(\mathcal{M} \equiv \mathcal{M}^{10,1}, g_{\mathcal{M}})$ given by the product of a four-dimensional Lorentzian manifold $(\widetilde{M} \equiv \widetilde{M}^{3,1}, \widetilde{g})$ and a seven-dimensional Riemannian manifold $(M \equiv M^7, g)$,

$$(\mathcal{M}, g_{\mathcal{M}}) = (\widetilde{M} \times M, \;\; g_{\mathcal{M}} := \widetilde{g} + g). \tag{2.1}$$

We assume that both $(\widetilde{M}, \widetilde{g})$ and (M, g) are oriented with volume forms $\mathrm{vol}_{\widetilde{M}}$ and vol_M, respectively. Then, the volume form on $\mathcal{M}$ is given by $\mathrm{vol}_{\mathcal{M}} := \mathrm{vol}_{\widetilde{M}} + \mathrm{vol}_M$ and $\mathcal{M}$ is oriented as well. Since $\dim \widetilde{M} = 4$, notice that any 4-form on $\widetilde{M}^4$ is closed. We mention that we do *not* assume any *homogeneity condition* for the Lorentzian manifold $\mathcal{M} = \widetilde{M} \times M$. However, we will assume that M^7 is compact and that the flux 4-form is given by

$$\mathcal{F}^4 := f \cdot \mathrm{vol}_{\widetilde{M}} + F^4, \tag{2.2}$$

for some closed 4-form F^4 on M and a constant $f \in \mathbb{R}$. Note that the last condition is equivalent to say that $\widetilde{F}^4$ is co-closed, i.e. $\mathrm{d} \star_4 \widetilde{F}^4 = 0$, where $\star_4 : \Omega^k(\widetilde{M}) \to \Omega^{4-k}(\widetilde{M})$ is the Hodge star operator on $\widetilde{M}$. Indeed, $\star_4^2\big|_{\Omega^k} = (-1)^{k(4-k)+1} \mathrm{Id}_{\Omega^k}$, with $\star_4 \mathrm{vol}_{\widetilde{M}^4} = (-1)^q = -1$ (since $q = 1$), and hence the relation $\widetilde{F}^4 := f \cdot \mathrm{vol}_{\widetilde{M}}$ yields $\star_4 \widetilde{F}^4 = -f$. Next we shall call 4-forms of type (2.2) *decomposable*.

On the closure condition ($\mathscr{C}$) and the Maxwell equation ($\mathscr{M}$). Let us focus now on the closure condition ($\mathscr{C}$) and the Maxwell equation ($\mathscr{M}$). We denote the Hodge star operators on $\mathcal{M}$ and M as $\star_{11} : \Omega^k(\mathcal{M}) \to \Omega^{11-k}(\mathcal{M})$ and $\star_7 : \Omega^k(M) \to \Omega^{7-k}(M)$, respectively. We need the following elementary result (which makes sense, appropriately reformulated, for any pseudo-Riemannian metric).

Lemma 2.1. *Consider the Lorentzian manifold* $(\mathcal{M}^{10,1} = \widetilde{M}^{3,1} \times M^7, g_{\mathcal{M}} = \widetilde{g} + g)$ *and let* $\widetilde{\alpha} \in \Omega^k(\widetilde{M})$ *and* $\alpha \in \Omega^\ell(M)$ *be some differential forms of* $\widetilde{M}$ *and* M*, respectively. Then, since* $T\mathcal{M} = T\widetilde{M} \oplus TM$ *defines a decomposition of the tangent bundle of* $\mathcal{M}$*, the following holds:*
(1)

$$g_{\mathcal{M}}(\widetilde{\alpha} \wedge \alpha, \widetilde{\alpha} \wedge \alpha) = \frac{(k+\ell)!}{k!\ell!}\, \widetilde{g}(\widetilde{\alpha}, \widetilde{\alpha}) \cdot g(\alpha, \alpha)\,.$$

and consequently,

$$\langle \widetilde{\alpha} \wedge \alpha, \widetilde{\alpha} \wedge \alpha \rangle_{\mathcal{M}} = \langle \widetilde{\alpha}, \widetilde{\alpha} \rangle_{\widetilde{M}} \cdot \langle \alpha, \alpha \rangle_M, \quad \|\widetilde{\alpha}^k \wedge \alpha^\ell\|_{\mathcal{M}} = \|\widetilde{\alpha}^k\|_{\widetilde{M}} \cdot \|\alpha^\ell\|_M.$$

(2) The action of the Hodge star operator $\star_{11} : \Omega^r(\mathcal{M}) \to \Omega^{11-r}(\mathcal{M})$ *on* $\widetilde{\alpha}^k \wedge \alpha^\ell$ *reads as*

$$\star_{11}(\widetilde{\alpha} \wedge \alpha) = (-1)^{\ell(p-k)} \star_p \widetilde{\alpha} \wedge \star_{11-p}\alpha.$$

Now we are ready to prove that

Proposition 2.2. *For the 4-form on* $\mathcal{M} = \widetilde{M} + M$ *given by the Ansatz (2.2) with* $f \in \mathbb{R}$*, the closure condition* $(\mathscr{C})$ *and the Maxwell equation* $(\mathscr{M})$ *are simultaneously satisfied, if and only if*

$$\mathrm{d}\, F^4 = 0, \quad \textit{and} \quad \mathrm{d} \star_7 F^4 = f \cdot F^4. \tag{2.3}$$

In the case where $f = 0$*, then the equations* $(\mathscr{C})$ *and* $(\mathscr{M})$ *are simultaneously satisfied if and only if the 4-form* F^4 *on* M^7 *is closed and co-closed,* $dF^4 = \mathrm{d} \star_7 F^4 = 0$*.*

Proof. Let us compute $\star_{11}\mathcal{F}$. We write $\mathcal{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} \wedge 1 + F^4 \wedge \widetilde{1}$. Thus, by Lemma 2.1 and since $\omega_1 \wedge \omega_2 = (-1)^{st}\omega_2 \wedge \omega_1$ for some s-form ω_1 and t-form ω_2, we conclude that

$$\begin{aligned}
\star_{11}\mathcal{F}^4 &= \star_4(f \cdot \mathrm{vol}_{\widetilde{M}} \wedge 1) + \star_7(F^4 \wedge \widetilde{1}) = f \cdot \star_4(\mathrm{vol}_{\widetilde{M}} \wedge 1) + \star_7(\widetilde{1} \wedge F^4) \\
&= f \cdot [(-1)^{0(4-4)} \star_4 \mathrm{vol}_{\widetilde{M}} \wedge \star_7 1] + [(-1)^{4(4-0)} \star_4 \widetilde{1} \wedge \star_7 F^4] \\
&= f \cdot [-\widetilde{1} \wedge \mathrm{vol}_M] + \mathrm{vol}_{\widetilde{M}} \wedge \star_7 F^4 \\
&= -f \cdot \mathrm{vol}_M + \mathrm{vol}_{\widetilde{M}} \wedge \star_7 F^4,
\end{aligned}$$

where we used the identity $\star_4 \widetilde{1} = \mathrm{vol}_{\widetilde{M}}$. Thus $\star_{11}\mathcal{F} = -f \cdot \mathrm{vol}_M + \mathrm{vol}_{\widetilde{M}} \wedge \star_7 F^4$ and consequently $\mathrm{d} \star_{11}\mathcal{F} = \mathrm{vol}_{\widetilde{M}} \wedge \mathrm{d} \star_7 F^4$. We also compute $\mathcal{F} \wedge \mathcal{F} = 2 \cdot f \cdot \mathrm{vol}_{\widetilde{M}} \wedge F^4$. Therefore, for our Ansatz (2.2) the Maxwell equation $\mathrm{d} \star_{11}\mathcal{F} = \frac{1}{2}\mathcal{F} \wedge \mathcal{F}$ is equivalent to $\mathrm{vol}_{\widetilde{M}} \wedge \mathrm{d} \star_7 F^4 = f \cdot \mathrm{vol}_{\widetilde{M}} \wedge F^4$, and our assertion is immediate. □

For the following, let us denote the 3-form $\star_7 F^4$ by $\phi := \star_7 F^4$. Since the square of the star operator $\star_7$ acts by $\star_7^2|_{\Omega^p(M^7)} = (-1)^{p(7-p)} \mathrm{Id}_{\Omega^p(M^7)}$, we get that $\star_7\phi = \star_7^2 F^4 = (-1)^{4(7-4)} F^4 = F^4$. Thus, by Proposition 2.2 we deduce that

Corollary 2.3. *The Maxwell equation* $(\mathscr{M})$ *for the 4-form* $\mathcal{F}$ *given by (2.2), i.e. the second relation in (2.3), is equivalent to the equation*

$$\mathrm{d}\,\phi = f \star_7 \phi, \tag{2.4}$$

for the 3-form $\phi := \star_7 F^4$*. Moreover, the closure condition* $(\mathscr{C})$ *is equivalent to the relation*

$$\mathrm{d} \star_7 \phi = 0. \tag{2.5}$$

This motivates us to introduce the following definition.

Definition 2.4. A 3-form $\phi \in \Omega^3(M)$ on a Riemannian 7-manifold (M, g) is called *special* if it is co-closed ($\mathrm{d} \star_7 \phi = 0$) and satisfies the relation $\mathrm{d}\,\phi = f \star_7 \phi$ for some constant $f \in \mathbb{R}$.

In terms of special 3-forms, Corollary 2.3 reads as follows:

Corollary 2.5. *The 4-form $\mathcal{F} = f \cdot \mathrm{vol}_{\widetilde{M}} + F^4 \in \Omega^4_{\mathrm{cl}}(\mathcal{M})$ for some constant f and closed 4-form $F^4 \in \Omega^4_{\mathrm{cl}}(M^7)$, is a solution of Maxwell equation $(\mathscr{M})$ if and only if $\phi := \star_7 F$ is a special 3-form on M^7.*

On the Einstein supergravity equation $(\mathscr{E})$. For the computations related to the right hand side of the Einstein supergravity equation $(\mathscr{E})$ we use the following basic lemma.

Lemma 2.6. *Let ϕ be a k-form on a smooth pseudo-Riemannian manifold $(M^{p,q}, g)$ of signature (p, q) with $p + q = n$. When $1 \leq k \leq n - 1$, we have*

$$(-1)^q \langle X \lrcorner \star \phi, Y \lrcorner \star \phi \rangle = \langle \phi, \phi \rangle \langle X, Y \rangle - \langle X \lrcorner \phi, Y \lrcorner \phi \rangle \,, \qquad \text{for all vector fields } X \text{ and } Y. \tag{2.6}$$

When $k = n$, we have

$$\langle X \lrcorner \phi, Y \lrcorner \phi \rangle = \langle \phi, \phi \rangle \langle X, Y \rangle \,, \qquad \text{for all vector fields } X \text{ and } Y. \tag{2.7}$$

Proof. It suffices to prove (2.6) and (2.7) by taking X and Y to be basis elements at a point. Let us fix an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ with $\langle e_i, e_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq p$, and $\langle e_i, e_j \rangle = -\delta_{ij}$ for $p+1 \leq i, j \leq p+q$. Denote by $\{e^i\}_{i=1,\dots,n}$ the corresponding dual basis such that the volume form is given by $\mathrm{vol} = e^1 \wedge \ldots \wedge e^n$. For any $1 \leq k \leq n$, the k-forms $\{e^{i_1} \wedge \ldots \wedge e^{i_k}\}$ constitute a basis for $\bigwedge^k TM$ orthonormal with respect to the natural extension $\langle \cdot, \cdot \rangle$ of the metric , i.e.

$$\langle e^{i_1} \wedge \ldots \wedge e^{i_k}, e^{i_1} \wedge \ldots \wedge e^{i_k} \rangle = (-1)^u \,,$$

where u is the number of timelike 1-forms among the $\{e^i\}$. In the following discussion, $\{i_1, \dots, i_n\}$ will denote an even permutation of $\{1 \dots n\}$. For any $1 \leq k \leq n$, set $I = \{i_1, \dots, i_k\}$ and $J = \{i_{k+1}, \dots, i_n\}$ so that $I \cap J = \emptyset$. Then

$$\star(e^{i_1} \wedge \ldots \wedge e^{i_k}) = (-1)^u e^{i_{k+1}} \wedge \ldots \wedge e^{i_n}.$$

Let us deal with the case $1 \leq k \leq n - 1$ first. By invariance, we may assume that

$$\langle X \lrcorner \star \phi, Y \lrcorner \star \phi \rangle = a \langle \phi, \phi \rangle \langle X, Y \rangle + b \langle X \lrcorner \phi, Y \lrcorner \phi \rangle$$

for some $a, b \in \mathbb{R}$. To determine a and b we choose ϕ be a basis element, i.e. $\phi = e^{i_1} \wedge \ldots \wedge e^{i_k}$. It is also clear that if X and Y are linearly independent, then each term of this expression vanishes. Hence, we may take $X = Y = e_r$ for any $1 \leq r \leq n$. Then, it is easy to check the following:

- If $r \in I$, then we have $0 = (-1)^u a + (-1)^u b$ when e_r is spacelike, and $0 = a(-1)^u(-1) + b(-1)^{u-1}$ when e_r is timelike, so we must deduce $a = -b$ in both cases.
- If $r \in J$, then we have $(-1)^{q-u} = a(-1)^u + 0$ when e_r is spacelike, and $(-1)^{q-u-1} = a(-1)^u(-1) + 0$ when e_r is timelike. Hence, in both cases, $a = (-1)^q$.

Therefore, $a = (-1)^q$ and $b = -(-1)^q$, which proves the claim. We leave it to the reader to check (2.7), which is completely analogous (here, one takes ϕ to be $e^1 \wedge \ldots \wedge e^n$). □

Applying Lemma 2.6 in our case, we obtain the following useful corollary.

Corollary 2.7. *The 4-forms $\widetilde{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} \in \Omega^4(\widetilde{M})$ and $F^4 = \star_7 \phi \in \Omega^4_{\mathrm{cl}}(M)$ satisfy the following relations*

$$\begin{aligned} \langle X \lrcorner \widetilde{F}, Y \lrcorner \widetilde{F} \rangle_{\widetilde{M}} &= f^2 \| \mathrm{vol}_{\widetilde{M}} \|^2_{\widetilde{M}} \widetilde{g}(X, Y) = -f^2 \widetilde{g}(X, Y), \quad \forall\ X, Y \in \Gamma(T\widetilde{M}), \\ \langle X \lrcorner F, Y \lrcorner F \rangle_M &= g(X, Y) \|\phi\|^2_M - \langle X \lrcorner \phi, Y \lrcorner \phi \rangle_M, \quad \forall\ X, Y \in \Gamma(TM). \end{aligned}$$

Moreover, $\|F\|^2_M = \| \star_7 \phi \|^2_M = \|\phi\|^2_M$ and

$$\|\mathcal{F}\|^2_{\mathcal{M}} = \langle \mathcal{F}, \mathcal{F} \rangle_{\mathcal{M}} = \langle f \cdot \mathrm{vol}_{\widetilde{M}} + F^4, f \cdot \mathrm{vol}_{\widetilde{M}} + F^4 \rangle_{\mathcal{M}} = -f^2 + \|F^4\|^2_M.$$

Now, for the Lorentzian manifold $(\mathcal{M}=\widetilde{M}\times M, g_{\mathcal{M}}=\widetilde{g}+g)$ the Levi-Civita connection $\nabla^{g_{\mathcal{M}}}$ splits as $\nabla^{g_{\mathcal{M}}}=\nabla^{\widetilde{g}}+\nabla^{g}$, where $\nabla^{\widetilde{g}}$ and ∇^{g} are the Levi-Civita connections on $(\widetilde{M},\widetilde{g})$ and (M,g), respectively. This effects on the Ricci tensor $\mathrm{Ric}^{g_{\mathcal{M}}}$ of $\nabla^{g_{\mathcal{M}}}$, which splits accordingly, i.e.

$$\begin{aligned}
&\mathrm{Ric}^{g_{\mathcal{M}}}(X,Y)=0\,, && \text{for any vector field } X \text{ on } \widetilde{M} \text{ and } Y \text{ on } M,\\
&\mathrm{Ric}^{g_{\mathcal{M}}}(X,Y)=\mathrm{Ric}^{\widetilde{g}}(X,Y)\,, && \text{for any vector field } X,Y \text{ on } \widetilde{M},\\
&\mathrm{Ric}^{g_{\mathcal{M}}}(X,Y)=\mathrm{Ric}^{g}(X,Y)\,, && \text{for any vector field } X,Y \text{ on } M.
\end{aligned}$$

Initially we examine the Einstein supergravity equation $(\mathscr{E})$ for some vector fields X,Y on $\widetilde{M}$. In this case for the Lorentzian 4-manifold $(\widetilde{M},\widetilde{g})$ we deduce that

Proposition 2.8. *Let $(\widetilde{M},\widetilde{g},\widetilde{F}^4=f\cdot\mathrm{vol}_{\widetilde{M}})$ be the four-dimensional Lorentzian manifold of an eleven-dimensional supergravity background of the form $(\mathcal{M}=\widetilde{M}\times M, g_{\mathcal{M}}=\widetilde{g}+g)$, where the flux 4-form $\mathcal{F}$ is given by (2.2), with $f\in\mathbb{R}$. Then, $(\widetilde{M},\widetilde{g})$ is Einstein with negative Einstein constant $\Lambda:=-\frac{1}{6}\left(2f^2+\|\phi\|^2\right)$. In particular, $\|\phi\|$ is constant.*

Proof. Since we can always write $F=\star_7\phi$ for some (co-closed) 3-form ϕ on M^7, the proof is based on the previous observations. In particular, a direct computation in combination with Corollary 2.7, shows that

$$\begin{aligned}
\mathrm{Ric}^{\widetilde{g}}(X,Y) &= \frac{1}{2}\langle f\cdot X\lrcorner\,\mathrm{vol}_{\widetilde{M}}, f\cdot Y\lrcorner\,\mathrm{vol}_{\widetilde{M}}\rangle_{\widetilde{M}}-\frac{1}{6}\widetilde{g}(X,Y)\Big(\|f\cdot\mathrm{vol}_{\widetilde{M}}\|^2_{\widetilde{M}}+\|F\|^2_M\Big)\\
&= -\frac{1}{2}f^2\widetilde{g}(X,Y)+\frac{1}{6}\widetilde{g}(X,Y)\left(f^2-\|F\|^2_M\right)\\
&= \frac{1}{6}\left(-2f^2-\|F\|^2_M\right)\widetilde{g}(X,Y)=\frac{1}{6}\left(-2f^2-\|\phi\|^2\right)\widetilde{g}(X,Y).
\end{aligned}$$

The constancy of $\|\phi\|$ follows easily. □

Therefore, the supergravity Einstein equation $(\mathscr{E})$ for the specific flux form $\mathcal{F}^4$ given by (2.2), forces the Lorentzian 4-manifold $(\widetilde{M},\widetilde{g})$ to be Einstein. We mention that this occurs independently of the closure condition $(\mathscr{C})$ for $\mathcal{F}$, or the Maxwell equation $(\mathscr{M})$, so it is independent of the notion of special 3-forms. However, it yields the constraint $\|\phi\|=$ constant.

Let us restrict now the supergravity Einstein equation $(\mathscr{E})$ on vector fields $X,Y\in\Gamma(TM^7)$. Since $F=\star_7\phi$, by Corollary 2.7 it follows that

$$\begin{aligned}
\mathrm{Ric}^{g}(X,Y) &= \frac{1}{2}\langle X\lrcorner F, Y\lrcorner F\rangle_M-\frac{1}{6}g(X,Y)\left(-f^2+\|F\|^2_M\right)\\
&= \frac{1}{2}\langle X\lrcorner\star_7\phi, Y\lrcorner\star_7\phi\rangle_M+\frac{1}{6}g(X,Y)\left(f^2-\|F\|^2_M\right)\\
&= \frac{1}{2}\Big(g(X,Y)\cdot\langle\phi,\phi\rangle_M-\langle X\lrcorner\phi, Y\lrcorner\phi\rangle_M\Big)+\frac{1}{6}g(X,Y)\left(f^2-\|F\|^2_M\right)\\
&= \frac{1}{2}g(X,Y)\|\phi\|^2_M-\frac{1}{2}\langle X\lrcorner\phi, Y\lrcorner\phi\rangle_M+\frac{1}{6}g(X,Y)\left(f^2-\|\phi\|^2_M\right)\\
&= -\frac{1}{2}\langle X\lrcorner\phi, Y\lrcorner\phi\rangle_M+\frac{1}{6}g(X,Y)\left(f^2+2\|\phi\|^2_M\right).
\end{aligned}$$

Thus, one can write

$$\mathrm{Ric}^{g}(X,Y)=\frac{1}{6}g(X,Y)\left(f^2+2\|\phi\|^2_M\right)+q_\phi(X,Y), \tag{2.8}$$

where $q_\phi(X,Y)$ is the symmetric bilinear form $q_\phi(X,Y):=-\frac{1}{2}\langle X\lrcorner\phi, Y\lrcorner\phi\rangle_M$.
Hence, motivated by the results in this paragraph, we introduce the following definition:

Definition 2.9. A Riemannian 7-manifold (M^7, g, ϕ) with a special 3-form ϕ is called a *special gravitational Einstein manifold* if the pair (g, ϕ) is a solution of the supergravity Einstein equation (2.8).

Remark 2.10. Note that a special gravitational Einstein 7-manifold is not necessarily an Einstein manifold, since q_ϕ is not necessarily a multiple of the metric tensor g. In particular, (2.8) is an extension of the Einstein equation by a stress-energy tensor associated to the 3-form ϕ.

By Proposition 2.2 (or Corollary 2.5) and Proposition 2.8, it is obvious that the pair

$$(g_{\mathcal{M}} = \widetilde{g} + g, \mathcal{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} + F^4),$$

where the closed 4-form F^4 is given by $F^4 = \star_7\phi$ for some special 3-form ϕ on M^7, g is a gravitational special Einstein metric and $\widetilde{g}$ a Lorentzian Einstein metric, induces solutions of eleven-dimensional supergravity on $\mathcal{M}^{10,1} = \widetilde{M}^{3,1} \times M^7$, which we shall call $(4,7)$-*decomposable solutions of eleven-dimensional supergravity.* In this case, $\mathcal{M} = \widetilde{M} \times M$ will be referred by the term $(4,7)$-*decomposable supergravity background*. We conclude that

Corollary 2.11. *Any $(4,7)$-decomposable solution $(\mathcal{M}^{10,1}, g_{\mathcal{M}}, \mathcal{F})$ of eleven-dimensional supergravity, is a product of Lorentzian Einstein 4-manifold $(\widetilde{M}^{3,1}, \widetilde{g})$ with negative Einstein constant and a gravitational special Einstein 7-manifold (M^7, g) with special 3-form $\phi \in \Omega^3(M^7)$. In particular, the flux 4-form is given by $\mathcal{F} = f \cdot \mathrm{vol}_{\widetilde{M}} + F^4$ for some closed 4-form $F^4 := \star_7\phi \in \Omega^4_{\mathrm{cl}}(M^7)$ and some constant $f \in \mathbb{R}$.*

2.2. **Three basic types of $(4,7)$-decomposable supergravity backgrounds.** We now consider three basic classes of special 3-forms on Riemannian 7-manifolds, namely

(I) trivial 3-form, i.e. $\phi = 0$ (and hence $F = 0$) but $f \neq 0$.
(II) non-zero harmonic 3-form, i.e. $\phi \neq 0$, $f = 0$.
(III) non-harmonic 3-form, i.e. $\phi \neq 0$, $f \neq 0$.

Let us examine the construction of solutions of the supergravity Einstein equation (2.8) for any of these types of special 3-forms (or equivalently the associated flux 4-forms, which will be referred by the same name), and present the corresponding special gravitational Einstein manifolds (M^7, g, ϕ) and some examples. We begin with the first type.

Corollary 2.12. *The equation (2.8) for special 3-forms of Type I reduces to the standard Einstein equation, i.e. $\mathrm{Ric}^g = (f^2/6)g$. Consequently, using the flux 4-form $\mathcal{F} = f \cdot \mathrm{vol}_{\widetilde{M}}$ we obtain a $(4,7)$-decomposable supergravity background, given by a product of a Lorentzian Einstein 4-manifold $(\widetilde{M}^{3,1}, \widetilde{g})$ with Einstein constant $-f^2/3$, and a Riemannian Einstein 7-manifold (M^7, g) with Einstein constant $f^2/6$.*

Therefore, flux forms of type $\mathcal{F} = f \cdot \mathrm{vol}_{\widetilde{M}}$ with $f \in \mathbb{R}^*$, induce $(4,7)$-decomposable supergravity backgrounds by choosing a Lorentzian Einstein 4-manifold $(\widetilde{M}^{3,1}, \widetilde{g})$ and a compact Einstein 7-manifold (M^7, g).

We treat now special 3-forms of Type II. In this case the flux form $\mathcal{F}$ is given by $\mathcal{F} = \star_7\phi =: F^4$.

Corollary 2.13. *The equation (2.8) for a special harmonic 3-form $\phi \neq 0$ on M^7 of Type II, reduces to the equation*

$$\mathrm{Ric}^g = \frac{1}{3}\|\phi\|^2_M g - \frac{1}{2}q_\phi, \quad q_\phi(X,X) = ||X \lrcorner \phi||^2_M.$$

Moreover, $(\widetilde{M}^{3,1}, \widetilde{g})$ is Einstein with Einstein constant $-\|\phi\|^2/6$.

Remark 2.14. Apriori, we may consider a generic Type II special 3-form ϕ. However, such a 3-form is parallel and in Section 3 we will show that it does not induce $(4,7)$-decomposable supergravity backgrounds.

Example 2.15. Consider the Riemannian product $(M^7 := Q^3 \times P^4, g = g_Q + g_P)$ between a 3-dimensional Riemannian manifold (Q^3, g_Q) and a 4-dimensional Riemannian manifold (P^4, g_P). Assume that M^7 admits a special 3-form ϕ, given by $\phi := \mathrm{vol}_Q$, where vol_Q is the is volume 3-form on the first factor, with $\|\phi\|^2 = \|\mathrm{vol}_Q\|^2 = 1$. Then $\langle X \lrcorner \mathrm{vol}_Q, Y \lrcorner \mathrm{vol}_Q \rangle = g_Q(X,Y)$ for any $X, Y \in \Gamma(TM^7)$. Hence the supergravity Einstein equation becomes

$$\mathrm{Ric}^g = \frac{1}{3}g - \frac{1}{2}g_Q,$$

and we conclude that $\mathrm{Ric}^{g_Q} = -\frac{1}{6}g_Q$ and $\mathrm{Ric}^{g_P} = \frac{1}{3}g_P$. Therefore, the manifolds Q, P must be Einstein manifolds with Einstein constant $-\frac{1}{6}$ and $\frac{1}{3}$, respectively. Assume now that our initial metric g is complete. Then, Q is a complete space of constant negative curvature (i.e. a quotient $\mathbb{R}H^3/\Gamma$ of the Lobachevski space $\mathbb{R}H^3$ by a lattice) and P is a compact Einstein 4-manifold. Note that the manifold M^7 is compact if Γ is a co-compact lattice. So we get an example of decomposable supergravity background of Type II, with internal space $M^7 = Q^3 \times P^4$ and space-time any Lorentzian Einstein 4-manifold $\widetilde{M}^{3,1}$ with Einstein constant $-1/6$.

The supergravity Einstein equation (2.8) for a 7-manifold (M^7, g, ϕ) where ϕ is a special 3-form of Type III, i.e. a non-harmonic 3-form, remains unchanged. In the next section we study this case under the assumption that ϕ is a generic 3-form.

3. (4,7)-decomposable supergravity backgrounds of Type III associated to G_2-geometries

Let us fix the decomposable flux form $\mathcal{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} + \star_7 \phi$, where $\phi = \star_7 F^4$ is a special 3-form. Here we examine the situation where ϕ is in addition *generic*. To this end, it will be useful to refresh some notions of G_2-structures (see [Br87, Br05, FKMS97, Jc00] for more details).

3.1. **The Lie group G_2 and G_2-structures.** A 7-dimensional oriented Riemannian manifold (M^7, g) is called a G_2-manifold whenever the structure group of its frame bundle $\mathrm{SO}(M, g)$ is reduced to the exceptional compact Lie group $G_2 \subset \mathrm{SO}_7$. Recall that the Lie group G_2 has dimension 14 and traditionally is defined as the automorphism group of the octonion algebra $\mathbb{O}$. It is also defined as the stabilizer $G_\omega := \{\alpha \in \mathrm{GL}_7(\mathbb{R}) : \alpha\omega = \omega\}$ of a *generic* 3-form ω on $\mathbb{R}^7 = \mathrm{Im}\mathbb{O}$, with respect to the natural action of the group $\mathrm{GL}_7(\mathbb{R})$.

Definition 3.1. A 3-form $\omega \in \bigwedge^3(\mathbb{R}^7)^*$ is called *generic* if its stabilizer G_ω in $\mathrm{GL}_7(\mathbb{R})$ is the Lie group G_2.

A differential 3-form ω on a 7-manifold M is generic if its value at any point is a generic 3-form. Since $\dim G_\omega = \dim \mathrm{GL}_7(\mathbb{R}) - \dim \bigwedge^3(\mathbb{R}^7)^* = 49 - 35 = 14$, the $\mathrm{GL}_7(\mathbb{R})$-orbit Ω^3_+ of a generic 3-form is open. Another open $\mathrm{GL}_7(\mathbb{R})$-orbit is the orbit Ω^3_- of a 3-form with stabilizer the normal real form G_2^* of G_2, which is defined in terms of splittable octonions (see [K98, Lê06] for more details on G_2^*). A generic 3-form ω determines an Euclidean metric g by the rule

$$g(X,Y)\,\mathrm{vol}_M = -\frac{1}{6}(X \lrcorner \omega) \wedge (Y \lrcorner \omega) \wedge \omega,$$

for any $X, Y \in \Gamma(TM)$. In terms of an appropriate g-orthonormal basis of co-vectors $\{e^i\}$, ω has the form

$$\omega^3 := e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}, \tag{3.1}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ denotes the wedge product of e^i, e^j, e^k. A generic 3-form ω on a 7-manifold M induces a G_2-structure, i.e. a subbundle of $\mathrm{SO}(M, g)$ which is defined by frames $\{e_i\}$ with respect to which ω has the above canonical form (3.1), and conversely any G_2-structure defines a generic 3-form. So, we may identify a G_2-structure with the corresponding generic 3-form ω. We finally recall that the existence of a G_2-structure implies the following restrictions on the topology of M^7:

Proposition 3.2. ([FKMS97, Prop. 3.2] or [Jc00][Prop. 3.6.2, Prop. 10.1.6]) *The existence of a G_2-structure on a connected 7-dimensional manifold M^7 is equivalent to the vanishing of the first and the second Stiefel-Whitney classes of M^7 and hence equivalent to the existence of a spin structure.*

Definition 3.3. A G_2-manifold (M^7, g, ω) is called

- *parallel,* if $d\omega = 0 = d\star_7\omega$,
- *weak* G_2*,* if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $\mathrm{d}\,\omega = \lambda\star_7\omega$ (and thus $\mathrm{d}\star_7\omega = 0$),
- *co-callibrated,* if $d\star_7\omega = 0$.

When (M^7, g, ω^3) is a parallel G_2-manifold, then there exists a ∇^g-parallel spinor and hence (M^7, g) is Ric^g-flat [Wng89]. On the other hand, the existence of a weak G_2-structure on a compact 7-manifold (M^7, g) is equivalent to the existence of a spin structure carrying a *real Killing spinor* [FKMS97], i.e. a non-trivial section $\varphi \in \Gamma(\Sigma^g M)$ of the spinor bundle $\Sigma^g M$ over M satisfying the equation $\nabla^g_X\varphi = \lambda X\cdot\varphi$, for any $X \in \Gamma(TM)$ and some $0 \neq \lambda \in \mathbb{R}$, where here ∇^g represents the spinorial Levi-Civita connection. Thus, compact weak G_2-manifolds are singled out by the fact that admit Killing spinors and hence are Einstein manifolds with positive scalar curvature, i.e. (see [FKMS97]),

$$\mathrm{Ric}^g(X, Y) = \frac{3}{8}\lambda^2 g(X, Y), \quad \forall\ X, Y \in \Gamma(TM^7). \tag{3.2}$$

Remark 3.4. Compact weak G_2-manifolds (M^7, φ, g) admit an equivalent description in terms of the metric cone $(\hat{M} = \mathbb{R}\times M^7, \hat{g} = \mathrm{d}\,r^2 + r^2 g)$ over M^7. Since (M^7, φ, g) admits Killing spinors, $(\hat{M}, \hat{g})$ admits parallel spinors and hence has holonomy group $\mathrm{Hol}(\hat{M}) \subset \mathrm{Spin}_7$. In particular, if (M^7, φ, g) is simply-connected and not isometric to the standard sphere, then the inclusions $\mathrm{Sp}_2 \subset \mathrm{SU}_4 \subset \mathrm{Spin}_7$ yield the following three natural classes of weak G_2-manifolds:

- If $\mathrm{Hol}(\hat{M}) = \mathrm{Sp}_2$, then M^7 is called 3-Sasakian and it has a 3-dimensional space of Killing spinors.
- If $\mathrm{Hol}(\hat{M}) = \mathrm{SU}_4$, then M^7 is called Sasaki-Einstein manifold and it has a 2-dimensional space of Killing spinors.
- If $\mathrm{Hol}(\hat{M}) = \mathrm{Spin}_7$, then M^7 is called proper weak G_2-manifold, with 1-dimensional space of Killing spinors.

3.2. $(4,7)$**-decomposable supergravity solutions induced by weak** G_2**-structures.** Let us explain now how the above theory applies in supergravity equations and gives rise to special $(4,7)$-decomposable supergravity backgrounds. Let $\phi \equiv \phi^3$ be a *generic* 3-form on M^7, i.e. assume that (M^7, ϕ) is a G_2-manifold. We will normalise ϕ such that $\|\phi\|^2_M = \langle\phi, \phi\rangle_M = 7$. Then the identity $\langle X\lrcorner\phi, Y\lrcorner\phi\rangle = 3g(X, Y)$ holds, see [Br05] . Therefore, equation (2.8) reduces to

$$\mathrm{Ric}^g(X, Y) = \frac{1}{6}\left(f^2 + 5\right) g(X, Y), \tag{3.3}$$

for any $X, Y \in \Gamma(TM^7)$. Based on the previous description of weak G_2-structures, Proposition 2.2 (or Corollary 2.5) and the relations (3.2) and (3.3), we check that when the associated flux 4-form $\mathcal{F} = f\cdot\mathrm{vol}_{\widetilde{M}} + \star_7\phi$ is a solution of the supergravity Einstein equations $(\mathscr{E})$, then it needs to hold $f = \pm 2$. Thus we obtain the following

Theorem 3.5. *Let $\mathcal{M}^{10,1}$ be the oriented Lorentzian manifold given by the product of a four-dimensional oriented Lorentzian manifold $(\widetilde{M}^{3,1}, \widetilde{g})$ with volume form $\mathrm{vol}_{\widetilde{M}}$ and a seven-dimensional* *oriented manifold M^7 admitting a* G_2*-structure* $\phi \in \Omega^3_+(M)$*, such that* $\|\phi\|^2 = 7$*. Define*

$$\mathcal{F}^4_{\pm} := \pm 2\,\mathrm{vol}_{\widetilde{M}} + \star_7\,\phi.$$

Then $(\mathcal{M}, g_{\mathcal{M}} = \widetilde{g} + g, \mathcal{F}^4_{\pm})$*, where g is the Riemannian metric on M corresponding to ϕ, gives rise to a pair of* $(4,7)$*-decomposable supergravity backgrounds if and only if* (M^7, ϕ) *is a weak* G_2*-manifold and* $(\widetilde{M}^{3,1}, \widetilde{g})$ *is Lorentz Einstein with negative Einstein constant* $\Lambda := -15/6$*.*

Let us also discuss the case where the special 3-form ϕ is generic and of Type II, i.e. $f = 0$. Then, the closure condition and the Maxwell equation imply that ϕ is both closed and co-closed, so it induces a parallel G_2-structure. Therefore (M^7, g) must be Ricci-flat, and by (3.3) we obtain

Proposition 3.6. *The 4-form* $\mathcal{F} = F = \star_7\phi$*, where ϕ is a parallel* G_2*-structure on* (M^7, g)*, i.e. ϕ is a generic special 3-form of Type II, cannot satisfy the supergravity equations for the Lorentzian manifold* $\mathcal{M}^{10,1} = \widetilde{M}^{3,1} \times M^7$*, endowed with the induced product metric.*

4. Classification of 7-dimensional homogeneous manifolds of a compact Lie group

In this section we classify all compact almost effective homogeneous 7-manifolds $M^7 = G/H$ of a compact connected Lie group G (up to a covering). We apply this to the description of invariant generic (special) 3-forms, and some invariant non-generic special 3-forms that solve the Maxwell equation. In particular, one can separate the examination of Type III invariant special 3-forms into the following two subclasses:

- Type IIIα, i.e. $\phi := \star_7 F^4$ is an invariant *generic special 3-form* and thus it induces a homogeneous co-callibrated weak G_2-structure on $M^7 = G/H$.
- Type IIIβ, i.e. $\phi := \star_7 F^4$ is an invariant *non-generic special 3-form* on $M^7 = G/H$.

4.1. **Classification of subalgebras of $\mathfrak{so}_7$.** So, consider a seven-dimensional compact connected homogeneous Riemannian manifold $(M^7 = G/H, g)$. We will always assume that the action of G is almost effective, that is the kernel of effectivity $C = \{g \in G : gx = x,\ \forall x \in M\}$ is finite. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$, such that $\mathfrak{m}$ is identified with the tangent space T_oM^7 of M, where $o := eH$. The isotropy representation $\chi : H \to \mathrm{SO}(\mathfrak{m}) \cong \mathrm{SO}_7$ is given by $\chi(h)X = \mathrm{Ad}_h X$, for any $h \in H$ and $X \in \mathfrak{m}$. Almost effectivity means that the differential $\chi_* : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ of the isotropy representation is exact, i.e. $\ker(\chi_*) = \{0\}$ (cf. [Bes86]). Hence, $\mathfrak{h}$ is isomorphic to the isotropy subalgebra $\chi_*(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m}) = \mathfrak{so}_7$.

The classification of almost effective homogeneous 7-manifolds of a compact Lie group G reduces to the description of all compact Lie algebras $\mathfrak{g}$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $\mathfrak{m} = T_oM^7$, whose isotropy representation χ_* is exact and such that $\mathfrak{h} = \chi_*(\mathfrak{h})$ generates a compact subgroup H of a compact Lie group G with the Lie algebra $\mathfrak{g}$. This procedure splits into two simple steps:

- Description of all subalgebras $\mathfrak{h}$ of the orthogonal Lie algebra $\mathfrak{so}_7$.
- Description of all compact Lie algebras $\mathfrak{g}$ which contain $\mathfrak{h}$ as a codimension 7 Lie subalgebra.

Since $\mathfrak{so}_7 = \mathfrak{b}_3$ is a rank 3 simple Lie algebra, any subalgebra $\mathfrak{h} \subseteq \mathfrak{so}_7$ is a compact Lie algebra of rank $r := \mathrm{rnk}\,\mathfrak{h} \leq 3$. The list of simple Lie algebras of rank ≤ 3 is given below (here the lower indices denote the rank, the upper indices denote the dimension): $\mathfrak{a}_1^3 = \mathfrak{b}_1^3 = \mathfrak{c}_1^3$, $\mathfrak{a}_2^8$, $\mathfrak{a}_3^{15} = \mathfrak{d}_3^{15}$, $\mathfrak{b}_2^{10} = \mathfrak{c}_2^{10}$, $\mathfrak{g}_2^{14}$, $\mathfrak{b}_3^{21}$, $\mathfrak{c}_3^{21}$. Using it, we write down the list of proper semisimple subalgebras of $\mathfrak{so}_7$: $\mathfrak{so}_3, 2\mathfrak{so}_3, 3\mathfrak{so}_3 = \mathfrak{so}_4 + \mathfrak{so}_3, \mathfrak{so}_5, \mathfrak{su}_4 = \mathfrak{so}_6, \mathfrak{su}_3$. Calculating the centralizer of these subalgebras, we get the following non-semisimple proper subalgebras of $\mathfrak{so}_7$: $\mathfrak{u}_1, 2\mathfrak{u}_1, 3\mathfrak{u}_1, \mathfrak{so}_3 + \mathfrak{u}_1, \mathfrak{so}_3 + 2\mathfrak{u}_1, \mathfrak{so}_5 + \mathfrak{u}_1, \mathfrak{u}_3$. Now, the several non-conjugate subalgebras of type $\mathfrak{so}_3$ can be described as follows. Let us denote by V^k the irreducible submodule of real dimension k and by $\ell\mathbb{R}$ the trivial ℓ-dimensional module. Let $V^3 := \mathbb{R}^3$ be the standard representation of $\mathfrak{so}_3$

and $V^4 := \mathbb{C}^2$ the standard representation of $\mathfrak{su}_2$. Recall that there are two injective homomorphisms $\mathfrak{so}_3 \to \mathfrak{so}_5$ of $\mathfrak{so}_3$ into $\mathfrak{so}_5$, the standard one $A \mapsto \mathrm{diag}(A, 0, 0)$ and the embedding which corresponds to the unique 5-dimensional representation $V^5 := \mathbb{R}^5 \cong \mathrm{Sym}^2_0(\mathbb{R}^3)$. Similarly, we shall write $V^7 := \mathbb{R}^7 \cong \mathrm{Sym}^3_0(\mathbb{R}^3)$ for the unique 7-dimensional irreducible representation of $\mathfrak{so}_3$.

Any $\mathfrak{so}_3$ subalgebra of $\mathfrak{so}_7$ is given by a 7-dimensional representation $\rho : \mathfrak{so}_3 \to \mathfrak{so}_7 \subset \mathfrak{gl}(\mathbb{R}^7)$ of $\mathfrak{so}_3$, which must be a direct sum of the irreducible representations $\mathbb{R}, V^3, V^4, V^5, V^7$. As before, we use upper indices to indicate dimension of irreducible representations of dimension > 1. Then, up to conjugation in SO_7, we get the following description of subalgebras of $\mathfrak{so}_7$ isomorphic to $\mathfrak{so}_3$.

Lemma 4.1. *A subalgebra of $\mathfrak{so}_3$ type inside $\mathfrak{so}_7$ coincides with one of the following:*

α_1) $\mathfrak{su}_2 = \mathfrak{so}_3^4$, *such that* $\mathbb{R}^7 = V^4 + 3\mathbb{R}$,
α_2) $\mathfrak{su}_2^c = \mathfrak{so}_3^{(4,3)}$, *such that* $\mathbb{R}^7 = V^4 + V^3$,
α_3) $\mathfrak{so}_3^3$, *such that* $V^3 + 4\mathbb{R}$,
α_4) $\mathfrak{so}_3^{(3,3)}$, *such that* $\mathbb{R}^7 = V^3 + V^3 + \mathbb{R}$,
α_5) $\mathfrak{so}_3^5$, *such that* $\mathbb{R}^7 = V^5 + 2\mathbb{R}$,
α_6) $\mathfrak{so}_3^7$, *such that* $\mathbb{R}^7 = V^7$.

Since $\mathfrak{so}_3^4 = \mathfrak{su}_2 = \mathfrak{sp}_1 \subset \mathfrak{so}_5 = \mathfrak{sp}_2$, the splitting of $\mathbb{R}^7$ in case α_1) coincides with the isotropy representation of the 7-sphere $\mathrm{S}^7 = \mathrm{Sp}_2 / \mathrm{Sp}_1$ (see [Zil82, LM10]). On the other hand, the isotropy representation of the Stiefel manifold $\mathbb{V}_{5,2} = \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{st}}$, where SO_3 is embedded in SO_5 diagonally, decomposes as $\mathbb{R}^7 = V^3 + V^3 + \mathbb{R}$ and V^7 coincides with the isotropy representation of the 7-dimensional Berger sphere $B^7 = \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{ir}}$ (see [Br87]). Notice that V^5 coincides with the isotropy representation of the symmetric space $\mathrm{SU}_3 / \mathrm{SO}_3$.

We treat now subalgebras of rank 2. Up to conjugation in SO_7 there are two subalgebras of type $\mathfrak{so}_4$ inside $\mathfrak{so}_7$. The first corresponds to the standard embedding $A \to \mathrm{diag}(A, 0, 0, 0)$ and we write $\mathfrak{so}_4 = \mathfrak{su}_2 + \mathfrak{su}_2'$, with decomposition $\mathbb{R}^7 = V^4 + 3\mathbb{R}$. Notice that $\mathfrak{su}_2$ and $\mathfrak{su}_2'$ are conjugate in SO_7. The second subalgebra of this type is denoted by $\mathfrak{so}_4^{(4,3)} = \mathfrak{su}_2 + \mathfrak{su}_2^c$ with $\mathbb{R}^7 = V^4 + V^3$. We proceed with non-conjugate subalgebras of type $\mathfrak{so}_3 + \mathfrak{u}_1$ inside $\mathfrak{so}_7$.

Lemma 4.2. *A subalgebra of $\mathfrak{so}_3 + \mathfrak{u}_1$ type inside $\mathfrak{so}_7$ coincides with one of the following:*

β_1) $\mathfrak{so}_3^4 + \mathfrak{u}_1^2 = \mathfrak{su}_2 + \mathfrak{u}_1^2$ *with* $\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$,
β_2) $\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2} = \mathfrak{su}_2 + \mathfrak{u}_1^{2,2} =: \mathfrak{u}_2$ *with* $\mathbb{R}^7 = V^4 + 3\mathbb{R}$,
β_3) $\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2,2} = \mathfrak{su}_2 + \mathfrak{u}_1^{2,2,2}$ *with* $\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$,
β_4) $\mathfrak{so}_3^{(4,3)} + \mathfrak{u}_1^{2,2} = \mathfrak{su}_2^c + \mathfrak{u}_1^{2,2} =: \mathfrak{u}_2^c$ *with* $\mathbb{R}^7 = V^4 + V^3$,
β_5) $\mathfrak{so}_3^3 + \mathfrak{u}_1^2$ *with* $\mathbb{R}^7 = V^3 + V^2 + 2\mathbb{R}$,
β_6) $\mathfrak{so}_3^3 + \mathfrak{u}_1^{2,2}$ *with* $\mathbb{R}^7 = V^3 + V^2 + V^2$,
β_7) $\mathfrak{so}_3^{(3,3)} + \mathfrak{u}_1^{2,2,2}$ *with* $\mathbb{R}^7 = V^3 \otimes V^2 + \mathbb{R}$,
β_8) $\mathfrak{so}_3^5 + \mathfrak{u}_1^2$ *with* $\mathbb{R}^7 = V^5 + V^2$.

Here $V^2 := \mathbb{C}^1$ states for the standard representation of $\mathfrak{u}_1$. Notice that in the third case β_3) the Lie algebra $\mathfrak{u}_1$ acts both on V^4 and V^2, in the second case β_2) it acts on V^4 and in the first case β_1) it acts only on V^2.

Proof. We use Lemma 4.1 and compute the centralizers of all subalgebras inside $\mathfrak{so}_7$ of type $\mathfrak{so}_3$. We see that

$$
\begin{array}{lll}
C_{\mathfrak{so}_7}(\mathfrak{so}_3^3) = \mathfrak{so}_4, & C_{\mathfrak{so}_7}(\mathfrak{su}_2) = \mathfrak{su}_2' + \mathfrak{so}_4, & C_{\mathfrak{so}_7}(\mathfrak{su}_2^c) = \mathfrak{su}_2', \\
C_{\mathfrak{so}_7}(\mathfrak{so}_3^{(3,3)}) = \mathfrak{u}_1^{2,2,2}, & C_{\mathfrak{so}_7}(\mathfrak{so}_3^5) = \mathfrak{u}_1^2, & C_{\mathfrak{so}_7}(\mathfrak{so}_3^7) = \{0\}.
\end{array}
$$

Hence we need to exclude $\mathfrak{so}_3^7 + \mathfrak{u}_1$ and our claim follows by considering the several possible actions of $\mathfrak{u}_1$ (the case arising by the decomposition $\mathbb{R}^7 = V^5 + 2\mathbb{R}$ cannot exist due to the $\mathfrak{u}_1$-action). $\square$

Concerning subalgebras of rank 3, we remark that $\mathfrak{so}_4 + \mathfrak{so}_2 = \mathfrak{su}_2 + \mathfrak{su}_2' + \mathfrak{u}_1$ belongs to $\mathfrak{so}_7$, but this is not true for the direct sum $\mathfrak{so}_4^{(4,3)} + \mathfrak{so}_2 = \mathfrak{su}_2 + \mathfrak{su}_2^c + \mathfrak{u}_1$. Indeed, in the first case one computes $C_{\mathfrak{so}_7}(\mathfrak{so}_4) = \mathfrak{su}_2$, while the centralizer of $\mathfrak{so}_4^{(4,3)}$ is trivial, i.e. $C_{\mathfrak{so}_7}(\mathfrak{so}_4^{(4,3)}) = \{0\}$. Let us summarise all the results (including Lemmas 4.1, 4.2) with some more information in Table 1.

Table 1. The Lie subalgebras of $\mathfrak{so}_7 = \mathfrak{b}_3$

$r = \operatorname{rnk} \mathfrak{h}$	$\mathfrak{h} = \mathfrak{h}^d$	$\mathfrak{g}^{d+7}$	$\mathfrak{h}$-decomposition of $\mathbb{R}^7$
$r = 0$	$\mathfrak{h} = $ trivial	$\mathfrak{g}^7$	
$r = 1$	$\mathfrak{u}_1$	$\mathfrak{g}^8$	$\mathbb{R}^7 = V^2 + 5\mathbb{R}$
	$\mathfrak{u}_1$	$\mathfrak{g}^8$	$\mathbb{R}^7 = 2V^2 + 3\mathbb{R}$
	$\mathfrak{u}_1$	$\mathfrak{g}^8$	$\mathbb{R}^7 = 3V^2 + \mathbb{R}$
	$\mathfrak{su}_2 = \mathfrak{so}_3^4$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^4 + 3\mathbb{R}$
	$\mathfrak{su}_2^c$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^4 + V^3$
	$\mathfrak{so}_3^3$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^3 + 4\mathbb{R}$
	$\mathfrak{so}_3^5$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^5 + 2\mathbb{R}$
	$\mathfrak{so}_3^{(3,3)}$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^3 + V^3 + \mathbb{R}$
	$\mathfrak{so}_3^7$,	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = V^7$
$r = 2$	$2\mathfrak{u}_1 = \operatorname{diag}(\mathfrak{u}_1 + \mathfrak{u}_1) + \mathfrak{u}_1'$	$\mathfrak{g}^9$	$\mathbb{R}^7 = V^2 \otimes \mathbb{R}^2 + (V')^2 + \mathbb{R}$
	$\mathfrak{so}_3^4 + \mathfrak{u}_1^2 = \mathfrak{su}_2 + \mathfrak{u}_1^2$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$
	$\mathfrak{u}_2 := \mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2} = \mathfrak{su}_2 + \mathfrak{u}_1^{2,2}$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^4 + 3\mathbb{R}$
	$\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2,2}$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$
	$\mathfrak{u}_2^c := \mathfrak{so}_3^{(4,3)} + \mathfrak{u}_1^{2,2} = \mathfrak{su}_2^c + \mathfrak{u}_1^{2,2}$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^4 + V^3$
	$\mathfrak{so}_3^3 + \mathfrak{u}_1^2$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^3 + V^2 + 2\mathbb{R}$
	$\mathfrak{so}_3^3 + \mathfrak{u}_1^{2,2}$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^3 + V^2 + V^2$
	$\mathfrak{so}_3^{(3,3)} + \mathfrak{u}_1^{2,2,2}$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^3 \otimes V^2 + \mathbb{R}$
	$\mathfrak{so}_3^5 + \mathfrak{u}_1^2$	$\mathfrak{g}^{11}$	$\mathbb{R}^7 = V^5 + V^2$
	$\mathfrak{so}_4 = \mathfrak{su}_2 + \mathfrak{su}_2'$	$\mathfrak{g}^{13}$	$\mathbb{R}^7 = V^4 + 3\mathbb{R}$
	$\mathfrak{so}_4^{(4,3)} = \mathfrak{su}_2 + \mathfrak{su}_2^c$	$\mathfrak{g}^{13}$	$\mathbb{R}^7 = V^4 + V^3$
	$\mathfrak{su}_3$	$\mathfrak{g}^{15}$	$\mathbb{R}^7 = V^6 + \mathbb{R}$
	$\mathfrak{so}_5 = \mathfrak{sp}_2$	$\mathfrak{g}^{17}$	$\mathbb{R}^7 = V^5 + 2\mathbb{R}$
	$\mathfrak{g}_2$	$\mathfrak{g}^{21}$	$\mathbb{R}^7 = V^7$
$r = 3$	$3\mathfrak{u}_1$	$\mathfrak{g}^{10}$	$\mathbb{R}^7 = 3V^2 + \mathbb{R}$
	$2\mathfrak{u}_1 + \mathfrak{su}_2 = \mathfrak{u}_2 + \mathfrak{u}_1$	$\mathfrak{g}^{12}$	$\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$
	$\mathfrak{so}_4 + \mathfrak{so}_2 = \mathfrak{su}_2 + \mathfrak{su}_2' + \mathfrak{u}_1$	$\mathfrak{g}^{14}$	$\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$
	$\mathfrak{u}_3$	$\mathfrak{g}^{16}$	$\mathbb{R}^7 = V^6 + \mathbb{R}$
	$\mathfrak{su}_2 + \mathfrak{su}_2' + \mathfrak{so}_3 = \mathfrak{so}_4 + \mathfrak{so}_3$	$\mathfrak{g}^{16}$	$\mathbb{R}^7 = V^4 + V^3$
	$\mathfrak{so}_5 + \mathfrak{u}_1 = \mathfrak{sp}_2 = \mathfrak{so}_2$	$\mathfrak{g}^{18}$	$\mathbb{R}^7 = V^5 + V^2$
	$\mathfrak{so}_6$	$\mathfrak{g}^{22}$	$\mathbb{R}^7 = V^6 + \mathbb{R}$
	$\mathfrak{so}_7$	$\mathfrak{g}^{28} = \mathfrak{d}_4$	$\mathbb{R}^7 = V^7$

4.2. **Classification of almost-effective compact homogeneous 7-manifolds.** Now, the classification of almost effective homogeneous 7-manifolds $M^7 = G/H$ of a compact Lie group G, reduces to an enumeration of all compact Lie algebras $\mathfrak{g} = \mathfrak{g}^{d+7}$ of dimension $d+7$, which contain a subalgebra $\mathfrak{h} = \mathfrak{h}^d$ from Table 1 and have as reductive decomposition $\mathfrak{g}^{d+7} = \mathfrak{h}^d + \mathfrak{m}$, one of the indicated isotropy representations. We present all such homogeneous 7-manifolds in Table 2, but initially it is convenient to use Lemma 4.2 and present a proof for the almost effective cosets $M^7 = G^{d+7}/H^d$ whose isotropy subalgebra $\mathfrak{h}^d \subset \mathfrak{so}_7$ is of type $\mathfrak{so}_3 + \mathfrak{u}_1$ (and hence $d = 4$). We mention that in Table 2 we omit the details for most of the embeddings $\mathfrak{h} \subset \mathfrak{so}_7$ which do not give rise to some almost effective coset and use the following notation: For a given direct product $M = G/H \times \mathrm{T}^k$ of a homogeneous space G/H (whose isotropy subgroup is given by $H = H' \times \mathrm{T}^\ell$) with a torus T^k, we shall denote by $M_\psi = G/H \widetilde{\times} \mathrm{T}^k$ the *twisted product* $M_\psi = G/H^\psi$, defined by a homomorphism $\psi : H = H' \times \mathrm{T}^\ell \to \mathrm{T}^k$, where $H^\psi := \{(h, \psi(h)) : h \in H\} \subset H \times \mathrm{T}^k$. It is remarkable that several cosets $M^7 = G/H$ is of this type.

Proposition 4.3. *Let $M^7 = G^{11}/H^4$ be an almost effective homogeneous 7-manifold of an eleven-dimensional compact Lie group G, whose stability subalgebra $\mathfrak{h} \equiv \mathfrak{h}^4$ is of type $\mathfrak{so}_3 + \mathfrak{u}_1$. Then M is diffeomorphic to one of the cosets appearing in Table 2, case $d = 4$.*

Proof. It is useful to split the examination of compact Lie algebras $\mathfrak{g}^{11}$ into two main cases:
Case A: $\mathfrak{g}^{11}$ is semisimple. Let us assume that $\mathfrak{g}^{11}$ is semisimple, i.e. $\mathfrak{g}^{11} = [\mathfrak{g}^{11}, \mathfrak{g}^{11}]$. The only semisimple eleven-dimensional Lie algebra is the direct sum $\mathfrak{a}_1 + \mathfrak{a}_2$, hence we set $\mathfrak{g}^{11} = \mathfrak{so}_3 + \mathfrak{su}_3 = \mathfrak{su}_2 + \mathfrak{su}_3$. The only subalgebras of type $\mathfrak{so}_3$ inside $\mathfrak{su}_3$ are the subalgebras $\mathfrak{su}_2 = \mathfrak{so}_3^4$ and $\mathfrak{so}_3^5$, whose centralizer in $\mathfrak{su}_3$ is $\mathfrak{u}_1$ and $\{0\}$, respectively. Therefore, the following cases appear:

1) If $\mathfrak{su}_2 \subset \mathfrak{su}_3$, then $\mathfrak{h} = \mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2$. This gives rise to the homogeneous space $M = \mathbb{C}P^2 \times \mathrm{S}^3 = (\mathrm{SU}_3 / \mathrm{U}_2) \times \mathrm{SU}_2$ with isotropy representation $\mathbb{R}^7 = V^4 + 3\mathbb{R}$.

2) If $\mathfrak{so}_3 \subset \mathfrak{su}_3$ and $\mathfrak{u}_1 \subset \mathfrak{so}_3 \subset \mathfrak{su}_2 + \mathfrak{su}_3$, then we deduce that there are two desired subalgebras of type $\mathfrak{so}_3 + \mathfrak{u}_1$. The first one is given by $\mathfrak{h} = \mathfrak{so}_3^4 + \mathfrak{u}_1^2$ and induces the coset $M = \mathrm{S}^2 \times \mathrm{S}^5 = (\mathrm{SU}_2 / \mathrm{U}_1) \times (\mathrm{SU}_3 / \mathrm{SU}_2)$, whose isotropy representation decomposes as $\mathbb{R}^7 = V^2 + V^4 + \mathbb{R}$. The second one coincides with $\mathfrak{h} = \mathfrak{so}_3^5 + \mathfrak{u}_1^2$ with corresponding coset $M = (\mathrm{SU}_2 / \mathrm{U}_1) \times (\mathrm{SU}_3 / \mathrm{SO}_3)$. Here, the isotropy representation is given by $\mathbb{R}^7 = V^2 + V^5$.

3) If $\mathfrak{so}_3 \subset \mathfrak{su}_3$ but $\mathfrak{u}_1 \not\subseteq \mathfrak{so}_3$, then $\mathfrak{h} = \mathfrak{su}_2 + \mathfrak{u}_1^{2,2,2}$ where $\mathfrak{su}_2 = \mathfrak{so}_3^4$ is the standard subgroup of $\mathfrak{su}_3$ and $\mathfrak{u}_1^{2,2,2} = \Delta\mathfrak{u}_1$ is the diagonal subgroup of $\mathfrak{u}_1 + \mathfrak{u}_1 \subset \mathfrak{su}_2 + \mathfrak{su}_3$. Then we get the homogeneous space $M = (\mathrm{SU}_3 \times \mathrm{SU}_2)/(\mathrm{SU}_2 \times \mathrm{U}_1) = \big((\mathrm{SU}_3 / \mathrm{SU}_2) \times \mathrm{SU}_2\big)/\Delta\, \mathrm{U}_1$, whose isotropy representation decomposes as follows: $\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$. Usually, the embedding of $\Delta\mathfrak{u}_1$ in $\mathfrak{u}_1 + \mathfrak{u}_1$ is indicated by two parameters a, b and it is classical to denote these manifolds by $N_{a,b}$.

4) If $\mathfrak{su}_2 \not\subseteq \mathfrak{su}_3$, then $\mathfrak{h} = \mathfrak{so}_3^{(4,3)} + \mathfrak{u}_1^{2,2} = \mathfrak{su}_2^c + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2^c$, where we identify $\mathfrak{su}_2^c$ with the diagonal subalgebra $\Delta\mathfrak{su}_2$ of $\mathfrak{su}_2 \oplus \mathfrak{su}_2{}' \subset \mathfrak{su}_2 \oplus \mathfrak{su}_3$, and $\mathfrak{u}_1 = \mathfrak{u}_1^{2,2}$ with the centralizer of $\mathfrak{su}_2'$ in $\mathfrak{su}_3$. This gives rise to the so-called exceptional Allof-Wallach spaces $W_{1,1} = (\mathrm{SU}_3 \times \mathrm{SU}_2)/(\mathrm{SU}_2^c \times \mathrm{U}_1)$, with isotropy representation $\mathbb{R}^7 = V^4 + V^3$. Note that here the Lie group SU_2^c can be viewed as the normalizer of $\Delta\, \mathrm{SU}_2$ inside $\mathrm{SU}_3 \times \mathrm{SU}_2$.

In order to complete Case A, we need to show that the subalgebra $\mathfrak{h} = \mathfrak{so}_3^3 + \mathfrak{u}_1^{2,2}$ does not induce some almost effective homogeneous 7-manifold. Indeed, since $\mathbb{R}^7 = V^3 + V^2 + V^2$, the eleven-dimensional Lie algebra $\mathfrak{g}^{11}$ must be without center, and thus we get $\mathfrak{g}^{11} = \mathfrak{su}_3 + \mathfrak{su}_2$. However, it must be $\mathfrak{so}_3^3 \subset \mathfrak{su}_3$ but only $\mathfrak{su}_2, \mathfrak{so}_3^5$ have non-trivial centralizer inside $\mathfrak{su}_3$ and our claim follows.
Case B: $\mathfrak{g}^{11}$ is non-semisimple. Assume now that $\mathfrak{g}^{11}$ is non-semisimple. Then the dimension of the center $Z(\mathfrak{g}^{11})$ must satisfy $1 \leq \dim Z(\mathfrak{g}^{11}) \leq 3$. Hence we need to consider three cases:

1) $\dim Z(\mathfrak{g}^{11}) = 1$. The unique candidate of a Lie algebra of type $\mathfrak{g}^{11} = \mathfrak{s} + \mathfrak{u}_1$ with $\mathfrak{s}$ simple, is the Lie algebra $\mathfrak{g}^{11} = \mathfrak{so}_5 + \mathfrak{u}_1 = \mathfrak{sp}_2 + \mathfrak{u}_1$. Inside $\mathfrak{so}_5$ the $\mathfrak{so}_3$-subalgebras $\mathfrak{so}_3^{(3,3)}$ and $\mathfrak{su}_2 \subset \mathfrak{u}_2$ have non trivial centralizer and the same holds for $\mathfrak{su}_2^c = \mathfrak{so}_3^{(4,3)}$ inside $\mathfrak{sp}_2$. Hence, in this case we find the following subalgebras of type $\mathfrak{so}_3 + \mathfrak{u}_1$ which induce almost effective homogeneous 7-manifolds:
• $\mathfrak{h} = \mathfrak{so}_3^4 + \mathfrak{u}_1^2$, with corresponding coset $M = (\mathrm{SO}_5 / \mathrm{U}_2) \widetilde{\times} \mathrm{S}^1 = \mathbb{C}P^3 \widetilde{\times}\, \mathrm{S}^1$ and $\mathbb{R}^7 = V^4 + V^2 + \mathbb{R}$.
• $\mathfrak{h} = \mathfrak{so}_3^{(4,3)} + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2^c$, which defines the squashed 7-sphere $\mathrm{S}^7 = (\mathrm{Sp}_2 \times \mathrm{U}_1)/(\mathrm{Sp}_1 \times \Delta\, \mathrm{U}_1)$. Here, the isotropy representation is such that $\mathbb{R}^7 = V^4 + V^3$.
• $\mathfrak{h} = \mathfrak{so}_3^{(3,3)} + \mathfrak{u}_1^{2,2,2}$, which induces the twisted product $\mathrm{Gr}_2(\mathbb{R}^5) \widetilde{\times}\, \mathrm{S}^1 = (\mathrm{SO}_5 / \mathrm{SO}_3 \times \mathrm{SO}_2) \widetilde{\times}\, \mathrm{S}^1$, where $\mathrm{Gr}_2(\mathbb{R}^5)$ is a Grassmann manifold. In this case the isotropy representation decomposes by $\mathbb{R}^7 = (V^3 \otimes V^2) + \mathbb{R}$, where we identify the irreducible representation $V^3 \otimes V^2$ with the isotropy representation of the six-dimensional symmetric space $\mathrm{Gr}_2(\mathbb{R}^5)$.

2) $\dim Z(\mathfrak{g}^{11}) = 2$. Then $\mathfrak{g}^{11} = 3\mathfrak{so}_3 + 2\mathfrak{u}_1 = 3\mathfrak{su}_2 + 2\mathfrak{u}_1$ and $\mathfrak{h} = \mathfrak{so}_3^3 + \mathfrak{u}_1^2$. In this case we obtain the space $M = (\mathrm{SO}_4 / \mathrm{SO}_3) \times (\mathrm{SU}_2 / \mathrm{U}_1) \widetilde{\times} \mathrm{T}^2 = \mathrm{S}^3 \times \mathrm{S}^2\, \widetilde{\times} \mathrm{T}^2$, with $\mathbb{R}^7 = V^3 + V^2 + 2\mathbb{R}$.

3) $\dim Z(\mathfrak{g}^{11}) = 3$. Then $\mathfrak{g}^{11} = \mathfrak{su}_3 + 3\mathfrak{u}_1$ and the isotropy subalgebra $\mathfrak{h}$ must be $\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2$. Thus here we get the coset $M = \mathbb{C}P^2 \widetilde{\times} \mathrm{T}^3$, whose isotropy representation decomposes as $\mathbb{R}^7 = V^4 + 3\mathbb{R}$. $\square$

Table 2. Compact almost effective homogeneous 7-manifolds $M^7 = G/H$.

d	$\mathfrak{h}$	$\mathfrak{g} \equiv \mathfrak{g}^{d+7}$	$M^7 = G^{d+7}/H^d$	$\mathrm{G}_2^{\mathrm{inv}}$	np $\mathrm{G}_2^{\mathrm{inv}}$	$\mathcal{E}_{\mathrm{inv}}$
$d=0$	$\{0\}$	$7\mathfrak{u}_1$	T^7	$\checkmark$	$\times$	$\times$
		$\mathfrak{su}_2 + 4\mathfrak{u}_1$	$\mathrm{SU}_2 \times \mathrm{T}^4 = \mathrm{S}^3 \times \mathrm{T}^4$	$\checkmark$	$\times$	$\times$
		$2\mathfrak{su}_2 + \mathfrak{u}_1$	$\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{T}^1 = \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$	$\checkmark$	$\times$	$\times$
$d=1$	$\mathfrak{u}_1$	$\mathfrak{su}_3$	$W_{k,l} := \dfrac{\mathrm{SU}_3}{\mathrm{U}_1^{k,l}}$ $(k,l \in \mathbb{Z}_{\geq 0},\ k \geq l \geq 0,\ kl > 1)$	$\checkmark$	$\checkmark$	2
		$2\mathfrak{su}_2 + 2\mathfrak{u}_1$	$\mathbb{V}_{4,2} \widetilde{\times} \mathrm{T}^2 = \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2}{\mathrm{U}_1} \widetilde{\times} \mathrm{T}^2 = \dfrac{\mathrm{SO}_4}{\mathrm{SO}_2} \widetilde{\times} \mathrm{T}^2$	$\checkmark$	$\times$	$\times$
		$\mathfrak{su}_2 + 5\mathfrak{u}_1$	$\mathbb{C}P^1 \widetilde{\times} \mathrm{T}^5 = \mathrm{S}^2 \widetilde{\times} \mathrm{T}^5 = \dfrac{\mathrm{SU}_2}{\mathrm{U}_1} \widetilde{\times} \mathrm{T}^5$	$\times$	$\times$	$\times$
$d=2$	$2\mathfrak{u}_1$	$\mathfrak{su}_2 + 6\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
		$2\mathfrak{su}_2 + 3\mathfrak{u}_1$	$\dfrac{\mathrm{SU}_2}{\mathrm{U}_1} \times \dfrac{\mathrm{SU}_2}{\mathrm{U}_1} \widetilde{\times} \mathrm{T}^3 = \mathrm{S}^2 \times \mathrm{S}^2 \widetilde{\times} \mathrm{T}^3$	$\times$	$\times$	$\times$
		$3\mathfrak{su}_2$	$M_{a,b,c} = \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{SU}_2}{\mathrm{U}_1 \times \mathrm{U}_1}$ $(a \geq b \geq c \geq 0,\ a > 0,\ \gcd(a,b,c) = 1)$	$\checkmark$	$\checkmark$	1 or 2
		$\mathfrak{su}_3 + \mathfrak{u}_1$	$\mathbb{F}_{1,2} \widetilde{\times} \mathrm{S}^1 = \dfrac{\mathrm{SU}_3}{\mathrm{T}_{\max}} \widetilde{\times} \mathrm{S}^1$	$\checkmark$	$\times$	$\times$
			$W_{k,l} := \dfrac{\mathrm{SU}_3}{\mathrm{U}_1^{k,l}}$ $(k,l$ arbitary$)$	$\checkmark$	$\checkmark$	2
$d=3$	$\alpha_1)$ $\mathfrak{su}_2 = \mathfrak{so}_3^4$	$\mathfrak{su}_2 + 7\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
		$\mathfrak{sp}_2$	$\mathrm{S}^7_{V^4 + 3\mathbb{R}} = \dfrac{\mathrm{Sp}_2}{\mathrm{Sp}_1}$	$\checkmark$	$\checkmark$	2
		$\mathfrak{su}_3 + 2\mathfrak{u}_1$	$\mathrm{S}^5_{V^4 + \mathbb{R}} \times \mathrm{T}^2 = \dfrac{\mathrm{SU}_3}{\mathrm{SU}_2} \times \mathrm{T}^2$	$\checkmark$	$\times$	$\times$
	$\alpha_2)$ $\mathfrak{su}_2^c = \mathfrak{so}_3^{(4,3)}$	$\mathfrak{g}^{10} \supset \mathfrak{su}_2^c$	no almost effective coset	$\times$	$\times$	$\times$
	$\alpha_3)$ $\mathfrak{so}_3^3$	$2\mathfrak{su}_2 + 4\mathfrak{u}_1$	$\mathrm{S}^3 \times \mathrm{T}^4 = \dfrac{\mathrm{SO}_4}{\mathrm{SO}_3} \times \mathrm{T}^4 = \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2}{\Delta\, \mathrm{SU}_2} \times \mathrm{T}^4$	$\checkmark$	$\times$	$\times$
	$\alpha_4)$ $\mathfrak{so}_3^{(3,3)}$	$3\mathfrak{su}_2 + \mathfrak{u}_1$	$\dfrac{\mathrm{SO}_3 \times \mathrm{SO}_3 \times \mathrm{SO}_3}{\Delta\, \mathrm{SO}_3} \times \mathrm{S}^1 = \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$	$\checkmark$	$\times$	$\times$
		$\mathfrak{so}_5$	$\mathbb{V}_{5,2} = \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{st}}$	$\checkmark$	$\checkmark$	1
	$\alpha_5)$ $\mathfrak{so}_3^5$	$\mathfrak{su}_3 + 2\mathfrak{u}_1$	$Q_1^7 = \dfrac{\mathrm{SU}_3}{\mathrm{SO}_3} \times \mathrm{T}^2$	$\times$	$\times$	$\times$
	$\alpha_6)$ $\mathfrak{so}_3^7$	$\mathfrak{so}_5$	$B^7 = \mathrm{SO}_5 / \mathrm{SO}_3^{\mathrm{ir}}$	$\checkmark$	$\checkmark$	1 g_{irr}
	$3\mathfrak{u}_1$	$3\mathfrak{su}_2 + \mathfrak{u}_1$	$\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \widetilde{\times} \mathrm{S}^1$	$\times$	$\times$	$\times$
$d=4$	$\beta_1)$ $\mathfrak{so}_3^4 + \mathfrak{u}_1^2$	$\mathfrak{su}_3 + \mathfrak{su}_2$	$\mathrm{S}^5_{V^4 + \mathbb{R}} \times \mathrm{S}^2 = \dfrac{\mathrm{SU}_3}{\mathrm{SU}_2} \times \dfrac{\mathrm{SU}_2}{\mathrm{U}_1}$	$\times$	$\times$	1 g_{sym}
		$\mathfrak{so}_5 + \mathfrak{u}_1$	$\mathbb{C}P^3 \widetilde{\times} \mathrm{S}^1 = \dfrac{\mathrm{SO}_5}{\mathrm{U}_2} \widetilde{\times} \mathrm{S}^1 = \dfrac{\mathrm{Sp}_2}{\mathrm{Sp}_1 \times \mathrm{U}_1} \widetilde{\times} \mathrm{S}^1$	$\checkmark$	$\times$	$\times$
	$\beta_2)$ $\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2$	$\mathfrak{su}_3 + \mathfrak{su}_2$	$\mathbb{C}P^2 \times \mathrm{S}^3 = \dfrac{\mathrm{SU}_3}{\mathrm{U}_2} \times \mathrm{SU}_2$	$\times$	$\times$	1 g_{sym}
		$\mathfrak{su}_3 + 3\mathfrak{u}_1$	$\mathbb{C}P^2 \widetilde{\times} \mathrm{T}^3 = \dfrac{\mathrm{SU}_3}{\mathrm{U}_2} \widetilde{\times} \mathrm{T}^3$	$\times$	$\times$	$\times$
	$\beta_3)$ $\mathfrak{so}_3^4 + \mathfrak{u}_1^{2,2,2}$	$\mathfrak{su}_3 + \mathfrak{su}_2$	$N_{a,b} = \dfrac{\mathrm{SU}_3 \times \mathrm{SU}_2}{\mathrm{SU}_2 \times \mathrm{U}_1} = \Big(\dfrac{\mathrm{SU}_3}{\mathrm{SU}_2} \times \mathrm{SU}_2\Big) / \Delta\, \mathrm{U}_1$	$\checkmark$	$\checkmark$	1
	$\beta_4)$ $\mathfrak{su}_2^c + \mathfrak{u}_1^{2,2} = \mathfrak{u}_2^c$	$\mathfrak{su}_3 + \mathfrak{su}_2$	$W_{1,1} = \dfrac{\mathrm{SU}_3 \times \mathrm{SU}_2}{\mathrm{SU}_2^c \times \mathrm{U}_1}$	$\checkmark$	$\checkmark$	2
		$\mathfrak{sp}_2 + \mathfrak{u}_1$	$\mathrm{S}^7_{V^4 + V^3} = \dfrac{\mathrm{Sp}_2 \times \mathrm{U}_1}{\mathrm{Sp}_1 \times \Delta\, \mathrm{U}_1}$	$\checkmark$	$\checkmark$	2
	$\beta_5)$ $\mathfrak{so}_3^3 + \mathfrak{u}_1^2$	$3\mathfrak{su}_2 + 2\mathfrak{u}_1$	$\dfrac{\mathrm{SO}_4}{\mathrm{SO}_3} \times \dfrac{\mathrm{SU}_2}{\mathrm{U}_1} \widetilde{\times} \mathrm{T}^2 = \mathrm{S}^3 \times \mathrm{S}^2 \widetilde{\times} \mathrm{T}^2$	$\times$	$\times$	$\times$
	$\beta_6)$ $\mathfrak{so}_3^3 + \mathfrak{u}_1^{2,2}$	$\mathfrak{so}_5 + \mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
	$\beta_7)$ $\mathfrak{so}_3^{(3,3)} + \mathfrak{u}_1^{2,2,2}$	$\mathfrak{so}_5 + \mathfrak{u}_1$	$\mathrm{Gr}_2(\mathbb{R}^5) \widetilde{\times} \mathrm{S}^1 = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_3 \times \mathrm{SO}_2} \widetilde{\times} \mathrm{S}^1$	$\times$	$\times$	$\times$
	$\beta_8)$ $\mathfrak{so}_3^5 + \mathfrak{u}_1^2$	$\mathfrak{su}_3 + \mathfrak{su}_2$	$Q_2^7 = \dfrac{\mathrm{SU}_3}{\mathrm{SO}_3} \times \dfrac{\mathrm{SU}_2}{\mathrm{U}_1} = \dfrac{\mathrm{SU}_3}{\mathrm{SO}_3} \times \mathrm{S}^2$	$\times$	$\times$	1 g_{sym}
	$4\mathfrak{u}_1$	$\mathfrak{g}^{11} \supset 4\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$

d	$\mathfrak{h}$	$\mathfrak{g} \equiv \mathfrak{g}^{d+7}$	$M^7 = G^{d+7}/H^d$	$\mathrm{G}_2^{\mathrm{inv}}$	$\mathrm{np}\,\mathrm{G}_2^{\mathrm{inv}}$	$\mathcal{E}_{\mathrm{inv}}$
$d > 4$	Then $r = 2, 3$ **Case (I)** : $r = 2$					
$d = 6$	$\mathfrak{so}_4 = \mathfrak{su}_2 + \mathfrak{su}_2'$	$3\mathfrak{su}_2 + 4\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
		$4\mathfrak{su}_2 + \mathfrak{u}_1$	$\dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2}{\Delta\,\mathrm{SU}_2} \times \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2}{\Delta\,\mathrm{SU}_2} \times \mathrm{S}^1$	✓	$\times$	$\times$
		$\mathfrak{su}_3 + 5\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
		$\mathfrak{so}_5 + 3\mathfrak{u}_1$	$\mathrm{S}^4 \times \mathrm{T}^3 = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_4} \times \mathrm{T}^3$	$\times$	$\times$	$\times$
		$\mathfrak{so}_5 + \mathfrak{su}_2$	$\mathrm{S}^4 \times \mathrm{S}^3 = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_4} \times \mathrm{SU}_2$	$\times$	$\times$	1 g_{sym}
	$\mathfrak{so}_4^{(4,3)} = \mathfrak{su}_2 + \mathfrak{su}_2^c$	$\mathfrak{sp}_2 + \mathfrak{sp}_1$	$\mathrm{S}^7_{V^4+\mathbb{R}^3} = \dfrac{\mathrm{Sp}_2 \times \mathrm{Sp}_1}{\mathrm{Sp}_1 \times \Delta\,\mathrm{Sp}_1}$	✓	✓	1
$d = 8$	$\mathfrak{su}_3$	$\mathfrak{su}_4 \supset \mathfrak{su}_3$	$\mathrm{S}^7_{V^6+\mathbb{R}} = \dfrac{\mathrm{SU}_4}{\mathrm{SU}_3}$	✓	✓	1 g_{stn}
		$\mathfrak{g}_2 + \mathfrak{u}_1$	$\mathrm{S}^6_{\mathrm{irr}} \times \mathrm{S}^1 = \dfrac{\mathrm{G}_2}{\mathrm{SU}_3} \times \mathrm{S}^1$	✓	$\times$	$\times$
$d = 10$	$\mathfrak{so}_5$	$\mathfrak{so}_6 + 2\mathfrak{u}_1$	$\mathrm{S}^5_{\mathrm{sym}} \times \mathrm{T}^2 = \dfrac{\mathrm{SO}_6}{\mathrm{SO}_5} \times \mathrm{T}^2$	✓	$\times$	$\times$
$d = 14$	$\mathfrak{g}_2$	$\mathfrak{so}_7 \supset \mathfrak{g}_2$	$\mathrm{S}^7_{\mathrm{irr}} = \dfrac{\mathrm{SO}_7}{\mathrm{G}_2}$	✓	✓	1 g_{irr}
	Case (II) : $r = 3$					
$d = 5$	$\mathfrak{su}_2 + 2\mathfrak{u}_1$	$\mathfrak{g}^{12} \supset \mathfrak{su}_2 + 2\mathfrak{u}_1$	no almost effective coset	$\times$	$\times$	$\times$
$d = 7$	$\mathfrak{so}_4 + \mathfrak{u}_1$	$\mathfrak{so}_5 + \mathfrak{su}_2 + \mathfrak{u}_1$	$\mathrm{S}^1 \widetilde{\times} \dfrac{\mathrm{SU}_2}{\mathrm{U}_1} \times \dfrac{\mathrm{SO}_5}{\mathrm{SO}_4} = \mathrm{S}^1 \widetilde{\times} \mathrm{S}^2 \times \mathrm{S}^4$	$\times$	$\times$	$\times$
$d = 9$	$\mathfrak{u}_3$	$\mathfrak{su}_4 + \mathfrak{u}_1$	$\dfrac{\mathrm{SU}_4}{\mathrm{U}_3} \widetilde{\times} \mathrm{S}^1 = \mathbb{C}P^3 \widetilde{\times} \mathrm{S}^1$	$\times$	$\times$	$\times$
	$3\mathfrak{su}_2 = \mathfrak{so}_4 + \mathfrak{su}_2$	$\mathfrak{so}_5 + \mathfrak{so}_4$	$\dfrac{\mathrm{SO}_5}{\mathrm{SO}_4} \times \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2}{\Delta\,\mathrm{SU}_2} = \mathrm{S}^4 \times \mathrm{S}^3$	$\times$	$\times$	1 g_{sym}
$d = 11$	$\mathfrak{so}_5 + \mathfrak{u}_1$	$\mathfrak{so}_6 + \mathfrak{so}_3$	$\dfrac{\mathrm{SO}_6}{\mathrm{SO}_5} \times \dfrac{\mathrm{SO}_3}{\mathrm{SO}_2} = \mathrm{S}^5_{\mathrm{sym}} \times \mathrm{S}^2$	$\times$	$\times$	1 g_{sym}
$d = 15$	$\mathfrak{su}_4 = \mathfrak{so}_6$	$\mathfrak{g}^{22} \supset \mathfrak{su}_4$	no almost effective coset	$\times$	$\times$	$\times$
$d = 28$	$\mathfrak{so}_7$	$\mathfrak{so}_8 \supset \mathfrak{so}_7$	$\mathrm{S}^7_{\mathrm{sym}} = \dfrac{\mathrm{SO}_8}{\mathrm{SO}_7}$	$\times$	$\times$	1 g_{sym}

Table 2 implies the following classification theorem.

Theorem 4.4. *A 7-dimensional compact connected almost effective homogeneous manifold $M^7 = G/H$ of a compact Lie group G, is diffeomorphic either to the flat tours T^7 or to a homogeneous manifold of the following list (up to covering)*

$$\mathrm{S}^7 = \frac{\mathrm{SO}_8}{\mathrm{SO}_7} = \frac{\mathrm{SU}_4}{\mathrm{SU}_3} = \frac{\mathrm{SO}_7}{\mathrm{G}_2} = \frac{\mathrm{Sp}_2}{\mathrm{Sp}_1} = \frac{\mathrm{Sp}_2 \times \mathrm{U}_1}{\mathrm{Sp}_1 \times \Delta\,\mathrm{U}_1} = \frac{\mathrm{Sp}_2 \times \mathrm{Sp}_1}{\mathrm{Sp}_1 \times \Delta\,\mathrm{Sp}_1}$$

$\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2 \widetilde{\times} \mathrm{S}^1$

$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$

$\mathrm{S}^4 \times \mathrm{S}^2 \widetilde{\times} \mathrm{S}^1$

$\mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{S}^2$

$\mathrm{S}^3 \times \mathrm{S}^2 \widetilde{\times} \mathrm{T}^2$

$\mathrm{S}^2 \times \mathrm{S}^2 \widetilde{\times} \mathrm{T}^3$

$\mathrm{S}^3 \times \mathrm{T}^4$

$\mathrm{S}^4 \times \mathrm{T}^3$

$\mathrm{S}^5 \times \mathrm{T}^2$

$\mathrm{S}^5 \times \mathrm{S}^2$

$\mathrm{S}^3 \times \mathrm{S}^4$

$\mathrm{S}^6 \times \mathrm{S}^1$

$$Q_1^7 = \frac{\mathrm{SU}_3}{\mathrm{SO}_3} \times \mathrm{T}^2$$

$\mathrm{S}^3 \times \mathbb{C}P^2$

$\mathbb{C}P^1 \widetilde{\times} \mathrm{T}^5$

$\mathbb{C}P^2 \widetilde{\times} \mathrm{T}^3$

$\mathbb{C}P^3 \widetilde{\times} \mathrm{S}^1$

$\mathbb{F}_{1,2} \widetilde{\times} \mathrm{S}^1$

$$W_{k,l} = \frac{\mathrm{SU}_3}{\mathrm{U}_1^{k,l}}$$

$$Q_2^7 = \frac{\mathrm{SU}_3}{\mathrm{SO}_3} \times \mathrm{S}^2$$

$\mathbb{V}_{4,2} \widetilde{\times} \mathrm{T}^2$

$\mathrm{Gr}_2(\mathbb{R}^5) \widetilde{\times} \mathrm{S}^1$

$$M_{a,b,c} = \frac{\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^3}{\mathrm{U}_1 \times \mathrm{U}_1}$$

$B^7 = \mathrm{SO}_5 \,/\, \mathrm{SO}_3^{\mathrm{ir}}$

$\mathbb{V}_{5,2} \cong T^1\,\mathrm{S}^3 = \mathrm{SO}_5 \,/\, \mathrm{SO}_3^{\mathrm{st}}$

$$N_{a,b} = \frac{\mathrm{SU}_2 \times \mathrm{SU}_3}{\mathrm{SU}_2 \times \mathrm{U}_1}$$

$$W_{1,1} = \frac{\mathrm{SU}_3 \times \mathrm{SU}_2}{\mathrm{SU}_2^c \times \mathrm{U}_1}$$

Notice that several manifolds in this list admit several presentations as homogeneous spaces, e.g. $\mathrm{S}^3, \mathrm{S}^5, \mathrm{S}^7$, $\mathbb{C}P^3$, $\mathbb{C}P^3 \widetilde{\times} \mathrm{S}^1$, $\mathrm{S}^5 \times \mathrm{S}^2$, $\mathbb{V}_{4,2} \widetilde{\times} \mathrm{T}^2$, $\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1$ *and other (for details see Table 2).*

4.3. $(4,7)$**-decomposable homogeneous supergravity backgrounds of Type III**α**.** The classification of compact simply-connected homogeneous weak G_2-manifolds [FKMS97] and that of homogeneous Lorentzian Einstein 4-manifolds [Km01, FeR06], together with Theorem 3.5 yield a large list of $(4,7)$-decomposable *homogeneous* supergravity backgrounds of type IIIα. Recall that a G_2-manifold (M^7, ω) is called homogeneous if there is a transitive Lie group G which leaves ω invariant. A classical result of Dynkin states that the Lie algebras $\mathfrak{so}_3^7$, $\mathfrak{so}_4^{(4,3)} = \mathfrak{su}_2 + \mathfrak{su}_2^c$ and $\mathfrak{su}_3$ exhaust (up to conjugation) all maximal subalgebras of $\mathfrak{g}_2$. Hence, a homogeneous manifold $M^7 = G/H$ admits an invariant G_2-structure ϕ if and only if $M^7 = \mathrm{Spin}_7 / \mathrm{G}_2$ or $\chi_*(\mathfrak{h})$ belongs to one of the subalgebras $\mathfrak{so}_3^7$, $\mathfrak{so}_4^{(4,3)}$ and $\mathfrak{su}_3$. Following the papers [LM10, Rd10] and [FKMS97] in Table 2 we also indicate which of the compact almost effective homogeneous 7-manifolds $M^7 = G/H$ admit an invariant G_2-structure and moreover an invariant weak G_2-structure. To track this information we use the notations "$\mathrm{G}_2^{\mathrm{inv}}$" and "np $\mathrm{G}_2^{\mathrm{inv}}$", respectively. For convenience, in the last column we also include the number $\mathcal{E}_{\mathrm{inv}}$ of non-isometric invariant Einstein metrics, see also [CR84, DfNP86, FKMS97, Nk04] and Remark 4.5 below. By "$\times$" we mean that the corresponding coset does not admit some of the aforementioned invariant objects.

Remark 4.5. (**Remarks on Table 2 about homogeneous Einstein metrics**) For the homogeneous spheres $\mathrm{S}^5, \mathrm{S}^6$ and S^7 in Table 2 we use a subscript with the decomposition of the associated tangent space into irreducible submodules, in particular the subscript "irr" characterises an irreducible isotropy representation (but not symmetric), while "sym" means that the corresponding sphere is a symmetric space (and similarly for the metrics). The space $M_{a,b,c}$ is diffeomorphic to $\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^3$ and is a circle bundles over $\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$. Details about the number of invariant Einstein metrics on $M_{a,b,c}$, which depends on the parameters (a,b,c), can be found in [Nk04], for example. The Berger sphere B^7 and the 7-spheres $\mathrm{Spin}_7 / \mathrm{G}_2$ or $(\mathrm{Sp}_2 \times \mathrm{Sp}_1)/(\mathrm{Sp}_1 \times \Delta\, \mathrm{Sp}_1)$ admit a unique invariant proper weak G_2-structure, see [Br87, Bär93, FKMS97] and a unique invariant Einstein metric. In fact, this structure on the squashed sphere $(\mathrm{Sp}_2 \times \mathrm{Sp}_1)/(\mathrm{Sp}_1 \times \Delta\, \mathrm{Sp}_1)$ is also invariant under the Lie group $\mathrm{Sp}_2 \times \mathrm{U}_1$. Recall now that the Allof-Wallach spaces $W_{k,l} = \mathrm{SU}_3 / \mathrm{U}_1^{k,l}$, where $\mathrm{U}_1^{k,l} = \mathrm{diag}(z^l, z^k, \bar{z}^{l+k}) \subset \mathrm{U}_2 \subset \mathrm{SU}_3$ with $z \in \mathrm{S}^1 = Z(\mathrm{U}_2)$, $k \geq 1$, $l \geq 1$, $\gcd(k,l) = 1$, admit (up to homothety) two SU_3-invariant weak G_2-structures and two invariant Einstein metrics, see [FKMS97, Nk04]. These Einstein metrics are isometric each other for the special case of $W_{1,0}$, in particular the weak G_2-structures on $W_{1,0}$ coincide. By [BoG94] it is also known that the exceptional Allof-Wallach space $W_{1,1} = (\mathrm{SU}_3 \times \mathrm{SU}_2)/(\mathrm{SU}_2^c \times \mathrm{U}_1)$ and the 7-sphere $\mathrm{S}^7 = \mathrm{Sp}_2 / \mathrm{Sp}_1$ exhaust all compact homogenous 3-Sasakian spaces in dimension seven. Note that a 7-dimensional 3-Sasakian manifold admits a second weak G_2-structure which is proper, with the corresponding Einstein metric to be a member of the canonical variation of the invariant 3-Sasakian Einstein metric, see [FKMS97]. Recall also that the Stiefel manifold $\mathbb{V}_{5,2}$ is an Einstein-Sasakian manifold and the unique SU_4-invariant Einstein metric on $\mathrm{SU}_4 / \mathrm{SU}_3$ is the standard one, g_{stn}, see [Jn73]. Finally notice that the homogeneous spaces $Q_1^7 = (\mathrm{SU}_3 / \mathrm{SO}_3) \times \mathrm{T}^2$ and $Q_2^7 = (\mathrm{SU}_3 / \mathrm{SO}_3) \times \mathrm{S}^2$ are products of the symmetric space $\mathrm{SU}_3 / \mathrm{SO}_3$ with the 2-torus T^2 and the 2-sphere S^2, respectively. The coset $\mathrm{SU}_3 / \mathrm{SO}_3$ belongs to the family $\mathrm{SU}_n / \mathrm{SO}_n$, which according to [ChG] is spin only for $n =$ even. Consequently, none of Q_1^7 and Q_2^7 are spin or admit a G_2-structure (see Proposition 3.2). A difference between the symmetric spaces Q_1^7, Q_2^7 is that Q_1^7 is not simply-connected neither Einstein, in contrast to Q_2^7 which satisfies both these properties (it admits a unique invariant Einstein metric given by the product of the Killing metrics).

4.4. **Non existence of invariant G_2-structures and invariant G_2^*-structures.** Let us describe now all compact almost effective homogeneous spaces $M^7 = G/H$ which admit no G-invariant G_2-structure and moreover no G_2-structure. This task is based on our classification Theorem 4.4, the column "$\mathrm{G}_2^{\mathrm{inv}}$" of Table 2 and Proposition 3.2. We conclude the following

Theorem 4.6. *1) Let $M^7 = G/H$ be a compact connected almost effective homogeneous 7-manifold of a compact Lie group G. The manifold M^7 admits no G-invariant G_2-structure (or equivalently, no G-invariant spin structure) if and only if it is diffeomorphic (up to covering) to one of the following cosets:*

spin	*non-spin*
$\mathrm{S}^3\times\mathrm{S}^4 = (\mathrm{SU}_2\times\mathrm{SU}_2/\Delta\,\mathrm{SU}_2)\times(\mathrm{SO}_5/\mathrm{SO}_4)$	$\mathbb{C}P^2\times\mathrm{S}^3 = (\mathrm{SU}_3/\mathrm{U}_2)\times\mathrm{SU}_2$
$\mathrm{S}^4\times\mathrm{T}^3 = (\mathrm{SO}_5/\mathrm{SO}_4)\times\mathrm{T}^3$	$\mathbb{C}P^2\widetilde{\times}\mathrm{T}^3 = (\mathrm{SU}_3/\mathrm{U}_2)\widetilde{\times}\mathrm{T}^3$
$\mathrm{S}^2\times\mathrm{S}^2\times\mathrm{S}^2\times\mathrm{S}^1 = (\mathrm{SU}_2/\mathrm{U}_1)^3\times\mathrm{S}^1$	$Q_1^7 = (\mathrm{SU}_3/\mathrm{SO}_3)\times\mathrm{T}^2$
$\mathrm{S}^2\times\mathrm{S}^5 = (\mathrm{SO}_3/\mathrm{SO}_2)\times(\mathrm{SO}_6/\mathrm{SO}_5)$	$Q_2^7 = (\mathrm{SU}_3/\mathrm{SO}_3)\times\mathrm{S}^2$
$\mathbb{C}P^1\widetilde{\times}\mathrm{T}^5 = (\mathrm{SU}_2/\mathrm{U}_1)\widetilde{\times}\mathrm{T}^5$	$\mathrm{Gr}_2(\mathbb{R}^5)\widetilde{\times}\mathrm{S}^1$
$\mathrm{S}^2\times\mathrm{S}^2\widetilde{\times}\mathrm{T}^3 = (\mathrm{SU}_2\times\mathrm{SU}_2/\mathrm{U}_1\times\mathrm{U}_1)\widetilde{\times}\mathrm{T}^3$	
$\mathrm{S}^3\times\mathrm{S}^2\widetilde{\times}\mathrm{T}^2 = (\mathrm{SU}_2\times\mathrm{SU}_2/\Delta\,\mathrm{SU}_2)\times(\mathrm{SU}_2/\mathrm{U}_1)\widetilde{\times}\mathrm{T}^2$	
$\mathrm{S}^4\times\mathrm{S}^2\widetilde{\times}\mathrm{S}^1 = (\mathrm{SO}_5/\mathrm{SO}_4)\times(\mathrm{SO}_3/\mathrm{SO}_2)\widetilde{\times}\mathrm{S}^1$	
$\mathbb{C}P^3\times\mathrm{S}^1 = (\mathrm{SU}_4/\mathrm{U}_3)\widetilde{\times}\mathrm{S}^1$	
$\mathrm{S}^7 = \mathrm{SO}_8/\mathrm{SO}_7$	

2) Manifolds from the left column admit a G_2-structure which is not invariant, or in other words, admit a generic 3-form which is not invariant. Inside the class of compact connected almost effective homogeneous 7-manifolds $M^7 = G/H$ only the manifolds from the right column doest not admit a G_2-structure.

Theorem 4.6 gives rise to the following natural questions for further research.

Question 1. What is the explicit form of the non-invariant spin structure, or equivalent, non-invariant G_2-structure assigned in Theorem 4.6?

Question 2. What is the symmetry group corresponding to such a structure?

These type of questions are in general difficult. To our knowledge, they have been examined for example in [Lê06] for the coset $\mathrm{S}^3\times\mathrm{S}^4$ and for G_2^*-structures. Below we also describe our conclusions for non-existence of G_2^*-structures. But firstly, let us analyse some example and enlighten the details of Theorem 4.6.

Example 4.7. The space $\mathrm{S}^3\times\mathrm{S}^4$ is a spin manifold and by Proposition 3.2, also a G_2-manifold. However, this G_2-structure is not invariant with respect to $G = \mathrm{SO}_5\times\mathrm{SU}_2$, where we identify $\mathrm{S}^3\times\mathrm{S}^4 \cong \mathrm{SU}_2\times(\mathrm{SO}_5/\mathrm{SO}_4)$. Indeed, a spin structure on a seven-dimensional oriented connected homogeneous Riemannian manifold $(M^7 = G/H, g)$ with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is *invariant* if the isotropy representation $\chi : H \to \mathrm{SO}(\mathfrak{m})$ lifts to $\mathrm{Spin}(\mathfrak{m}) \cong \mathrm{Spin}_7$, i.e. there exists a homomorphism $\hat{\chi} : H \to \mathrm{Spin}(\mathfrak{m})$ which makes the following diagram commutative

$$\begin{tikzcd} & \mathrm{Spin}_7 \arrow[d, "\mathrm{Ad}"] \\ K \arrow[ur, "\hat{\chi}"] \arrow[r, "\chi"'] & \mathrm{SO}_7\,. \end{tikzcd}$$

Here, $\mathrm{Ad} : \mathrm{Spin}_7 \to \mathrm{SO}_7$ is the double covering. Conversely, if G is simply-connected and $(M^7 = G/H, g)$ has a spin structure, then χ lifts to $\mathrm{Spin}(\mathfrak{m})$, i.e. the spin structure is G-invariant (see [ChGT93, Thm.1, p. 146]). Hence in this case there is a bijective correspondence between the set of spin structures on $(M^7 = G/H, g)$ and the set of lifts of χ onto $\mathrm{Spin}(\mathfrak{m})$. If in addition $M = G/K$ is simply-connected and such a lift exists, then it will be unique. For the product $\mathrm{S}^3 \times \mathrm{S}^4 = \mathrm{SU}_2 \times (\mathrm{SO}_5 / \mathrm{SO}_4)$ the full isometry group $G = \mathrm{SO}_5 \times \mathrm{SU}_2$ is not simply-connected, so the spin structure which admits $\mathrm{S}^3 \times \mathrm{S}^4$ does not lift to a G-invariant spin structure, or in other words the corresponding G_2-structure is not G-invariant. All the spaces in Theorem 4.6 which are spin can be justified in a similar way.

Results about G_2^*-structures. Recall that in a line with a G_2-structure, a compact manifold M^7 admits a G_2^*-structure if and only if M^7 is orientable and spin, see [Lê07, Main Theorem]. On the other hand, recall that SO_4 is the unique maximal compact subgroup of G_2^*, but also a maximal subgroup G_2. Therefore, in the homogeneous setting we see that a G-invariant G_2^*-structure on a compact homogeneous space $M^7 = G/H$ induces also a G-invariant G_2 structure. However, the converse does not always true, since given a compact connected coset $M^7 = G/H$ such that $\chi(H) \subset \mathrm{G}_2$, then we may have $\chi(H) \nsubseteq \mathrm{G}_2^*$. In fact, this is the case for the invariant G_2-structures on the cosets

$$B^7 = \frac{\mathrm{SO}_5}{\mathrm{SO}_3^{\mathrm{ir}}}, \quad \frac{\mathrm{Spin}_7}{\mathrm{G}_2}, \quad \frac{\mathrm{SU}_4}{\mathrm{SU}_3}, \quad \frac{\mathrm{G}_2}{\mathrm{SU}_3} \times \mathrm{S}^1 \,. \tag{4.1}$$

In [Lê06] one obtains the non-existence of invariant G_2^*-structures on the product $\mathrm{S}^3 \times \mathrm{S}^4$. Next we classify all compact almost effective homogeneous spaces $M^7 = G/H$ which can be characterised by the same non-existence.

Corollary 4.8. *1) A seven-dimensional compact connected almost effective homogenous manifold $(M^7 = G/H, g)$ of a connected compact Lie group G which admits no G-invariant G_2^*-structure is diffeomorphic (up to covering) to one of the cosets given in Theorem 4.6, 1), or one of the cosets given in (4.1).*
2) Inside the class of compact connected almost effective homogeneous 7-manifolds $M^7 = G/H$ only the manifolds $\mathbb{C}P^2 \times \mathrm{S}^3$, $\mathbb{C}P^2 \widetilde{\times} \mathrm{T}^3$, $\mathrm{Gr}_2(\mathbb{R}^5) \widetilde{\times} \mathrm{S}^1$ and Q_1^7, Q_2^7 do not admit a G_2^*-structure.

5. Some solutions of the Maxwell equation for non generic 3-forms

Next we present examples of compact homogeneous Riemannian manifolds $(M^7 = G/H, g)$ which admit *non-generic* invariant special 3-forms, that means 3-forms ϕ which satisfy the Maxwell equation $\mathrm{d}\,\phi = f \star_7 \phi$ and are of type IIIβ.

5.1. Solution of Type IIIβ for the Maxwell equation on $M^7 = \mathbb{CP}^2 \times \mathrm{S}^3$.

The simply-connected homogeneous manifold $M^7 = \mathbb{CP}^2 \times \mathrm{S}^3 = (\mathrm{SU}_3 / \mathrm{U}_2) \times \mathrm{SU}_2$ has no spin structure. Hence there are not exist generic 3-forms. However, here we will show that it is endowed with invariant (non-generic) special 3-forms.

The Lie algebra $\mathfrak{g} = \mathfrak{su}_3 + \mathfrak{su}_2$ admits the reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathfrak{h} = \mathfrak{u}_2, \quad \mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 = \mathbb{R}^4 + \mathfrak{su}_2.$$

The tangent space at the identity in M^7 can be identified with $\mathfrak{m}$. Dually, we have $\mathfrak{g}^* = \mathfrak{m}_1^* + \mathfrak{m}_2^* + \mathfrak{h}^*$ where we identify $\mathfrak{m}^* = \mathfrak{m}_1^* + \mathfrak{m}_2^*$ with the cotangent space at the identity. One can choose a basis adapted to this decomposition of $\mathfrak{g}^*$: $\mathfrak{m}_1^* = \mathrm{span}(\alpha^i)_{i=1,\dots,4}$, $\mathfrak{m}_2^* = \{\beta^i\}_{i=1,\dots,3}$, $\mathfrak{h}^* = \{\gamma^i\}_{i=1,\dots,4}$. Note that $\mathrm{Ann}(\mathfrak{m}_1) = \mathfrak{m}_2^* + \mathfrak{h}^*$, $\mathrm{Ann}(\mathfrak{m}_2) = \mathfrak{m}_1^* + \mathfrak{h}^*$ and $\mathrm{Ann}(\mathfrak{h}) = \mathfrak{m}_1^* + \mathfrak{m}_2^*$. The structure equations then read

$$\begin{aligned}
&\mathrm{d}\,\alpha^1 = -\alpha^2\wedge\gamma^3 - \alpha^3\wedge\big(3\,\gamma^1-\gamma^2\big) - \alpha^4\wedge\gamma^4\,, \quad \mathrm{d}\,\gamma^1 = -\alpha^1\wedge\alpha^3 - \alpha^2\wedge\alpha^4\,,\\
&\mathrm{d}\,\alpha^2 = \alpha^1\wedge\gamma^3 - \alpha^3\wedge\gamma^4 - \alpha^1\wedge\big(3\,\gamma^1+\gamma^2\big)\,, \quad \mathrm{d}\,\gamma^2 = \alpha^1\wedge\alpha^3 - \alpha^2\wedge\alpha^4 - 2\,\gamma^3\wedge\gamma^4\,,\\
&\mathrm{d}\,\alpha^3 = \alpha^1\wedge\big(3\,\gamma^1-\gamma^2\big) + \alpha^2\wedge\gamma^4 - \alpha^4\wedge\gamma^2\,, \quad \mathrm{d}\,\gamma^3 = -\alpha^1\wedge\alpha^2 - \alpha^3\wedge\alpha^4 - 2\,\gamma^4\wedge\gamma^2\,,\\
&\mathrm{d}\,\alpha^4 = \alpha^1\wedge\gamma^4 + \alpha^2\wedge\big(3\,\gamma^1+\gamma^2\big) - \alpha^3\wedge\gamma^3\,, \quad \mathrm{d}\,\gamma^4 = -\alpha^1\wedge\alpha^4 - \alpha^2\wedge\alpha^3 - 2\,\gamma^2\wedge\gamma^3\,,\\
&\mathrm{d}\,\beta^1 = -\beta^2\wedge\beta^3\,, \quad \mathrm{d}\,\beta^2 = -\beta^3\wedge\beta^1\,, \quad \mathrm{d}\,\beta^3 = -\beta^1\wedge\beta^2\,.
\end{aligned}$$

Any U_2-invariant metric on M^7 has the form $g = g_4 + g_3$ where $g_4 = a\sum_{i=1}^4 \alpha^i\otimes\alpha^i$ is proportional to the Fubin-Strudy metric and g_3 is any Euclidean metric on $\mathfrak{su}_3$. Without loss of generality, we may assume that $g_3 = \sum_{i=1}^3 c_i\,\beta^i\otimes\beta^i$, for some positive constants c_i (see [Mln76]). Denote by $\mathrm{vol}_4 = a^2\cdot(\alpha^1\wedge\alpha^2\wedge\alpha^3\wedge\alpha^4)$ the volume form induced from g_4 on $\mathbb{CP}^2$ and by $\mathrm{vol}_3 = \sqrt{c_1c_2c_3}\cdot(\beta^1\wedge\beta^2\wedge\beta^3)$ the volume form on S^3 induced from g_3. Then, the metric-compatible volume form is given by $\mathrm{vol}_7 = \mathrm{vol}_4\wedge\mathrm{vol}_3$.

Now, the most general U_2-invariant 3-form on M^7 is given by

$$\phi = \omega\wedge\theta + b\cdot\mathrm{vol}_3, \tag{5.1}$$

where $\omega = a\cdot\big(\alpha^1\wedge\alpha^3 + \alpha^2\wedge\alpha^4\big)$ is the Kähler form on $\mathbb{CP}^2$, θ is an arbitrary SU_2-invariant 1-form on S^3 and b a constant. It is straightforward to check that ω is anti-self-dual, i.e. $\star_4\omega = -\omega$. In particular, we have $\star_7\phi = -\omega\wedge\star_3\theta + b\cdot\mathrm{vol}_4$. Computing the exterior derivatives, we find

$$\mathrm{d}\star_7\phi = -\omega\wedge\mathrm{d}\star_3\theta, \quad \mathrm{d}\,\phi = \omega\wedge\mathrm{d}\,\theta\,.$$

From the structure equations we also see that any 2-form on SU_2 is closed and thus θ must be co-closed, i.e. $\mathrm{d}\star_3\theta = 0$. Hence, the equation $\mathrm{d}\star_7\phi = 0$ is always satisfied. Now, the Maxwell equation $\mathrm{d}\,\phi = f\star_7\phi$ reads as

$$\omega\wedge\mathrm{d}\,\theta = f\cdot(-\omega\wedge\star_3\theta + b\cdot\mathrm{vol}_4)\,.$$

Matching each side of the equation yields the following conditions:

$$\mathrm{d}\,\theta = -f\star_3\theta, \quad f\cdot b\cdot\mathrm{vol}_4 = 0\,.$$

Taking the components of the first of these equations leads to

$$\left(-\sqrt{\frac{c_1}{c_2c_3}} + f\right)\theta_1 = 0, \quad \left(-\sqrt{\frac{c_2}{c_3c_1}} + f\right)\theta_2 = 0, \quad \left(-\sqrt{\frac{c_3}{c_1c_2}} + f\right)\theta_3 = 0. \tag{5.2}$$

Thus, there are two non-trivial cases to examine:

- If $f = 0$, then we automatically get $\mathrm{d}\,\theta = 0$, which implies $\theta = 0$ by the last system of equations. Thus, (5.1) reduces to $\phi = b\cdot\mathrm{vol}_3$.
- If $f\neq 0$, then we obtain $b = 0$ so that (5.1) reduces to $\phi = \omega\wedge\theta$.

Proposition 5.1. *The only invariant solutions of the Maxwell equation on $M^7 = \mathbb{CP}^2\times\mathrm{S}^3$ are the following:*

- *if $f = 0$, $\phi = b\cdot\mathrm{vol}_3$, $b = const$,*
- *if $f\neq 0$, $\phi = \omega\wedge\theta$ where ω is the Kähler form of $\mathbb{C}P^2$ and the components of the 1-form θ and of the metric are subject to (5.2).*

In both cases, one can check that these special 3-forms do not satisfy the supergravity Einstein equation with respect to the metric g, hence M^7 does not provide us with a special gravitational 7-manifold.

5.2. **Solution of of Type IIIβ for the Maxwell equation on the Lie group $G = \mathrm{S}^3 \times \mathrm{T}^4$.** We choose a left invariant metric g on G such that the decomposition $\mathfrak{g} = \mathfrak{su}_2 + \mathfrak{t}$ is orthogonal, where we indentify the tangent space of $\mathrm{S}^3 = \mathrm{SU}_2$ with the Lie algebra $\mathfrak{su}_2$ and similarly for the 4-torus T^4, i.e. $\mathfrak{t} = T_e\mathrm{T}^4$. Then we may choose and orthogonal basis ω_α of 1-forms on $\mathfrak{su}_2$ such that $d\omega^\alpha = \omega^\beta \wedge \omega^\gamma$, where (α, β, γ) is a cyclic permutation of $(1,2,3)$, and moreover an orthonormal basis ρ_i, $i = 1,2,3,4$ of $\mathfrak{t}$ such that $d\rho_i = 0$. Set

$$\bigwedge^{p,q} = \bigwedge^{p}(\mathfrak{su}_2^*) \wedge \bigwedge^{q}(\mathfrak{t}^*).$$

Then $\mathrm{d}\bigwedge^{p,q} \subset \bigwedge^{p+1,q}$ and $\star_7 \bigwedge^{p,q} \subset \bigwedge^{3-p,4-q}$. This show that any solution of Maxwell equation belongs to

$$\bigwedge^{1,2} = \mathfrak{su}_2^* \wedge \bigwedge^{2}(\mathfrak{t}^*).$$

Now, the space $\bigwedge^2(\mathfrak{t}^*) = \bigwedge^+ + \bigwedge^-$ is the direct sum of self-dual forms $\bigwedge^+$ and anti-self-dual forms $\bigwedge^-$, which are the $\pm$ eigenspaces of the Hodge operator $\star_4$. Set $\phi = \omega \wedge \sigma \in \bigwedge^{1,2}$, where ω is a left-invariant 1-form on SU_2 and $\sigma \in \bigwedge^2(\mathfrak{t}^*)$ is a left-invariant 2-form on the torus T^4. Then we get

$$\mathrm{d}\,\phi = d\omega \wedge \sigma, \quad \star_7\phi = \star_3\omega \wedge \star_4\sigma.$$

Now, we may assume that $g(\omega^\alpha, \omega^\beta) = (\lambda^\alpha)^{-2}\delta^{\alpha,\beta}$. In this case it is easy to see that $\tilde{\omega}^\alpha = \lambda^\alpha\omega$ is an orthonormal basis and moreover

$$\star_3\omega^\alpha = \frac{\lambda^\beta\lambda^\gamma}{\lambda^\alpha}\omega^\beta \wedge \omega^\gamma.$$

Therefore, $\phi = \omega^\alpha \wedge \sigma$ satisfies the Maxwell equation if and only if

$$\star_4\sigma = \pm\sigma, \quad \text{and} \quad \lambda^\beta\lambda^\gamma = \pm\lambda^\alpha.$$

This implies that $\lambda^\alpha = \pm 1$. More precisely, $(\lambda^1, \lambda^2, \lambda^3) = (\pm 1, \pm 1, \pm 1)$. Note that if σ is self-dual the number of units in this triple must be odd and if σ is an anti-self-dual the corresponding number is even. For example, assume that $\lambda^\alpha = 1$, $\alpha = 1,2,3$. Then, any self-dual 2 form $\sigma \in \bigwedge^+$ defines a solution of Type IIIβ for the Maxwell, given by $\phi = \omega \wedge \sigma$, where ω is any unit 1-form in $\mathfrak{su}_2^*$.

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