HOMOGENEOUS ALMOST KÄHLER MANIFOLDS AND THE CHERN-EINSTEIN EQUATION

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Abstract. Given a non compact semisimple Lie group G we describe all homogeneous spaces G/L carrying an invariant almost Kähler structure (ω, J). When L is abelian and G is of classical type, we classify all such spaces which are Chern-Einstein, i.e. which satisfy $\rho = \lambda\omega$ for some $\lambda \in \mathbb{R}$, where ρ is the Ricci form associated to the Chern connection.

1. Introduction

Given an almost Kähler manifold (M, g, J), that is an Hermitian manifold with closed Kähler form $\omega = g \circ J$, the Ricci form ρ of the associated Chern connection D is a closed 2-form which represents the cohomology class $2\pi c_1(M, J)$ in $H^2(M, \mathbb{R})$. The Chern-Einstein equation $\rho = \lambda\omega$ for some $\lambda \in \mathbb{R}$ gives a very natural generalization of the Einstein condition. This equation has been considered in [AD] and more recently in [DV]. More generally, we can consider a symplectic manifold (M, ω) and study the existence of a compatible almost complex structure J so that the Chern- Einstein equation $\rho = \lambda\omega$ is satisfied. In [DV] several examples of non-compact homogeneous examples of Chern-Einstein almost Kähler manifolds are given and some structure theorems are proved.

In this work we focus on non-compact symplectic manifolds (M, ω) which admit a (non-compact) semisimple Lie group G of transitive symplectomorphisms with compact isotropy subgroup. Our first result is stated in Theorem 2.2 and it shows that there exists a unique G-homogeneous almost complex structure J which is compatible with ω. This result has to be contrasted to the well-known case when G is compact and the homogeneous almost complex structure is necessarily integrable (see e.g. [WG]). We then study the Chern-Einstein equation in the homogeneous setting and we can establish a full classification under the additional condition that the isotropy subgroup is abelian (hence a maximal torus in G). The main result is summarized in the following

Theorem 1.1. *Let $(M = G/L, \omega)$ be a homogeneous symplectic manifold of a non compact semisimple Lie group and L compact. Then there exists a unique invariant almost complex structure J compatible with ω so that (M, ω, J) is almost Kähler.*

If the group G is simple non compact of classical type and L is abelian, then (M, ω, J) is Chern Einstein, say $\rho = \lambda\omega$, if and only if one of the following occurs:

i) $\lambda < 0$ *and* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$
ii) $\lambda = 0$ *and* $\mathfrak{g} = \mathfrak{su}(p + 1, p)$, $p \geq 1$.

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While the full classification of the Chern-Einstein non compact homogeneous almost Kähler manifolds remains out of reach by this time, we can give some simple examples when the isotropy has one dimensional center (see section 4.2). We also remark that the case when G is semisimple can be easily deduced from the main result, as the manifold will split as an almost Kähler product of G_i- homogeneous manifolds where G_i are the simple factors of G.

In section 2 we describe the homogeneous almost Kähler non-compact manifolds, which are acted on transitively by a semisimple Lie group. In section 3 we describe the Chern connection in the homogeneous setting and give a formula for the Ricci form ρ, which is analogue to the standard formula in the compact case (see e.g. [AP], [BFR]). In the last section we describe the general strategy to prove the classification claimed in our main theorem and we prove it going through each simple Lie algebra of classical type. We also indicate how to produce homogeneous Chern-Einstein manifolds with non abelian compact isotropy with one dimensional center.

Notation. Lie groups and their Lie algebras will be indicated by capital and gothic letters respectively. We will denote the Cartan-Killing form by B. If a Lie group G acts on a manifold M, for every $X \in \mathfrak{g}$ we will denote by X^* the corresponding vector field induced by the one-parameter subgroup $\exp(tX)$.

2. Homogeneous almost Kähler non-compact manifolds of semisimple Lie groups

Let G be a non compact semisimple Lie group with Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K and $\mathfrak{p}$ is an $\mathrm{Ad}(K)$-invariant complement with $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$.

We consider a homogeneous symplectic manifold of the form G/L where $L \subseteq K$ is the centralizer $C_G(t_o)$ for some $t_o \in \mathfrak{k}$. Any G-homogeneous symplectic manifold with compact stabilizer is simply connected and it has this form (see e.g. [BFR]).

The reductive decomposition of the manifold $M = G/L$ is given by

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{m} = \mathfrak{l} + \mathfrak{n} + \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{l} + \mathfrak{n}.$$

The manifold G/L admits a G-equivariant fibration $G/L \to G/K$ over the non-compact symmetric space $S := G/K$ with typical fibre given by the flag manifold $F := K/L$.

Any invariant symplectic form ω is defined by a closed non degenerate $Ad(L)$-invariant element $\omega_{\mathfrak{m}}$ in $\Lambda^2(\mathfrak{m}^*)$. Any such form $\omega_{\mathfrak{m}}$ can be written as $\omega_{\mathfrak{m}} = d\eta$, where the 1-form $\eta = B \circ t_o$ for some element $t_o \in \mathfrak{t}$ so that $C_{\mathfrak{g}}(t_o) = \mathfrak{l}$. We fix an invariant symplectic form ω which is associated to an element $t_o \in \mathfrak{l}$.

We denote by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$ and we set $\mathfrak{h} := i\mathfrak{t} \subset \mathfrak{g}^c$ together with $z_o := it_o \in \mathfrak{h}$. The complexification $\mathfrak{g}^{\mathbb{C}}$ has the root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where $R \subset \mathfrak{h}^*$ is the root system w.r.t. to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$. The root system R can be split as $R = R_{\mathfrak{l}} \cup R_{\mathfrak{m}} = R_{\mathfrak{l}} \cup (R_c \cup R_{nc})$ where

$$\mathfrak{l} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,\ \mathfrak{n}^{\mathbb{C}} = \bigoplus_{\alpha \in R_c} \mathfrak{g}_\alpha,\ \mathfrak{p}^{\mathbb{C}} = \bigoplus_{\alpha \in R_{nc}} \mathfrak{g}_\alpha$$

The roots in R_c are called *compact*, while roots in R_{nc} are said to be *non-compact*.

We select a standard Chevalley basis for root spaces, namely a set of root vectors $\{E_\alpha\}_{\alpha \in R}$ so that $\mathfrak{g}_\alpha = \mathbb{C} \cdot E_\alpha$ for every $\alpha \in R$ and

$$B(E_\alpha, E_{-a}) = 1, \quad [E_\alpha, E_{-\alpha}] = H_\alpha \in \mathfrak{h},$$

where H_α denotes the coroot in $\mathfrak{h}$ so that $B(H_\alpha, H) = \alpha(H)$ for every $H \in \mathfrak{h}$. If we now consider the antilinear involution σ of $\mathfrak{g}^{\mathbb{C}}$ corresponding to the real form $\mathfrak{g}$ (for brevity we will write the conjugation τ as $^-$), then

$$\overline{E_\alpha} = -E_{-\alpha}, \qquad \alpha \in R_c$$
$$\overline{E_\alpha} = E_{-\alpha}, \qquad \alpha \in R_{nc}.$$

We consider the vectors in $\mathfrak{g}$ defined as follows

$$v_\alpha := E_\alpha + \bar{E}_\alpha = \begin{cases} E_\alpha - E_{-\alpha}, & \alpha \in R_c \\ E_\alpha + E_{-\alpha}, & \alpha \in R_{nc} \end{cases}$$

$$w_\alpha := i(E_\alpha - \bar{E}_\alpha) = \begin{cases} i(E_\alpha + E_{-\alpha}), & \alpha \in R_c \\ i(E_\alpha - E_{-\alpha}), & \alpha \in R_{nc}. \end{cases}$$

Then

$$\mathfrak{n} = \bigoplus_{\alpha \in R_c} \mathrm{Span}_{\mathbb{R}}\{v_\alpha, w_\alpha\}, \qquad \mathfrak{p} = \bigoplus_{\beta \in R_{nc}} \mathrm{Span}_{\mathbb{R}}\{v_\beta, w_\beta\}.$$

We now consider G-invariant almost complex structures J on G/L, i.e. $\mathrm{Ad}(L)$-invariant endomorphisms $J \in \mathrm{End}(\mathfrak{m})$ with $J^2 = -Id$, or equivalently $\mathrm{ad}(\mathfrak{l}^{\mathbb{C}})$-invariant endomorphisms $J \in \mathrm{End}(\mathfrak{m}^{\mathbb{C}})$ with $J^2 = -Id$ and commuting with the conjugation σ.

Since $\mathfrak{l}^{\mathbb{C}}$ contains a Cartan subalgebra, the $\mathrm{ad}(\mathfrak{l})$-invariance of J implies that J preserves every root space and therefore for every $\alpha \in R_{\mathfrak{m}}$ we have

$$JE_\alpha = i\epsilon_\alpha \cdot E_\alpha, \qquad \epsilon_\alpha = \pm 1.$$

Therefore we can decompose

$$R_c = R_c^{10} \cup R_c^{01}, \qquad R_{nc} = R_{nc}^{10} \cup R_{nc}^{01},$$

where

$$R_{c/nc}^{10} = \{\alpha \in R_{c/nc} |\ \epsilon_\alpha = 1\},\ R_{c/nc}^{01} = \{\alpha \in R_{c/nc} |\ \epsilon_\alpha = -1\}.$$

Since J commutes with σ, we have $\epsilon_{-\alpha} = -\epsilon_\alpha$ for every $\alpha \in R_{\mathfrak{m}}$, hence

$$R_c^{01} = -R_c^{10}, \qquad R_{nc}^{01} = -R_{nc}^{10}.$$

The $\mathrm{ad}(\mathfrak{l}^{\mathbb{C}})$-invariance of J means that

$$(R_{\mathfrak{l}} + R_c^{10}) \cap R \subseteq R_c^{10},\ (R_{\mathfrak{l}} + R_{nc}^{10}) \cap R \subseteq R_{nc}^{10}. \tag{2.1}$$

We now consider the invariant pseudo-Riemannian metric g which is defined by the symmetric form

$$g(u, v) = \omega(u, Jv),\ u, v \in \mathfrak{m}.$$

Lemma 2.1. *The pseudo-Riemannian metric g defined above is J-Hermitian and it is positive definite if and only if the following conditions are satisfied*

$$\alpha(z_o) > 0 \quad \text{for } \alpha \in R_c^{10}, \qquad \alpha(z_o) < 0 \quad \text{for } \alpha \in R_{nc}^{10}.$$

Proof. In order to prove that g is J-Hermitian, we note that $\omega(E_\alpha, E_\beta) \neq 0$ if and only if $\alpha + \beta = 0$ and in this case we have

$$\omega(JE_\alpha, JE_{-\alpha}) = -\epsilon_{-\alpha} \cdot \epsilon_\alpha \cdot \omega(E_\alpha, E_{-\alpha}) = \omega(E_\alpha, E_{-\alpha}).$$

Clearly $g(E_\alpha, E_\beta) = 0$ if $\alpha + \beta \neq 0$ and g is positive definite if and only if $g(v_\alpha, v_\alpha) > 0$ for every $\alpha \in R_{\mathfrak{m}}$. Now if $\alpha \in R_c^{10}$ we have

$$0 < g(v_\alpha, v_\alpha) = \omega(E_\alpha - E_{-\alpha}, J(E_\alpha - E_{-\alpha})) = i\,\omega(E_\alpha - E_{-\alpha}, E_\alpha + E_{-\alpha}) = 2i\,\omega(E_\alpha, E_{-\alpha}) =$$
$$= 2i\ B([t_o, E_\alpha], E_{-\alpha}) = 2\alpha(z_o),$$

while if $\alpha \in R_{nc}^{10}$ we have

$$0 < g(v_\alpha, v_\alpha) = \omega(E_\alpha + E_{-\alpha}, J(E_\alpha + E_{-\alpha})) = i\,\omega(E_\alpha + E_{-\alpha}, E_\alpha - E_{-\alpha}) = -2i\,\omega(E_\alpha, E_{-\alpha}) =$$
$$= -2i\ B([t_o, E_\alpha], E_{-\alpha}) = -2\alpha(z_o).$$

□

We summarize the above arguments in the following Theorem.

Theorem 2.2. *Let G be a semisimple non-compact Lie group, L a compact subgroup of G given by the centralizer in G of some element $t_o \in \mathfrak{g}$. Let $\omega = \omega_{t_o}$ be the invariant symplectic form associated to t_o. Then there exists a unique extension of the homogeneous symplectic manifold (M, ω) to a homogeneous almost Kähler manifold (M, ω, J_{t_o}) where the invariant almost complex structure J_{t_o} is defined by the holomorphic space $\mathfrak{m}^{10}$*

$$\mathfrak{m}^{10} = \mathfrak{g}(R^{10}), \quad R^{10} = R_c^{10} \cup R_{nc}^{10},$$
$$R_c^{10} = \{\alpha \in R_c |\ \alpha(z_o) > 0\}, \quad R_{nc}^{10} = \{\alpha \in R_{nc} |\ \alpha(z_o) < 0\}.$$

The almost complex structure J_{t_o} is integrable, hence (M, ω, J_{t_o}) is Kähler if and only if the symmetric space G/K is Hermitian.

The last assertion follows from the following observations. Indeed, $(R_c^{01} + R_{nc}^{10}) \cap R \subset R_{nc}^{10}$ and K-invariance of $J|_{\mathfrak{p}}$ is equivalent to the integrability condition $(R_c^{10} + R_{nc}^{10}) \cap R \subset R_{nc}^{10}$.

Remark 2.3. (1) Note that the restriction of the almost complex structure J_{t_o} to the (complex) fibre F is integrable and $(\omega|_F, J|_F)$ is a Kähler structure.

(2) If we decompose $G = G_1 \cdot G_2 \cdot \ldots \cdot G_k$ as the product of its simple factors, then the homogenous almost Kähler space G/L splits accordingly as $\Pi_{i=1}^k G_i/L_i$ where $L_i = L \cap G_i$ and each factor is a homogeneous almost Kähler space. Therefore we can always assume that G is simple.

Let $M = G/L \to G/K$ be the G-equivariant fibering of an almost Kähler homogeneous manifold $(M = G/L, \omega, J)$ over the symmetric space G/K. We can construct an integrable complex structure $\tilde{J}$ which coincides with J along the fibre F and is the opposite to J on the orthogonal space. We call $\tilde{J}$ the complex structure associated to the almost Kähler homogeneous manifold $(M = G/L, \omega, J)$. Note that $(\omega, \tilde{J}_{t_o})$ is an invariant pseudo-Kähler structure on $M = G/L$.

3. The Chern-Einstein equation for almost Kähler homogeneous manifolds

Now we describe the expression for the Chern connection D on the homogeneous almost Kähler manifold G/L with invariant symplectic structure ω_{z_o} and corresponding invariant almost complex structure J (see also [P]).

It is well known that the Chern connection is the unique connection D that leaves g and J parallel and whose torsion T satisfies the property

$$T(JX, Y) = T(X, JY) = JT(X, Y).$$

We are mainly interested in the (first) Ricci form ρ which is a defined as

$$\rho(X, Y) = \mathrm{Tr}\ J \circ R_{XY},$$

where R denotes the curvature tensor. It is known that the Ricci form ρ is a closed 2-form whose cohomology class $[\rho]$ represents $2\pi c_1(M, \omega)$ (see e.g. [AD], §7).

We recall that any invariant connection D on the homogeneous space G/L with reductive decomposition $\mathfrak{g} = \mathfrak{l} + \mathfrak{m}$ can be described by the $\mathrm{ad}(\mathfrak{l})$-equivariant Nomizu's map $\Lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ satisfying the condition

$$\Lambda_X = \mathrm{ad}_X|_{\mathfrak{m}}, \qquad X \in \mathfrak{l}.$$

Under the identification $\mathfrak{m} \cong T_oG/L$ we have

$$\Lambda_X Y = (D_X Y^* - [X^*, Y^*])|_o,$$

where X^*, Y^* denote the vector fields on M corresponding to $X, Y \in \mathfrak{m}$. Then in terms of Nomizu's operator, the torsion T and the curvature R at $o \in M$ are given by

$$T(X, Y) = \Lambda_X Y - \Lambda_Y X - [X, Y]_{\mathfrak{m}},$$

$$R_{XY} = [\Lambda_X, \Lambda_Y] - \Lambda_{[X,Y]}.$$

If D is the Chern connection on the homogeneous space G/L we can compute its Ricci form in terms of the root space decomposition.

Proposition 3.1. *For every root $\alpha, \beta \in R_{\mathfrak{m}}$ we have*

$$\rho(E_\alpha, E_\beta) = 0 \quad \textit{if } \alpha + \beta \neq 0,$$

$$\rho(E_\alpha, E_{-\alpha}) = -2i \sum_{\beta \in R^{10}_{\mathfrak{m}}} \langle \alpha, \beta \rangle.$$

Proof. Using the expression for the curvature R and the fact that Λ commutes with J, we see that for $X, Y \in \mathfrak{m}$

$$\rho(X, Y) = Tr(J\Lambda_X\Lambda_Y - J\Lambda_Y\Lambda_X) - Tr(J\Lambda_{[X,Y]}) =$$

$$= Tr([\Lambda_X, J\Lambda_Y]) - Tr(J\Lambda_{[X,Y]}) = -Tr(J\Lambda_{[X,Y]}).$$

For $\alpha, \beta \in R_{\mathfrak{m}}$ and $H \in \mathfrak{t}$ the $\mathrm{ad}(\mathfrak{l})$-invariance implies that

$$0 = \rho([H, E_\alpha], E_\beta) + \rho(E_\alpha, [H, E_\beta]) = (\alpha + \beta)(H) \cdot \rho(E_\alpha, E_\beta),$$

so that $\rho(E_\alpha, E_\beta) = 0$ unless $\alpha + \beta = 0$.

Now for $\alpha \in R_{\mathfrak{m}}$

$$\rho(E_\alpha, E_{-\alpha}) = -Tr|_{\mathfrak{m}^C} J\,\mathrm{ad}(H_\alpha) = -2i \sum_{\beta \in R^{10}_{\mathfrak{m}}} \langle \alpha, \beta \rangle.$$

□

We introduce the Koszul's form

$$\delta := 2 \sum_{\gamma \in R^{10}_{\mathfrak{m}}} \gamma \in \mathfrak{h}^*$$

so that we have the following corollary

Corollary 3.2. *The Ricci form ρ is given by*

$$\rho = i\ d\delta,$$

where for every $X, Y \in \mathfrak{m}$

$$d\delta(X,Y) = -\delta([X,Y]_{\mathfrak{t}}).$$

The Ricci form ρ does not depend on the metric, but only on the almost complex structure J

Note that the map $d : \mathfrak{t}^* \to \Lambda^2(\mathfrak{m}^*)$ is injective and ρ is $\mathrm{Ad}(L)$-invariant, so that δ belongs to the center $\mathfrak{z}$ of $\mathfrak{l}$.

Definition 3.3. An almost Kähler manifold (M, ω, J) is called Chern-Einstein if its Ricci form ρ satisfies

$$\rho = \lambda\omega$$

for some constant $\lambda \in \mathbb{R}$.

Remark 3.4. We remark that when L is maximal compact in G, i.e. the homogeneous space G/L is Hermitian symmetric, the center of L is one-dimensional and there exists only one invariant symplectic structure up to a multiple. It follows that the corresponding invariant almost complex structure is integrable and the manifold is Kähler. In this case the Chern connection coincides with the Levi Civita connection and the manifold is Kähler-Einstein.

More generally, it is known that any non-compact homogeneous Kähler manifold G/L with non compact simple G is Kähler if a maximal compact subgroup $K \supset L$ of G has 1-dimensional center. Then $G/L \to G/K$ is a G-equivariant fibering over the Hermitian symmetric space G/K. Moreover, G/K is the only Kähler-Einstein homogeneous manifold of the group G, see e.g. [BFR].

Let $\tilde{J}$ be the integrable complex structure, associated with $(M = G/L, \omega, J)$. We set

$$\tilde{\delta} = 2 \sum_{\alpha \in R_{\mathfrak{m}}, \alpha(z_0) > 0} \alpha.$$

Then $\tilde{\rho} = id\tilde{\delta}$ defines an invariant non-degenerate representative $\tilde{\rho}$ of the Chern class $c_1(\tilde{J})$. Hence for any $\lambda \neq 0$, $\tilde{\omega}_\lambda := -(\lambda)^{-1}\tilde{\rho}$ defines an invariant pseudo-Kähler structure $(\tilde{\omega}_\lambda, \tilde{J})$ on $M = G/L$ which satisfies the Einstein equation $\tilde{\rho} = \lambda\tilde{\omega}_\lambda$.

Proposition 3.5. *Let $(M = G/L, \omega, J)$ be a homogeneous almost Kähler manifold of a semisimple Lie group G. Then $(\tilde{\omega}_\lambda, \tilde{J})$ is an invariant pseudo-Kähler-Einstein structure on $M = G/L$.*

4. Chern-Einstein almost Kähler homogeneous spaces

4.1. **General approach for classification.** Let $(M = G/L, \omega, J)$ be a homogeneous almost Kähler manifold with G simple non-compact Lie group. The description of all almost Kähler Chern Einstein homogeneous manifolds with $\rho = \lambda\omega$ ($\lambda \in \mathbb{R}$) reduces to the solutions of the following equation

$$\delta = \lambda(B \circ z_o).$$

Recall that a real simple Lie algebra is either the real form of a complex simple Lie algebra or it is the realification of a complex simple Lie algebra. The following Lemma shows that the last possibility does not occur.

Lemma 4.1. *If $M = G/L$ with G simple admits an invariant almost Kähler structure, then $\mathfrak{g}$ is the real form of a complex simple Lie algebra.*

Proof. If $\mathfrak{g} = \mathfrak{s}_{\mathbb{R}}$, where $\mathfrak{s}$ is a simple complex Lie algebra, then a Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{q} + i\mathfrak{q}$, where $\mathfrak{q}$ is a compact real form of $\mathfrak{s}$. If $z \in \mathfrak{g}$ is an element whose centralizer $\mathfrak{l}$ is a compact subalgebra, then there is an automorphism ϕ of $\mathfrak{g}$ such that $\phi(z) \in \phi(\mathfrak{l}) \subseteq \mathfrak{q}$. But then the centralizer of $\phi(z)$ in $\mathfrak{g}$ is given by $\phi(\mathfrak{l}) + i\phi(\mathfrak{l})$, a contradiction. □

Therefore we will suppose that $\mathfrak{g}$ is a real form of the complex simple Lie algebra $\mathfrak{g}^c$.

Step 1. As a first step we consider the list of all inner symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ of non-compact type with $\mathfrak{g}$ simple. Using the notation as in [He], p. 126, we obtain Table 1.

Step 2. We fix a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{k}$ and we choose an *admissible* element $z \in \mathfrak{h} := i\mathfrak{t}$, i.e. such that

$$C_{\mathfrak{k}}(t_o) = C_{\mathfrak{g}}(t_o) := \mathfrak{l}.$$

Step 3 We define

$$R_c^+(z) := R_c^{10}(z) = \{\alpha \in R_c |\ \alpha(z) > 0\},$$

$$R_{nc}^+(z) := R_{nc}^{01}(z) = \{\alpha \in R_{nc} |\ \alpha(z) > 0\}$$

and we set

$$\delta_c(z) := 2 \sum_{\alpha \in R_c^+(z)} \alpha, \qquad \delta_{nc}(z) := 2 \sum_{\alpha \in R_{nc}^+(z)} \alpha,$$

$$\delta(z) := \delta_c(z) - \delta_{nc}(z).$$

Step 4. We solve the equation

$$\delta(z) = \lambda B z. \tag{4.1}$$

Type	$\mathfrak{g}$	$\mathfrak{k}$	conditions
A	$\mathfrak{su}(p,q)$	$\mathfrak{su}(p)+\mathfrak{su}(q)+\mathbb{R}$	$p\geq q\geq 1$
B	$\mathfrak{so}(2p+1,2q)$	$\mathfrak{so}(2p+1)+\mathfrak{so}(2q)$	$p\geq 0, q\geq 1$
C	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{su}(n)+\mathbb{R}$	$n\geq 1$
C	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p)+\mathfrak{sp}(q)$	$p,q\geq 1$
D	$\mathfrak{so}(2n)^*$	$\mathfrak{su}(n)+\mathbb{R}$	$n\geq 3$
D	$\mathfrak{so}(2p,2q)$	$\mathfrak{so}(2p)+\mathfrak{so}(2q)$	$p,q\geq 1, p+q\geq 3$
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(2)$	
F	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$	
F	$\mathfrak{f}_{4(4)}$	$\mathfrak{su}(2)+\mathfrak{sp}(3)$	
E	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(2)+\mathfrak{su}(6)$	
E	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10)+\mathbb{R}$	
E	$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$	
E	$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(2)+\mathfrak{so}(12)$	
E	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6+\mathbb{R}$	
E	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$	
E	$\mathfrak{e}_{8(-24)}$	$\mathfrak{su}(2)+\mathfrak{e}_7$	

TABLE 1. Inner symmetric pairs $(\mathfrak{g},\mathfrak{k})$ of non-compact type with $\mathfrak{g}$ simple.

4.2. **Examples of Chern Einstein manifolds.** In the special case when $\mathfrak{l}$ has one-dimensional center $\mathfrak{z}=i\mathbb{R}z$, then the equation 4.1 is automatically satisfied for some $\lambda\in\mathbb{R}$ since $\delta(z)$ belongs to the center of $\mathfrak{l}$. In the particular case when $\mathfrak{l}=\mathfrak{k}$, the space is Kähler and irreducible Hermitian symmetric, hence automatically Chern-Einstein. We may then start with a non-Hermitian symmetric pair $(\mathfrak{g},\mathfrak{k})$ out of Table 1 and all simple factors of $\mathfrak{k}$ will be included in $\mathfrak{l}$ except one, say $\mathfrak{k}_1$. In $\mathfrak{k}_1$ we take $\mathfrak{l}_1$ so that the pair $(\mathfrak{k}_1,\mathfrak{l}_1)$ is a flag manifold whose corresponding painted Dynkin diagram has only one black node (see e.g. [AP], [BFR]). We then have to restrict ourselves to the cases when the centralizer in $\mathfrak{g}$ of the center of $\mathfrak{l}_1$ coincides with $\mathfrak{l}$. It is not difficult to see that for classical $\mathfrak{g}$ the only pairs $(\mathfrak{g},\mathfrak{l})$ with $\dim\mathfrak{z}(\mathfrak{l})=1$ are given in Table 2.

Type	$\mathfrak{g}$	$\mathfrak{l}$	conditions
B	$\mathfrak{so}(2p+1,2q)$	$\mathbb{R}+\mathfrak{so}(2p+1)+\mathfrak{su}(q)$	$p\geq 0, q\geq 1$
C	$\mathfrak{sp}(p,q)$	• $\mathbb{R}+\mathfrak{su}(p)+\mathfrak{sp}(q)$ • $\mathbb{R}+\mathfrak{sp}(p)+\mathfrak{su}(q)$	$p,q\geq 1$
D	$\mathfrak{so}(2p,2q)$	• $\mathbb{R}+\mathfrak{su}(p)+\mathfrak{so}(2q)$ • $\mathbb{R}+\mathfrak{so}(2p)+\mathfrak{su}(q)$	$p,q\geq 1, p+q\geq 3$

TABLE 2. Pairs $(\mathfrak{g},\mathfrak{l})$ with $\mathfrak{g}$ simple of classical type, $\mathfrak{l}\subsetneq\mathfrak{k}$ and $\dim\mathfrak{z}(\mathfrak{l})=1$.

Note also that in [DV], Theorem 5, the homogeneous manifolds $M = \mathrm{SO}(2p,q)/\mathrm{U}(p) \times \mathrm{SO}(q)$ are shown to be Chern-Einstein with constant $\lambda = 2(p-q-1)$, which therefore may vanish or assume a positive/negative sign.

4.3. **The case of abelian L.** We now restrict ourselves to the case when the isotropy L is a maximal torus. We recall that for any admissible $z \in \mathfrak{h}$ the set

$$R_c^+(z) \cup R_{nc}^+(z)$$

gives a sytem of positive roots of $\mathfrak{g}^C$ and therefore there exists a unique element w in the Weyl group W that maps the standard system of positive root R_o^+ into $R_c^+(z) \cup R_{nc}^+(z)$.

Then

$$\delta_c(z) = 2 \sum_{\alpha \in w(R_o^+) \cap R_c} \alpha, \quad \delta_{nc}(z) = 2 \sum_{\alpha \in w(R_o^+) \cap R_{nc}} \alpha.$$

Note that the equation (4.1) with $\lambda \neq 0$ admits a solution if and only

$$\forall \alpha \in w(R_o^+) \quad \langle \alpha, \delta(z) \rangle > 0 \text{ (or } < 0) \qquad \text{if } \lambda > 0 \ (\lambda < 0 \text{ resp.}). \tag{4.2}$$

We remark that the computation of $\delta(z)$ depends only on the root system of $\mathfrak{g}^C$ and its decomposition into compact and non-compact roots together with the action of Weyl group.

In the next sections we will go through the classical Lie algebras of type A,B,C,D.

4.4. **Proof of the main Theorem in case $\mathfrak{g}$ of type A_n.** According to Table 1, we analyze the case $\mathfrak{g} = \mathfrak{su}(p,q)$, $p \geq q \geq 1$. The standard Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ gives rise to the root system R of $\mathfrak{g}^c = \mathfrak{sl}(n+1,\mathbb{C})$, where $p+q = n+1$, given by $\{\epsilon_i - \epsilon_j |\ i,j = 1,\ldots,n+1,\ i \neq j\}$. The standard system of positive roots R_o^+ is given by $\{\epsilon_i - \epsilon_j |\ i < j\}$ and the Weyl group W is given by the full group of permutations $\mathcal{S}_{n+1}$.

If we put $P := \{1,\ldots,p\}$ and $Q := \{p+1,\ldots,n+1\}$, we have

$$R_c = \{\epsilon_i - \epsilon_j |\ i,j \in P \text{ or } i,j \in Q, i \neq j\}, \quad R_{nc} = R \setminus R_c.$$

We consider an element σ of the Weyl group, i.e. $\sigma \in \mathcal{S}_{n+1}$, and we define $P_\sigma = \sigma^{-1}(P)$, $Q_\sigma := \sigma^{-1}(Q)$. Then

$$\sigma(R_o^+) \cap R_c = \{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} |\ i,j \in P_\sigma,\ i<j\} \cup \{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} |\ i,j \in Q_\sigma,\ i<j\},$$

$$\sigma(R_o^+) \cap R_{nc} = \{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} |\ i<j,\ i \in P_\sigma, j \in Q_\sigma\} \cup \{\epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} |\ i<j,\ i \in Q_\sigma, j \in P_\sigma\}.$$

For $i \in \{1,\ldots,n+1\}$ we set

$$\begin{aligned} k_Q(i) &:= |\{k \in Q_\sigma |\ k > i\}|, \quad k_P(i) := |\{k \in P_\sigma |\ k > i\}|, \\ \bar{k}_Q(i) &:= |\{k \in Q_\sigma |\ k < i\}|, \quad \bar{k}_P(i) := |\{k \in P_\sigma |\ k < i\}|. \end{aligned}$$

We have

$$\sum_{\alpha \in \sigma(R^+) \cap R_c} \alpha = \sum_{\substack{i<j \\ i,j \in P_\sigma}} \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} \ + \sum_{\substack{i<j \\ i,j \in Q_\sigma}} \epsilon_{\sigma(i)} - \epsilon_{\sigma(j)} =$$

$$= \sum_{i \in P_\sigma} (k_P(i) - \bar{k}_P(i))\ \epsilon_{\sigma(i)} + \sum_{i \in Q_\sigma} (k_Q(i) - \bar{k}_Q(i))\ \epsilon_{\sigma(i)}$$

and similarly we obtain

$$\sum_{\alpha\in\sigma(R^+)\cap R_{nc}} \alpha = \sum_{i\in P_\sigma} (k_Q(i) - \bar{k}_Q(i))\ \epsilon_{\sigma(i)} + \sum_{i\in Q_\sigma} (k_P(i) - \bar{k}_P(i))\ \epsilon_{\sigma(i)}.$$

Therefore for any admissible $z \in \mathfrak{h}$ with corresponding system of positive roots given by $\sigma(R_o^+)$ we obtain

$$\frac{1}{2}\delta(z) = \sum_{i\in P_\sigma} (k_P(i) - \bar{k}_P(i) - k_Q(i) + \bar{k}_Q(i))\ \epsilon_{\sigma(i)} + \\ + \sum_{i\in Q_\sigma} (k_Q(i) - \bar{k}_Q(i) - k_P(i) + \bar{k}_P(i))\ \epsilon_{\sigma(i)}. \tag{4.3}$$

Lemma 4.2. *(i) If $i \in P_\sigma$ we have*

$$k_P(i) - \bar{k}_P(i) - k_Q(i) + \bar{k}_Q(i) = 4k_P(i) + 2i - n - 2p.$$

(ii) If $i \in Q_\sigma$ we have

$$k_Q(i) - \bar{k}_Q(i) - k_P(i) + \bar{k}_P(i) = -4k_P(i) - 2i + n + 2p + 2.$$

Proof. We start noting the following trivial equalities for all $i = 1, \dots, n+1$

$$k_P(i) + k_Q(i) = n + 1 - i, \quad \bar{k}_P(i) + \bar{k}_Q(i) = i - 1. \tag{4.4}$$

If $i \in P_\sigma$ then $k_P(i) + \bar{k}_P(i) = p - 1$ and therefore the first claim follows using (4.4). The second claim follows similarly using that $k_P(i) + \bar{k}_P(i) = p$ for $i \in Q_\sigma$. □

We now solve the equation (4.1) for $\lambda \neq 0$.

Case $\lambda > 0$. Condition (4.2) together with (4.3) and Lemma 4.2 imply the following: if $i, j \in P_\sigma$, $i < j$,

$$2k_P(i) + i > 2k_P(j) + j, \tag{4.5}$$

while if $i, j \in Q_\sigma$, $i < j$, we have

$$2k_P(j) + j > 2k_P(i) + i. \tag{4.6}$$

Now suppose $i, j \in P_\sigma$ with $i < j$ and any $i < a < j$ belongs to Q_σ. Then $k_P(i) = k_P(j) + 1$ and (4.5) implies $2 > j - i$, i.e. $j = i + 1$. This means that P_σ is made of consecutive numbers. If we repeat the same argument with Q_σ we obtain that also Q_σ is made of consecutive numbers. Therefore we are left with the following two possibilities:

(a) $P_\sigma = \{1, \dots, p\}$, $Q_\sigma = \{p+1, \dots, n+1\}$,
(b) $P_\sigma = \{q+1, \dots, n+1\}$, $Q_\sigma = \{1, \dots, q\}$.

We now consider condition (4.2) with $\alpha \in \sigma(R_o^+) \cap R_{nc}$. In case (a) we choose $\alpha = \epsilon_{\sigma(p)} - \epsilon_{\sigma(p+1)}$ and (4.2) gives $-2n > 0$, a contradiction. In case (b) we choose $\alpha = \epsilon_{\sigma(q)} - \epsilon_{\sigma(q+1)}$ and (4.2) gives again $-2n > 0$, a contradiction.

Case $\lambda < 0$. For every $i, j \in P_\sigma$, $i < j$, we have

$$2k_P(i) + i < 2k_P(j) + j.$$

We claim that $p = 1$. Indeed, suppose there exist $i < j$ in P_σ with $(i, j) \cap P_\sigma = \emptyset$. Then $k_P(i) = k_P(j) + 1$ and therefore $j - i > 2$. This means that there exist at least two consecutive numbers, say $l, l+1$ in Q_σ. Then $k_P(l) = k_P(l+1)$ and condition (4.2) with $\alpha = \epsilon_{\sigma(l)} - \epsilon_{\sigma(l+1)}$ gives the contradiction $l > l+1$. This implies that P_σ has only one element, i.e. $p = 1$. Hence $1 = q \leq p = 1$. In this case $R_c = \emptyset$ and $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{l} = \mathbb{R}$. Since $R = R_{nc}$ consists of one root up to sign, the equation $\rho = \lambda\omega$ is satisfied for some $\lambda < 0$.

Case $\lambda = 0$. We recall that $\sum_{i=1}^{n+1} \epsilon_i = 0$ is the only relation among the $\{\epsilon_i\}_{i=1,\dots,n+1}$, so that the equation $\delta(z) = 0$ implies that all the coefficients in the right-hand side of (4.3) are mutually equal.

Lemma 4.3. *Two elements in P_σ or Q_σ are not consecutive.*

Proof. Indeed, suppose $i, i+1 \in P_\sigma$. Then $k_P(i) = k_P(i+1) + 1$ and

$$4k_P(i+1) + 2(i+1) = 4k_P(i) + 2i,$$

yielding a contradiction. A similar argument applies for Q_σ. □

As a corollary, we see that n is even. Indeed, consider $i \in P_\sigma$ and $i+1 \in Q_s$. Then $k_P(i) = k_P(i+1) + 1$ and using Lemma 4.2 we have

$$8k(i) + 4i - 2n - 4p - 4 = 0,$$

that implies n is even. Now P_σ and Q_σ consist precisely of the sets of even or odd integers in $\{1, \dots, n+1\}$. Since n is even, the set of even (odd) numbers less or equal to $n+1$ has cardinality $\frac{n}{2}$ ($\frac{n}{2}+1$ resp.) and since we supposed $p \geq q$, we have $p = q+1$ and

$$P_\sigma = \{1, 3, 5, \dots, n+1\}, \quad Q_\sigma = \{2, 4, 6, \dots, n\}.$$

Viceversa it is immediate to check that with the above choice of P_σ, Q_σ the equation $\delta(z) = 0$ is satisfied. Notice that a suitable permutation σ is given by $\sigma(2k) = p+k, \sigma(2k+1) = k+1$ for $k = 1, \dots, q$.

Example 4.4. We consider $\mathfrak{g} = \mathfrak{su}(3,2)$. A permutation σ that produces a Chern-Ricci flat almost Kähler structure on $\mathrm{SU}(3,2)/\mathrm{T}^4$ is given by the cycle (2453) and the corresponding system of positive roots is given as follows (here $\epsilon_{ij} := \epsilon_i - \epsilon_j$ for the sake of brevity):

$$R_c^+(z) = \{\epsilon_{12}, \epsilon_{13}, \epsilon_{23}, \epsilon_{45}\},\ R_{nc}^+(z) = \{\epsilon_{14}, \epsilon_{15}, \epsilon_{25}, \epsilon_{42}, \epsilon_{43}, \epsilon_{53}\}.$$

4.5. **Proof of the main Theorem in case $\mathfrak{g}$ of type B_n.** According to Table 1, when $\mathfrak{g}^c = \mathfrak{so}(2n+1,\mathbb{C})$, $n \geq 2$, we consider the subalgebras $\mathfrak{g} = \mathfrak{so}(2p+1, 2q)$ with $p+q = n$, $p, q \geq 1$ and $\mathfrak{g} = \mathfrak{so}(1, 2n)$ separately. The standard root system R is given by $R = \{\pm\epsilon_i,\ \pm\epsilon_i \pm \epsilon_j,\ 1 \leq i \neq j \leq n\}$ ($\pm$ independent), with $R_o^+ = \{\epsilon_i,\ \epsilon_i \pm \epsilon_j,\ 1 \leq i < j \leq n\}$.

We start considering the case $\mathfrak{g} = \mathfrak{so}(2p+1, 2q)$ with $p+q = n$, $p, q \geq 1$. The compact roots are given by

$$R_c = \{\pm\epsilon_i,\ i = 1, \dots, p\} \cup \{\pm\epsilon_i \pm \epsilon_j,\ 1 \leq i \neq j \leq p\} \cup \{\pm\epsilon_i \pm \epsilon_j,\ p+1 \leq i \neq j \leq n\}.$$

An element w in the Weyl group $W \cong (Z_2)^n \rtimes \mathcal{S}_n$ acts as $w(\epsilon_i) = \phi_i \epsilon_{\sigma(i)}$, where $\phi_i \in \{1, -1\}$ can be chosen independently. Therefore an easy computation shows that

$$\frac{1}{2}\delta(z) = \sum_{i \in P_\sigma} (4k_P(i) + 2i - 2n + 1)\phi_i \epsilon_{\sigma(i)} + \sum_{i \in Q_\sigma} (4k_Q(i) + 2i - 2n - 1)\phi_i \epsilon_{\sigma(i)}, \tag{4.7}$$

where $P_\sigma := \sigma^{-1}\{1, \dots, p\}$, $Q_\sigma := \sigma^{-1}\{p+1, \dots, n\}$ and $k_{P/Q}(i) := |\{j \in P_\sigma/Q_\sigma|\ j > i\}|$.

Case $\lambda > 0$. If $i \in Q_\sigma$, then $\alpha := \phi_i\epsilon_{\sigma(i)} \in w(R_o^+)$ and $\langle\delta(z),\alpha\rangle > 0$ implies $4k_Q(i)+2i > 2n+1$. If i_Q is the maximum element in Q_σ, then $k_Q(i_Q) = 0$ and therefore $2i_Q > 2n+1$, a contradiction.

Case $\lambda < 0$. The maps $c_P : P_\sigma \ni i \mapsto 4k_P(i) + 2i - 2n + 1$ and $c_Q : Q_\sigma \ni j \mapsto 4k_Q(i) + 2i - 2n - 1$ are negative and strictly increasing. It follows that if $i < j$ are two numbers both in P_σ or both in Q_σ then $j - i > 2$, contradicting $P_\sigma \cup Q_\sigma = \{1, \ldots, n\}$. This implies that $p = q = 1$. Since $2 = n \notin P_\sigma$ (otherwise $c_P(2) = 1$), we see that $P_\sigma = \{1\}$ and $Q_\sigma = \{2\}$. In this case $c_P(1) = -1 = c_Q(2)$ and therefore we contradict (4.2) using $\alpha = \phi_1\epsilon_1 - \phi_2\epsilon_2$.

Case $\lambda = 0$. This cannot occur as all the coefficients of $\frac{1}{2}\delta(z)$ are odd numbers.

We now deal with the case $\mathfrak{g} = \mathfrak{so}(1,n)$, where the compact roots are given by $R_c = \{\pm\epsilon_i \pm \epsilon_j,\ 1 \le i \ne j \le n\}$. It is immediate to compute

$$\frac{1}{2}\delta(z) = \sum_{i=1}^{n}(2n - 2i - 1)\phi_i\epsilon_{\sigma(i)}.$$

Since the coefficients $2n - 2i - 1$ do not have a constant sign for $i = 1, \ldots, n$ and cannot vanish, we see that equation (4.1) has no solution.

4.6. **Proof of the main Theorem in case $\mathfrak{g}$ of type C_n.** When $\mathfrak{g}$ is a real form of $\mathfrak{sp}(n,\mathbb{C})$, $n \ge 3$, we need to consider two subcases according to Table 1, namely when $\mathfrak{k} = \mathfrak{u}(n)$ or $\mathfrak{k} = \mathfrak{sp}(p) + \mathfrak{sp}(q)$, $n = p + q$.

The standard root system of $\mathfrak{sp}(n,\mathbb{C})$ is given by $R = \{\pm\epsilon_i \pm \epsilon_j,\ 1 \le i < j \le n\} \cup \{\pm 2\epsilon_i,\ i = 1, \ldots, n\}$ with $R_o^+ = \{\epsilon_i \pm \epsilon_j, 2\epsilon_i,\ i < j,\ i,j = 1, \ldots, n\}$. The Weyl group of $\mathfrak{g}^c$ is a semidirect product $(\mathbb{Z}_2)^n \rtimes \mathcal{S}_n$ and any $w \in W$ acts as $w(\epsilon_i) = \phi_i\epsilon_{\sigma(i)}$, where $\phi_i \in \{1, -1\}$ can be chosen independently.

4.6.1. *Case* $\mathfrak{k} = \mathfrak{u}(n)$. We have

$$R_c = \{\epsilon_i - \epsilon_j,\ i \ne j = 1, \ldots, n\}.$$

Given $w \in W$ we have

$$w(R_o^+) \cap R_c = \{\phi_i\epsilon_{\sigma(i)} + \phi_j\epsilon_{\sigma(j)}|\ i < j,\ \phi_i\phi_j < 0\} \cup \{\phi_i\epsilon_{\sigma(i)} - \phi_j\epsilon_{\sigma(j)}|\ i < j,\ \phi_i\phi_j > 0\}$$

and therefore, if we denote $A := \{i|\ \phi_i = 1\}$, $B := \{i|\ \phi_i = -1\}$,

$$\sum_{\alpha\in w(R_o^+)\cap R_c}\alpha = \sum_{\substack{i<j\\ \phi_i\phi_j<0}}\phi_i\epsilon_{\sigma(i)} + \sum_{\substack{i<j\\ \phi_i\phi_j<0}}\phi_j\epsilon_{\sigma(j)} + \sum_{\substack{i<j\\ \phi_i\phi_j>0}}\phi_i\epsilon_{\sigma(i)} - \sum_{\substack{i<j\\ \phi_i\phi_j>0}}\phi_j\epsilon_{\sigma(j)} =$$

$$= \sum_{i\in A}\sum_{B\ni j>i}\epsilon_{\sigma(i)} - \sum_{i\in B}\sum_{A\ni j>i}\epsilon_{\sigma(i)} + \sum_{j\in A}\sum_{B\ni i<j}\epsilon_{\sigma(j)} - \sum_{j\in B}\sum_{A\ni i<j}\epsilon_{\sigma(j)} +$$
$$+ \sum_{i\in A}\sum_{A\ni j>i}\epsilon_{\sigma(i)} - \sum_{i\in B}\sum_{B\ni j>i}\epsilon_{\sigma(i)} - \sum_{j\in A}\sum_{A\ni i<j}\epsilon_{\sigma(j)} + \sum_{j\in B}\sum_{B\ni i<j}\epsilon_{\sigma(j)}.$$

We set

$$k_B(i) := |\{k \in B|k > i\}|,\ k_A(i) := |\{k \in A|k > i\}|,$$
$$\bar{k}_B(i) := |\{k \in B|k < i\}|,\ \bar{k}_A(i) := |\{k \in A|k < i\}|.$$

Hence

$$\sum_{\alpha\in w(R_o^+)\cap R_c}\alpha = \sum_{i\in A} k_B(i)\ \epsilon_{\sigma(i)} - \sum_{i\in B} k_A(i)\ \epsilon_{\sigma(i)} + \sum_{j\in A}\bar k_B(j)\epsilon_{\sigma(j)} - \sum_{j\in B}\bar k_A(j)\epsilon_{\sigma(j)}+$$

$$+\sum_{i\in A} k_A(i)\ \epsilon_{\sigma(i)} - \sum_{i\in B} k_B(i)\ \epsilon_{\sigma(i)} - \sum_{j\in A}\bar k_A(j)\ \epsilon_{\sigma(j)} + \sum_{j\in B}\bar k_B(j)\epsilon_{\sigma(j)} =$$

$$= \sum_{i\in A}(k_B(i)+\bar k_B(i)+k_A(i)-\bar k_A(i))\ \epsilon_{\sigma(i)} + \sum_{i\in B}(\bar k_B(i)-k_A(i)-\bar k_A(i)-k_B(i))\ \epsilon_{\sigma(i)} =$$

$$= \sum_{i\in A}(-2\bar k_A(i)+n-1)\ \epsilon_{\sigma(i)} + \sum_{i\in B}(2\bar k_B(i)-n+1)\ \epsilon_{\sigma(i)}.$$

Using the fact that $\sum_{\alpha\in R_o^+} w\alpha = 2\sum_{i=1}^n (n-i+1)\ \phi_i\epsilon_{\sigma(i)}$, we obtain

$$\frac{1}{2}\delta(z) = \sum_{i\in A}[-4\bar k_A(i)+2i-4]\ \epsilon_{\sigma(i)} + \sum_{i\in B}[4\bar k_B(i)-2i+4]\ \epsilon_{\sigma(i)}.$$

Denote by $c_A := -4\bar k_A(i)+2i-4$ and $c_B := 4\bar k_B(i)-2i+4$ the two coefficients. We now discuss the equation (4.1).

First suppose $\lambda > 0$: using the root $\phi_i\epsilon_{\sigma(i)} \in w(R_o^+)$ and (4.2), we see that $c_A(i) > 0$ if $i \in A$ and $c_B(i) < 0$ if $i \in B$. Now, if $1 \in A$, then $\bar k_A(1) = 0$ and $c_A(1) < 0$, while if $1 \in B$ then $c_B(1) > 0$, showing that $1 \notin A \cup B$, a contradiction.

If $\lambda < 0$, then similarly as above we have $c_A(i) < 0$ if $i \in A$ and $c_B(i) > 0$ if $i \in B$. Suppose A is not empty and let i_A be the minimum element in A: then $i_A < 2$, i.e. $i_A = 1$. Similarly, if B is not empty, its minimum point is 1, showing $1 \in A \cap B$, a contradiction. Then either A or B is empty. If $B = \emptyset$, then $\bar k_A(i) = i-1$ and $c_A(i) = -2i$. Using the roots $\alpha = \epsilon_{\sigma(i)} - \epsilon_{\sigma(i+1)}$ and (4.2), we see that c_A is increasing, a contradiction. Similarly $A = \emptyset$ leads to a contradiction.

Finally, $\lambda = 0$ is also impossible, as $c_A(i) = 0,\ i \in A$ and $c_B(i) = 0,\ i \in B$ force $i \in A\cup B$ to be even, a contradiction.

4.6.2. *Case* $\mathfrak{k} = \mathfrak{sp}(p) + \mathfrak{sp}(q),\ p \geq q \geq 1$. If we denote $P := \{1,\ldots,p\}$ and $Q := \{p+1,\ldots,n\}$ we have

$$R_c = \{\pm\epsilon_i \pm \epsilon_j,\ \pm 2\epsilon_i \mid i,j \in P\} \cup \{\pm\epsilon_i \pm \epsilon_j,\ \pm 2\epsilon_i \mid i,j \in Q\}$$

and therefore, if $P_\sigma := \sigma^{-1}(P), Q_\sigma := \sigma^{-1}(Q)$,

$$w(R_o^+)\cap R_c = \{\phi_i\epsilon_{\sigma(i)} \pm \phi_j\epsilon_{\sigma(j)},\ 2\phi_i\epsilon_{\sigma(i)} |\ i,j \in P_\sigma, i<j\} \cup$$

$$\cup\ \{\phi_i\epsilon_{\sigma(i)} \pm \phi_j\epsilon_{\sigma(j)},\ 2\phi_i\epsilon_{\sigma(i)} |\ i,j \in Q_\sigma, i<j\}.$$

Therefore

$$\sum_{\alpha\in w(R_o^+)\cap R_c}\alpha = 2\sum_{i\in P_\sigma}(k_P(i)+1)\phi_i\epsilon_{\sigma(i)} + 2\sum_{i\in Q_\sigma}(k_Q(i)+1)\phi_i\epsilon_{\sigma(i)},$$

where $k_{P/Q}(i) := |\{k \in P_\sigma/Q_\sigma |\ k > i\}|$. We then obtain

$$\frac{1}{2}\delta(z) = \sum_{i\in P_\sigma}(4k_P(i)+2i-2n+2)\phi_i\epsilon_{\sigma(i)} + \sum_{i\in Q_\sigma}(4k_Q(i)+2i-2n+2)\phi_i\epsilon_{\sigma(i)}.$$

Denote by $c_P := 4k_P(i) + 2i - 2n + 2$ and $c_Q := 4k_Q(i) + 2i - 2n + 2$ the two coefficients for $i \in P_\sigma, Q_\sigma$ respectively. We now discuss the equation (4.1).

First suppose $\lambda > 0$: we have $c_P, c_Q > 0$ and if we denote by i_P the maximum element in P_σ we have $i_P > n-1$, i.e. $i_P = n$. Similarly $n \in Q_\sigma$, contradicting the fact that P_σ and Q_σ are disjoint.

Suppose now $\lambda < 0$. Then c_P, c_Q are negative and strictly increasing. If $i < j$ are two numbers both in P_σ or both in Q_σ then $j - i > 2$, contradicting $P_\sigma \cup Q_\sigma = \{1, \ldots, n\}$. This implies that $p = q = 1$. Then $c_P(i) = 2i - 2 < 0$ for $i \in P_\sigma$ and $c_Q(i) = 2i - 2 < 0$ for $i \in Q_\sigma$, a contradiction.

If now $\lambda = 0$, we see that $c_P = 0$ and $c_Q = 0$ imply that $i \equiv n - 1 \pmod 2$ for every $i = 1, \ldots, n$, a contradiction.

4.7. **Proof of the main Theorem in case $\mathfrak{g}$ of type D_n, $n \geq 3$.** When $\mathfrak{g}$ is a real form of $\mathfrak{so}(2n, \mathbb{C})$, $n \geq 3$, we need to consider two subcases according to Table 1, namely when $\mathfrak{k} = \mathfrak{u}(n)$ or $\mathfrak{k} = \mathfrak{so}(2p) + \mathfrak{so}(2q)$, $n = p + q$.

The standard root system of $\mathfrak{so}(2n, \mathbb{C})$ is given by $R = \{\pm\epsilon_i \pm \epsilon_j,\ 1 \leq i < j \leq n\}$ with $R_o^+ = \{\epsilon_i \pm \epsilon_j,\ i < j,\ i, j = 1, \ldots, n\}$. The Weyl group of $\mathfrak{g}^c$ is a semidirect product $(\mathbb{Z}_2)^n \rtimes \mathcal{S}_n$ and any $w \in W$ acts as $w(\epsilon_i) = \phi_i \epsilon_{\sigma(i)}$, where $\phi_i \in \{1, -1\}$ satisfies $\prod_{i=1}^n \phi_i = 1$.

4.7.1. *Case* $\mathfrak{k} = \mathfrak{u}(n)$. This case can be dealt with similar arguments as in the subsection 4.6.1 for $\mathfrak{g}^c$ of type C_n and $\mathfrak{k} = \mathfrak{u}(n)$ and we will omit the detailed computations, keeping the same notation. We have

$$\frac{1}{2}\delta(z) = \sum_{i \in A} [-4\bar{k}_A(i) + 2i - 2]\ \epsilon_{\sigma(i)} + \sum_{i \in B} [4\bar{k}_B(i) - 2i + 2]\ \epsilon_{\sigma(i)}.$$

Denote by $c_A := -4\bar{k}_A(i) + 2i - 2$ and $c_B := 4\bar{k}_B(i) - 2i + 2$ the two coefficients.

If $\lambda > 0$, then $c_A(i) > 0$ if $i \in A$ and $c_B(i) < 0$ if $i \in B$. It follows that $1 \notin A \cup B$, a contradiction.

If $\lambda < 0$, then $c_A(i) < 0$ if $i \in A$. If i_A is the minimum point of A, then $\bar{k}_A(i_p) = 0$ and therefore $i_p < 1$, a contradiction. Then A is empty. Similarly the minimum point i_B of B satisfies $i_B < 1$, showing that B is empty, a contradiction.

Finally, $\lambda = 0$ is also impossible, as $c_A(i) = 0,\ i \in A$ and $c_B(i) = 0,\ i \in B$ force every $i \in A \cup B$ to be odd, a contradiction.

4.7.2. *Case* $\mathfrak{k} = \mathfrak{so}(2p) + \mathfrak{so}(2q)$, $p \geq q \geq 1$, $p + q \geq 3$. If we denote $P := \{1, \ldots, p\}$ and $Q := \{p+1, \ldots, n\}$ we have

$$R_c = \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in P\} \cup \{\pm\epsilon_i \pm \epsilon_j,\ \mid i, j \in Q\}$$

and therefore, if $P_\sigma := \sigma^{-1}(P), Q_\sigma := \sigma^{-1}(Q)$ and $k_{P/Q}$ have the same meaning as in the previous subsections, we obtain

$$\frac{1}{2}\delta(z) = \sum_{i \in P_\sigma} (4k_P(i) + 2i - 2n)\phi_i \epsilon_{\sigma(i)} + \sum_{i \in Q_\sigma} (4k_Q(i) + 2i - 2n)\phi_i \epsilon_{\sigma(i)}.$$

If $\lambda > 0$ and i_P is the maximum element in P_σ then $4k_P(i_P) + 2i_P - 2n > 0$ implies $i_P > n$, a contradiction.

If $\lambda < 0$, the maps $P_\sigma/Q_\sigma \ni i \mapsto 4k_{P/Q}(i) + 2i - 2n$ are negative and strictly increasing. This implies that two elements $i < j$ both in P_σ or both in Q_σ satisfy $j - i > 2$, forcing $p = q = 1$.

If $\lambda = 0$ then $i \equiv n \pmod 2$ for every $i = 1, \ldots, n$, a contradiction.

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