CR-GEOMETRY AND SHEARFREE LORENTZIAN GEOMETRY

DMITRI V. ALEKSEEVSKY, MASOUD GANJI, AND GERD SCHMALZ

1. Introduction

Abstract. We study higher dimensional versions of shearfree null-congurences in conformal Lorentz manifolds. We show that such structures induce a subconformal structure and a partially integrable almost CR-structure on the leaf space and we classify the Lorentz metrics that induce the same subconformal structure. In the last section we survey some known applications of the correspondence between almost CR-structures and shearfree null-congurences in dimension 4.

It is well known that CR-manifolds are intimately related with conformal Lorentzian manifolds by the Fefferman metric [1, 4, 7]. However there exist also other natural constructions of Lorentzian metrics on sphere or line bundles over CR-manifolds associated with the underlying CR-structure. Such correspondences have been used in both ways: to describe special algebraic solutions to Einstein's equation in 4-dimensional Lorentz space using CR-structures and also to interpret CR-phenomena in terms of general relativity.

More precisely, let M be a 3-dimensional CR-manifold with contact distribution H and CR-structure $J\colon H \to H$. Following Cartan, the CR-stucture of M can be (locally) encoded as a choice of a real 1-form λ and a complex 1-form μ such that

(i) $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$
(ii) $H = \ker \lambda$
(iii) $\mu|_H \circ J = \mathrm{i}\, \mu|_H$ for all $X \in H$.

Then any other pair (λ', μ') of 1-forms defines the same CR-structure if it is related to (λ, μ) by

$$\lambda' = f\lambda, \qquad \mu' = \phi\mu + \psi\lambda$$

where f is a non-vanishing real function, ϕ is a non-vanishing complex function and ψ is an arbitrary complex function.

We assume that the CR-structure $(M, H, J) = (M, [\lambda, \mu])$ is Levi-nondegenerate, i.e. $d\lambda \wedge \lambda \neq 0$. In this case we can choose the pair of forms such that

$$\begin{aligned} d\lambda &= \mathrm{i}\, \mu \wedge \bar{\mu} + c\mu \wedge \lambda + \bar{c}\bar{\mu} \wedge \lambda \\ d\mu &= a\mu \wedge \lambda + b\bar{\mu} \wedge \lambda. \end{aligned} \tag{1}$$

Recall that the Fefferman metric is a conformal class of Lorentzian metrics defined on the circle bundle $\mathfrak{M} = H^{1,0}/\mathbb{R}^+$ where $H^{1,0}$ is the i-eigenbundle of J on $H \otimes C$. Using a

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coframe (μ, λ) that satisfies (1) and the trivialisation $\mathfrak{M} \ni m|_p = \mathrm{e}^{-t-\mathrm{i}\, r}\, \partial|_p \mapsto (p, r)$ where $r \in [0, 2\pi)$, the Fefferman metric is defined by

$$g = \mathrm{e}^{2t}\left[\mu\bar{\mu} + \frac{1}{3}\lambda\left(2dr - \mathrm{i}\, c\mu + \mathrm{i}\, \bar{c}\bar{\mu} - \left(\frac{c_{\bar{\mu}} + \bar{c}_{\mu}}{4} - \frac{3\,\mathrm{i}(a - \bar{a})}{4}\right)\lambda\right)\right] \tag{2}$$

where, by abuse of notation, $\mu, \bar{\mu}, \lambda$ denote the pull-backs of the corresponding 1-forms on M, dr is the differential of the coordinate function r, a, c are the pull-backs of the structure functions from (1), e^{2t} is a conformal scaling factor and

$$dc = c_\mu \mu + c_{\bar{\mu}}\bar{\mu} + c_\lambda \lambda, \qquad d\bar{c} = \bar{c}_\mu \mu + \bar{c}_{\bar{\mu}}\bar{\mu} + \bar{c}_\lambda \lambda.$$

It can be shown, e.g. by using the canonical Cartan connection, that the conformal Fefferman metric (2) does not depend on the choice of the pair (λ, μ). It is obvious that the fundamental vector field ∂_r is a conformal Killing null vector field of the Fefferman metric.

The Fefferman construction is universal in the sense that it provides a unique conformal Lorentz space for any Levi-nondegenerate CR-manifold, but, in general, the resulting conformal metrics cannot be rescaled to Einstein metrics.

An alternative, more flexible approach has been introduced by physicists. They consider a line bundle $\mathfrak{M}$ over a CR-manifold M together with a family of Lorentz metrics, which takes in a trivialising chart $M \times \mathbb{R}$ the form

$$g = \mathrm{e}^{2t}\left[\mu\bar{\mu} + \lambda\left(dr + W\mu + \bar{W}\bar{\mu} + H\lambda\right)\right] \tag{3}$$

where r is the fibre variable and t, W, H are arbitrary functions.

The fundamental vector field $p = \partial_r$ is again null with respect to all metrics from the family g but it is not, in general, conformal Killing. Instead it satisfies the following condition of shearfreeness:

$$\mathcal{L}_p g = \rho g + g(p, \cdot) \vee \psi$$

where ρ is some function and ψ is some 1-form. This condition controls only the change of g on $\ker \lambda$ and hence is somewhat weaker than p being conformal Killing.

In this paper we describe a generalisation of the correspondence between higher dimensional CR-geometry, subconformal geometry and shearfree Lorentz geometry.

2. Subconformal and CR-manifolds

Throughout this paper we use the notion of a CR-manifold as a shorthand for a Levi-nondegenerate, partially integrable almost CR-manifold of hypersurface type as defined below:

Definition 1. *A CR-manifold M is a contact manifold with contact distribution H and a smooth family of endomorphisms $J_x \colon H_x \to H_x$ with $J_x^2 = -\,\mathrm{id}$. We assume that (M, H, J) is partially integrable, i.e. the complex eigen-distribution $H^{1,0} \subset H \otimes \mathbb{C}$ of J with eigenvalue* i *satisfies*

$$[H^{1,0}, H^{1,0}] \subseteq H \otimes \mathbb{C}.$$

Remark. Partial integrability is equivalent to the following property: If λ is a contact form for the contact distribution H then

$$d\lambda(JX, JY) = d\lambda(X, Y)$$

for any sections X, Y of H. This property is also equivalent to $d\lambda(\cdot, J\cdot)$ being symmetric.

Definition 2. *A subconformal manifold is a contact manifold M with contact distribution H that is endowed with a conformal class of subRiemannian metrics $[g_H]$.*

In this article we will only consider orientable subconformal manifolds. In this case there exists a global contact 1-form λ such that $H = \ker\lambda$.

For $\dim M = 3$ subconformal manifolds are essentially the same as CR-manifolds. More precisely, the conformal metric on the contact distribution induces two mutually conjugate complex structures that rotate vectors by an angle $\frac{\pi}{2}$. Vice versa, the conformal structure can be recovered from either of these complex structures by making multiplication by complex numbers conformal mappings on the distribution.

In higher dimensions the relation between subconformal and CR-manifolds is less obvious.

Theorem 1. *Let $(M, H, [g_H])$ be an orientable subconformal manifold. Then M inherits two mutually conjugated partially integrable almost CR structures J and $-J$.*

Proof. Choose a contact form λ. Let $A = g^{-1}d\lambda|_H$, i.e. $d\lambda|_H = g(A\cdot, \cdot)|_H$. Then A is non-degenerate and skew-symmetric, hence A^2 is symmetric and negative definite. Define $J = \sqrt{-A^{-2}}A$. It follows that J depends smoothly on the coordinates of M. A different choice of the contact form λ affects only the sign of J. We show that J, and hence $-J$, define partially integrable almost CR-structures.

Since A and A^2 commute, the eigenspaces of A^2 at each point are invariant for A and, since $\sqrt{-A^{-2}}$ is diagonalisable with the same eigenspaces as A^2, A commutes with $\sqrt{-A^{-2}}$.

Therefore,

$$J^2 = \sqrt{-A^{-2}}A\sqrt{-A^{-2}}A = -A^{-2}A^2 = -\operatorname{id}.$$

To prove partial integrability let X, Y be two sections of H. Since $\sqrt{-A^{-2}}$ is symmetric, then

$$\begin{aligned} d\lambda(JX, JY) &= g(A\sqrt{-A^{-2}}AX, \sqrt{-A^{-2}}AY) = g(-A^{-2}A^2X, AY) \\ &= -g(X, AY) = -g(AY, X) = -d\lambda(Y, X) = d\lambda(X, Y). \end{aligned}$$

This proves partial integrability. □

The Theorem above indicates that CR-structures in higher dimensions are weaker structures than subconformal ones. There are many different conformal structures that induce the same almost CR-structure. E.g. different subconformal structures can be obtained from a strictly pseudoconvex CR-structure (M, H, J) by additionally prescribing different $d\lambda$-orthogonal decompositions of the distribution H

$$H = \oplus H_j$$

and positive functions α_j. Then let $A|_{H_j} = \alpha_j J|_{H_j}$ and $g = d\lambda \circ A^{-1}$.

The extremal choices of the decomposition of H are on the one hand the trivial decomposition $H = H$ and on the other hand the decomposition into complex one-dimensional H_j. The former choice is equivalent to the CR-structure while the latter one induces a much more rigid geometric structure.

3. Shearfree congruences

Definition 3. *A shearfree congruence is a (2n+2)-dimensional Lorentz-manifold $(\mathfrak{M}, g)$ equipped with a foliation into integral curves of a nowhere vanishing vector field p such that*

(i) *p is null, i.e. $g(p,p) = 0$*

(ii) *$\mathcal{L}_p g = \rho g + \theta \vee \psi$, where $\theta = g(p, \cdot)$, ρ is a function and ψ is a 1-form. This condition means that the local flow of p preserves the distribution*

$$p^{\perp} = \{X \in T\mathfrak{M} \colon g(X,p) = 0\}$$

and the degenerate subconformal metric the metric $[g|_{p^{\perp}}]$ on $p^{\perp}$.

We call p a shearfree vector field (with respect to $(\mathfrak{M}, g)$) if it satisfies conditions (i) and (ii) above.

It can be shown that the conditions (i) and (ii) in the definition above imply that the vector field p is geodetic, i.e.

$$\nabla_p p = \beta p,$$

where ∇ is the Levi-Civita connection of g and β is some function. (See Proposition 1 below.) Hence a shearfree congruence is in fact a foliation of $\mathfrak{M}$ into null-geodesics, which can be interpreted as light rays.

Notice that shearfreeness of p depends only on the conformal class of g and is preserved under scaling of p.

We define also a global conformal version of shearfree congruences.

Definition 4. *Let $(\mathfrak{M}, [g])$ be a $2n+2$-dimensional conformal Lorentz manifold with a shearfree vector field p and assume that the flow of p generates a free action of $G = \mathbb{R}$ or $G = S^1$ so that the orbit space by $M = \mathfrak{M}/G$ is a manifold and the canonical projection $\pi \colon \mathfrak{M} \to M$ is a principal G-bundle. We call the $(\mathfrak{M}, [g], p, M)$ a Robinson-Trautman space (RT-space) of type G.*

Examples. 1. For a Lorentzian metric g any conformal Killing null vector field p is shearfree.
2. For the Lorentzian metrics (3) $p = \partial_r$ is a shearfree vector field on the trivial $\mathbb{R}$-bundle $M \times \mathbb{R}$.

Definition 5. *A shearfree congruence is called diverging if the function ρ in (ii) does not vanish; it is called distinguished in the opposite case, i.e. if $\rho = 0$.*
A shearfree vector field p is said to be autoparallel if

$$\nabla_p p = 0.$$

By rescaling the Lorentzian metric g, a shearfree congruence can be made distinguished, locally, and by rescaling the shearfree vector field p it can be made autoparallel at the same time. If $\mathfrak{M}$ is an RT space of type $\mathbb{R}$ then this can be achieved globally.

We summarise some properties of shearfree congruences.

Proposition 1. (i) *A shearfree vector field p is geodetic, i.e. $\nabla_p p = \beta p$.*

(ii) *Locally, a Lorentzian metric g and a shearfree vector field p can be rescaled, so that g becomes distinguished and p becomes autoparallel at the same time. On an RT-space of type $\mathbb{R}$ this can be achieved globally.*

(iii) *Being autoparallel is equivalent to $p \lrcorner d\theta = 0$, which in turn is equivalent to*

$$\mathcal{L}_p \theta = d(p \lrcorner \theta) + p \lrcorner d\theta = 0.$$

Sketch of the proof. For the proof of (i) and (iii) the notion of the Nomizu operator is convenient. For any vector field X the Nomizu operator L_X is defined as

$$L_X \colon Y \mapsto -\nabla_Y X,$$

where ∇ is the covariant derivative of the Levi-Civita connection for g. It is well known that

$$g^{-1}\mathcal{L}_X g = -L_X - L_X^* \tag{4}$$

$$2g^{-1} d\theta = -L_X + L_X^*, \tag{5}$$

where $\theta = g(X, \cdot)$ and L_X^* is the g-adjoint of L_X.

The Nomizu operator satisfies

$$L_p^* p = 0 \tag{6}$$

for any null vector field p, because of

$$g(L_p^* p, X) = g(p, L_p X) = g(p, -\nabla_X p) = -\frac{1}{2} X g(p,p) = 0.$$

Now, statement (i) follows from

$$\begin{aligned} g(\nabla_p p, X) = -g(L_p p, X) = -g(L_p p + L_p^* p, X) = \mathcal{L}_p g(p, X) = \rho g(p, X) + \psi(p)\theta(X) \\ = g(\rho p, X) + \psi(p) g(p, X) = g((\rho + \psi(p))p, X), \end{aligned}$$

for all X, hence $\nabla_p p = \beta p$ with $\beta = \rho + \psi(p)$.

The condition that the shearfree congruence is distinguished can be achieved by scaling g by a factor t that is a solution of $\partial_p \log t = -\rho$. The condition $\nabla_p p = 0$ can be achieved by scaling p by a factor s that is a solution of $\partial_p \log s = -\beta$. These PDE can be solved locally, or globally in the case of an RT-space of type $\mathbb{R}$.

The equivalence of $\nabla_p p = 0$ and $p \lrcorner d\theta = 0$ follows from (6) because $\nabla_p p = -L_p p = 0$ can be written as $0 = -g(L_p p, X) = -g(L_p p - L_p^* p, X) = d\theta(p, X)$, that is $p \lrcorner d\theta = 0$, which is equivalent to

$$\mathcal{L}_p \theta = p \lrcorner d\theta + d(p \lrcorner \theta) = p \lrcorner d\theta = 0. \qquad \square$$

4. Shearfree congruences and their orbit spaces

In this section let $(\mathfrak{M}, [g], p)$ be an RT-space.

Definition 6. *We say that* $(\mathfrak{M}, [g], p)$ *is* twisting *if* $(d\theta)^n \wedge \theta \neq 0$, *where* $\theta = g(p, \cdot)$.

Notice that the notion of being twisting is invariant under scaling of p and g and therefore it is well-defined. Indeed, both scalings result in a scaling of θ, hence

$$d(\alpha\theta) \equiv \alpha \, d\theta (\operatorname{mod} \theta),$$

and

$$(d\alpha\theta)^n \wedge \alpha\theta = \alpha^{n+1}(d\theta)^n \wedge \theta.$$

Since the notion of shearfreeness of p is invariant with respect to rescalings of p we can replace p in the definition of a twisting RT-space by its equivalence class $[p]$.

We will show that the orbit space M of a twisting RT-space carries a canonical subconformal structure and hence a CR-stucture.

Definition 7. *An RT-structure* $(\mathfrak{M}, [g], [p])$ *and a subconformal structure* $(H, [g_H])$ *with contact distribution* H *and subconformal metric* $[g_H]$ *on the orbit space* M *are called compatible if for any contact form* λ *on* M *with Reeb vector field* Z

(i) $\ker \pi^*\lambda = p^\perp = \{X \in T\mathfrak{M} \colon g(X, p) = 0\}$ *and*

(ii) $\pi^* g_H^\lambda|_{p^\perp}$ *is conformally equivalent to* $g|_{p\perp}$. *Here* g_H^λ *is the extension of* g_H *to the degenerate metric on* M *with* $Z = \ker g_H^\lambda$. *That is*

$$g = P^2(\pi^* g_H^\lambda + g(p, \cdot) \vee \psi)$$

for some positive function P^2 *and some 1-form* ψ.

Theorem 2. *Let* $(\mathfrak{M}, [g], [p])$ *be a twisting RT-space. Then there exists a unique compatible subconformal structure on the orbit space* M.

Proof. Let $U \in T_Q M$. Then we call $u \in T_q\mathfrak{M}$ a lift of U if $\pi(q) = Q$ and $\pi_* u = U$. A compatible contact distribution $H_Q \subset T_Q M$ must satisfy the condition $\theta(u) = g(p, u) = 0$ for any lift u of any $U \in H_Q$. This proves the uniqueness of the contact structure. We show that this condition does not depend on the choice of the lift. Let u_0 and u_1 be two lifts at q_0 and q_1, respectively, connected by a path $u(t)$, where t is the time parameter of the flow of the vector field p. Then, with respect to some local trivialisation,

$$u(t) = U + \alpha(t)p$$

and

$$\frac{d}{dt}\theta(u(t)) = \mathcal{L}_p g(u(t), p) = \rho g(u(t), p) + \theta(u(t))\psi(p) = (\rho + \psi(p))\theta(u(t)).$$

It follows that $\theta(u(t)) = Ce^{\int \rho + \psi(p) dt}$ and therefore either equals zero for all t or nowhere.

We show that H is a contact distribution. Let λ be a form that annihilates H. Then $\pi^*\lambda = \alpha\theta$, where α is a non-vanishing function. Since $\pi^* d\lambda^n \wedge \lambda = \alpha^{n+1} d\theta^n \wedge \theta \neq 0$ it follows $d\lambda^n \wedge \lambda \neq 0$. The conformal metric g_H on H_Q is uniquely determined by

$$g_H(U, V) = g(u, v)$$

for $U, V \in H_Q$ and any lifts $u, v \in p^\perp$ at the same base point q. We show that this definition does not depend on the choice of the lifts. Let

$$u(t) = U + \alpha(t)p, \qquad v(t) = V + \beta(t)p$$

be two paths connecting two pairs of lifts (u_0, v_0) and u_1, v_1 with respect to some trivialisation. Then,

$$\frac{d}{dt} g(u(t), v(t)) = \mathcal{L}_p g(u(t), v(t)) = \rho g(u(t), v(t)),$$

where ρ depends on t but not on $u(t)$ and $v(t)$ It follows that $g(u(t), v(t))$ scales along the path by a multiplier that does not depend on the path. Hence $g_H(U, V)$ is well-defined as a conformal metric. □

The theorem below describes the RT-structures that are compatible with a given subconformal structure on their orbit space.

Theorem 3. *Let $\pi\colon \mathfrak{M} \to M$ be a line bundle over a subconformal manifold $(M, H, [g_H])$ and p any non-vanishing vertical vector field. Then $(\mathfrak{M}, [g], [p])$ is a twisting RT-structure compatible with $(M, H, [g_H])$ if and only if*

$$g = P^2(\pi^* g_H^\lambda + \pi^*\lambda \vee \psi) \tag{7}$$

where λ is a contact form on M, P is a positive function on $\mathfrak{M}$ and ψ is a 1-form on $\mathfrak{M}$.

Proof. Assume g has the form (7). Then

(i) g is Lorentzian, and $g|_{p^\perp}$ is conformally equivalent to $\pi^* g_H^\lambda|_{p^\perp}$
(ii) p is null, and
(iii) $\mathcal{L}_p g = 2P\frac{\partial P}{\partial t}(g_H + \pi^*\lambda \vee \psi) + P^2(\pi^*\lambda \vee \mathcal{L}_p\psi) = 2P\frac{\partial P}{\partial v} g + \pi^*\lambda \vee \tilde{\psi}$, i.e. p is shearfree for g.

Therefore, $(\mathfrak{M}, [g], [p])$ is an RT-space compatible with $(M, H, [g_M])$.

It remains to show that any conformal Lorentzian metric that satisfies (i)-(iii) has the form (7). Condition (i) means that there exists a positive function P on M such that

$$g|_{p^\perp} = P^2 \pi^* g_H^\lambda|_{p^\perp}.$$

Consider the symmetric 2-form

$$T = g - P^2 \pi^* g_H^\lambda$$

for some choice of the contact form λ on M. Then $T(u, v) = 0$ for any $u, v \in T_q M$ such that $g(v, p) = 0$. Let z be a lift of the Reeb vector field Z. We can choose z such that $g(z, z) = 0$.

Consider the 1-forms

$$\theta = g(p, \cdot) = \gamma \pi^*\lambda, \qquad \psi' = g(z, \cdot).$$

We have $\theta(z) = g(z, p) = \gamma\lambda(Z) = \gamma$.

If $u = u' + \alpha z$ is the decomposition of a vector field u on M such that $u' \in p^\perp$ then

$$\theta(u) = \alpha g(p, z) = \alpha\gamma\pi^*(Z) = \alpha\gamma,$$

hence

$$\alpha = \frac{1}{\gamma}\theta(u) = \pi^*\lambda(u).$$

For two vector fields u, v on $\mathfrak{M}$ with $u = u' + \alpha z$, $v = v' + \beta z$ where $u', v' \in p^\perp$ we have

$$T(u,v) = \alpha g(z,v) + \beta g(u,z) = \frac{1}{\gamma}(\theta(u)\psi'(v) + \theta(v)\psi'(u)) = \pi^*\lambda \vee \psi'(u,v).$$

It follows

$$g = P^2\pi^* g_H^\lambda + T = P^2(\pi^* g_H^\lambda + \pi^*\lambda \vee \psi)$$

where $\psi = \frac{1}{P^2}\psi'$. □

5. Applications of shearfree congruences in dimension 4

In this section we survey some applications of shearfree congruences in dimension 4. The correspondence between 4-dimensional shearfree congruences and 3-dimensional CR-manifolds has been known by physicists and has been exploited in both directions (see, e.g., [5, 10] and references therein).

In [3] a 3-parametric family of Ricci flat Lorentzian 4-manifolds with shear free congruence, which include the Kerr metric, have been constructed. It is given by

$$g = P^2\mu\bar{\mu} + \lambda(dr + W\mu + \bar{W}\bar{\mu} + H\lambda),$$

where

$$\begin{aligned}
\mu &= dz \\
\lambda &= du - 2\,\mathrm{Im}\,\frac{((a+b)|z|^2 + b)dz}{z(1+|z|^2)^2} \\
P^2 &= \frac{r^2}{(1+|z|^2)^2} + \frac{(b-a)+(b+a)|z|^2}{(1+|z|^2)^4} \\
W &= \frac{2\,\mathrm{i}\,az}{(1+|z|^2)^2} \\
H &= \frac{2(mr+b^2)(1+|z|^2)^2 - 2ab(1-|z|^4)}{r^2(1+|z|^2)^2 + (b-a+(b+a)|z|^2)^2} - 1.
\end{aligned}$$

Here $z = x + \mathrm{i}\,y, u, r$ are coordinates in $\mathbb{R}^4$ and a, b, m are real parameters. The metric g is singular for $z = 0$ if $b \neq 0$ and for $r = 0$ and $|z|^2 = \frac{a-b}{a+b}$ if $|b| \leq |a|$. The corresponding RT-space $(\mathfrak{M}, [g], [\partial_r])$ is twisting, unless $a = b = 0$. For $b = 0$, the metric g is the Kerr rotating black hole with mass m and the angular momentum parameter a; if $a = b = 0$ the metric g describes the Schwarzschild black hole with mass m. For $m = a = 0$ this is the Taub-NUT vacuum metric. The orbit spaces M can be identified with $\mathbb{C} \times \mathbb{R}$ with coordinates (z, u). If $b \neq 0$ we have to delete the singular line $z = 0$. The induced subconformal structures are $(M, [\lambda], [\mu\bar{\mu}])$ and the CR-structures are defined by $(M, [\lambda, \mu])$. Notice that the parameter

m only appears in the function H and does not affect the family of CR-manifolds. All resulting CR-manifolds can be embedded into $\mathbb{C}^2$ with coordinates (z, w) as

$$v = \operatorname{Im} w = \frac{-2a}{1+|z|^2} + 2b \log \frac{|z|^2}{1+|z|^2}.$$

This is the trivial Levi-flat CR-manifold $v = 0$ for the Schwarzschild solution, a spherical CR-manifold (with singularity at 0) in the Taub-NUT case and a non-spherical Sasakian manifold for the Kerr solution.

It is a natural question to ask, how analytic properties of a CR-manifold are reflected in a corresponding shearfree congruence. The papers [5, 8] by Lewandowski, Nurowski, Tafel and Hill, Lewandowski, Nurowski feature a fascinating approach to the local embeddability problem for 3-dimensional CR-manifolds: Let M be a 3-dimensional manifold with a CR-structure that is given by a pair of 1-forms (μ, λ) as above. Then the local embeddability problem reduces to finding two functionally independent CR-functions f, g, i.e. functions that satisfy

$$\bar\partial f = \bar\partial g = 0, \text{ and } df \wedge dg \neq 0,$$

where $(\partial, \bar\partial, \partial_0)$ is a dual frame to the coframe $(\mu, \bar\mu, \lambda)$. Using a Frobenius type result (see e.g. [3, 5]), one CR-function is constructed from a complex 1-form ϕ such that

$$d\phi \wedge \phi = 0 \text{ and } \phi \wedge \bar\phi \neq 0.$$

Such 1-form can be obtained as a structure form of the Levi-Civita connection as a consequence of the vanishing of certain components of the complexified Ricci curvature. The latter condition is, in a sense, vanishing of a $\bar\partial$ derivative.

According to a result by Jacobowitz [6], the existence of a second, functionally independent, CR-function can be related to a non-vanishing closed section of the canonical bundle of M. Here, the canonical bundle is simply the complex rank 1 line bundle spanned by the 2-form $\mu \wedge \lambda$. It is clear that this does not depend on the choice of the pair (μ, λ). The theorem below is a special case of Jacobowitz's result.

Theorem 4 (Jacobowitz,1987)**.** *If near some point x, (M^3, H, J) has a non-constant CR function and its canonical bundle has a closed non-zero section then (M^3, H, J) is embeddable on some neighbourhood of x.*

The condition of the existence of a non-zero closed section of the canonical bundle has a nice interpretation in general relativity. Recall that a 2-form F is a solution of the Maxwell equation in vacuum if

$$dF = d * F = 0.$$

The solution F is called a plane wave if it is null, i.e. $g(F, F) = g(F, *F) = 0$. Let $\mathcal{F} = F - \mathrm{i} *F$. According to a result by Robinson [9] the 2-forms $\mathcal{F}$ that correspond to plane waves have a representation

$$\mathcal{F} = \theta \wedge \epsilon$$

where $\theta = g(p, \cdot)$, $\epsilon = g(e, \cdot)$, p is a shearfree null vector field and $e = e_1 + \mathrm{i}\, e_2$ is a complex vector field with $g(p, e) = 0$, $g(e_i, e_j) = \delta_{ij}$. Vice versa, if p is a shearfree vector field and

e a complex vector field as above, then

$$\theta \wedge \epsilon$$

is a plane wave.

A plane wave solution $\mathcal{F} = \theta \wedge \epsilon$ is said to be aligned with the shearfree congruence if $\mathcal{F} = \Phi\mu \wedge \lambda$, where μ and λ are the lifts of the corresponding forms on the underlying CR-manifold. Hence the existence of a nonzero closed section of the canonical bundle translates into the existence of an aligned plane wave solution of the Maxwell equation.

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D.V.A.: Institute for Information Transmission Problems of the Russian Academy of Sciences, Moscow, Russia

M.G. & G.S.: School of Science and Technology, University of New England, Armidale, Australia