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INFORMATION PROCESSES

Structure of Extremal Trajectories of Discrete Linear Systems and the Finiteness Conjecture[1]

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Abstract—In 1995 J. C. Lagarias and Y. Wang conjectured that the generalized spectral radius of a finite set of square matrices can be attained on a finite product of matrices. The first counterexample to this Finiteness Conjecture was given in 2002 by T. Bousch and J. Mairesse and their proof was based on measure-theoretical ideas. In 2003 V. D. Blondel, J. Theys and A. A. Vladimirov proposed another proof of a counterexample to the Finiteness Conjecture which extensively exploited combinatorial properties of permutations of products of positive matrices.

In the control theory, so as in the general theory of dynamical systems, the notion of generalized spectral radius is used basically to describe the rate of growth or decrease of the trajectories generated by matrix products. In this context, the above mentioned methods are not enough satisfactory (from the point of view of the author, of course) since they give no description of the structure of the trajectories with the maximal growing rate (or minimal decreasing rate).

In connection with this, in 2005 the author presented one more proof of the counterexample to the Finiteness Conjecture fulfilled in the spirit of the theory of dynamical systems. Unfortunately, the developed approach did not cover the class of matrices considered by Blondel, Theys and Vladimirov. The goal of the present paper is to compensate for this deficiency in the previous approach.

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1. INTRODUCTION

Let $\boldsymbol{A} = \{A_1, \ldots, A_r\}$ be a finite set of real $m \times m$ matrices, and $\|\cdot\|$ be a norm in $\mathbb{R}^m$. Associate with any finite sequence $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \{1, \ldots, r\}^n$ the matrix

$$A_{\boldsymbol{\sigma}} = A_{\sigma_n} \cdots A_{\sigma_2} A_{\sigma_1},$$

and define for any $n \geq 1$ two quantities:

$$\rho_n(\boldsymbol{A}) = \max_{\boldsymbol{\sigma} \in \{1,\ldots,r\}^n} \|A_{\boldsymbol{\sigma}}\|^{1/n}, \qquad \bar{\rho}_n(\boldsymbol{A}) = \max_{\boldsymbol{\sigma} \in \{1,\ldots,r\}^n} \rho(A_{\boldsymbol{\sigma}})^{1/n}.$$

Then there exists the limit

$$\rho(\boldsymbol{A}) = \limsup_{n \to \infty} \rho_n(\boldsymbol{A}),$$

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which does not depend on the choice of the norm $\|\cdot\|$. This limit is called *the joint spectral radius* of the matrix set $\boldsymbol{A}$. Analogously, the limit

$$\bar{\rho}(\boldsymbol{A}) = \limsup_{n\to\infty} \bar{\rho}_n(\boldsymbol{A}),$$

is called *the generalized spectral radius* of the matrix set $\boldsymbol{A}$. As is shown in [2], for finite matrix sets $\boldsymbol{A}$ the quantities $\rho(\boldsymbol{A})$ and $\bar{\rho}(\boldsymbol{A})$ coincide with each other, and for any n the following inequalities hold

$$\bar{\rho}_n(\boldsymbol{A}) \le \bar{\rho}(\boldsymbol{A}) = \rho(\boldsymbol{A}) \le \rho_n(\boldsymbol{A}). \tag{1}$$

In [17] J. C. Lagarias and Y. Wang conjectured that $\bar{\rho}(\boldsymbol{A})$ in fact coincides with $\rho(A_{\boldsymbol{\sigma}})^{1/n}$ for some n and $\boldsymbol{\sigma} \in \{1,\ldots,r\}^n$. The first counterexample to this conjecture (which got the name *the Finiteness Conjecture*) was proposed in [5], and the corresponding proof was essentially based on the measure-theoretical ideas. Later, another proof [3, 4] of the counterexample to the Finiteness Conjecture appeared which extensively exploited combinatorial properties of permutations of products of positive matrices.

In [11], one more proof of the counterexample to the Finiteness Conjecture fulfilled in a rather traditional manner of the theory of dynamical systems was given. The proof was based on the technique of the so called Barabanov norms [1] (closely related with the usage of functionals Mañé in [5]) and associated with them extremal trajectories for analysis of "the fastest growing trajectories" generated by matrix sets.

Unfortunately, proofs suggested in [11, 12] did not cover the "boundary" situation investigated in [3, 4]. In a private discussion Vladimirov conjectured that for the matrix sets studied in [3, 4] the generalized spectral radius may be attained on infinite non-periodic matrix products different from those described in [11, 12]. In the present work a modified proof of constructions from [11, 12] is proposed which fully covers the matrix sets considered in [3, 4] and so disproves the conjecture by Vladimirov.

The structure of the paper is as follows. In Section 2 we recall basics of the theory of difference equations and inclusions, so as simplest properties of the irreducible matrix sets. Section 3 is devoted to the study of general facts related to the Barabanov norms. Here we prove compactness and uniform equivalence of all the Barabanov norms corresponding to the irreducible matrix sets $\boldsymbol{A}$; the corresponding proofs have something in common with those from [21]. In Section 3 we introduce the key notion of the paper, the notion of the extremal trajectories corresponding to the Barabanov norms, i.e., such trajectories which provide the maximal rate of growth $\bar{\rho}(\boldsymbol{A}) = \rho(\boldsymbol{A})$ amongst all the trajectories generated by the matrix set $\boldsymbol{A}$. Here it is shown also that the extremal trajectories of a matrix set $\boldsymbol{A}$ may be obtained as the trajectories of some nonlinear discontinuous map called "the generator of extremal trajectories." In Section 5 for the matrix sets $\boldsymbol{A} = \{A_0, A_1\}$ consisting of a pair of two-dimensional matrices of a special kind with non-negative entries some extra properties of the Barabanov norms are established. In particular, here we study the structure of the unit ball in a Barabanov norm and prove that Barabanov norms are monotone with respect to the cone of vectors with non-negative coordinates. Then, in the same manner as in [11, 12], with the use of the technique of the Gram symbols borrowed from [5], we investigate the structure of the so-called "switching sets" of Barabanov norms playing the principal role in description of the properties of extremal trajectories. In Section 6 we recall fundamental facts of the technique of the so-called rotation numbers for discontinuous orientation preserving circle maps [6, 9, 10], with the help of which we fulfil the analysis of the frequency properties of the extremal trajectories. As a result, we succeed to show that for the extremal trajectories it is well defined the frequency $\sigma(\boldsymbol{A})$ of applying the matrix A_1 in construction of the extremal trajectory. Moreover, this frequency is a continuous invariant of the matrix set $\boldsymbol{A}$. At last, in Section 7 it is shown that the value $\sigma(\boldsymbol{A})$ takes rational values if and only if the generalized spectral radius $\bar{\rho}(\boldsymbol{A}) = \rho(\boldsymbol{A})$ is attained on a periodic matrix sequence the existence of a matrix set $\boldsymbol{A}$ with an irrational frequency $\sigma(\boldsymbol{A})$. After this, to show that the Finiteness Conjecture is generally not valid it remains only to prove existence of at least one matrix set $\boldsymbol{A}$ for which the value of $\sigma(\boldsymbol{A})$ is irrational. The proofs of all the statements from Sections 2–7 are relegated to the Appendix.

2. TRAJECTORIES OF MATRIX SETS

One of the important problem in the study of properties of matrix sets $\boldsymbol{A} = \{A_1, \ldots, A_r\}$ is how the joint (generalized) spectral radius $\rho(\boldsymbol{A})$ is related with the rate of growth of solutions of the difference inclusion

$$x_{n+1} \in \{A_1, \ldots, A_r\} x_n, \tag{2}$$

in which the value of x_{n+1} is chosen from the set of vectors $\{A_1 x_n, \ldots, A_r x_n\}$. Notice that each solution of inclusion (2) is defined for all $n \geq 0$ and, with some choice of the index sequence $\{\sigma_n\}$, satisfies the equation

$$x_{n+1} = A_{\sigma_n} x_n, \qquad \sigma_n \in \{1, \ldots, r\}. \tag{3}$$

Clearly, the converse is also true, which means that each solution of the difference equation of the type (3) corresponding to some index sequence $\{\sigma_n\}$ is a solution of inclusion (2). To formulate further properties of the solutions of inclusion (2) we recall some definitions and commonly known facts.

In what follows solutions of inclusion (2) will be referred to as *trajectories* defined by the matrix set $\boldsymbol{A}$ or simply trajectories of the matrix set $\boldsymbol{A}$. The set of all trajectories of the matrix set $\boldsymbol{A}$ will be denoted as $\mathcal{T}(\boldsymbol{A})$, the set of all trajectories $\boldsymbol{x} = \{x_n\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ satisfying the initial condition $x_0 = x$ will be denoted as $\mathcal{T}(\boldsymbol{A}, x)$. In general, for $r > 1$ the map

$$x \mapsto \mathcal{T}(\boldsymbol{A}, x)$$

is set-valued. In connection with this recall some definitions and basic facts of the theory of set-valued maps (see, e.g., [16, §18]).

Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces and f be a map associating with each element $x \in \mathbb{X}$ a set $f(x) \subseteq \mathbb{Y}$. Then the map f is called set-valued or multi-valued. The map f is called *upper semi-continuous* at a point $x \in \mathbb{X}$ if for any open set $\mathcal{U} \ni f(x)$ there is an open set $\mathcal{V} \ni x$ such that $f(\mathcal{V}) \subseteq \mathcal{U}$.[1] *The graph of the map* f is the set

$$\mathrm{Gr}(f) = \{(x, y) : \ x \in \mathbb{X}, y \in f(x)\} \subseteq \mathbb{X} \times \mathbb{Y}.$$

The map f is called *closed* (*compact*) if for any closed (compact) set $\mathcal{G} \subseteq \mathbb{X}$ the set $f(\mathcal{G}) \subseteq \mathbb{Y}$ is also closed (compact). Clearly, each compact map is closed.

Recall without proofs some commonly known properties of set-valued maps.

Lemma 1. *Let $x \in \mathbb{X} \mapsto f(x) \subseteq \mathbb{Y}$ be a set-valued map and let the space $\mathbb{Y}$ be regular.*[2] *Then the following statements are valid:*

(i) if the map f is closed and upper semi-continuous then its graph is closed in $\mathbb{X} \times \mathbb{Y}$;

(ii) if the map f is compact and its graph is closed then it is upper semi-continuous;

(iii) the map f is compact and upper semi-continuous if and only if, given a converging sequence $\{x_n \in \mathbb{X}\}$, any sequence $\{y_n \in \mathbb{Y}\}$ satisfying $y_n \in f(x_n)$ is compact and the limiting elements x_* and y_* of the sequences $\{x_n\}$ and $\{y_n\}$, respectively, are bounded by the inclusion $y_* \in f(x_*)$.

[1] Here, the notation $f(\mathcal{V})$ is used to denote the set $\cup_{y \in \mathcal{V}} f(y)$.

[2] A topological space $\mathbb{X}$ is called regular if for any its closed set $\mathcal{G}$ and point $x \notin \mathcal{G}$ there are open sets $\mathcal{U}$ and $\mathcal{V}$ such that $x \in \mathcal{U}$, $\mathcal{G} \subset \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. For example, any metric space is regular. In particular, spaces $\mathbb{R}^m$ and $\mathcal{M}_{m,r}$ are regular.

Denote the set of all ordered r-tuples $\boldsymbol{A} = \{A_1, \dots, A_r\}$ of real $m \times m$ matrices by $\mathcal{M}_{m,r}$. Then the set $\mathcal{M}_{m,r}$ may be identified in a natural way with $\mathbb{R}^{rm^2}$ if to treat entries of the matrices from $\boldsymbol{A}$ enumerated in some predefined order as coordinates in $\mathbb{R}^{rm^2}$. This allows to treat $\mathcal{M}_{m,r}$ as a topological or, when needed, a metric space.

Denote the space of sequences $\{x_n\}_{n=0}^{\infty}$ endowed with the topology of point-wise convergence by $\Omega(\mathbb{R}^m)$. At last, the subset

$$\Omega_n = \{x :\ \exists\, \boldsymbol{x} = \{x_n\}_{n=0}^{\infty} \in \Omega :\ x_n = x\}.$$

of $\mathbb{R}^m$ consisting of n-th elements of the sequences from the set $\Omega \subseteq \Omega(\mathbb{R}^m)$ will be called *the n-section* of the set Ω. Point out that the set Ω is compact in the space $\Omega(\mathbb{R}^m)$ provided that each its section Ω_n is bounded.

Now, we are able to formulate properties of the trajectories of matrix sets needed in what follows.

Lemma 2. *For any matrix set $\boldsymbol{A}$ the set of trajectories $\mathcal{T}(\boldsymbol{A})$ is closed in the space $\Omega(\mathbb{R}^m)$, and the map $(\boldsymbol{A}, x) \mapsto \mathcal{T}(\boldsymbol{A}, x)$ is compact and upper semi-continuous.*

This Lemma is a simple corollary of the compactness criterium in the sequence space $\Omega(\mathbb{R}^m)$, so its proof is omitted.

In what follows, our prime point of interest will be the so-called irreducible matrix sets. In connection with this, recall that the matrix set $\boldsymbol{A}$ is called *irreducible* if the matrices from $\boldsymbol{A}$ have no common invariant spaces except $\{0\}$ and $\mathbb{R}^m$. In [13–15] such a matrix set was called *quasi-controllable*.

3. BARABANOV NORMS: GENERAL CASE

In the analysis of the properties of the joint spectral radius ideas introduced by N. E. Barabanov in [1] play an important role. These ideas were further developed in a number of publications amongst which we distinguish [20].

Theorem (Barabanov). *Let the matrix set $\boldsymbol{A} = \{A_1, \dots, A_r\}$ be irreducible. Then the quantity ρ is the joint (generalized) spectral radius of $\boldsymbol{A}$ if and only if there exists a norm $\|\cdot\|$ in $\mathbb{R}^m$ such that*

$$\rho\|x\| = \max\left\{\|A_0 x\|, \|A_1 x\|, \dots, \|A_r x\|\right\}. \tag{4}$$

A norm satisfying (4) will be called *a Barabanov norm* corresponding to the matrix set $\boldsymbol{A}$. Clearly, if $\|\cdot\|$ is a Barabanov norm than it is *an extremal norm*, i.e., it satisfies the relations

$$\|A_i x\| \le \rho\|x\|, \qquad \forall\, A_i \in \boldsymbol{A}.$$

So as the Barabanov norm, the extremal norms play an important role in different problems arising during study of matrix products (see., e.g., [13–15, 20]). Notice, that for an arbitrary norm $\|\cdot\|_0$ in $\mathbb{R}^m$ and for any irreducible matrix set $\boldsymbol{A}$ the formula

$$\|x\| = \sup_{n\ge 0} \frac{1}{\rho^n(\boldsymbol{A})} \max_{\boldsymbol{\sigma}\in\{1,\dots,r\}^n} \|A_{\boldsymbol{\sigma}} x\|_0$$

defines [8, 15] an extremal norm $\|x\|$, while the formula

$$\|x\| = \limsup_{n\to\infty} \frac{1}{\rho^n(\boldsymbol{A})} \max_{\boldsymbol{\sigma}\in\{1,\dots,r\}^n} \|A_{\boldsymbol{\sigma}} x\|_0$$

defines [1] a Barabanov norm $\|x\|$. Unfortunately, the formulae presented above have almost no practical use.

Remark that in the Barabanov Theorem it is sufficient to suppose that $\|\cdot\|$ in (4) is not a norm but only a semi-norm. The validity of this statement follows from the next Lemma.

Lemma 3. *Let the matrix set $\boldsymbol{A}$ be irreducible. Then any semi-norm $\|\cdot\|$ satisfying (4) is a norm provided that it does not equal identically to zero.*

To prove this Lemma note that the kernel of the semi-norm $\|\cdot\|$ is a subspace $\mathcal{L}$ which, due to the supposition that the semi-norm is not identically zero, does not coincide with the whole space $\mathbb{R}^m$, i.e., $\mathcal{L} \neq \mathbb{R}^m$. If additionally $\mathcal{L} \neq \{0\}$ then from the irreducibility of the matrix set $\boldsymbol{A}$ it follows the existence of such a vector $x_* \in \mathcal{L}$ for which $A_i x_* \notin \mathcal{L}$ for some i. Then by the definition of the subspace $\mathcal{L}$ the following two relations should be valid simultaneously: $\|x_*\| = 0$ and $\|A_i x_*\| \neq 0$, which contradicts to (4). From the obtained contradiction it follows that $\mathcal{L} = \{0\}$, and so $\|\cdot\|$ is a norm.

Note that the set of Barabanov norms possesses a variety of strong properties which will be shown below. Denote by $N_{\mathrm{Bar}}(\boldsymbol{A}, x_0)$, where $x_0 \neq 0 \in \mathbb{R}^m$, the set of all Barabanov norms $\|\cdot\|$ corresponding to the matrix set $\boldsymbol{A}$ which satisfy the calibrating condition $\|x_0\| = 1$. The notation $C_{\mathrm{loc}}(\mathbb{R}^m)$ will be used for the linear topological space of continuous functions defined on $\mathbb{R}^m$ with the topology of uniform convergence on bounded subsets from $\mathbb{R}^m$.

Theorem 1. *Let $x_0 \neq 0 \in \mathbb{R}^m$ and let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there exists a compact neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that the map $\boldsymbol{A}' \mapsto N_{\mathrm{Bar}}(\boldsymbol{A}', x_0)$ where $\boldsymbol{A}' \in \mathcal{A}$ is compact and upper semi-continuous.*

The proof of Theorem 1 is given in the Appendix. Theorem 2 presented below is an important supplement to Theorem 1.

Let $\|\cdot\|_0$ be a norm in $\mathbb{R}^m$ which will play the role of a calibrating norm, i.e., such a norm with which all other norms in $\mathbb{R}^m$ are compared.

As is known, all norms in $\mathbb{R}^m$ are equivalent, so for any norm $\|\cdot\|$ there are constants $\Delta, \delta > 0$ such that

$$\delta \|x\|_0 \leq \|x\| \leq \Delta \|x\|_0.$$

Clearly, in general, there are no universal constants $\Delta, \delta > 0$ since for any given constants $\Delta, \delta > 0$ the multiplication of the norm $\|\cdot\|$ by a number easily breaks the above inequalities. Hence, it is meaningful to compare with $\|\cdot\|_0$ only such norms which are calibrated beforehand, i.e., which, for example, take the same values at a some predefined point $x_0 \neq 0$.

In this case the following question may be posed: are there constants $\Delta, \delta > 0$ for which the inequalities

$$\delta \|x\|_0 \leq \frac{\|x\|}{\|x_0\|} \leq \Delta \|x\|_0$$

hold? Still, even in this case the question posed above has the negative answer for arbitrary norms $\|\cdot\|$. At the same time, if we consider only Barabanov norms $\|\cdot\|$ then *the universal constants* $\Delta, \delta > 0$ *exist!* The corresponding fact is a corollary of the Theorem presented below and which is formulated in a universal form independent from the choice of an auxiliary vector x_0.

Theorem 2. *For any irreducible set of $m \times m$ matrices $\boldsymbol{A}$ there are neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ and constants $0 < \delta \leq \Delta < \infty$ such that for any pair of Barabanov norms $\|\cdot\|'$ and $\|\cdot\|''$ corresponding* *to the matrix sets* $\boldsymbol{A}', \boldsymbol{A}'' \in \mathcal{A}$, *respectively, the following estimates*

$$\frac{\delta^2}{\Delta^2}\frac{\|x\|''}{\|y\|''} \leq \frac{\|x\|'}{\|y\|'} \leq \frac{\Delta^2}{\delta^2}\frac{\|x\|''}{\|y\|''} \qquad \forall x, y \neq 0 \in \mathbb{R}^m,$$

are valid.

The proof of Theorem 2 is given in the Appendix.

Note in conclusion that in addition to topological properties formulated above, Barabanov norms possess also some algebraic structure.

Lemma 4. *Let* $\|\cdot\|'$ *and* $\|\cdot\|''$ *be Barabanov norms corresponding to a matrix set. Then* $\|x\| = \max\{\|x\|', \|x\|''\}$ *is also a Barabanov norm corresponding to the same matrix set.*

Proof of this Lemma is evident.

4. EXTREMAL TRAJECTORIES AND THEIR GENERATORS

Introduce some notions. A trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ will be called *characteristic* if there are constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \leq \rho^{-n}(\boldsymbol{A})\|x_n\| \leq c_2 \qquad \forall n.$$

Remark that the definition of a characteristic trajectory does not depend on the choice of the norm $\|\cdot\|$ in $\mathbb{R}^m$. An important particular case of characteristic trajectories are so-called extremal trajectories. A trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ will be called *extremal (B-extremal)* if in some extremal (Barabanov) norm $\|\cdot\|$ the following identities hold:

$$\rho^{-n}(\boldsymbol{A})\|x_n\| \equiv \text{const.} \tag{5}$$

In contrast to the definition of characteristic trajectories the definition of extremal trajectories depends on the choice of the extremal norm. So, a trajectory extremal in one norm may be not extremal in another. Nevertheless, as will be shown below in Theorem 3 for an irreducible matrix set one can always find extremal trajectories which are *universal* in the that such trajectories are extremal in each extremal norm.

Now we prove that the set of B-extremal trajectories, and consequently the corresponding sets of extremal and characteristic trajectories are not empty in the case when the matrix set $\boldsymbol{A}$ is irreducible.

Lemma 5. *For any vector* $x \neq 0 \in \mathbb{R}^m$ *and any Barabanov norm* $\|\cdot\|$ *there is an extremal trajectory* $\{x_n\}$ *satisfying* $x_0 = x$.

To prove this Lemma construct recursively the trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ satisfying $x_0 = x$. Suppose that the element x_n is already found. Then, by the definition of the Barabanov norm, the following equality is valid:

$$\rho(\boldsymbol{A})\|x_n\| = \max\{\|A_0x_n\|, \|A_1x_n\|, \ldots, \|A_rx_n\|\}.$$

Hence, there exists an index σ_n for which $\rho(\boldsymbol{A})\|x_n\| = \|A_{\sigma_n}x_n\|$. So, in order to satisfy conditions (2), (5) it is sufficient to define the element x_{n+1} by the equality $x_{n+1} = A_{\sigma_n}x_n$.

Corollary. *If the matrix set* $\boldsymbol{A}$ *is irreducible then the sets of its extremal, B-extremal and characteristic trajectories are nonempty.*

The proof of this Corollary immediately follows from the Barabanov Theorem asserting that for an irreducible matrix set the set of Barabanov norms is not empty, and from Lemma 5 according to which in this case the set of corresponding B-extremal trajectories is also nonempty.

Above, it was mentioned that the definition of extremal trajectories depends on the choice of the extremal norm. Nevertheless, as shows the next Theorem, amongst all extremal trajectories there are in a sense “universal” trajectories.

Theorem 3. *For any irreducible matrix set $\boldsymbol{A}$ there are trajectories which are extremal with respect to any norm extremal for the matrix set $\boldsymbol{A}$.*

The proof of Theorem 3 is given in the Appendix.

¿From now on the main attention will ba paid to investigation of properties of the B-extremal trajectories. Denote by $\mathcal{E}_{\mathrm{Bar}}(\boldsymbol{A}, x)$ the set of all B-extremal trajectories $\{x_n\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ satisfying the initial condition $x_0 = x \neq 0$.

Theorem 4. *Let $\mathcal{X} \subset \mathbb{R}^m$ be a compact set which does not contain the origin and let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there is a compact neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that the map $(\boldsymbol{A}', x) \mapsto \mathcal{E}_{\mathrm{Bar}}(\boldsymbol{A}', x)$ where $\boldsymbol{A}' \in \mathcal{A}$, $x \in \mathcal{X}$, is compact and upper semi-continuous.*

The proof of Theorem 4 is given in the Appendix.

In order to describe completely a B-extremal trajectory $\boldsymbol{x} = \{x_n\}$ one should know not only the information about the sequence $\{x_n\}$ but also the information about the related index sequence $\{\sigma_n\}$. Below, it will be proposed a construction which determines B-extremal trajectories as all possible trajectories of some set-valued nonlinear dynamical system. Such a construction will allow us to avoid the necessity to describe explicitly the index sequence $\{\sigma_n\}$.

Let $\rho = \rho(\boldsymbol{A})$ and let $\|\cdot\|$ be a Barabanov norm corresponding to the matrix set $\boldsymbol{A} = \{A_1, \ldots, A_r\}$. Denote for each $x \in \mathbb{R}^m$ the map $g(x)$ by setting

$$g(x) := \{w : \exists i \in \{1, \ldots, r\} \text{ for which } w = A_i x, \text{ with } \|A_i x\| = \rho\|x\|\}.$$

By the definition of a Barabanov norm the set $g(x)$ for each $x \in \mathbb{R}^m$ is not empty and consists of no more than m elements. Note also that each map $g(x)$ has a closed graph and for it the following identity is valid

$$\|g(x)\| \equiv \rho\|x\|. \tag{6}$$

Lemma 6. *The sequence $\boldsymbol{x} = \{x_n\}$ is extremal for the matrix set $\boldsymbol{A}$ in the Barabanov norm $\|\cdot\|$ if and only if it satisfies the inclusions*

$$x_{n+1} \in g(x_n) \qquad \forall n.$$

The proof of this Lemma immediately follows from the definitions of the Barabanov norm and the map g.

According to Lemma 6 each trajectory of the set-valued map $g(\cdot)$ is extremal in the Barabanov norm $\|\cdot\|$. This motivates us to call the map $g(\cdot)$ as *the generator of B-extremal trajectories*. In general, the map $g(\cdot)$ can not be described explicitly. Nevertheless, in Section 6 for the sets of 2×2 matrices we will be able to obtain a rather detailed description of the properties of the generators of B-extremal trajectories.

5. BARABANOV NORMS: THE CASE OF A PAIR OF TWO-DIMENSIONAL MATRICES

In this Section, for the case of matrix sets consisting of two 2×2 matrices some additional properties of Barabanov norms and B-extremal trajectories are established.

Consider the pair of matrices

$$A_0 = \alpha \left\| \begin{matrix} a & b \\ 0 & 1 \end{matrix} \right\|, \quad A_1 = \beta \left\| \begin{matrix} 1 & 0 \\ c & d \end{matrix} \right\|, \tag{7}$$

where $\alpha, \beta > 0$ and

$$bc \geq 1 \geq a, d > 0. \tag{8}$$

Associate the ray $t(x_0, x_1)$, $t > 0$, passing the point $(x_0, x_1) \neq 0$, $x_0, x_1 \geq 0$, with the number $\xi = x_1/(x_0 + x_1) \in [0, 1]$. Under such association the semi-axis of abscissas, i.e., the ray $t(1, 0)$, is represented by the number $\xi = 0$, while the semi-axis of ordinates, i.e., the ray $t(0, 1)$, is represented by the number $\xi = 1$. Then the matrix A_0 maps the ray with the coordinate ξ at the ray with the coordinate $\varphi_0(\xi)$, where

$$\varphi_0(\xi) = \frac{\xi}{a(1-\xi) + b\xi + \xi}, \tag{9}$$

while the matrix A_1 maps the ray with the coordinate ξ at the ray with the coordinate $\varphi_1(\xi)$:

$$\varphi_1(\xi) = \frac{c(1-\xi) + d\xi}{c(1-\xi) + d\xi + 1 - \xi}. \tag{10}$$

Consider also the pair of matrices conjugate to the matrices A_0 and A_1:

$$A_0' = \alpha \left\| \begin{matrix} a & 0 \\ b & 1 \end{matrix} \right\|, \quad A_1' = \beta \left\| \begin{matrix} 1 & c \\ 0 & d \end{matrix} \right\|.$$

Then the matrix A_0' maps the ray with the coordinate ξ at the ray with the coordinate $\psi_0(\xi)$, where

$$\psi_0(\xi) = \frac{b(1-\xi) + \xi}{a(1-\xi) + b(1-\xi) + \xi},$$

while the matrix A_1' maps the ray with the coordinate ξ at the ray with the coordinate $\psi_1(\xi)$:

$$\psi_1(\xi) = \frac{d\xi}{1 - \xi + c\xi + d\xi}.$$

Under condition (8) for any $0 \leq \xi, \zeta \leq 1$ the inequalities $\varphi_1(\xi) \geq \varphi_0(\zeta)$ hold, while the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ strictly increase. Hence, the graphs of the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ look like those plotted in Fig. 1. Analogously, the graphs of the functions $\psi_0(\xi)$ and $\psi_1(\xi)$ look like those plotted in Fig. 2.

Denote by $\mathcal{M}^\sharp \subset \mathcal{M}_{2,2}$ the set of all matrix sets $\boldsymbol{A}$ consisting of the matrices A_0 and A_1 of the form (7) satisfying conditions (8). Then, from the description of the invariant spaces for the matrices A_0 and A_1, we immediately get the following Lemma.

Lemma 7. *Each matrix set* $\boldsymbol{A} \in \mathcal{M}^\sharp$ *is irreducible.*

Given some Barabanov norm $\|\cdot\|$ in $\mathbb{R}^2$ corresponding to $\boldsymbol{A}$, denote by $\mathbb{S}$ the unit ball in the norm $\|\cdot\|$. Recall that the linear functional $l(x)$, $x \in \mathbb{R}^2$ is called *the support functional* for the unit ball $\mathbb{S}$ if

$$\sup_{x \in \mathbb{S}} |l(x)| \leq 1, \qquad \text{and} \quad \exists\, u_* \in \mathbb{S}: \quad l(u_*) = 1.$$

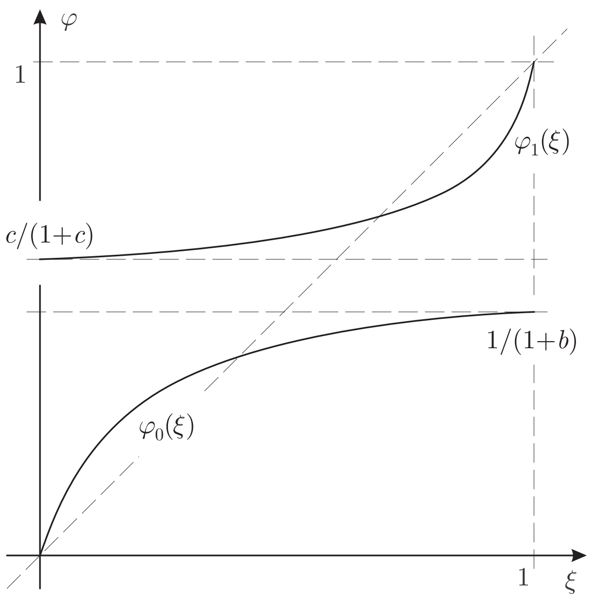

Fig. 1. Plots of functions $\varphi_0(\xi)$, $\varphi_1(\xi)$

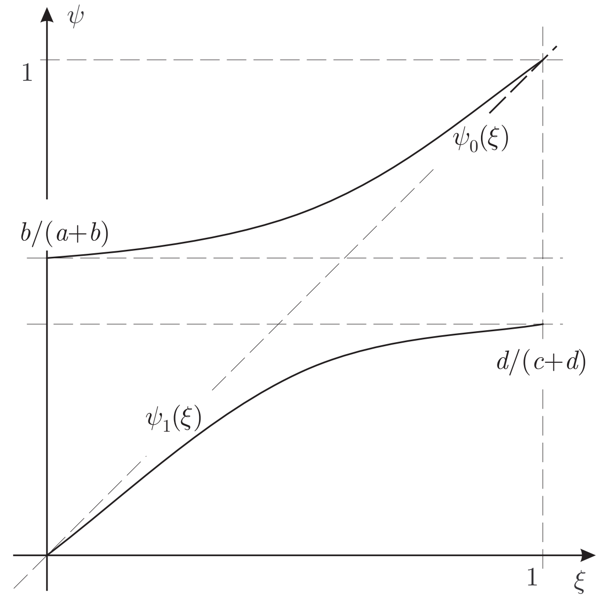

Fig. 2. Plots of functions $\psi_0(\xi)$, $\psi_1(\xi)$

By the Khan-Banach theorem for any point $u_* \in \mathbb{S}$, $\|u_*\| = 1$, there is a support functional l_* for which $l_*(u_*) = 1$. Remark that each functional $l(x)$ can be represented by a linear form:

$$l(x) \equiv \langle l, x\rangle := l_0x_0 + l_1x_1, \quad \text{where} \quad l = (l_0, l_1),\ x = (x_0, x_1) \in \mathbb{R}^2,$$

and so, the values l_0, l_1 may be treated as the coordinates of the functional $l(x)$.

Lemma 8. *Let $\|\cdot\|$ be a Barabanov norm corresponding to the matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$. Then for any vector $u \in \mathbb{S}$, $\|u\| = 1$, with non-negative coordinates the support functional $l(x) = \langle l, x\rangle$ satisfying $l(u) = 1$ is also has non-negative coordinates. In other words, the unit ball in the norm $\|\cdot\|$ in the first quadrant has the form like that presented in Fig. 3.*

The proof of Lemma 8 is given in the Appendix.

Call the norm $\|\cdot\|$ *monotone* (with respect to the cone of the vectors with non-negative coordinates) if for any pair of vectors u and v the relations $v \geq u \geq 0$, where the inequalities are understood coordinate-wise, imply the inequality $\|v\| \geq \|u\|$. Then from the description of the structure of the boundary of the unit ball of the Barabanov norm given in Lemma 8, and from Fig. 3 in which the point set $\{v : v \geq u\}$ is plotted we obtain the next Lemma.

Lemma 9. *Any Barabanov norm corresponding to a matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$ is monotone.*

Define the sets

$$X_0 = \{x :\ \|A_0x\| = \rho\|x\|\}, \qquad X_1 = \{x :\ \|A_1x\| = \rho\|x\|\}. \tag{11}$$

Each of these sets is closed, conic (i.e., contains any vector tx along with the vector $x \neq 0$), and by the definition of a Barabanov norm $X_0 \cup X_1 = \mathbb{R}^2$. The set $\varTheta = X_0 \cap X_1$ will be called *the switching set* of the Barabanov norm $\|\cdot\|$.

Theorem 5. *Let $\boldsymbol{A} = \{A_0, A_1\}$ be the matrix set defined by equalities (7) and satisfying conditions (8), and let $\|\cdot\|$ be a Barabanov norm of the matrix set $\boldsymbol{A}$. Then each of the sets $X_0 \cap K_+$ and*

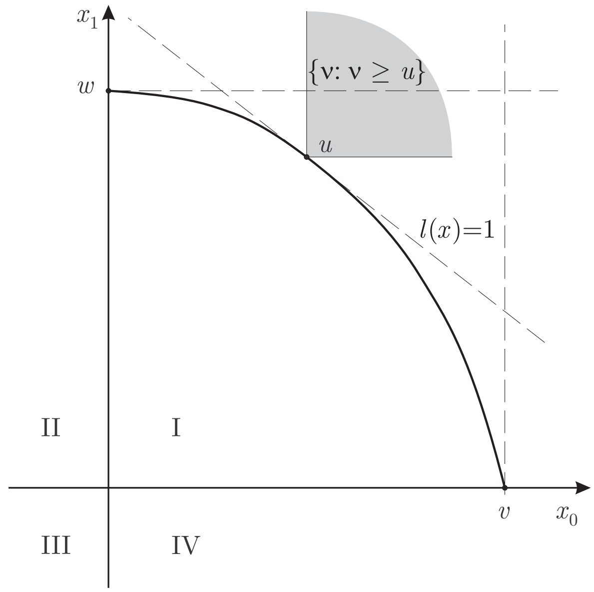

Fig. 3. The unit ball of a Barabanov norm

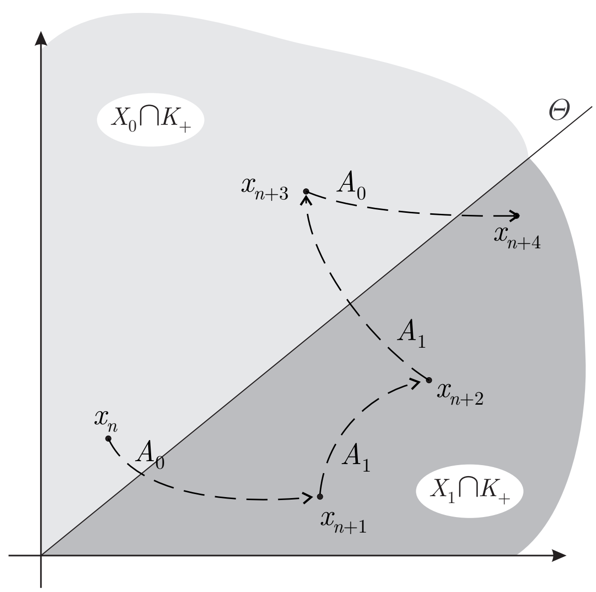

Fig. 4. Location of the sectors $X_0 \cap K_+$ and $X_1 \cap K_+$ and of typical extremal trajectory

$X_1 \cap K_+$ is a sector with non-empty interior, the set $X_1 \backslash X_0$ has a nonempty intersection with the abscissa axis and the set $X_0 \backslash X_1$ has a nonempty intersection with the ordinate axis. Intersection of the sectors $X_0 \cap K_+$ and $X_1 \cap K_+$ is the ray

$$\Theta = X_0 \cap X_1 \cap K_+ = \{t\vartheta :\ t \in \mathbb{R}_+\} \tag{12}$$

passing a normed vector $\vartheta \in K_+$ (see Fig. 4) which is uniquely determined by the system of equations

$$\|A_0\vartheta\| = \|A_1\vartheta\|, \quad \|\vartheta\| = 1, \qquad \vartheta \in K_+, \tag{13}$$

and continuously depends on the matrices A_0, A_1 and the norm $\|\cdot\|$.

The proof of Theorem 5 is given in the Appendix. Remark that according to Lemma 6 and Theorem 5 for the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ the extremal trajectories $\{x_n\}$ are the sliding modes of some linear system of variable structure (see., e.g., [18, 19]): to obtain the next element x_{n+1} of the extremal trajectory one need apply the matrices A_0 or A_1 to the element x_n depending on whether x_n belongs to the sector $X_0 \cap K_+$ or $X_1 \cap K_+$, respectively (see Fig. 4).

6. FREQUENCY PROPERTIES OF EXTREMAL TRAJECTORIES: THE CASE OF TWO-DIMENSIONAL MATRICES

In this Section, the analysis of the properties of the B-extremal trajectories of the matrix sets $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ will be continued. Our prime goal will be to prove the following statement.

Theorem 6 (on the switching frequency). *For any B-extremal trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ determined by the equation*

$$x_{n+1} = A_{\sigma_n} x_n, \qquad n = 0, 1, \dots\ ,$$

it is defined the frequency (the switching frequency of the trajectory)

$$\sigma = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sigma_i}{n}$$

of applying the matrix A_1 in the process of computation of the trajectory $\{x_n\}$.

The frequency σ does not depend on the choice of the extremal trajectory $\{x_n\}$ or on the index sequence $\{\sigma_n\}$, and so, it may be denoted as $\sigma(\boldsymbol{A})$. In addition, $\sigma(\boldsymbol{A})$ depends continuously on matrices of the matrix set $\boldsymbol{A}$ and takes rational values if and only if the matrix set $\boldsymbol{A}$ has a B-extremal trajectory corresponding to a periodic index sequence $\{\sigma_n\}$.

To prove Theorem 6, we will need auxiliary statements and constructions.

6.1. Generator of Extremal Trajectories

Fix in $\mathbb{R}^2$ a Barabanov norm $\|\cdot\|$ corresponding to the matrix set $\boldsymbol{A}$, and denote by X_0 and X_1 sets (11) determined by this norm. In this case, the generator of B-extremal trajectories $g(\cdot)$ (see the definition in Section 4) will take the form

$$g(x) = \begin{cases} A_0 x, & \text{if} \quad x \in X_0 \backslash X_1 \\ A_1 x, & \text{if} \quad x \in X_1 \backslash X_0 \\ \{A_0 x, A_1 x\}, & \text{if} \quad x \in X_0 \cap X_1. \end{cases} \tag{14}$$

Let us study the structure of the map $g(\cdot)$ in the first quadrant, i.e., in the cone $K_+ := \{x = (x_1, x_2) :\ x_1, x_2 \geq 0\}$, in more details. Introduce in K_+ the coordinate system (λ, ξ) by setting

$$\lambda(x) = \|x\|, \quad \xi(x) = \frac{x_2}{x_1 + x_2}, \qquad x \neq 0 \in K_+. \tag{15}$$

As was noted above (see (6)), for the map $g(\cdot)$ the identity $\|g(x)\| \equiv \|x\|$ is valid. Besides, by Theorem 5 the sets $X_0 \cap K_+$, $X_1 \cap K_+$ and $X_0 \cap X_1 \cap K_+$ are transferred by the map $\xi(\cdot)$ in the intervals $[\theta, 1]$, $[0, \theta]$ and a point $\theta \in (0, 1)$, respectively, i.e.,

$$\xi(X_1 \cap K_+) = [0, \theta], \qquad \xi(X_0 \cap K_+) = [\theta, 1], \qquad \xi(X_0 \cap X_1 \cap K_+) = \theta.$$

Then in the coordinate system (λ, ξ) the map g takes the form of a map with separable variables

$$g : (\lambda, \xi) \mapsto (\rho\lambda, \varPhi), \tag{16}$$

where $\rho = \rho(\boldsymbol{A})$ and

$$\varPhi = \varPhi_\theta(\xi) = \begin{cases} \varphi_1(\xi), & \text{if} \quad \xi \in [0,\theta) \\ \{\varphi_0(\theta), \varphi_1(\theta)\}, & \text{if} \quad \xi = \theta \\ \varphi_0(\xi), & \text{if} \quad \xi \in (\theta, 1]. \end{cases} \tag{17}$$

Here the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ are defined by (9) and (10), and have the appearance plotted in Fig 1. The graph of the set-valued function $\varPhi_\theta(\xi)$ is presented in Fig. 5.

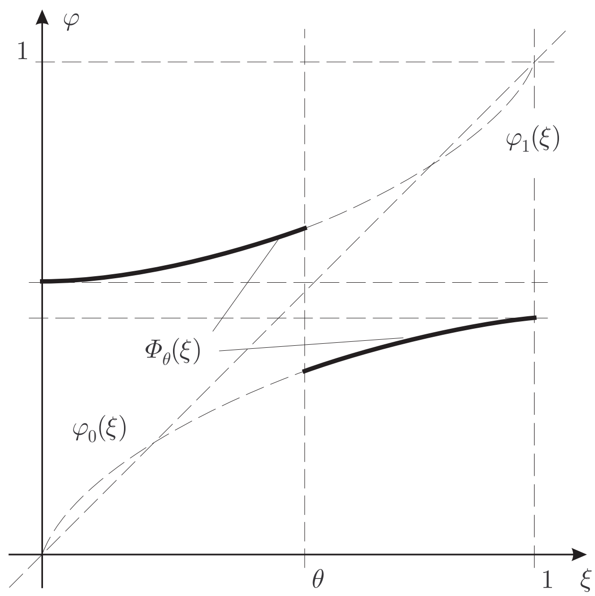

Fig. 5. Graph of the direction function $\varPhi_\theta(\xi)$

Remark that the coordinate $\lambda(x)$ characterizes "remoteness" of the vector x from the origin of coordinates, while the coordinate $\xi(x)$ characterizes the "direction" of the vector x. In accordance with this, $\varPhi_\theta(\xi)$ can be treated as *the direction function* of the generator of B-extremal trajectories.

¿From Lemma 6, Theorem 5 and representation (16), (17) of the map $g(\cdot)$ one can get the following description of the B-extremal trajectories.

Lemma 10. *A nonzero trajectory $\{x_n\} \subseteq K_+$ is extremal for the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ in the Barabanov norm $\|\cdot\|$ if and only if its elements in the coordinate system (λ, ξ) can be represented in the form $x_n = (\lambda_n, \xi_n)$, where $\lambda_n \equiv \lambda_0$, and $\{\xi_n\}$ is a trajectory of the set-valued map $\varPhi_\theta(\cdot)$, i.e.,*

$$\xi_{n+1} \in \varPhi_\theta(\xi_n), \qquad n = 0, 1, \dots ,$$

whose parameter θ satisfies the inclusion $\theta \in (0,1)$.

In addition, the trajectory $\{x_n\}$ satisfies the equation

$$x_{n+1} = A_{\sigma_n} x_n, \qquad n = 0, 1, \dots ,$$

with some index sequence $\{\sigma_n\}$ if and only if the trajectory $\{\xi_n\}$ satisfies the equation

$$\xi_{n+1} = \varphi_{\sigma_n}(\xi_n), \qquad n = 0, 1, \dots .$$

Remark that in spite of the fact that the Barabanov norm $\|\cdot\|$ is, in general, not known explicitly, the direction function $\varPhi_\theta(\xi)$ of the generator of B-extremal trajectories is "defined rather unambiguously" which means that according to (17) it is uniquely defined by the triplet $(\varphi_0, \varphi_1, \theta)$ with the only unknown parameter θ.

When it will be needed to emphasize the dependance of the function $\Phi_\theta(\xi)$ on the triplet $(\varphi_0, \varphi_1, \theta)$, we will use the notation

$$\Phi_\theta(\xi) = \Phi[\varphi_0, \varphi_1, \theta](\xi). \tag{18}$$

In its turn, the triplet $(\varphi_0, \varphi_1, \theta)$ depends on the choice of the matrix set $\boldsymbol{A}$ and the corresponding Barabanov norm $\|\cdot\|$. Therefore, consider in more details the question on how the direction function $\Phi_\theta(\xi)$ depends on the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ and the related Barabanov norm $\|\cdot\|$.

According to (9) and (10), the function φ_0 is uniquely determined by entries of the matrix A_0, while the function φ_1 is completely determined by entries of the matrix A_1. To point out this dependance we will use the notation

$$\varphi_0(\xi) = \varphi_0[A_0](\xi), \qquad \varphi_1(\xi) = \varphi_1[A_1](\xi).$$

At the same time, by Theorem 5 and relations (14), (17) the parameter θ is a single-valued function of the matrix set $\boldsymbol{A}$ and the related Barabanov norm $\|\cdot\|$, i.e.,

$$\theta = \theta[\boldsymbol{A}, \|\cdot\|]. \tag{19}$$

¿From (18), (19) one can see that the direction function $\Phi_\theta(\xi)$ is determined, in the long run, by the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ and by the Barabanov norm $\|\cdot\|$ corresponding to this set; in the cases when we need stress this dependance it will be used the notation

$$\Phi_\theta(\xi) = \Phi[\boldsymbol{A}, \|\cdot\|](\xi).$$

As will be shown in Lemma 11 below, the direction function $\Phi[\boldsymbol{A}, \|\cdot\|]$ depends continuously on the matrix set $\boldsymbol{A}$ and the Barabanov norm $\|\cdot\|$. To make said above meaningful, define first the notion of closeness between set-valued functions on the interval $[0,1]$.

Denote by $\mathcal{F} = \mathcal{F}([0,1])$ the set of all set-valued functions $f : [0,1] \mapsto 2^{\mathbb{R}}$ with the closed graphs. In this case the graph $\mathrm{Gr}(f)$ of the function f is a closed bounded subset of the set $[0,1] \times \mathbb{R} \subset \mathbb{R}^2$, and hence, for any pair of functions $f, g \in \mathcal{F}$ it is defined and finite the value

$$\chi(f,g) = \max\{\sup_{x\in\mathrm{Gr}(f)} \inf_{y\in\mathrm{Gr}(g)} |x-y|, \sup_{y\in\mathrm{Gr}(g)} \inf_{x\in\mathrm{Gr}(f)} |x-y|\},$$

where $|\cdot|$ is some norm in $\mathbb{R}^2$. The value χ is called *the Hausdorff distance* between the graphs of the maps f and g, it is a metric in the space $\mathcal{F}$. In its turn, the space $\mathcal{F}$, being equipped with the metric χ, is complete.

Lemma 11. *Let $x_0 \in \mathbb{R}^2$ be a nonzero vector. Then for any pair $(\boldsymbol{A}, \|\cdot\|)$, where $\boldsymbol{A} \in \mathcal{M}^\sharp$ and $\|\cdot\| \in N_{\mathrm{Bar}}(\boldsymbol{A}, x_0)$, the map $(\boldsymbol{A}, \|\cdot\|) \mapsto \Phi[\boldsymbol{A}, \|\cdot\|]$, is uniquely defined and continuous by the metric of the space $\mathcal{F}$.*

The proof of Lemma 11 is given in the Appendix.

Properties of maps, graphs of which are like those presented in Fig. 5, are studied below in more details.

6.2. Orientation Preserving Discontinuous Circle Maps

Maps of the interval $[0, 1)$ in itself is convenient to treat as maps of the circle $\mathbb{S} \equiv \mathbb{R}/\mathbb{Z}$. Below, we will study, primarily, discontinuous maps of the interval $[0, 1)$. Such maps were studied by different authors (see, e.g., [6, 9, 10] and the bibliography therein), but unfortunately no one of results, known to the author, can be immediately applied to the analysis of the properties of the map $\Phi_\theta(\xi)$. For example, in [6] main results are established for the set-valued maps with connected images while in [9, 10] properties of the single-valued discontinuous maps are investigated, whereas in our case $\Phi_\theta(\xi)$ is a set-valued map with disconnected images. In connection with this, in what follows we will recall basic facts of the theory of orientation preserving discontinuous circle maps, following primarily to the work [6], and then deduce from these results properties of the map $\Phi_\theta(\xi)$ needed below.

Let $\eta : [0, 1) \to [0, 1)$ be some, in general, discontinuous, set-valued function. The function $h : \mathbb{R} \to \mathbb{R}$ is called *the lift* of η if it satisfies conditions

$$h(\xi + 1) \equiv h(\xi) + 1, \tag{20}$$

and

$$\eta(\xi) = h(\xi) \pmod 1 \qquad \xi \in [0, 1). \tag{21}$$

As is easy to see, each circle map has a lift, and conversely, each map h of the straight line in itself satisfying (20) is a lift of the circle map $\eta(\cdot)$ defined by equality (21). Note that there is a plenty of properties of the circle maps which are more convenient to formulate in terms of corresponding lifts than in terms of the original circle maps.

The map $\eta : [0, 1) \to [0, 1)$, treated as a map of the circle $\mathbb{S} \equiv \mathbb{R}/\mathbb{Z}$ in itself, will be called *orientation preserving* if it has a strictly increasing lift[3]. A strictly increasing lift h of the map η will be called *standard* if it satisfies $h(0) = \eta(0)$. The orientation preserving map $\eta : [0, 1) \to [0, 1)$ will be called *closed* or *connectedly closed* if it has a strictly increasing lift with the closed graph, or the graph of some of its strictly increasing lift is a connected and closed set, respectively.

To illustrate notions introduced above, associate with the strictly increasing lift h of the map η the auxiliary maps

$$h_+(\xi) = \lim_{\bar\xi \downarrow \xi} h(\bar\xi), \qquad h_-(\xi) = \lim_{\bar\xi \uparrow \xi} h(\bar\xi),$$

where notations $\bar\xi \downarrow \xi$ and $\bar\xi \uparrow \xi$ are used to denote convergence of the variable $\bar\xi$ to ξ strictly from above or from below, correspondingly. Define also the maps

$$h_*(\xi) = \{h_-(\xi), h_+(\xi)\}, \qquad h_c(\xi) = [h_-(\xi), h_+(\xi)].$$

Directly from the definitions of the maps $h_+(\xi)$, $h_-(\xi)$, $h_*(\xi)$ and $h_c(\xi)$ it follows that all these maps are strictly increasing. The maps $h_+(\xi)$ and $h_-(\xi)$ are single-valued, and the map $h_+(\xi)$ is continuous from the right at each point, while the map $h_-(\xi)$ is continuous from the left at each point. The maps $h_*(\xi)$ and $h_c(\xi)$ are, in general, set-valued and their values coincide with the values of the map $h(\xi)$ at the points, in which the map $h(\xi)$ is single-valued and continuous. In all other points the values of $h_*(\xi)$ consist of exactly two points while the values of $h_c(\xi)$ consist of closed intervals. Besides, the graphs of the both maps $h_*(\xi)$ and $h_c(\xi)$ are closed. It should be noted also that

$$h_+(\xi), h_-(\xi) \in h_*(\xi) \subseteq h_c(\xi) \qquad \forall \xi.$$

[3] Remark that the lift of a circle map is determined non-uniquely. Nevertheless, just as is in the case of continuous lifts of the circle homeomorphisms, any two strictly increasing lifts of the same circle map (provided that they exist) can differ from each other only on an integer constant [9, Lemma 2]. A detailed description of the structure of single-valued discontinuous orientation preserving circle maps and their lifts can be found in [9, 10]. The role of the demand of strict increasing of a lift is discussed in Remark 1.

In addition, if the graph of the map $h(\xi)$ is closed then $h_*(\xi) \subseteq h(\xi) \subseteq h_c(\xi)$. Therefore, it is natural to call the map $h_*(\xi)$ the *minimal closure* of the map $h(\xi)$ while the map $h_c(\xi)$ can be called the *connected* or *maximal closure* of the map $h(\xi)$. Respectively, the map $h(\xi)$ will be called *minimally closed* if $h(\cdot) = h_*(\cdot)$, and it will be called *connectedly* or *maximally closed* if $h(\cdot) = h_c(\cdot)$.

Theorem 7 (see [6]). *Let $\eta : [0,1) \to [0,1)$ be an orientation preserving circle map with a connectedly closed lift h. Let $\{\xi_n\}$ be a trajectory of the map h, i.e.,*

$$\xi_{n+1} \in h(\xi_n), \qquad n = 0, 1, \dots \ . \tag{22}$$

Then the following assertions are valid:

(i) there is a number τ, not depending on the initial value ξ_0, for which the estimates hold

$$\left| \frac{\xi_n}{n} - \tau \right| \le \frac{1}{n}, \qquad n = 1, 2, \dots \ ,$$

and hence

$$\tau = \lim_{n\to\infty} \frac{\xi_n}{n};$$

(ii) if the number τ is rational of the form $\tau = p/q$ with coprime p and q then the map $\eta(\cdot)$ has a periodic point of period q, and any trajectory (22) converges to a periodic trajectory of period q as $n \to \infty$;

(iii) if the number τ is irrational then all trajectories (22) have the same limiting set which is either coincide with the whole circle or is the Cantor set;

(iv) the number τ depends continuously on the graph of the map h in the Hausdorff metric[4].

According to this Theorem the number τ is uniquely determined by the map h and does not depend neither on the choice of the initial point ξ_0 of the trajectory $\{\xi_n\}$ nor on arbitrariness in the construction of the trajectory $\{\xi_n\}$ by formula (22). So, it is reasonable to denote the number τ by $\tau(h)$; this number is called *the rotation number* of the lift h. The value $\tau(h)$ is often called also the rotation number of the circle map η. One should only bear in mind that the rotation number for a circle map is defined modulo integer additives since lifts of the circle map are also defined modulo integer additives. Therefore, sometimes the rotation number of a circle map is defined as $\tau(h) \pmod 1$.

Remark 1. An orientation preserving circle map was defined above as such a circle map which has a strictly increasing lift. Theorem 7 will be no longer valid if to omit the requirement that the corresponding lift increases strictly, which follows from the fact that in this case a circle map may have simultaneously periodic points of different coprime periods as is plotted in Fig. 6 and 7.

The next Remark shows that in Theorem 7 the requirement of the connectedness of the graph of the lift h is not essential. What is important is the closeness of the graph.

Remark 2. All the statements of Theorem 7 continue to be valid for any circle map possessing a strictly increasing closed lift.

[4] The statement means that for any orientation preserving circle map $\hat{\eta}$ with a connectedly closed lift $\hat{h}$ the values of $\hat{\tau}$ tend to τ when the graph of the map $\hat{h}$ tends to the graph of the map h by the Hausdorff metric. Point out that due to condition (20) the Hausdorff distance between the maps h and $\hat{h}$ is defined correctly in spite of the fact that the graphs of these maps are not bounded.

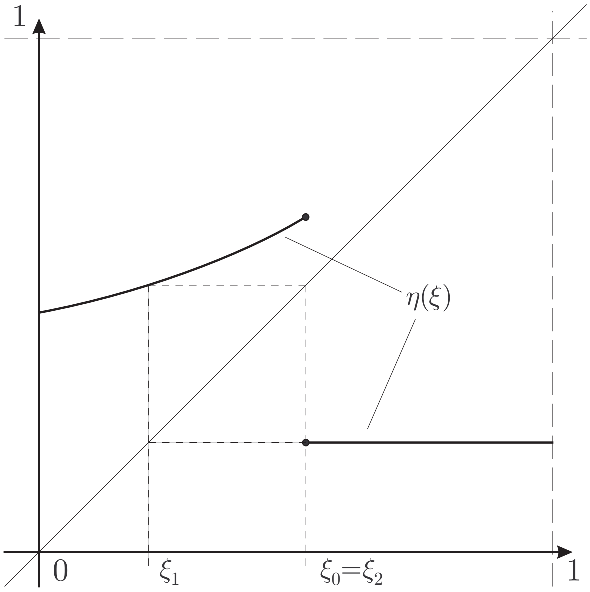

Fig. 6. Periodic point of period 2

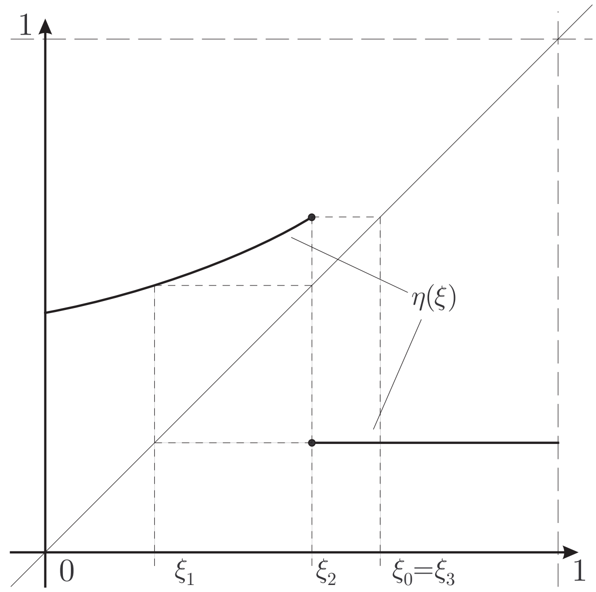

Fig. 7. Periodic point of period 3

To prove this Remark suppose that the circle map $\eta(\xi)$ has a strictly increasing closed lift $h(\xi)$. Consider the connected closure $h_c(\xi)$ of the map $h(\xi)$. Then from the inclusions $h(\xi) \subseteq h_c(\xi)$ valid for any $\xi \in \mathbb{R}$ it follows that each trajectory $\{\xi_n\}$ of the map $h(\xi)$ is also a trajectory of the map $h_c(\xi)$. Hence, the rotation number $\tau(h)$ of the map h is correctly defined and coincides with $\tau(h_c)$, and besides, the limiting set of the trajectory $\{\xi_n\}$ does not depend on the choice of the trajectory in the case when $\tau(h)$ is irrational. If the number $\tau(h)$ is rational then the trajectory $\{\xi_n\}$ of the map h, being at the same time a trajectory of the map h_c, by assertion (iii) of Theorem 7 converges to some periodic trajectory of the map h_c. But in view of closeness of the graph of the map h the corresponding limiting trajectory will be a trajectory of the map h, from which assertion (iii) of Theorem 7 for the map h follows. At last, assertion (iv) of Theorem 7 for the map h follows from the already established identity $\tau(h) \equiv \tau(h_c)$ and from the remark that for any two strictly increasing maps h and $\hat{h}$ with the closed graphs the Hausdorff distance between their graphs coincide with the Hausdorff distance between the graphs of the maps h_c and $\hat{h}_c$.

One of weak points in the definition of the rotation number $\tau(\eta)$ for the circle map $\eta(\cdot)$ is that one need perform intermediate steps (such as to construct the lift $h(\cdot)$ and to build the trajectory $\{\xi_n\}$ of the map $h(\cdot)$) to calculate the limit $\tau(\eta) = \lim_{n\to\infty} \xi_n/n$. It is desirable to find a method to calculate the rotation number $\tau(\eta)$ directly in terms of the map η and its trajectories. To do it, we first investigate in more details properties of the orientation preserving circle maps (cf. [9, Lemma 1]).

At last, we are able to present the definition of the rotation number of the circle map $\eta(\cdot)$ directly in terms of the map $\eta(\cdot)$ (to be precise, the definition of the rotation number of the standard lift $h(\cdot)$ of the map $\eta(\cdot)$).

Theorem 8. *Let $\eta : [0,1) \to [0,1)$ be an orientation preserving circle map with the closed standard lift h. Let $\{\zeta_n\}$ be a trajectory of the map η, i.e.,*

$$\zeta_{n+1} \in \eta(\zeta_n), \qquad n = 0, 1, \dots .$$

Then the uniform estimates hold

$$\left| \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n} - \tau(h) \right| \leq \frac{2}{n}, \qquad n = 1, 2, \dots , \tag{23}$$

and so,

$$\tau(h) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n},$$

where

$$\nu(\xi) = \begin{cases} 1 & \text{if } \; 0 \le \xi < \omega \\ 0 & \text{if } \; \omega \le \xi < 1, \end{cases} \tag{24}$$

with $\omega = \min\{y: \; y \in \eta(0)\}$[5].

The proof of Theorem 8 is given in the Appendix.

6.3. Frequency Properties of the Direction Function

In this Section, we make use of the properties of the circle maps obtained in Section 6.2 to analyze the properties of the direction function Φ_θ of the generator of B-extremal trajectories introduced in Section 6 (see (17)).

Note that the function $\Phi_\theta(\xi)$ differs from a function representing an orientation preserving circle map only in that it is defined on the closed interval $[0,1]$ but not on the semiopen one $[0,1)$ as is the case for a circle map. Let us show that the indicated difference is not essential, and for the function $\Phi_\theta(\xi)$ the notion of the rotation number can be defined with all the "good" properties intrinsical to the rotation number of the circle maps.

Theorem 9. *Let $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ be the set of 2×2 matrices (7) satisfying conditions (8), let Φ_θ be the direction function (17) of some generator of B-extremal trajectories for the matrix set $\boldsymbol{A}$ and let $\nu(\cdot)$ be the function defined by equality (24). Then for any trajectory $\{\xi_n\}_{n=0}^{\infty}$ of the map Φ_θ there are valid the non-equalities $\xi_n \ne 0, 1$, where $n \ge 1$, and there is defined the frequency*

$$\tau = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\xi_i)}{n} \tag{25}$$

with which the elements of the trajectory $\{\xi_n\}$ hit the interval $[0,\omega)$, where $\omega = \varphi_0(1)$.

The frequency τ does not depend neither on the choice of the trajectory $\{\xi_n\}$ nor on the choice of the function Φ_θ. So the frequency τ may be denoted as $\tau(\boldsymbol{A})$. In addition, for $\tau(\boldsymbol{A})$ assertions (i)–(iii) of Theorem 7 are valid, and besides, $\tau(\boldsymbol{A})$ depends continuously on the matrices of the set $\boldsymbol{A}$.

The proof of Theorem 9 is given in the Appendix.

Now, all is ready to prove Theorem 6. Let $\{x_n\}$ be a B-extremal trajectory of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ and let $\{\sigma_n\}$ be the corresponding index sequence, i.e., the sequences $\{x_n\}$ and $\{\sigma_n\}$ satisfy the equalities $x_{n+1} = A_{\sigma_n} x_n$ for $n = 0, 1, \ldots$. Then by Lemma 10 the numerical sequence $\xi_n = \xi(x_n)$, where the function $\xi(\cdot)$ is defined by equality (15), satisfies the relations

$$\xi_{n+1} = \varphi_{\sigma_n}(\xi_n) \in \Phi_\theta(\xi_n), \qquad n = 0, 1, \ldots \;,$$

with some direction function Φ_θ. At the same time, by Theorem 9 there is defined the frequency

$$\tau = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\xi_i)}{n},$$

[5] Remark that the function $\nu(\xi)$ is identically equal to zero if $\omega = 0$. In this case $h(\xi) \equiv \eta(\xi)$ on the interval $[0,1)$, and so, the function $\eta(\xi)$ strictly increases on $[0,1)$.

and besides, $\xi_n \neq 0, 1$ for $n \geq 1$. Therefore, for $n \geq 1$ the value $\xi_{n+1} \in (0,1)$ is obtained from $\xi_n \in (0,1)$ by the formula $\xi_{n+1} = \varphi_0(\xi_n)$ if and only if $0 < \xi_{n+1} < \varphi_0(1)$ or, what is the same, if and only if $\nu(\xi_{n+1}) = 1$. Consequently, $\sigma_n = 1 - \nu(\xi_{n+1})$ for $n \geq 1$ and by Theorem 9 there is the limit

$$\sigma(\boldsymbol{A}) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sigma_i}{n} = 1 - \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\xi_{i+1})}{n} = 1 - \tau(\boldsymbol{A}).$$

Now, all the assertions of Theorem 6 follow from analogous assertions of Theorem 9.

7. CONSTRUCTION OF THE COUNTEREXAMPLE

At last, start constructing the counterexample to the Finiteness Conjecture. The key tool here will be the next Theorem.

Theorem 10 (on unattainability of the generalized spectral radius). *Let the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ be such that the number $\sigma(\boldsymbol{A})$ is irrational. Then for any finite sequence of indices $\sigma_k \in \{0,1\}$, $k = 1, 2, \ldots, n$, the strict inequality $\rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1}) < \rho^n(\boldsymbol{A})$ is valid.*

According to this Theorem in order to construct the counterexample to the Finiteness Conjecture it is sufficient to prove existence of at least one of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ for which $\sigma(\boldsymbol{A})$ is irrational. The facts needed to prove existence of the required matrix set will be obtained in the next Lemma.

Lemma 12. *Let a, b, c, d be a fixed set of parameters satisfying (8), and let $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ be the corresponding matrix set (7) in which parameters α and β may vary. Then the following assertions are valid:*

a) if $a < 1$ then $\sigma(\boldsymbol{A}) = 0$ for all sufficiently large values of α/β and $\sigma(\boldsymbol{A}) > 0$ for $\alpha/\beta < 1$;

b) if $d < 1$ then $\sigma(\boldsymbol{A}) = 1$ for all sufficiently small values of α/β and $\sigma(\boldsymbol{A}) < 1$ for $\alpha/\beta > 1$;

c) if $a = d = 1$ then $\sigma(\boldsymbol{A}) = \frac{1}{2}$ for $\alpha = \beta$ and $\sigma(\boldsymbol{A}) \neq \frac{1}{2}$ for all sufficiently large and sufficiently small values of α/β.

Now, fix some set of numbers a, b, c, d satisfying conditions (8), and consider the family of the matrix sets $\boldsymbol{A}$ depending on α and β as on parameters. Then by Lemma 12 $\sigma(\boldsymbol{A})$ is not a constant function, i.e., it takes different values as α/β varies from zero to infinity. But by Theorem 6 the value $\sigma(\boldsymbol{A})$ depends continuously on the matrix set $\boldsymbol{A}$, and then on α and β. Hence, for some α and β the quantity $\sigma(\boldsymbol{A})$ takes an irrational value. Then by Theorem 10 for such α and β the generalized spectral radius $\rho(\boldsymbol{A})$ can not be attained on finite products of matrices from the set $\boldsymbol{A}$.

For $a = b = c = d = 1$ we get the proof of the counterexample to the Finiteness Conjecture for the case studied in [4].

APPENDIX

A.1. PROOF OF THEOREMS 1 AND 2

Establish first one auxiliary statement characterizing the property of irreducibility of matrix sets.

Let $x \in \mathbb{R}^m$. Denote the n-section of the set $\mathcal{T}(\boldsymbol{A}, x)$ by $\mathcal{T}_n(\boldsymbol{A}, x)$. Also, define for any $n = 0, 1, 2 \ldots$ the sets

$$\mathcal{T}_n^*(\boldsymbol{A}, x) = \bigcup_{k=0}^{n} \mathcal{T}_k(\boldsymbol{A}, x).$$

Recall that above every finite sequence $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \{1, \ldots, r\}^n$ was associated with the matrix $A_{\boldsymbol{\sigma}} = A_{\sigma_n} \cdots A_{\sigma_2} A_{\sigma_1}$, and it was supposed implicitly that $n \geq 1$. It is convenient to extend the notation $A_{\boldsymbol{\sigma}}$ on the case $n = 0$ in which the sequence $\boldsymbol{\sigma}$ is empty, i.e., consists of zero amount of elements. So, we set $\{1, \ldots, r\}^0 = \emptyset$. In this case it is naturally to identify $\boldsymbol{\sigma} \in \{1, \ldots, r\}^0$ with the empty set and to denote $A_\emptyset = I$.

Lemma A.1. *The set $\mathcal{T}_n^*(\boldsymbol{A}, x)$ coincides with the set of all possible vectors of the form $A_{\boldsymbol{\sigma}} x$, where $\boldsymbol{\sigma} \in \{1, \ldots, r\}^k$ for some, possibly zero, integer $k \leq n$.*

If $\boldsymbol{A}$ is an irreducible set of $m \times m$ matrices and $x \neq 0$ then the set $\mathcal{T}_n^*(\boldsymbol{A}, x)$ contains at least $\min\{n+1, m\}$ linearly independent elements one of which may be assumed to be coinciding with x.

Proof. Only the second claim of this Lemma has to be proved. Denote the linear hull of the set $\mathcal{T}_n^*(\boldsymbol{A}, x)$ by $\mathcal{L}_n(\boldsymbol{A}, x)$. Then the dimension of the subspace $\mathcal{L}_n(\boldsymbol{A}, x)$ will be equal to the amount of linearly independent vectors in the set $\mathcal{T}_n^*(\boldsymbol{A}, x)$. Since, in addition, for any $n \geq 0$ the inclusion $\mathcal{T}_n^*(\boldsymbol{A}, x) \subseteq \mathcal{T}_{n+1}^*(\boldsymbol{A}, x)$ holds then $\mathcal{L}_n(\boldsymbol{A}, x) \subseteq \mathcal{L}_{n+1}(\boldsymbol{A}, x)$. So,

$$1 = \dim \mathcal{L}_0(\boldsymbol{A}, x) \leq \dim \mathcal{L}_1(\boldsymbol{A}, x) \leq \ldots \leq \dim \mathcal{L}_n(\boldsymbol{A}, x) \leq \ldots .$$

Consequently, Lemma A.1 will be proved if we show that

$$\dim \mathcal{L}_n(\boldsymbol{A}, x) \geq n + 1, \qquad n = 0, 1, \ldots, m - 1. \tag{A.1}$$

Prove inequalities (A.1) by induction. For $n = 0$ inequality (A.1) holds since the subspace $\mathcal{L}_0(\boldsymbol{A}, x)$ coincides with the linear hull of the vector x, and so $\dim \mathcal{L}_0(\boldsymbol{A}, x) = 1$. Suppose that the assertion of Lemma A.1 is valid for some $n = k < N - 1$, i.e., $\dim \mathcal{L}_k(\boldsymbol{A}, x) \geq k + 1$. Then, due to the supposition about irreducibility of the matrix set $\boldsymbol{A}$, the subspace $\mathcal{L}_n(\boldsymbol{A}, x)$ can not be invariant for all the matrices $A_1, \ldots, A_r$. Therefore, there is a matrix A_i such that $A_i \mathcal{L}_n(\boldsymbol{A}, x) \not\subseteq \mathcal{L}_n(\boldsymbol{A}, x)$. Hence $\mathcal{L}_{k+1}(\boldsymbol{A}, x) \neq \mathcal{L}_k(\boldsymbol{A}, x)$. From this it follows that $\dim \mathcal{L}_{k+1}(\boldsymbol{A}, x) \geq \dim \mathcal{L}_{k+1}(\boldsymbol{A}, x) + 1 \geq k + 2$. So, the induction step is justified, and the proof of Lemma A.1 is completed. $\square$

Choose an arbitrary nonzero vector $x_0 \in \mathbb{R}^m$ and an irreducible set of $m \times m$ matrices $\boldsymbol{A}$. Then, by Lemma A.1, the set $\mathcal{T}_{m-1}^*(\boldsymbol{A}, x_0)$ contains N linearly independent vectors $x_0, x_1, \ldots, x_{m-1}$. Then the balanced convex set[6]

$$S_\sharp = \mathrm{co}\{\pm x_0, \pm x_1, \ldots, \pm x_{m-1}\} \tag{A.2}$$

contains the origin in its interior and so it may be treated as the unit ball in the norm $\|\cdot\|_\sharp$ in $\mathbb{R}^m$ determined by the inequality

$$\|x\|_\sharp = \inf\{t : \ t > 0, \ x \in t S_\sharp\}, \tag{A.3}$$

i.e., $S_\sharp = \{x : \ \|x\|_\sharp \leq 1\}$.

Lemma A.2. *Let $\|\cdot\|_\sharp$ be the norm introduced in (A.2)–(A.3) and determined by an irreducible set of $m \times m$ matrices $\boldsymbol{A}$ and by a vector $x_0 \neq 0$. Then for any Barabanov norm $\|\cdot\|$ corresponding to the matrix set $\boldsymbol{A}$ the following estimate holds:*

$$\frac{\|x\|}{\|x_0\|} \leq (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x\|_\sharp \qquad \forall x \in \mathbb{R}^m. \tag{A.4}$$

Proof. By Lemma A.1 each of the vectors $x_0, x_1, \ldots, x_{m-1}$ in (A.2) may be represented in the form

$$x_i = A_{\boldsymbol{\sigma}(i)} x_0, \qquad i = 0, 1, \ldots, m - 1,$$

[6] Recall that a set in a linear space is called balanced if with each its element x it contains also the element $-x$.

where $\boldsymbol{\sigma}(i) \in \{1, \dots, r\}^{k_i}$ for some, possibly zero, integer $k_i \le m-1$. Therefore, for arbitrary Barabanov norm $\|\cdot\|$ corresponding to the matrix set $\boldsymbol{A}$ the inequalities

$$\|x_i\| \le (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x_0\|, \qquad i = 0, 1, \dots, m-1.$$

are valid. The obtained inequalities show that

$$S_\sharp \subseteq \{x : \ \|x\| \le (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x_0\|\}$$

from which estimate (A.4) follows. Lemma A.2 is proved. □

Now, we are able to prove the right-hand side of inequalities

$$\delta\|x\|_0 \le \frac{\|x\|}{\|x_0\|} \le \Delta\|x\|_0, \tag{A.5}$$

even in a more strong form.

Lemma A.3. *Let $\|\cdot\|_0$ be a norm and $x_0 \neq 0$ be a vector in $\mathbb{R}^m$. Let also $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there is a number $\Delta < \infty$ and a neighborhood $\mathcal{A}$ of the matrix set $\boldsymbol{A}$ such that for any Barabanov norm $\|\cdot\|'$ corresponding to the matrix set $\boldsymbol{A}' \in \mathcal{A}$ the following estimate is valid*

$$\frac{\|x\|'}{\|x_0\|'} \le \Delta\|x\|_0 \qquad \forall x \in \mathbb{R}^m. \tag{A.6}$$

Proof. Let $\boldsymbol{A} = \{A_1, \dots, A_r\}$. Then by Lemma A.1 the set $\mathcal{T}^*_{m-1}(\boldsymbol{A}, x_0)$ contains the linearly independent vectors $x_0, x_1, \dots, x_{m-1}$ of the form

$$x_i = x_i(\boldsymbol{A}) = A_{\boldsymbol{\sigma}(i)} x_0, \qquad i = 0, 1, \dots, m-1,$$

where $\boldsymbol{\sigma}(i) \in \{1, \dots, r\}^{k_i}$ for some, possibly zero, $k_i \le m-1$. In this case, for any matrix set $\boldsymbol{A}' = \{A'_1, \dots, A'_r\}$ from a sufficiently small neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ the vectors

$$x'_i = x_i(\boldsymbol{A}') = A'_{\boldsymbol{\sigma}(i)} x_0, \qquad i = 0, 1, \dots, m-1,$$

are also linearly independent.

For each $\boldsymbol{A}' \in \mathcal{A}$ we denote by $S_\sharp(\boldsymbol{A}')$ the balanced convex set

$$S_\sharp(\boldsymbol{A}') = \mathrm{co}\{\pm x_0(\boldsymbol{A}'), \pm x_1(\boldsymbol{A}'), \dots, \pm x_{m-1}(\boldsymbol{A}')\},$$

which contains the origin in its interior. As it was noted above, such a set may be treated as the unit ball in the norm $\|\cdot\|'_\sharp$ in $\mathbb{R}^m$ determined by the equation

$$\|x\|'_\sharp = \inf\{t : \ t > 0, \ x \in tS_\sharp(\boldsymbol{A}')\}.$$

Then from Lemma A.2 it follows that

$$\frac{\|x\|}{\|x_0\|} \le (\max\{1, \rho(\boldsymbol{A}(\lambda))\})^{m-1} \|x\|'_\sharp \qquad \forall x \in \mathbb{R}^m, \quad \forall \boldsymbol{A}' \in \mathcal{A}. \tag{A.7}$$

To complete the proof, it remains only to note that the vectors $x_0(\boldsymbol{A}')$, $x_1(\boldsymbol{A}')$, $\dots$, $x_{m-1}(\boldsymbol{A}')$ depend continuously on $\boldsymbol{A}'$ and are linearly independent at the point $\boldsymbol{A}' = \boldsymbol{A}$. Hence the intersection of the sets $S_\sharp(\boldsymbol{A}')$ with $\boldsymbol{A}' \in \mathcal{A}$ has a nonempty interior to which the origin belongs. Therefore, there exists a constant μ such that

$$\{x : \ \|x\|_0 \le 1\} \subseteq \mu \bigcap_{\boldsymbol{A}' \in \mathcal{A}} S_\sharp(\boldsymbol{A}'),$$

and then

$$\|x\|'_\sharp \le \mu\|x\|_0 \qquad \forall x \in \mathbb{R}^m, \quad \forall \boldsymbol{A}' \in \mathcal{A}. \tag{A.8}$$

¿From (A.7) and (A.8) we readily obtain the statement of Lemma A.3 with the constant Δ defined as

$$\Delta = \mu \sup_{\boldsymbol{A}' \in \mathcal{A}} \left(\max\{1, \rho(\boldsymbol{A}')\}\right)^{m-1}.$$

Here, one can assume that the constant Δ is finite since the supremum in the right hand part of the latter formula is bounded in any bounded neighborhood of the matrix set $\boldsymbol{A}$, while the neighborhood $\mathcal{A}$ is supposed to be rather small and hence bounded. Lemma A.3 is proved. $\square$

Now, all is ready to prove Theorem 1.

Proof of Theorem 1. Given a norm $\|\cdot\|_0$ in $\mathbb{R}^m$, define $\mathcal{A}$ as such a compact neighborhood of the matrix set $\boldsymbol{A}$ whose existence has been established Lemma A.3. Introduce the set of norms

$$\mathcal{N} := \bigcup_{\boldsymbol{A}' \in \mathcal{A}} N_{\mathrm{Bar}}(\boldsymbol{A}', x_0),$$

and show that this set is compact in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$.

Indeed, by Lemma A.3 for some $\Delta < \infty$ the following estimates hold

$$\|x\| \le \Delta\|x\|_0 \qquad \forall x \in \mathbb{R}^m, \quad \forall \|\cdot\| \in \mathcal{N},$$

and so the values of the norms from $\mathcal{N}$ are uniformly bounded on each bounded set from $\mathbb{R}^m$. Besides, again by Lemma A.3 we have

$$|\|x\| - \|y\|| \le \|x - y\| \le \Delta\|x - y\|_0 \qquad \forall x, y \in \mathbb{R}^m, \quad \forall \|\cdot\| \in \mathcal{N},$$

and hence the norms from $\mathcal{N}$ are functions satisfying a uniform Lipschitz condition on $\mathbb{R}^m$. Thus, the norms from $\mathcal{N}$ form a set of uniformly bounded and equicontinuous functions on each closed bounded set from $\mathbb{R}^m$, from which by the Arzela-Ascoli theorem the compactness of the set $\mathcal{N}$ in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$ follows.

Now we prove that the graph of map

$$\boldsymbol{A}' \mapsto N_{\mathrm{Bar}}(\boldsymbol{A}', x_0) \tag{A.9}$$

is closed in the space $\mathcal{A} \times C_{\mathrm{loc}}(\mathbb{R}^m)$. Let $\{(\boldsymbol{A}^{(n)}, \|\cdot\|^{(n)})\}$, where $\boldsymbol{A}^{(n)} \in \mathcal{A}$, be a sequence of elements belonging to the graph of map (A.9) and converging to some element $(\boldsymbol{A}^*, \nu(\cdot)) \in \mathcal{M}_{m,r} \times C_{\mathrm{loc}}(\mathbb{R}^m)$. Then the compactness of $\mathcal{A}$ implies the inclusion $\boldsymbol{A}^* \in \mathcal{A}$. At the same time, we may state that the function $\nu(\cdot)$, being a limit in $C_{\mathrm{loc}}(\mathbb{R}^m)$ of a sequence of norms $\|\cdot\|^{(n)}$, is only a semi-norm.

¿From the definition of the sequence $\{(\boldsymbol{A}^{(n)}, \|\cdot\|^{(n)})\}$ it follows that $\|\cdot\|^{(n)} \in N_{\mathrm{Bar}}(\boldsymbol{A}^{(n)}, x_0)$ for each value of n and therefore

$$\rho(\boldsymbol{A}^{(n)})\|x\|^{(n)} = \max\left\{\|A_1^{(n)}x\|^{(n)}, \dots, \|A_r^{(n)}x\|^{(n)}\right\} \qquad \forall x \in \mathbb{R}^m \ , \forall n. \tag{A.10}$$

Here, due to the assumption about the irreducibility of the matrix set $\boldsymbol{A}$, without loss of generality, one can assume that each of the matrix sets $\boldsymbol{A}^{(n)}$ is also irreducible. In this case it holds (see [7]) $\rho(\boldsymbol{A}^{(n)}) \to \rho(\boldsymbol{A}^*)$ and, by passing to limit in (A.10), we obtain

$$\rho(\boldsymbol{A}^*)\nu(x) = \max\{\nu(A_0^*x), \nu(A_1^*x), \dots, \nu(A_r^*x)\} \qquad \forall x \in \mathbb{R}^m,$$

with $\nu(x_0) = \lim_{n\to\infty} \|x_0\|^{(n)} = 1$. Hence, the semi-norm ν satisfies condition (4) for the irreducible matrix set $\boldsymbol{A}^*$ and does not equal identically to zero. Then by Lemma 3 this semi-norm is in fact a Barabanov norm, i.e., $\nu(\cdot) = \|\cdot\|^* \in N_{\mathrm{Bar}}(\boldsymbol{A}^*, x_0)$, which means that the graph of map (A.9) is closed.

So, it is proved that the graph of map (A.9) is closed and that the set $\mathcal{N}$ is compact. From this we get by Lemma 1 the compactness and upper semi-continuity of map (A.9). Theorem 1 is proved. $\square$

Now, we are able to prove the left-hand side of inequalities (A.5).

Lemma A.4. *Given a norm $\|\cdot\|_0$ and a vector $x_0 \neq 0$ in $\mathbb{R}^m$, let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there exist a number $\delta > 0$ and a neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that for any Barabanov norm $\|\cdot\|'$ corresponding to the matrix set $\boldsymbol{A}' \in \mathcal{A}$ the following estimate hold:*

$$\delta\|x\|_0 \leq \frac{\|x\|'}{\|x_0\|'} \qquad \forall x \in \mathbb{R}^m.$$

Proof. Define $\mathcal{A}$ as the neighborhood of the matrix set $\boldsymbol{A}$ determined by Theorem 1. Then, supposing that Lemma A.4 in not true, one may choose matrix sets $\boldsymbol{A}^{(n)} \in \mathcal{A}$ and corresponding to them Barabanov norms $\|\cdot\|^{(n)} \in N_{\mathrm{Bar}}(\boldsymbol{A}^{(n)}, x_0)$ as well as vectors $x^{(n)}$ such that $\|x^{(n)}\|_0 = 1$ and

$$\frac{\|x^{(n)}\|^{(n)}}{\|x_0\|^{(n)}} \to 0 \quad \text{as} \quad n \to \infty. \tag{A.11}$$

By Theorem 1 one can suppose that the sequences $\{\boldsymbol{A}^{(n)}\}$ and $\{\|\cdot\|^{(n)}\}$ are convergent, i.e., $\boldsymbol{A}^{(n)} \to \boldsymbol{A}^* \in \mathcal{A}$ and $\|\cdot\|^{(n)} \to \|\cdot\|^* \in N_{\mathrm{Bar}}(\boldsymbol{A}^*, x_0)$. The sequence $\{x^{(n)}\}$ also can considered as convergent: $x^{(n)} \to x^* \neq 0$. Then, by passing in (A.11) to limit, we obtain the equality

$$\frac{\|x_*\|^*}{\|x_0\|^*} = 0, \qquad x_*, x_0 \neq 0,$$

which is impossible since $\|\cdot\|^*$ is a norm. The contradiction completes the proof of Lemma A.4. $\square$

The proof of Theorem 2 now directly follows from Lemmas A.3 and A.4.

A.2. PROOF OF THEOREMS 3 AND 4

Proof of Theorem 3. Let $\{x_n\}$ be a trajectory of the matrix set $\boldsymbol{A}$ which is extremal in some extremal norm $\|\cdot\|_0$. Consider the sequence of "shifted" trajectories $\boldsymbol{x}_k = \{x_n^{(k)}\}$ defined as follows

$$x_n^{(k)} = \rho^{-k} x_{n+k}, \qquad n = 0, 1, 2, \ldots\ .$$

Then for each fixed $n = 0, 1, \ldots$ the set of elements $\{x_n^{(k)}\}$ is uniformly bounded

$$\|x_n^{(k)}\|_0 = \|\rho^{-k} x_{n+k}\|_0 = \rho^n \|x_0\|, \qquad k = 0, 1, \ldots\ ,$$

where $\rho = \rho(\boldsymbol{A})$. Hence, by Lemma 2 the sequence of trajectories $\boldsymbol{x}_k$ is compact in the space $\Omega(\mathbb{R}^m)$. Therefore, without loss of generality one may suppose that for each $n = 0, 1, \ldots$ there exists the limit

$$x_n^* = \lim_{k\to\infty} x_n^{(k)} = \lim_{k\to\infty} \rho^{-k} x_{n+k}. \tag{A.12}$$

Note that by Lemma 2 the set of all trajectories of the matrix set $\boldsymbol{A}$ is closed in the space $\Omega(\mathbb{R}^m)$, and so the limiting sequence $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is also a trajectory of the matrix set $\boldsymbol{A}$.

At last, show that the trajectory $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is extremal in any extremal norm of the matrix set $\boldsymbol{A}$. Fix an arbitrary norm $\|\cdot\|_*$ extremal for the matrix set $\boldsymbol{A}$. Then, by the definition of an extremal norm, for the trajectory $\{x_n\}$ the following inequalities hold

$$\|x_0\|_* \geq \rho^{-1}\|x_1\|_* \geq \rho^{-2}\|x_2\|_* \geq \ldots \geq \rho^{-n}\|x_n\|_* \geq \ldots \geq c_1 > 0.$$

Hence, the sequence $\{\rho^{-n}\|x_n\|_*\}$ monotonously decreases and consequently there exists limit

$$\lim_{n\to\infty}\rho^{-n}\|x_n\|_* = \omega \geq c_1 > 0.$$

Together with (A.12) the latter relation implies

$$\rho^{-n}\|x_n^*\|_* = \lim_{k\to\infty}\rho^{-(n+k)}\|x_{n+k}\|_* = \omega, \qquad n = 0, 1, \dots\ .$$

So, the trajectory $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is extremal in the norm $\|\cdot\|_*$. Theorem 3 is proved. $\square$

Proof of Theorem 4. Let $\mathcal{A}$ be such a closed neighborhood of the matrix set $\boldsymbol{A}$ whose existence is asserted by Lemma A.3. Consider the sets

$$\mathcal{E}_{\mathrm{Bar}} = \bigcup_{\boldsymbol{A}'\in\mathcal{A}}\bigcup_{x\in\mathcal{X}}\mathcal{E}_{\mathrm{Bar}}(\boldsymbol{A}', x), \qquad \mathcal{T} = \bigcup_{\boldsymbol{A}'\in\mathcal{A}}\bigcup_{x\in\mathcal{X}}\mathcal{T}(\boldsymbol{A}', x),$$

and observe that by Lemmas 1 and 2 the set $\mathcal{E}_{\mathrm{Bar}}$ is a subset of the compact set $\mathcal{T} \subseteq \Omega(\mathbb{R}^m)$. Therefore the set $\mathcal{E}_{\mathrm{Bar}}$ is also compact in the space $\Omega(\mathbb{R}^m)$.

Now we show that the graph of map

$$(\boldsymbol{A}', x) \mapsto \mathcal{E}_{\mathrm{Bar}}(\boldsymbol{A}', x), \qquad \boldsymbol{A}' \in \mathcal{A},\ x \in \mathcal{X}, \tag{A.13}$$

is closed in $\mathcal{A}\times\mathcal{X}\times\Omega(\mathbb{R}^m)$. Choose a sequence of elements $\big(\boldsymbol{A}^{(k)}, x^{(k)}, \boldsymbol{x}^{(k)}\big)$ with $\boldsymbol{A}^{(k)} \in \mathcal{A}$ and $x^{(k)} \in \mathcal{X}$ belonging to the graph of map (A.13) and converging to some element $\big(\boldsymbol{A}^*, x^*, \boldsymbol{x}^*\big) \in \mathcal{A}\times\mathcal{X}\times\Omega(\mathbb{R}^m)$. Then the sequence $\big(\boldsymbol{A}^{(k)}, x^{(k)}, \boldsymbol{x}^{(k)}\big)$ belongs also to the graph of the map $\mathcal{T}(\boldsymbol{A}, x)$. In this case, due to the compactness and upper semi-continuity of the map $\mathcal{T}(\boldsymbol{A}, x)$ (see Lemma 2), the limiting element $\big(\boldsymbol{A}^*, x^*, \boldsymbol{x}^*\big)$ also belongs to the graph of the map $\mathcal{T}(\boldsymbol{A}, x)$:

$$\boldsymbol{x}^* \in \mathcal{T}(\boldsymbol{A}^*, x^*).$$

Hence, $\boldsymbol{x}^*$ is a trajectory of the matrix set $\boldsymbol{A}^* \in \mathcal{A}$ satisfying the initial condition $x^* \in \mathcal{X}$. It remains only to prove that the trajectory $\boldsymbol{x}^*$ is B-extremal.

By construction, $\boldsymbol{x}^{(k)} = \{x_n^{(k)}\}$ is a trajectory of the matrix set $\boldsymbol{A}^{(k)}$ which is extremal in some Barabanov norm $\|\cdot\|^{(k)}$. Then

$$\|x_0^{(k)}\|^{(k)} = \rho^{-1}(\boldsymbol{A}^{(k)})\|x_1^{(k)}\|_k = \dots = \rho^{-n}(\boldsymbol{A}^{(k)})\|x_n^{(k)}\|^{(k)} = \dots\ , \tag{A.14}$$

where by Theorem 1 one may assume that the sequence of Barabanov norms $\|\cdot\|^{(k)}$ converges to some Barabanov norm $\|\cdot\|^*$ of the matrix set $\boldsymbol{A}^*$. Therefore, taking the limit in (A.14) we obtain[7]:

$$\|x_0^*\|^* = \rho^{-1}(\boldsymbol{A}^*)\|x_1^*\|^* = \dots = \rho^{-n}(\boldsymbol{A}^*)\|x_n^*\|^* = \dots\ .$$

The obtained relations justify that the trajectory $\boldsymbol{x}^* = \{x_n^*\}$ of the matrix set $\boldsymbol{A}^*$ is extremal in the Barabanov norm $\|\cdot\|^*$.

So, it is proved that the graph of map (A.13) is closed and that the set $\mathcal{E}_{\mathrm{Bar}}$ is compact. By Lemma 1 this implies the compactness and upper semi-continuity of map (A.13). Theorem 4 is proved. $\square$

[7] Remark that $\rho(\boldsymbol{A}^{(k)}) \to \rho(\boldsymbol{A}^*)$ as $k \to \infty$ since the generalized spectral radius depends continuously [7] on the irreducible matrix set.

A.3. PROOF OF THEOREM 5

To start the proof, obtain some auxiliary estimates of the generalized spectral radius. Let $\boldsymbol{A}$ be a set of matrices A_0 and A_1 of the form (7) satisfying (8), and let $\rho(\boldsymbol{A})$ be the joint spectral radius of the matrix set $\boldsymbol{A}$. Then in virtue of (1)

$$\rho(\boldsymbol{A}) \geq (\rho(A_0^n A_1))^{\frac{1}{n+1}}, \ \ (\rho(A_0 A_1^n))^{\frac{1}{n+1}}, \qquad \forall n \geq 0,$$

from which we get

$$\rho(\boldsymbol{A}) \geq \max\{\alpha, \beta\}. \tag{A.15}$$

Remark that formula (A.15) is generally unimprovable. Nevertheless, in what follows we will need estimates which sometimes will be more accurate.

Lemma A.5. *For each $n \geq 0$ hold the estimates*

$$\rho(\boldsymbol{A}) \geq \max\left\{\alpha a \left(\frac{(n+1)\beta}{\alpha}\right)^{\frac{1}{n+1}}, \ \beta d \left(\frac{(n+1)\alpha}{\beta}\right)^{\frac{1}{n+1}}\right\}. \tag{A.16}$$

Proof. As is easy to see,

$$A_0^n A_1 = \alpha^n \beta \begin{Vmatrix} a^n + (1 + \cdots + a^{n-1})bc & (1 + \cdots + a^{n-1})bd \\ c & d \end{Vmatrix},$$

and the characteristic polynomial $p(\lambda)$ of the matrix $(\alpha^n\beta)^{-1} A_0^n A_1$ has the form:

$$p(\lambda) = \lambda^2 - (d + a^n + (1 + \cdots + a^{n-1})bc)\lambda + a^n d.$$

In this case

$$p((n+1)a^n) \ = \ (n+1)^2 a^{2n} - (d + a^n + (1 + \cdots + a^{n-1})bc)(n+1)a^n + a^n d.$$

Here due to (8) $(1 + \cdots + a^{n-1})bc \geq na^n$, and therefore, $p((n+1)a^n) \leq -na^n d \leq 0$. Hence the maximal root of the polynomial $p(\lambda)$ is not less than $(n+1)a^n$, and then $\rho\left(A_0^n A_1\right) \geq (n+1)\alpha^n\beta a^n$, from which it follows that

$$\rho(\boldsymbol{A}) \geq \left(\rho\left(A_0^n A_1\right)\right)^{\frac{1}{n+1}} \geq \alpha a \left(\frac{(n+1)\beta}{\alpha a}\right)^{\frac{1}{n+1}} \geq \alpha a \left(\frac{(n+1)\beta}{\alpha}\right)^{\frac{1}{n+1}}.$$

Analogously, it can be shown that $\rho(\boldsymbol{A})$ is not less than the second member under the maximum sign in (A.16). Lemma A.5 is proved. $\square$

Corollary. *For any $\alpha, \beta > 0$ there is such a $\gamma(\alpha, \beta) > 1$, that*

$$\rho(\boldsymbol{A}) > \gamma(\alpha, \beta) \max\{\alpha a, \ \beta d\}.$$

Proof. The required estimate immediately follows from Lemma A.5 if to note that the quantities

$$\left(\frac{(n+1)\beta}{\alpha}\right)^{\frac{1}{n+1}}, \qquad \left(\frac{(n+1)\alpha}{\beta}\right)^{\frac{1}{n+1}}$$

in (A.16) are strictly greater than 1 when $n + 1 > \max\{\alpha/\beta, \beta/\alpha\}$. $\square$

It seems, that in a majority of situations this Corollary is even weaker than the statement of Lemma A.5. Nevertheless, it will be needed below namely in such a form. Besides, this Corollary implies that $\rho(\boldsymbol{A}) > \max\{\alpha, \beta\}$ for a and b sufficiently close to 1.

Now, we are ready to prove Lemma 8 clarifying to some extent the structure of the unit ball of a Barabanov norm.

Proof of Lemma 8. Let $\|\cdot\|$ be a Barabanov norm corresponding to the matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$, and let $w = (0, w_1)$, $w_1 > 0$, be a point lying on the boundary of the ball $\mathbb{S} = \{x : \ \|x\| = 1\}$, i.e., $\|w\| = 1$. Show that in this case for any point $x = (x_1, x_2) \in \mathbb{S}$ lying in the first quadrant (i.e., such that $x_0, x_1 \geq 0$) the following relation holds

$$x_1 \leq w_1.$$

Suppose the contrary, i.e., that there is a point $z = (z_0, z_1)$, $\|z\| = 1$, for which $z_1 > w_1$, $z_0 > 0$, and show that this is impossible. In what follows the "vertical" coordinate z_1 of the point z is supposed to be maximal, i.e., it is supposed that the unit sphere in the norm $\|\cdot\|$ has no points with the vertical coordinate exceeding z_1. Clearly, this additional assumption does not restricts generality.

Since $\|\cdot\|$ is a Barabanov norm then $\rho\|w\| = \max\{\|A_0 w\|, \|A_1 w\|\}$ where $\rho = \rho(\boldsymbol{A})$. Hence, either $\|A_0 w\| = \rho\|w\|$ or $\|A_0 w\| < \rho\|w\|$. Consider both of this cases.

Let first $\|A_0 w\| = \rho\|w\|$. Represent the vector $A_0 w$ as a linear combination of the vectors ρw and $A_0 z$:

$$A_0 w = s\rho w + t A_0 z. \tag{A.17}$$

Accounting that $A_0 w = \alpha\,(b w_1, w_1)$ and $A_0 z = \alpha\,(a z_0 + b z_1, z_1)$ the equality (A.17) can be rewritten in the following equivalent coordinate form:

$$\alpha b w_1 = t\alpha\,(a z_0 + b z_1)\,, \qquad \alpha w_1 = s\rho w_1 + t\alpha z_1. \tag{A.18}$$

The first equality (A.18) implies that $t = b w_1/(a z_0 + b z_1)$. Then from the relations $z_1 > w_1 > 0$ and $z_0 > 0$ we get:

$$0 < t = \frac{b w_1}{a z_0 + b z_1} < 1.$$

Now, substituting the obtained value for t in the second equality (A.18), we get:

$$s = \frac{\alpha}{\rho}\frac{w_1 - t z_1}{w_1} = \frac{\alpha}{\rho}\frac{w_1 - \frac{b w_1}{a z_0 + b z_1} z_1}{w_1} = \frac{\alpha}{\rho}\frac{a w_1 z_0}{w_1} = \frac{\alpha}{\rho} a z_0 > 0.$$

At last, taking into account that in view of (A.15) $\rho = \rho(\boldsymbol{A}) \geq \rho\,(A_0) \geq \alpha > 0$, from the second equation (A.18) and from the estimate $z_1 > w_1 > 0$ we obtain the chain of relations

$$\alpha w_1 = s\rho w_1 + t\alpha z_1 > \alpha w_1 = s\alpha w_1 + t\alpha w_1,$$

from which $s + t < 1$.

So, it is shown that

$$s, t > 0, \qquad s + t < 1.$$

Then from (A.17) and from the relations $\|w\| = 1$ and $\|z\| = 1$ we get the chain of inequalities

$$\|A_0 w\| \leq s\rho\|w\| + t\|A_0 z\| \leq s\rho\|w\| + t\rho\|z\| < \rho,$$

contradicting to the supposition that $\|A_0 w\| = \rho\|w\| = \rho$. In other words, the case $\|A_0 w\| = \rho\|w\|$ is impossible.

Let now $\|A_0 w\| < \rho\|w\|$. Since $\|\cdot\|$ is a Barabanov norm then in this case $\|A_1 w\| = \rho\|w\| = \rho$ where $A_1 w = \beta dw$ by the definition of the vector w. Then $\beta b = \rho$, and the following representation holds:

$$\frac{1}{\rho}A_1 z = \left(\frac{\beta}{\rho}z_0, \frac{\beta}{\rho}cz_0 + \frac{\beta}{\rho}bz_1\right) = \left(\frac{\beta}{\rho}z_0, \frac{\beta}{\rho}cz_0 + z_1\right).$$

Since $\|\cdot\|$ is a Barabanov norm then here $\left\|\frac{1}{\rho}A_1 z\right\| \le \|z\| = 1$. On the other hand the vertical coordinate of the vector $\frac{1}{\rho}A_1 z$, which is equal to $\frac{\beta}{\rho}cz_0 + z_1$, is turned to be strictly greater than z_1 which contradicts to the definition of the vector z. In other words, the case $\|A_0 w\| < \rho\|w\|$ is also impossible.

So, it is shown that the intersection of the ball $\mathbb{S}$ with the first quadrant lays entirely below the straight horizontal line crossing the point w (see Fig. 3).

Analogously can be shown that if the vector $v = (v_0, 0)$, $v_0 > 0$, is such that its norm equals to one, then the intersection of the ball $\mathbb{S}$ with the first quadrant lays entirely below the straight horizontal line crossing the point v (see Fig. 3). Now, from the convexity of the ball $\mathbb{S}$ immediately follows the statement of the Lemma. $\square$

To further analyze the structure of the unit ball of a Barabanov norm we will need some constructions involving the so-called Gram symbol. Given a pair of vectors $x, y \in \mathbb{R}^2$ and a pair of linear functionals

$$u(w) = \langle u, w\rangle, \quad v(w) = \langle v, w\rangle, \qquad u, v, w \in \mathbb{R}^2.$$

Then *the Gram symbol* of the ordered four-tuple $\{u, v, x, y\}$ is the expression

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} := u(x)v(y) - u(y)v(x) \equiv \langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle. \tag{A.19}$$

Lemma A.6.

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} = 0 \qquad \Longleftrightarrow \qquad u = tv \quad \text{or} \quad x = ty,$$

and

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} \ge 0 \qquad \text{if} \quad x = u,\ y = v, \tag{A.20}$$

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} \le 0 \qquad \text{if} \quad x = v,\ y = u. \tag{A.21}$$

Proof. By definition (A.19), the Gram symbol of the four-tuple $\{u, v, x, y\}$ vanishes if and only if

$$\langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle = \langle u, x\langle v, y\rangle - y\langle v, x\rangle\rangle = 0.$$

If here $x\langle v, y\rangle - y\langle v, x\rangle = 0$ then $x = ty$, and Lemma A.6 is proved. Therefore, we will suppose that $w = x\langle v, y\rangle - y\langle v, x\rangle \ne 0$. Then

$$\langle u, w\rangle = 0, \qquad w \ne 0, \tag{A.22}$$

and besides,

$$\langle v, w\rangle = \langle v, x\langle v, y\rangle - y\langle v, x\rangle\rangle \equiv 0. \tag{A.23}$$

Clearly, in a two-dimensional space equalities (A.22), (A.23) with nonzero w may be valid only in the case when $u = tv$.

Inequalities (A.20) and (A.21) immediately follow from the following relations

$$\left\{\begin{matrix} u & u \\ v & v \end{matrix}\right\} = -\left\{\begin{matrix} u & v \\ v & u \end{matrix}\right\} = \langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 \geq 0.$$

Lemma A.6 is proved. $\square$

Lemma A.6 implies that, under non-degenerate deformations of ordered pairs of the vectors $\{u, v\}$ and $\{x, y\}$ satisfying $u \neq tv$ and $x \neq ty$, the sign of the Gram symbol does not change. Moreover, each ordered pair of the vectors $\{u, v\}$ and $\{x, y\}$ may be deformed either at the ordered pair of the vectors $\{e_1, e_2\}$, or at the pair $\{e_2, e_1\}$, where

$$e_1 = (1, 0), \qquad e_2 = (0, 1).$$

So, the geometrical sense of the Gram symbol is that *the ordered pair of the vectors $\{x, y\}$ has the same orientation as the ordered pair of the vectors $\{u, v\}$ if and only if the Gram symbol of the corresponding ordered four-tuple of the vectors $\{u, v, x, y\}$ is positive.*

Now, we a ready to fulfil the final steps of the proof of Theorem 5. Let $\mathbb{S}$ be the unit ball of a Barabanov norm $\|\cdot\|$ and let

$$\mathbb{S}' = \{u \in \mathbb{R}^2 : \sup_{x\in\mathbb{S}} |\langle u, x\rangle| \leq 1\}.$$

Denote by K_+ the cone of vectors in $\mathbb{R}^2$ with the non-negative coordinates.

Lemma A.7. *Let $\|\cdot\|'$ be the norm the unit ball of which coincides with $\mathbb{S}'$. Then*

$$|\langle u, x\rangle| \leq \|x\|\|u\|'. \tag{A.24}$$

Moreover, for each vector $x \neq 0$ there is a vector $u \neq 0$ such that $\langle u, x\rangle = \|x\|\cdot\|u\|'$, and furthermore, if $x \in K_+$ then $u \in K_+$.

Proof. Inequality (A.24) is a well known fact in the theory of topological vector spaces and it directly follows from the definition of the dual norm $\|u\|'$. The fact that the equality $\langle u, x\rangle = \|x\| \cdot \|u\|'$ with $x \in K_+$ is valid for some $u \in K_+$ follows from Lemma 8. $\square$

Now, let $x, y \neq 0$ be a pair of the vectors satisfying $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$. Then due to the non-negativity of entries of the matrices A_0 and A_1,

$$A_0x \in K_+, \quad \|A_0x\| = \rho\|x\|, \qquad A_1y \in K_+, \quad \|A_1y\| = \rho\|y\|,$$

and by Lemma A.7 such vectors $u, v \in K_+$ can be found for which

$$\langle u, A_0x\rangle = \|u\|'\|A_0x\| = \rho\|u\|'\|x\|, \tag{A.25}$$

$$\langle v, A_1y\rangle = \|v\|'\|A_1y\| = \rho\|v\|'\|y\|. \tag{A.26}$$

On the other hand, (A.24) and the definition of a Barabanov norm imply

$$\langle u, A_0y\rangle \leq \|u\|'\|A_0y\| = \rho\|u\|'\|y\|, \tag{A.27}$$

$$\langle v, A_1x\rangle \leq \|v\|'\|A_1x\| = \rho\|v\|'\|x\|. \tag{A.28}$$

¿From (A.25), (A.26), (A.27) and (A.28) we get

$$\langle u, A_0x\rangle\langle v, A_1y\rangle = \rho^2\|u\|'\|v\|'\|x\|'\|y\| \geq \langle u, A_0y\rangle\langle v, A_1x\rangle.$$

Then

$$\left\{\begin{matrix} A_0'u & x \\ A_1'v & y \end{matrix}\right\} = \langle A_0'u, x\rangle\langle A_1'v, y\rangle - \langle A_0'u, y\rangle\langle A_1'v, x\rangle \geq 0. \tag{A.29}$$

So, we have proved the following

Lemma A.8. *Let $x, y \neq 0$ be a pair of the vectors satisfying $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$. Then there are such nonzero vectors $u, v \in K_+$ for which relation (A.29) is valid.*

This Lemma is a key point in the analysis of the structure of the sets $X_0 \cap K_+$ and $X_1 \cap K_+$.

Proof of Theorem 5. Let $v = (v_0, 0)$, $v_0 > 0$, be a vector of unit norm, and let $\rho = \rho(\{A_0, A_1\})$. Then $A_0 v = \alpha(av_0, 0) = \alpha a(v_0, 0) = \alpha a v$, and by the corollary from Lemma A.5 $\|A_0 v\| < \rho\|v\|$. Since $\|\cdot\|$ is a Barabanov norm then from here it follows that $\|A_1 v\| = \rho\|v\|$ and therefore $v \in X_1 \backslash X_0$. Moreover, the inequality $\|A_0 v\| < \rho\|v\|$ means that for all vectors $\bar{v}$ from some neighborhood of v it is also valid the inequality $\|A_0 \bar{v}\| < \rho\|\bar{v}\|$ which, as was shown above, implies $\bar{v} \in X_1 \backslash X_0$. So, the set $X_1 \backslash X_0$ has a nonempty intersection with the abscissa axis, and the interior of the set $X_1 \backslash X_0$ is nonempty. Analogously can be shown that the set $X_0 \backslash X_1$ has a nonempty intersection with the ordinate axis, and the interior of the set $X_0 \backslash X_1$ is nonempty.

By Lemma A.8 for a pair of nonzero vectors $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$ non-proportional to each other there are such nonzero vectors $u, v \in K_+$ for which the Gram symbol of the four-tuple $\{A_0' u, A_1' v, x, y\}$ is non-negative. This means that the ordered pair of vectors $\{x, y\}$ has the same orientation as the pair $\{A_0' u, A_1' v\}$. On the other hand, under conditions (8) for the pair of matrices A_0, A_1 the ordered pair of vectors $\{A_0' u, A_1' v\}$ *is always oriented negatively*, i.e., the vector $A_1' v$ can be obtained by rotating the vector $A_0' u$ counter clockwise on the angle not exceeding π and by appropriate stretching or contracting. Therefore, the ordered pair of vectors $\{x, y\}$ should also be oriented negatively.

So, any ordered pair of nonzero vectors $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$ not proportional to each other is negatively oriented. Since in addition, the sets $X_0 \cap K_+$ and $X_1 \cap K_+$ are closed and conic, i.e., contain with each its nonzero element the whole ray passing this element, then they should be such as it is asserted in Theorem 5.

The fact that the vector ϑ is the only solution of the system of equations (13) directly follows from definitions (11), (12) of the sets X_0, X_1, Θ and from the fact that the set Θ is a ray. Therefore, to complete the proof of Theorem 5 we need only show that the vector ϑ depends continuously on the matrices A_0, A_1 and the norm $\|\cdot\|$.

Let $\{A_0^{(n)}\}$, $\{A_1^{(n)}\}$ be sequences of matrices (7) satisfying (8), and let $\{\|\cdot\|^{(n)}\}$ be a sequence of the Barabanov norms corresponding to these matrices. Suppose that

$$A_0^{(n)} \to A_0^{(0)}, \qquad A_1^{(n)} \to A_1^{(0)}, \qquad \|\cdot\|^{(n)} \to \|\cdot\|^{(0)},$$

where convergence of the norms is understood as convergence in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$. Denote by $\{\vartheta^{(n)}\}$ the sequence of vectors satisfying the system of equations

$$\|A_0^{(n)} \vartheta^{(n)}\|^{(n)} = \|A_1^{(n)} \vartheta^{(n)}\|^{(n)}, \quad \|\vartheta^{(n)}\|^{(n)} = 1, \qquad \vartheta^{(n)} \in K_+, \tag{A.30}$$

To prove that $\vartheta^{(n)} \to \vartheta^{(0)}$ it is sufficient to show that any limiting point ϑ^* of the sequence $\{\vartheta^{(n)}\}$ coincides with the element $\vartheta^{(0)}$. But it is really so, since by passing to limit in (A.30) one can be convinced readily that ϑ^* satisfies the equations

$$\|A_0^{(0)} x\|^{(0)} = \|A_1^{(0)} x\|^{(0)}, \quad \|x\|^{(0)} = 1, \qquad x \in K_+.$$

Since the only solution of the latter system is, by the definition, the vector $\vartheta^{(0)}$ then $\vartheta^* = \vartheta^{(0)}$.

So, the continuous dependance of the vector ϑ on the matrices A_0, A_1 and the norm $\|\cdot\|$ is established, and the proof of Theorem 5 is completed. □

A.4. PROOF OF THEOREMS 6–9

Proof of Lemma 11. The fact that the map

$$(\boldsymbol{A}, \|\cdot\|) \mapsto \Phi[\boldsymbol{A}, \|\cdot\|]. \tag{A.31}$$

is defined uniquely by the pair $(\boldsymbol{A}, \|\cdot\|)$, directly follows from (18), (19).

Remark now that by definition (17) of the direction function Φ, continuity of map (A.31) will be established if we show that both of the functions $\varphi_0 = \varphi_0[A_0]$ and $\varphi_1 = \varphi_1[A_1]$ depend continuously on the matrices A_0 and A_1 in the metric of the space $C[0,1]$, while the parameter $\theta = \theta[\boldsymbol{A}, \|\cdot\|]$ depends continuously on $\boldsymbol{A}$ and $\|\cdot\|$.

Continuous dependance of the functions $\varphi_0 = \varphi_0[A_0]$ and $\varphi_1 = \varphi_1[A_1]$ on defining them matrices immediately follows from definitions (9), (10). Besides, continuous dependency of the parameter $\theta = \theta[\boldsymbol{A}, \|\cdot\|]$ on $\boldsymbol{A}$ and $\|\cdot\|$ follows from the fact that θ is the ξ-coordinate (see (15)) of the vector ϑ defined in Theorem 5, which depends continuously on $\boldsymbol{A}$ and $\|\cdot\|$ by Theorem 5.

So, map (A.31) is continuous, and the proof of Lemma 11 is completed. □

To prove Theorem 8 we need an auxiliary statement.

Lemma A.9. *Let η be a closed orientation preserving circle map and let h be its standard lift. Then for any $\xi \in [0,1)$ and any pair of elements $\eta_\xi \in \eta(\xi)$, $h_\xi \in h(\xi)$ satisfying $\eta_\xi = h_\xi \pmod 1$ the following relation is valid:*

$$h_\xi = \eta_\xi + \nu(\eta_\xi), \tag{A.32}$$

where $\nu(\xi)$ is function (24) (see Fig. 8).

Conversely, if for a pair of elements $\eta_\xi \in \eta(\xi)$ and h_ξ relation (A.32) holds then $h_\xi \in h(\xi)$.

Proof. Fix a point $\xi \in [0,1)$ and choose a pair of elements $\eta_\xi \in \eta(\xi)$ and $h_\xi \in h(\xi)$ satisfying the relation $\eta_\xi = h_\xi \pmod 1$. Since, by Lemma's conditions, $h(\cdot)$ ia a standard lift of the map $\eta(\cdot)$ then $h(0) = \eta(0) \in [0,1)$. Then from the fact that the map $h(\cdot)$ is strictly increasing we obtain the estimates

$$0 \le \eta(0) = h(0) \le h_\xi < h(1) = h(0) + 1 = \eta(0) + 1 < 2, \qquad \xi \in [0,1),$$

i.e., $h_\xi \in [0,2)$.

If $h_\xi \in [0,1)$ then the equality $\eta_\xi = h_\xi \pmod 1$ implies $\eta_\xi = h_\xi$, and by monotony of the function $h(\cdot)$

$$\omega = \min\{y :\ y \in \eta(0)\} = \min\{y :\ y \in h(0)\} \le h_\xi = \eta_\xi < 1.$$

Hence, in this case $\nu(\eta_\xi) = 0$ from which we obtain that $h_\xi = \eta_\xi + \nu(\eta_\xi)$.

But if $h_\xi \in [1,2)$ then the equality $\eta_\xi = h_\xi \pmod 1$ implies $\eta_\xi = h_\xi - 1$. In this case by monotony of the function $h(\cdot)$ the following relations take place

$$\begin{aligned} 0 \le \eta_\xi &= h_\xi - 1 < \min\{y :\ y \in h(1)\} - 1 = \min\{y :\ y \in h(0) + 1\} - 1 = \\ &= \min\{y :\ y \in h(0)\} = \min\{y :\ y \in \eta(0)\} = \omega. \end{aligned}$$

Hence $\nu(\eta_\xi) = 1$ which again implies $h_\xi = \eta_\xi + \nu(\eta_\xi)$. So, in one direction Lemma A.9 is proved.

Now, let $\eta_\xi \in \eta(\xi)$ and h_ξ be elements for which relation (A.32) is fulfilled. By the definition of the lift of a circle map, the sets $\eta(\xi)$ and $h(\xi)$ satisfy the relation $\eta(\xi) = h(\xi) \pmod 1$. Consequently, the set $h(\xi)$ contains such an element h_* that $\eta_\xi = h_* \pmod 1$. But then, due to the already proven first part of Lemma, the relation $h_* = \eta_\xi + \nu(\eta_\xi)$ should be valid. But by supposition, for the elements η_ξ and h_ξ the analogous relation (A.32) is also true, i.e., $h_\xi = \eta_\xi + \nu(\eta_\xi)$, from which we immediately obtain $h_\xi = h_* \in h(\xi)$. Lemma A.9 is completely proved. □

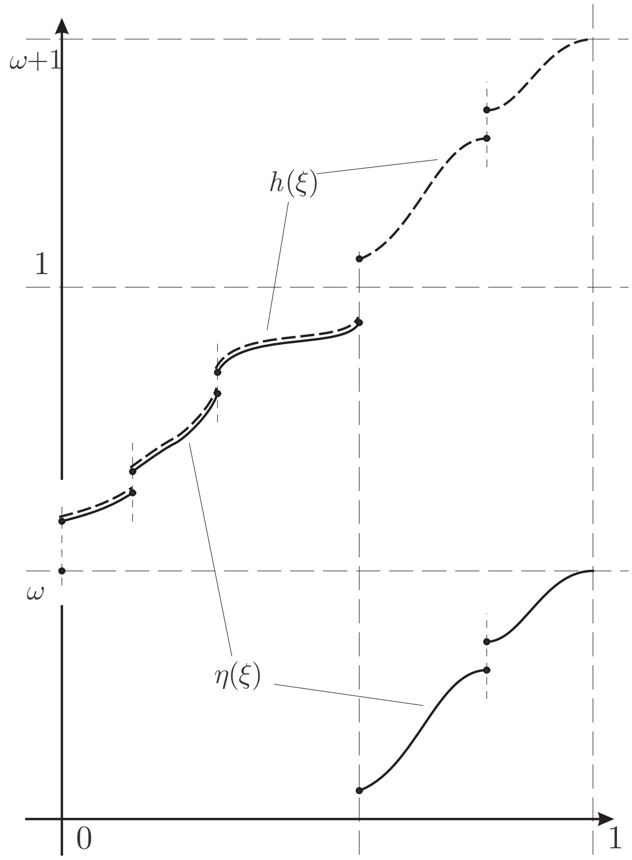

Fig. 8. Orientation preserving closed circle map $\eta(\xi)$ and its standard lift $h(\xi)$.

At last, we are able to present the definition of the rotation number of the circle map $\eta(\cdot)$ directly in terms of the map $\eta(\cdot)$ (to be precise, the definition of the rotation number of the standard lift $h(\cdot)$ of the map $\eta(\cdot)$).

Proof of Theorem 8. Define the sequence $\{\xi_n\}_{n=0}^{\infty}$ by setting $\xi_0 = \zeta_0$ and

$$\xi_n = \zeta_n + \sum_{i=1}^{n} \nu(\zeta_i), \qquad n = 1, 2, \dots \ .$$

Prove by induction that $\{\xi_n\}$ satisfies the inclusions

$$\xi_{n+1} \in h(\xi_n), \qquad n = 0, 1, \dots \ , \tag{A.33}$$

and so, it is a trajectory of the map h.

Indeed, by the definition, $\xi_1 = \zeta_1 + \nu(\zeta_1)$, where $\zeta_1 \in \eta(\zeta_0)$. Therefore, by Lemma A.9 $\xi_1 \in h(\zeta_0) = h(\xi_0)$, and the statement of Theorem 8 is true for $n = 0$.

Perform the step of induction. Suppose that the statement of Theorem 8 is valid for $n = k \geq 0$ and show that this imply its validity for $n = k + 1$. By the definition of the element ξ_{k+1},

$$\xi_{k+1} = \zeta_{k+1} + \sum_{i=1}^{k+1} \nu(\zeta_i)$$

or, what is the same,

$$\xi_{k+1} - \sum_{i=1}^{k} \nu(\zeta_i) = \zeta_{k+1} + \nu(\zeta_{k+1}).$$

Since here, by the definition of the trajectory $\{\zeta_n\}$, the inclusion $\zeta_{k+1} \in \eta(\zeta_k)$ with $\zeta_k \in [0,1)$ holds, then by Lemma A.9 $\zeta_{k+1} + \nu(\zeta_{k+1}) \in h(\zeta_k)$. Hence,

$$\xi_{k+1} - \sum_{i=1}^{k} \nu(\zeta_i) \in h(\zeta_k)$$

or, what is the same,

$$\xi_{k+1} \in h(\zeta_k) + \sum_{i=1}^{k} \nu(\zeta_i) = h(\zeta_k + \sum_{i=1}^{k} \nu(\zeta_i)).$$

Here, by the supposition of induction, the argument of the function h in the right-hand part coincides with ξ_k which implies $\xi_{k+1} \in h(\xi_k)$.

So, the step of induction is justified and inclusions (A.33) are proved. To complete the proof of Theorem 8 it remains to note only that by Theorem 7 and Remark 2 for the trajectory $\{\xi_n\}$ the estimates hold

$$\left|\frac{\xi_n}{n} - \tau(h)\right| \le \frac{1}{n}, \qquad n = 1, 2, \dots ,$$

while by the definition of trajectory $\{\xi_n\}$ it is valid the equality

$$\frac{\xi_n}{n} = \frac{\zeta_n}{n} + \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n},$$

where $\zeta_n \in [0,1)$. Estimates (23) now directly follow from the latter relations. Theorem 8 is proved. □

Proof of Theorem 9. Now all is ready to prove Theorem 6. Let $\{x_n\}$ be a B-extremal trajectory of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ and let $\{\sigma_n\}$ be the corresponding index sequence, i.e., the sequences $\{x_n\}$ and $\{\sigma_n\}$ satisfy the equalities $x_{n+1} = A_{\sigma_n} x_n$ for $n = 0, 1, \dots$. Then by Lemma 10 the numerical sequence $\xi_n = \xi(x_n)$, where the function $\xi(\cdot)$ is defined by equality (15), satisfies the relations

$$\xi_{n+1} = \varphi_{\sigma_n}(\xi_n) \in \varPhi_\theta(\xi_n), \qquad n = 0, 1, \dots ,$$

with some direction function $\varPhi_\theta$. At the same time, by Theorem 9 there is defined the frequency

$$\tau = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\xi_i)}{n},$$

and besides, $\xi_n \neq 0, 1$ for $n \ge 1$. Therefore, for $n \ge 1$ the value $\xi_{n+1} \in (0,1)$ is obtained from $\xi_n \in (0,1)$ by the formula $\xi_{n+1} = \varphi_0(\xi_n)$ if and only if $0 < \xi_{n+1} < \varphi_0(1)$ or, what is the same, if and only if $\nu(\xi_{n+1}) = 1$. Consequently, $\sigma_n = 1 - \nu(\xi_{n+1})$ for $n \ge 1$ and by Theorem 9 there is the limit

$$\sigma(\boldsymbol{A}) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \sigma_i}{n} = 1 - \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\xi_{i+1})}{n} = 1 - \tau(\boldsymbol{A}).$$

Now, all the assertions of Theorem 6 follow from analogous assertions of Theorem 9. □

Construct the map $\eta_\theta(\cdot)$ of the semiopen interval $[0,1)$ in itself with the help of following equalities

$$\eta_\theta(\xi) = \begin{cases} \varphi_1(0) \cup \varphi_0(1), & \text{if} \quad \xi = 0 \\ \varphi_1(\xi), & \text{if} \quad \xi \in (0, \theta] \\ \varphi_0(\xi), & \text{if} \quad \xi \in [\theta, 1). \end{cases}$$

This map can be treated as an orientation preserving circle map with the closed graph since it has the strictly increasing lift with a closed graph $h_\theta(\cdot)$ defined for $\xi \in [0,1)$ by the relation[8]

$$h_\theta(\xi) = \begin{cases} \varphi_1(0) \cup \varphi_0(1), & \text{if} \quad \xi = 0 \\ \varphi_1(\xi), & \text{if} \quad \xi \in (0, \theta] \\ \varphi_0(\xi) + 1, & \text{if} \quad \xi \in [\theta, 1). \end{cases}$$

[8] The lift $h_\theta(\cdot)$ can be extended on other values of $\xi \in \mathbb{R}$ with the preservation of the identity $h_\theta(\xi + 1) \equiv h_\theta(\xi) + 1$.

Point out that the map $\eta_\theta(\cdot)$ takes two values at each of the points $\xi = 0, \theta$.

Now, let $\{\xi_n\}$ be a trajectory of the map Φ_θ, i.e.,

$$\xi_{n+1} \in \Phi_\theta(\xi_n), \qquad n = 0, 1, \dots \ . \tag{A.34}$$

By Lemma 10 the parameter θ of the map $\Phi_\theta(\xi)$ satisfies the inclusion $\theta \in (0, 1)$. Then, as can be seen, e.g., from Fig. 5, the values of the function $\Phi_\theta(\xi)$ are separated from 0 and 1, i.e., one can find such $\mu > 0$ for which for all the elements of the trajectory $\{\xi_n\}$ will be valid the estimates $\mu \leq \xi_n \leq 1 - \mu$, except maybe for the element ξ_0. ¿From here and from (A.34), and taking into account that the values of the functions $\Phi_\theta(\xi)$ and $\eta_\theta(\xi)$ coincide with each other for $0 < \xi < 1$, we deduce that the trajectory $\{\xi_n\}$ satisfies the inclusions $\xi_{n+1} \in \eta_\theta(\xi_n)$ for $n = 1, 2, \dots$. Defining now the sequence $\{\zeta_n\}$ by setting

$$\zeta_n = \begin{cases} \xi_0 \pmod 1, & \text{if} \quad n = 0 \\ \xi_n, & \text{if} \quad n \geq 1, \end{cases}$$

one can easily verify that this sequence satisfies the inclusions $\zeta_{n+1} \in \eta_\theta(\zeta_n)$ for $n = 1, 2, \dots$. From here by Theorem 8 it follows the existence of such a number τ for which the estimates hold

$$\left| \frac{\sum_{i=1}^{n} \nu(\xi_i)}{n} - \tau \right| = \left| \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n} - \tau \right| \leq \frac{2}{n}, \qquad n = 1, 2, \dots \ , \tag{A.35}$$

where the function $\nu(\cdot)$ by Lemma A.9 has the form

$$\nu(\xi) = \begin{cases} 1 & \text{if} \quad 0 \leq \xi < \omega \\ 0 & \text{if} \quad \omega \leq \xi < 1. \end{cases}$$

Here $\omega = \min\{h_\theta(0)\} = \Phi_\theta(1)$ which means that in a formal sense the number ω depends on θ. But since by Lemma 10 the number θ satisfies the inclusion $\theta \in (0, 1)$ then $\Phi_\theta(1) \equiv \varphi_0(1)$. Therefore, in fact the number ω, as well as the function $\nu(\cdot)$, does not depend on θ.

Estimates (A.35) imply the existence of limit (25). Note that for a given direction function Φ_θ the number τ by Theorem 8 does not depend on the choice of the trajectory $\{\xi_n\}$, and so, τ is a function of the only argument θ, i.e., $\tau = \tau(\theta)$. Show that in fact *the number τ does not depend on θ*, too, but it is uniquely determined by the matrix set $\boldsymbol{A}$, i.e., $\tau = \tau(\boldsymbol{A})$.

Let $\Phi_{\theta_1}(\xi)$ and $\Phi_{\theta_2}(\xi)$ be the direction functions of some generators of B-extremal trajectories $g_1(x)$ and $g_2(x)$ corresponding to different Barabanov norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then by Theorem 3 there is a trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ which is extremal as in the norm $\|\cdot\|_1$ as in the norm $\|\cdot\|_2$. By the definition of a generator of B-extremal trajectories, this trajectory should satisfy as the inclusions

$$x_{n+1} \in g_1(x_n), \qquad n = 0, 1, 2, \dots \ ,$$

as the inclusions

$$x_{n+1} \in g_2(x_n), \qquad n = 0, 1, 2, \dots \ .$$

Then, by the definition of the direction function, the sequence $\{\xi_n\}$ defined by

$$\xi_n = \frac{x_{1,n}}{x_{1,n} + x_{2,n}}, \qquad n = 0, 1, 2, \dots \ ,$$

should satisfy as the inclusions

$$\xi_{n+1} \in \Phi_{\theta_1}(\xi_n), \qquad n = 0, 1, 2, \dots \ , \tag{A.36}$$

as the inclusions

$$\xi_{n+1} \in \Phi_{\theta_2}(\xi_n), \qquad n = 0, 1, 2, \dots\ . \tag{A.37}$$

We can use now formula (25) to calculate the number $\tau(\theta_1)$ for the sequence $\{\xi_n\}$ treating the latter as the sequence satisfying (A.36). Analogously, we can use the same formula (25) to calculate the number $\tau(\theta_2)$ for the same sequence $\{\xi_n\}$ but this time treating it as the sequence satisfying (A.37). Since in the both cases calculations (25) are performed with the same sequence $\{\xi_n\}$ then we conclude that $\tau(\theta_1) = \tau(\theta_2)$, from which independence of the number τ from θ follows.

Validity of assertions (i)–(iii) for $\tau(\boldsymbol{A})$ follows from the definition of the number $\tau(\boldsymbol{A})$ and from Theorem 7. Therefore, to complete the proof of Theorem 9 it remains only to establish the continuous dependance of the function $\tau(\boldsymbol{A})$ on the matrix set $\boldsymbol{A}$. Let $\{\boldsymbol{A}^{(n)} \in \mathcal{M}^\sharp\}$ be a sequence of matrix sets converging to the matrix set $\boldsymbol{A}^* \in \mathcal{M}^\sharp$. Fix a vector $x_0 \neq 0 \in \mathbb{R}^2$ and choose for each n a Barabanov norm $\|\cdot\|^{(n)} \in N_{\mathrm{Bar}}(\boldsymbol{A}^{(n)}, x_0)$, and then build the direction function $\Phi_{\theta^{(n)}}$ of the generator of B-extremal trajectories corresponding to the matrix set $\boldsymbol{A}^{(n)}$ and the norm $\|\cdot\|^{(n)}$.

By Theorem 1 one can suppose that the sequence $\{\|\cdot\|^{(n)}\}$ converges in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$ to some Barabanov norm $\|\cdot\|^*$ corresponding to the matrix set $\boldsymbol{A}^*$. Then by Lemma 11 the sequence $\{\Phi_{\theta^{(n)}}\}$ converges by the metric of the space $\mathcal{F}$ to the direction function Φ_{θ^*} of the generator of B-extremal trajectories corresponding to the matrix set $\boldsymbol{A}^*$ and the norm $\|\cdot\|^*$. Hence

$$\tau(\boldsymbol{A}^{(n)}) = \tau(\Phi_{\theta^{(n)}}) \to \tau(\Phi_{\theta^*}) = \tau(\boldsymbol{A}^*). \tag{A.38}$$

Here, convergence of the numerical sequence $\{\tau(\Phi_{\theta^{(n)}})\}$ to $\tau(\Phi_{\theta^*})$ follows from convergence of the sequence of functions $\{\Phi_{\theta^{(n)}}\}$ to the function Φ_{θ^*} by the metric of the space $\mathcal{F}$ (i.e., in the sense of convergence of the graphs of these functions in the Hausdorff metric) and from Theorems 7 and 8. Equalities in (A.38) follows from the already proven fact that the number $\tau(\boldsymbol{A})$ does not depend on the choice of the direction function of the generator of B-extremal trajectories of the matrix set $\boldsymbol{A}$.

Thus, continuous dependance of the number $\tau(\boldsymbol{A})$ on the matrix set $\boldsymbol{A}$ is proved, and so the proof of Theorem 9 is completed. □

A.5. PROOF OF THEOREM 10 AND LEMMA 12

Proof of Theorem 10. Since the matrices A_0 and A_1 are non-negative then by the Perron-Frobenius theorem there is a vector x_0 with non-negative coordinates such that

$$\rho^n x_0 = A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1} x_0, \tag{A.39}$$

where $\rho = \rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1})$.

Extend the finite index sequence $\{\sigma_k\}_{k=1}^n$ to the infinite periodic one with period n and then consider the corresponding sequence $\{x_k\}_{k=0}^\infty$:

$$x_1 = A_{\sigma_1} x_0, \quad \dots, \quad x_{n-1} = A_{\sigma_{n-1}} x_{n-2}, \quad x_n = A_{\sigma_n} x_{n-1}, \quad \dots\ .$$

Then from (A.39) we get $x_n = \rho^n x_0$, and in any Barabanov norm $\|\cdot\|$ the following inequalities will be valid

$$\|x_1\| \le \rho(\boldsymbol{A}) \|x_0\|, \quad \dots, \quad \|x_n\| = \rho^n \|x_0\| \le \rho(\boldsymbol{A}) \|x_{n-1}\|, \quad \dots\ , \tag{A.40}$$

from which $\rho \le \rho(\boldsymbol{A})$. Here, the equality $\rho = \rho(\boldsymbol{A})$ may take place only in the case when each of inequalities (A.40) is in fact equality, i.e., when the sequence $\{x_n\}$ is extremal in the Barabanov norm $\|\cdot\|$. However, by Theorem 9 periodicity of the index sequence of at least one of the B-extremal trajectories of the matrix set $\boldsymbol{A}$ implies the rationality of the number $\sigma(\boldsymbol{A})$ which contradicts the condition of Theorem. The obtained contradiction is caused by the supposition that $\rho = \rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1}) = \rho(\boldsymbol{A})$. Theorem 10 is proved. □

Proof of Lemma 12. First prove assertion a). Denote by $\mathcal{K}$ the set (cone) of all vectors from the first quadrant lying between the straight lines $L_0 = \{(x_0, x_1) : bx_1 = (1-a)x_0\}$ and $L_1 = \{(x_0, x_1) : (1-d)x_1 = cx_0\}$, i.e.,

$$\mathcal{K} := \{(x_0, x_1) :\ x_0, x_1 \geq 0,\ (1-a)x_0 \leq bx_1,\ (1-d)x_1 \leq cx_0\}.$$

Then the direct verification shows that

$$A_0\mathcal{K} \subseteq \mathcal{K}, \qquad A_1\mathcal{K} \subseteq \mathcal{K}. \tag{A.41}$$

Fix a number $\gamma > 1$, and show that

$$A_0 x > \gamma A_1 x, \qquad \forall\ x \neq 0 \in \mathcal{K}, \tag{A.42}$$

as soon as the quotient α/β is sufficiently large (inequality (A.42) is understood coordinate-wise). Indeed, the vector inequality (A.42) is equivalent to the pair of scalar inequalities

$$\alpha(ax_0 + bx_1) > \gamma\beta x_0, \quad \alpha x_1 \geq \gamma\beta(cx_0 + dx_1), \qquad \forall\ x \neq 0 \in \mathcal{K},$$

or, what is the same,

$$\frac{\alpha}{\beta} > \gamma \sup_{x \neq 0 \in \mathcal{K}} \left\{ \frac{x_0}{ax_0 + bx_1},\ \frac{cx_0 + dx_1}{x_1} \right\}. \tag{A.43}$$

But as is easy to see, supremum in the right-hand part of (A.43) is finite, from which it follows that inequality (A.42) is valid for sufficiently large values of the quotient α/β.

Show now that $\sigma(\boldsymbol{A}) = 0$ for all sufficiently large values of α/β. Let $\|\cdot\|$ be an arbitrary Barabanov norm for the matrix set $\boldsymbol{A}$, let x^* be a nonzero vector from the cone $\mathcal{K}$, and let the parameters α and β be such that inequality (A.42) holds. Then by Lemma 5 there is a B-extremal trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ which starts from the point x^*, i.e., $x^{(0)} = x^* \neq 0 \in \mathcal{K}$ and

$$x^{(n+1)} = A_{\sigma_n} x^{(n)}, \quad \|x^{(n+1)}\| = \rho\|x^{(n)}\| \qquad n = 0, 1, \ldots\ .$$

Moreover, (A.41) implies that $x^{(n)} \in \mathcal{K}$ for $n = 0, 1, \ldots$. Show that in this case the index sequence $\{\sigma_n\}$ satisfies the identity $\sigma_n \equiv 0$.

Indeed, in the opposite case $\sigma_{n_0} = 1$ for some n_0. Then by the definition of the B-extremal trajectory

$$\|x^{(n_0+1)}\| = \|A_1 x^{(n_0)}\| = \rho\|x^{n_0}\|, \tag{A.44}$$

where $\rho = \rho(\boldsymbol{A})$, and at the same time the inequality

$$\|A_0 x^{(n_0)}\| \leq \rho\|x^{n_0}\|, \tag{A.45}$$

should be valid. But by Lemma 9 the Barabanov norm $\|\cdot\|$ is monotone and then by (A.42)

$$\|A_0 x^{(n_0)}\| \geq \|\gamma A_1 x^{(n_0)}\|,$$

where $\gamma > 1$, which contradicts to relations (A.44) and (A.45).

So, it is shown that $\sigma_n \equiv 0$, from which by Theorem 6 $\sigma(\boldsymbol{A}) = 0$.

To complete the proof of assertion a) it remains to note that in view of (A.15) $\rho(\boldsymbol{A}) > \beta$ for $\alpha/\beta < 1$. Therefore the generalized spectral radius cannot be attained on a B-extremal trajectory with the index sequence $\sigma_n \equiv 0$. But this means that $\sigma(\boldsymbol{A}) > 0$ for $\alpha/\beta < 1$. The proof of assertion a) is completed.

The proof of assertion b) can be provided analogously to that of assertion a). Therefore, it remains only to prove assertion c). To emphasize that $\sigma(\boldsymbol{A})$ is the frequency of applying of the matrix A_1 in the process of construction of the B-extremal trajectory for the ordered matrix set $\boldsymbol{A} = \{A_0, A_1\}$, until the end of the proof of Lemma the quantity $\sigma(\boldsymbol{A})$ will be denoted by $\sigma(A_0, A_1)$. Note now that the entries b and c of the matrices A_0 and A_1 can be treated equal to each other since this may be achieved by the change of variables $\tilde{x}_0 = tx_0$, $\tilde{x}_1 = x_1$ with $t = \sqrt{c/b}$. Therefore, without restriction in generality one may suppose that the matrices A_0 and A_1 have the form

$$A_0 = \alpha \begin{Vmatrix} 1 & s \\ 0 & 1 \end{Vmatrix}, \quad A_1 = \beta \begin{Vmatrix} 1 & 0 \\ s & 1 \end{Vmatrix},$$

where $s = \sqrt{bc} \geq 1$.

Now, change the variables by setting $\tilde{x}_0 = x_1$, $\tilde{x}_1 = x_0$. Then in the new coordinates the matrices A_0 and A_1 take the form $\tilde{A}_0$ and $\tilde{A}_1$ where

$$\tilde{A}_0 = \alpha \begin{Vmatrix} 1 & 0 \\ s & 1 \end{Vmatrix}, \quad \tilde{A}_1 = \beta \begin{Vmatrix} 1 & s \\ 0 & 1 \end{Vmatrix}.$$

Clearly, any change of variables transforms extremal trajectories of the pair of matrices $\{A_0, A_1\}$ in extremal trajectories of the pair of matrices $\{\tilde{A}_0, \tilde{A}_1\}$. Hence, $\sigma(A_0, A_1) = \sigma(\tilde{A}_0, \tilde{A}_1)$. Note now that for $\alpha = \beta$ we have the equalities $\tilde{A}_0 = A_1$, $\tilde{A}_1 = A_0$, and therefore the following chain of equalities is valid:

$$\sigma(A_0, A_1) = \sigma(\tilde{A}_0, \tilde{A}_1) = \sigma(A_1, A_0) = 1 - \sigma(A_0, A_1).$$

The latter relations immediately implies the equality $\sigma(\boldsymbol{A}) = \sigma(A_0, A_1) = \frac{1}{2}$.

It remains to prove that $\sigma(\boldsymbol{A}) \neq \frac{1}{2}$ for all sufficiently large values of α/β. This will be shown if we prove that for sufficiently large values of α/β it is valid the inequality

$$\rho\left(A_0^2 A_1\right)^{\frac{1}{3}} > \rho\left(A_0 A_1\right)^{\frac{1}{2}}, \tag{A.46}$$

since the latter inequality means that for given α and β a B-extremal trajectory of the matrix set $\{A_0, A_1\}$ cannot has the index sequence of period 2, and so, $\sigma(A_0, A_1) \neq \frac{1}{2}$.

Direct calculations show that $A_0^2 A_1 = \alpha^2 \beta R$, $A_0 A_1 = \alpha\beta S$, where the matrices R and S have the form:

$$R = \begin{Vmatrix} 1 + 2s^2 & 2s \\ s & 1 \end{Vmatrix}, \quad S = \begin{Vmatrix} 1 + s^2 & s \\ s & 1 \end{Vmatrix}.$$

Then inequality (A.46) is equivalent to the inequality $\left(\alpha^2\beta\rho(R)\right)^{\frac{1}{3}} > \left(\alpha\beta\rho(S)\right)^{\frac{1}{2}}$, and so, inequality (A.46) is valid for

$$\frac{\alpha}{\beta} > \frac{\rho^2(R)}{\rho^3(S)}.$$

Analogously can be shown that $\rho\left(A_0 A_1^2\right)^{\frac{1}{3}} > \rho\left(A_0 A_1\right)^{\frac{1}{2}}$ as soon as α/β is sufficiently small, which implies validity of the non-equality $\sigma(A_0, A_1) \neq \frac{1}{2}$ for all sufficiently small values of α/β.

So, assertion c), and the Lemma with it, are completely proved. $\square$

REFERENCES

1. Barabanov, N.E., Lyapunov Indicator of Discrete Inclusions. I-III, *Autom. Remote Control*, 1988, vol. 49, no. 2, pp. 152–157; vol. 49, no. 3, pp. 283–287; vol. 49, no. 5, pp. 558–565 .
2. Berger, M.A. and Wang, Y., Bounded Semigroups of Matrices, *Lin. Algebra Appl.*, 1992, vol. 166, pp. 21–27.
3. Blondel, V.D., Theys, J., and Vladimirov, A.A., Switched Systems That Are periodically Stable May Be Unstable, in *Proc. of the Symposium MTNS*, Notre-Dame, USA, 2002.
4. Blondel, V.D., Theys, J., and Vladimirov, A.A., An Elementary Counterexample to the Finiteness Conjecture, *SIAM J. Matrix Analysis*, 2003, vol. 24, no. 4, pp. 963–970.
5. Bousch, T. and Mairesse, J., Asymptotic Height Optimization for Topical IFS, Tetris Heaps, and the Finiteness Conjecture, *J. Am.. Math. Soc.*, 2002, vol. 15, no. 1, pp. 77–111.
6. Brette, R., Rotation Numbers of Discontinuous Orientation-preserving Circle Maps, *Set-Valued Analysis*, 2003, vol. 11, no. 4, pp. 359–371.
7. Heil, C. and Strang, G., Continuity of the Joint Spectral Radius: Application to Wavelets, in *Linear Algebra for Signal Processing*, Bojanczyk, A. and Cybenko G., Eds., Minneapolis: IMA, 1992.
8. Kozyakin, V.S., Algebraic Unsolvability of Problem of Absolute Stability of Desynchronized Systems, *Autom. Remote Control*, 1990, vol. 51, no. 6, pp. 754–759.
9. Kozyakin, V.S., Sturmian Sequences Generated by Order Preserving Circle Maps, *Preprint of Boole Centre for Research in Informatics, University College Cork, National University of Ireland*, Cork, 2003, no. 11.
10. Kozyakin, V.S., Discontinuous Order Preserving Circle Maps Versus Circle Homeomorphisms, *Preprint of Boole Centre for Research in Informatics, University College Cork, National University of Ireland*, Cork, 2003, no. 12.
11. Kozyakin, V.S., Extremal Norms, Discontinuous Circle Maps and a Counterexample to the Finiteness Conjecture, *Information Processes*, 2005, vol. 5, no. 4, pp. 301–335.
12. Kozyakin, V.S., Proof of a Counterexample to the Finiteness Conjecture in the Spirit of the Theory of Dynamical Systems, *Preprint of Weierstraß-Institut für Angewandte Analysis und Stochastik*, Berlin, 2005, no 105.
13. Kozyakin, V.S. and Pokrovskii, A.V., The Role of Controllability-type Properties in the Study of the Stability of Desynchronized Dynamical Systems, *Soviet Phys. Dokl.*, 1992, vol. 37, no. 5, pp. 213–215.
14. Kozyakin, V.S. and Pokrovskii, A.V., Estimates of Amplitudes of Transient Regimes in Quasi-controllable Discrete Systems, *CADSEM Report 96–005*, Deakin University, Geelong, Australia, 1996.
15. Kozyakin, V.S. and Pokrovskii, A.V., Quasi-controllability and Estimation of the Amplitudes of Transient Regimes in Discrete Systems, *Trans. Russ. Acad. Natural Sci., ser. MMMIC*, 1997, vol. 1, no. 3, pp. 128–150.
16. Kuratowski, K, *Topology*, New York: Academic Press, 1966, vol. 1.
17. Lagarias, J.C. and Wang, Y., The Finiteness Conjecture for the Generalized Spectral Radius of a Set of Matrices, *Lin. Algebra Appl.*, 1995, vol. 214, pp. 17–42.
18. Miloslavljevich, C., General Conditions for the Existence of Quasisliding Mode on the Switching Hyperplane in Discrete Variable Structure Systems, *Autom. Remote Control*, 1985, vol. 46, no. 3, pp. 307–314.
19. Misawa, E.A., Discrete-time Sliding Mode Control: The Linear Case, *ASME J. Dynamical Syst., Measur., Control*, 1997, vol. 119, no. 4, pp. 819–821.
20. Wirth, F., The Generalized Spectral Radius and Extremal Norms, *Lin. Algebra Appl.*, 2002, vol. 342, pp. 17–40.
21. Wirth, F., On the Structure of the Set of Extremal Norms of a Linear Inclusion, in *Proc. of the 44th IEEE Conf. on Decision and Control, and the Eur. Control Conf.*, Seville, Spain, 2005, pp. 3019–3024.