DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **37**, Number **5**, May **2017**

doi:10.3934/dcds.2017103

pp. **2375–2393**

HOMOGENIZATION OF TRAJECTORY ATTRACTORS OF 3D NAVIER–STOKES SYSTEM WITH RANDOMLY OSCILLATING FORCE

Kuanysh A. Bekmaganbetov

Kazakhstan Branch of M.V.Lomonosov Moscow State University
Kazhymukan st., 11, Astana, 010010, Kazakhstan

Gregory A. Chechkin*

M.V.Lomonosov Moscow State University
Moscow, 119991, Russia

Vladimir V. Chepyzhov

Institute for Information Transmission Problems
Russian Academy of Sciences
Bolshoy Karetniy 19, Moscow 127051, Russia
&
National Research University Higher School of Economics
Myasnitskaya Street 20, Moscow 101000, Russia

Andrey Yu. Goritsky

M.V.Lomonosov Moscow State University
Moscow, 119991, Russia

(Communicated by Roger Temam)

Abstract. We consider the 3D Navier–Stokes systems with randomly rapidly oscillating right–hand sides. Under the assumption that the random functions are ergodic and statistically homogeneous in space variables or in time variables we prove that the trajectory attractors of these systems tend to the trajectory attractors of homogenized 3D Navier–Stokes systems whose right–hand sides are the average of the corresponding terms of the original systems. We do not assume that the Cauchy problem for the considered 3D Navier–Stokes systems is uniquely solvable.

Bibliography: 44 titles.

2010 *Mathematics Subject Classification.* Primary: 35B40, 35B41, 35B45, 35Q30.

Key words and phrases. Attractors, random homogenization, Navier–Stokes equations, nonlinear equations.

Work of KAB is partially supported by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (grant no. 0816/GF4). Work of GAC, VVC, and AYG is partially supported by the Russian Foundation for Basic Research (projects 15-01-07920 and 14-01-00346).

* Corresponding author: G.A.Chechkin.

1. **Introduction.** Asymptotic analysis and homogenization (cf., for example, [34, 6, 4, 37, 32, 35, 10]) are basic methods for studying mathematical problems appearing in theories of media with nontrivial microstructure. These methods allow to simplify modeling of composites, skeletons, reinforced structures, perforated materials (porous media), cell-structures, bodies with concentrated masses, stratified flows (of Newtonian and non-Newtonian fluids), mixtures of fluids with different viscosities (densities) or mixtures of fluids and gases (fluids and solid particles), and many others.

Homogenization methods allow to consider media with periodic microstructure as well as with random one (for random case cf., for instance, [11, 12, 13, 1, 14, 15]).

In this paper we study the attractors of differential equations that model the long time behaviour of such media with complicated microstructure.

Attractors describe the final behaviour of solutions of dissipative nonlinear evolution equations on large time intervals and in the limit as time tends to infinity. It is also convenient to study, using attractors, the stability and instability of the limiting structures of the corresponding dynamical systems that are finite-dimensional for ODEs and infinite dimensional for PDEs. Attractors make it possible to single out the most essential limit sets of trajectories, which characterize the whole dynamics of the complicated model described by evolution equations (see, for examples, monographes [3, 20, 38] and the references therein).

More precisely, our interest is the asymptotic behavior of trajectory attractors of 3D Navier–Stokes systems with randomly oscillating outer forces.

Along the lines of the Bogolyubov averaging principle [8], the first results related to attractors of evolution equations with rapidly, but non-randomly oscillating terms of periodic or almost periodic kind, can be found in the papers [29, 30, 31]. The averaging of global attractors of autonomous and non-autonomous 2D Navier–Stokes equations has been studied in [20, 21, 23, 39]. Some problems related to the homogenization and the averaging of uniform global attractors for dissipative wave equations has been considered in [16, 24, 30, 40, 44], in presence of time oscillations, and in [20, 36, 39, 43], in presence of oscillations in space. For parabolic equations with oscillating parameters, similar problems have been considered in [20, 25, 26, 27, 28]. Papers [17, 18, 22, 23, 41] deal with partial differential equations containing singular oscillating terms.

The methods of trajectory attractors for evolution partial differential equations were developed in [19, 20] (see also the review [42]). This approach is very fruitful in the study of the long time behaviour of solutions to evolution equations for which the uniqueness theorem of the corresponding initial-value problem is not proved yet (e.g., for the inhomogeneous 3D Navier–Stokes system) or does not hold. Some homogenization problem for trajectory attractors of evolution equation with rapidly oscillating terms were studied in [20, 39] see also a paper [5] for random homogenization of attractors.

In this paper we consider autonomous and non-autonomous 3D Navier-Stokes systems and we assume that the right–hand sides $g\left(x,\frac{x}{\varepsilon},\omega\right)$ or $g\left(x,\frac{t}{\varepsilon},\omega\right)$ of the systems are random functions, which oscillates rapidly with respect to the spatial or time variables. Here ω is an element of a standard probability space $(D,\mathcal{B},\mu)$. The parameter $\varepsilon>0$ characterizes the oscillation frequency. Along with such systems we also consider the corresponding homogenized 3D Navier–Stokes system with external force $g^{hom}(x)$, where $g^{hom}(x)$ is the mathematical expectation of $g\left(x,\frac{x}{\varepsilon},\omega\right)$ or $g\left(x,\frac{t}{\varepsilon},\omega\right)$ as $\varepsilon\to 0$.

We prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the system with randomly oscillating term converges almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized system in an appropriate functional space.

We do not assume that the corresponding Cauchy problem for the 3D Navier–Stokes system with the external force $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ or $g\left(x, \frac{t}{\varepsilon}, \omega\right)$ has the unique solution. Under the assumption that the random function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ or $g\left(x, \frac{t}{\varepsilon}, \omega\right)$ is statistically homogeneous and ergodic with smooth realizations (for detailed definitions see below), we prove that the mathematical expectation coincides with deterministic spacial mean.

In Section 1 we formulate the problem and give necessary definitions of randomness. In Section 2 we give the main notions and theorems concerning the trajectory attractors of autonomous and non-autonomous evolution equations. Section 3 is devoted to the study of the averaging of attractors of autonomous and non-autonomous 3D Navier–Stokes systems with randomly rapidly oscillating external forces.

2. **Notation and settings.**

Consider the 3D Navier-Stokes system in the domain $D \Subset \mathbb{R}^3$:

$$\partial_t u_\varepsilon + \nu L u_\varepsilon + B(u_\varepsilon) = g, \quad \operatorname{div} u_\varepsilon = 0, \quad u_\varepsilon|_{\partial D} = 0, \tag{1}$$

where $x = (x_1, x_2, x_3) \in D,\ g = (g^1, g^2, g^3),$ and $u_\varepsilon = u_\varepsilon(x,t) = (u_\varepsilon^1, u_\varepsilon^2, u_\varepsilon^3)$ is the unknown velocity–vector and $\varepsilon > 0$ is a small parameter.

Using the standard approach, we exclude the pressure from the 3D Navier–Stokes system assuming from the very beginning that $\operatorname{div} g = 0$.

We study two cases:

1. *autonomous* system (1) with rapid oscillations *in space* having the external force $g = g\left(x, \frac{x}{\varepsilon}, \omega\right)$, where $g^j = g^j(x, \xi, \omega), x \in D, \xi \in \mathbb{R}^3,\ j = 1, 2, 3;$
2. *non-autonomous* system (1) with rapid oscillations *in time* having the external force $g = g\left(x, \frac{t}{\varepsilon}, \omega\right)$, where $g^j = g^j(x, \tau, \omega), x \in D, \tau \in \mathbb{R},\ j = 1, 2, 3.$

We assume that g is a random statistically homogeneous (in space or in time) ergodic function with smooth realizations, ω is an element of a standard probability space $(\Omega, \mathcal{B}, \mu)$ (for the detailed definitions see below).

In the system (1), L is the 3D Stokes operator, $Lu = -P\Delta u$, and

$$B(u) = B(u,u), \quad B(u,v) := P(u, \nabla)\, v = P \sum_{i=1}^{3} u_i\, \partial_{x_i} v.$$

The Laplace operator $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ acts in x-space. The parameter $\nu > 0$ stands for the kinematic viscosity, while the density of the fluid is assumed to be constant and equal to 1. Throughout the paper we shall omit the subindex ε of functions u_ε.

We denote by H and H^1 the closure in $(L_2(D))^3$ and $(H^1(D))^3$ of the set

$$\mathcal{V}_0 = \{v \mid v \in (C_0^\infty(D))^3,\ \operatorname{div} v = 0\}.$$

P denotes the Leray–Helmholtz orthogonal projector in $(L_2(D))^3$ onto the Hilbert space H. The scalar products in H and in H^1 are

$$(u,v) := \int_D (u(x), v(x))\, dx \quad \text{and} \quad (u,v)_1 := \langle Lu, v \rangle = \int_D (\nabla u(x), \nabla v(x))\, dx$$

and the norms are $\|u\| := (u,u)^{1/2}$ and $\|u\|_1 := \langle Lu, u \rangle^{1/2}$, respectively.

Randomness. Assume that $(\Omega, \mathcal{B}, \mu)$ is a probability space, i.e., the set Ω is endowed with a σ-algebra $\mathcal{B}$ of its subsets and a σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

Definition 2.1. A family of measurable maps $T_\xi : \Omega \to \Omega$, $\xi \in \mathbb{R}^3$ ($T_\tau : \Omega \to \Omega$, $\tau \in \mathbb{R}$) is called a *dynamical system* if the following properties hold:

1) *group property*: $T_{\xi_1+\xi_2} = T_{\xi_1} T_{\xi_2}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^3$; $T_0 = Id$ ($T_{\tau_1+\tau_2} = T_{\tau_1} T_{\tau_2}$, $\forall \tau_1, \tau_2 \in \mathbb{R}$; $T_0 = Id$) (Id is the identity mapping on Ω);

2) *isometry property* (the mappings T_ξ (T_τ) preserve the measure μ on Ω): $T_\xi B \in \mathcal{B}$, $\mu(T_\xi B) = \mu(B)$, $\forall \xi \in \mathbb{R}^3$, $\forall B \in \mathcal{B}$ ($T_\tau B \in \mathcal{B}$, $\mu(T_\tau B) = \mu(B)$, $\forall \tau \in \mathbb{R}$, $\forall B \in \mathcal{B}$);

3) *measurability*: for any measurable function $\psi(\omega)$ on Ω, the function $\psi(T_\xi \omega)$ ($\psi(T_\tau \omega)$) is measurable on $\Omega \times \mathbb{R}^3$ (on $\Omega \times \mathbb{R}$) and continuous in ξ (in τ).

Let $L_q(\Omega, \mu)$ ($q \geq 1$) be the space of measurable functions on Ω whose absolute value at the power q is integrable with respect to the measure μ. If $T_\xi : \Omega \to \Omega$ ($T_\tau : \Omega \to \Omega$) is a dynamical system, then on the space $L_2(\Omega, \mu)$ we define a parameter dependent group of operators $\{T_\xi\}, \xi \in \mathbb{R}^3$ ($\{T_\tau\}, \tau \in \mathbb{R}$) (we keep the same notation), by the formula $(T_\xi \psi)(\omega) := \psi(T_\xi \omega)$ ($(T_\tau \psi)(\omega) := \psi(T_\tau \omega)$), $\psi \in L_2(\Omega, \mu)$.

Condition 3) in the definition implies that the group T_ξ (T_τ) is strongly continuous, i.e., we have $\lim\limits_{\xi \to 0} \|T_\xi \psi - \psi\|_{L_2(\Omega,\mu)} = 0$ ($\lim\limits_{\tau \to 0} \|T_\tau \psi - \psi\|_{L_2(\Omega,\mu)} = 0$) for any $\psi \in L_2(\Omega, \mu)$.

Definition 2.2. Suppose that $\psi(\omega)$ is a measurable function on Ω. The function $\mathbb{R}^3 \ni \xi \mapsto \psi(T_\xi \omega) \in \mathbb{R}$ ($\mathbb{R} \ni \tau \mapsto \psi(T_\tau \omega) \in \mathbb{R}$) for fixed $\omega \in \Omega$ is called the *realization of the function* ψ.

The following assertion is proved, for instance, in [32] and [10].

Proposition 1. *If $\psi \in L_q(\Omega, \mu)$, then ω-almost all realizations $\xi \mapsto \psi(T_\xi \omega)$ ($\tau \mapsto \psi(T_\tau \omega)$) belong to $L_q^{loc}(\mathbb{R}^3)$ (to $L_q^{loc}(\mathbb{R})$).*

If the sequence $\{\psi_k\} \subset L_q(\Omega, \mu)$ converges in $L_q(\Omega, \mu)$ to the function ψ, then there exists a subsequence $\{\psi_{k'}\}$ such that ω-almost all realizations $\xi \mapsto \psi_{k'}(T_\xi \omega)$ ($\tau \mapsto \psi_{k'}(T_\tau \omega)$) converge in $L_q^{loc}(\mathbb{R}^3)$ ($L_q^{loc}(\mathbb{R})$) to the realization $\xi \mapsto \psi(T_\xi \omega)$ ($\tau \mapsto \psi(T_\tau \omega)$).

Definition 2.3. A measurable function $\psi(\omega)$ on Ω is called *invariant*, if $\psi(T_\xi \omega) = \psi(\omega)$ ($\psi(T_\tau \omega) = \psi(\omega)$) for any $\xi \in \mathbb{R}^3$ ($\tau \in \mathbb{R}$) and almost all $\omega \in \Omega$.

Definition 2.4. The dynamical system T_ξ (T_τ) is called *ergodic*, if any invariant function is ω-almost everywhere a constant.

We denote by $\mathcal{R}$ the natural Borel σ-algebra of subsets of $\mathbb{R}^3$ (of $\mathbb{R}$). Suppose that $\mathcal{F}(\xi) \in L_1^{loc}(\mathbb{R}^3)$ ($\mathtt{F}(\tau) \in L_1^{loc}(\mathbb{R})$).

Definition 2.5. We say that the function $\mathcal{F}(\xi)$ ($\mathtt{F}(\tau)$) has a SPACE AVERAGE (AVERAGE IN TIME), if the limit

$$M(\mathcal{F}) := \lim_{\varepsilon \to 0} \frac{1}{|R|} \iiint\limits_R \mathcal{F}\left(\frac{x}{\varepsilon}\right) \, dx \qquad \left(M(\mathtt{F}) := \lim_{\varepsilon \to 0} \frac{1}{|R|} \int\limits_R \mathtt{F}\left(\frac{t}{\varepsilon}\right) \, dt \right)$$

exists for any bounded Borel set $R \in \mathcal{R}$ and does not depend on the choice of R. The number $M(\mathcal{F})$ ($M(\mathtt{F})$) is called the SPATIAL MEAN VALUE (MEAN VALUE IN TIME) of the function $\mathcal{F}$ ($\mathtt{F}$).

Equivalently, the space average is defined by

$$M(\mathcal{F}) := \lim_{s\to+\infty} \frac{1}{|B_s|} \iiint\limits_{B_s} \mathcal{F}(\xi)\, d\xi, \quad \text{where} \quad B_s = \left\{\xi \in \mathbb{R}^3 \,\middle|\, \frac{\xi}{s} \in B\right\}$$

$$\left(M(\mathtt{F}) =: \lim_{s\to+\infty} \frac{1}{|B_s|} \int\limits_{B_s} \mathtt{F}(\tau)\, d\tau, \quad \text{where} \quad B_s = \left\{\tau \in \mathbb{R} \,\middle|\, \frac{\tau}{s} \in B\right\}\right).$$

The following statement can be found, for instance, in [10].

Proposition 2. *Let the function* $\mathcal{F}(\xi)$ *have a space mean value in* $\mathbb{R}^3$*, and suppose that the family* $\{\mathcal{F}\left(\frac{\zeta}{\varepsilon}\right),\ 0 < \varepsilon \le 1\}$ *is bounded in* $L_q(\mathcal{K})$ *for some* $q \ge 1$*, where* $\mathcal{K}$ *is an arbitrary compact set in* $\mathbb{R}^3$*. Then* $\mathcal{F}\left(\frac{\zeta}{\varepsilon}\right) \rightharpoonup M(\mathcal{F})$ *weakly in* $L_q^{loc}(\mathbb{R}^3)$ *as* $\varepsilon \to 0$*.*

Let the function $\mathtt{F}(\tau)$ *have a mean value in time, and suppose that the family of functions* $\mathtt{F}\left(\frac{\varsigma}{\varepsilon}\right),\ 0 < \varepsilon \le 1\}$ *is bounded in* $L_q(\mathcal{K})$ *for some* $q \ge 1$*, where* $\mathcal{K}$ *is an arbitrary compact set in* $\mathbb{R}$*. Then* $\mathtt{F}\left(\frac{\varsigma}{\varepsilon}\right) \rightharpoonup M(\mathtt{F})$ *weakly in* $L_q^{loc}(\mathbb{R})$ *as* $\varepsilon \to 0$*.*

Throughout the paper we use the Birkhoff theorem (see [7] and [2]) in the following form (see, for instance, [32] and [10]):

Theorem 2.6. ***(Birkhoff ergodic theorem)*** *Let the dynamical system* $T_\xi, \xi \in \mathbb{R}^3$*, satisfy Definition 2.1 and assume that* $\psi \in L_q(\Omega,\mu), q \ge 1$*. Then, for almost all* $\omega \in \Omega$*, the realization* $\psi(T_\xi\omega)$ *has the space mean value* $M(\psi(T_\xi\omega))$*. Moreover, the space mean value* $M(\psi(T_\xi\omega))$ *is a conditional mathematical expectation of the function* $\psi(\omega)$ *with respect to the* σ*-algebra of invariant subsets. Hence,* $M(\psi(T_\xi\omega))$ *is an invariant function and*

$$\mathbb{E}(\psi) \equiv \int\limits_\Omega \psi(\omega)\, d\mu = \int\limits_\Omega M(\psi(T_\xi\omega))\, d\mu.$$

In particular, if the dynamical system T_ξ *is ergodic then, for almost all* $\omega \in \Omega$*, we have the identity*

$$\mathbb{E}(\psi) = M(\psi(\xi)).$$

Remark 1. The formulation of the Birkhoff ergodic theorem for mean value in time is completely the same and for an ergodic dynamical system $T_\tau, \tau \in \mathbb{R}$, we have

$$\mathbb{E}(\psi) = M(\psi(\tau)).$$

Definition 2.7. A random function $\psi(\xi,\omega)$, $\xi \in \mathbb{R}^3$, $\omega \in \Omega$, is called *statistically homogeneous*, if the representation $\psi(\xi,\omega) = \Psi(T_\xi\omega)$ is valid for some measurable function $\Psi : \Omega \to \mathbb{R}$, where T_ξ is a dynamical system in Ω.

A random function $\phi(\tau,\omega)$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, is called *statistically homogeneous*, if the representation $\phi(\tau,\omega) = \Phi(T_\tau\omega)$ is valid for some measurable function $\Phi : \Omega \to \mathbb{R}$, where T_τ is a dynamical system in Ω.

3. **Trajectory attractors of evolution equations.**

In this section we give a scheme for the construction of trajectory attractors of autonomous and non-autonomous evolution equations. In the next section we shall apply this scheme to the study of trajectory attractors of the concrete evolution equations with rapidly oscillating coefficients and the corresponding averaged equations.

To begin with we consider an abstract autonomous evolution equation

$$\partial_t u = A(u), \quad t \geq 0. \tag{2}$$

Here $A(\cdot) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. For instance, $A(u) = a\Delta u - f(u) + g$ (see also Section 4).

We are going to study solutions $u(s)$ of equation (2) as functions of $s \in \mathbb{R}_+$ as a whole. Here $s \equiv t$ denote the time variable. The set of solutions of (2) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (2). Let us describe the trajectory space $\mathcal{K}^+$ in greater detail.

At first, we consider solutions $u(s)$ of (2) defined on a fixed time interval $[t_1, t_2]$ from $\mathbb{R}$. We study solutions of (2) in a Banach space $\mathcal{F}_{t_1,t_2}$ that depends on t_1 and t_2. The space $\mathcal{F}_{t_1,t_2}$ consists of functions $f(s), s \in [t_1, t_2]$ such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, $\mathcal{F}_{t_1,t_2}$ can be the space $C([t_1, t_2]; E)$, or $L_p([t_1, t_2]; E)$, for $p \in [1, \infty]$, or the intersection of such spaces (see Section 4). We assume that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathcal{F}t_1,t_2} \leq \|f\|_{\mathcal{F}\tau_1,\tau_2}, \quad \forall f \in \mathcal{F}_{\tau_1,\tau_2}, \tag{3}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1, t_2]$.

Let $S(h)$ for $h \in \mathbb{R}$ denote the translation operator

$$S(h)f(s) = f(h + s).$$

Evidently, if the argument s of $f(\cdot)$ belongs $[t_1, t_2]$, then the argument s of $S(h)f(\cdot)$ can be taken form $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}t_1,t_2}, \quad \forall f \in \mathcal{F}_{t_1,t_2}. \tag{4}$$

This assumption is fairly natural.

We assume that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a larger Banach space, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\partial_t f(t)$ is a distribution with values in $E_0, \partial_t f(s) \in D'((t_1, t_2); E_0)$ and we assume that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is said to be a *solution* of (2) from the space $\mathcal{F}_{t_1,t_2}$ (on the interval (t_1, t_2)) if $\partial_t u(s) = A(u(s))$ in the distributional sense of the space $D'((t_1, t_2); E_0)$.

We also define the space

$$\mathcal{F}_+^{loc} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}. \tag{5}$$

For example, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is called a solution of (2) from $\mathcal{F}_+^{loc}$ if $\Pi_{t_1,t_2} u(s) \in \mathcal{F}_{t_1,t_2}$ and this function is a solution of (2) for every $[t_1, t_2] \subset \mathbb{R}_+$.

We denote by $\mathcal{K}^+$ a set of solutions of (2) from $\mathcal{F}_+^{loc}$. Notice, that $\mathcal{K}^+$ is not necessarily the set of *all* solutions from $\mathcal{F}_+^{loc}$. The elements of $\mathcal{K}^+$ are called *trajectories* and the set $\mathcal{K}^+$ is called the *trajectory space* of the equation (2).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h + s) \in \mathcal{K}^+$ for every $h \geq 0$. This is a very natural assumption for solutions of autonomous equations.

We now consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s + h), \quad h \geq 0.$$

It is clear that the mappings $\{S(h), h \geq 0\}$ form a semigroup in $\mathcal{F}_+^{loc}$: $S(h_1)S(h_2) = S(h_1 + h_2)$ for $h_1, h_2 \geq 0$ and $S(0)$ is the identity operator. We change the variable h into the time variable t. The semigroup $\{S(t), t \geq 0\}$ is called the *translation semigroup*. By our assumption the translation semigroup maps the trajectory space $\mathcal{K}^+$ to itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+, \quad \forall t \geq 0. \tag{6}$$

We shall study attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We define a topology in the space $\mathcal{F}_+^{loc}$.

Let metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ be defined on $\mathcal{F}_{t_1,t_2}$ for all $[t_1,t_2] \subset \mathbb{R}$. Similar to (3) and (4) we assume that

$$\rho_{t_1,t_2}\left(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g\right) \leq \rho_{\tau_1,\tau_2}\left(f, g\right), \quad \forall f, g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$
$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g), \quad \forall f, g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

Denote by Θ_{t_1,t_2} the corresponding metric spaces on $\mathcal{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of the Banach space $\mathcal{F}_{t_1,t_2}$. However, usually in application ρ_{t_1,t_2} generate the topologies Θ_{t_1,t_2} that are weaker than the strong convergence topology of the Banach spaces $\mathcal{F}_{t_1,t_2}$.

The inductive limit of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, i.e., by definition, a sequence $\{f_n(s)\} \subset \mathcal{F}_+^{loc}$ converges to $f(s) \in \mathcal{F}_+^{loc}$ as $n \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2} f_n, \Pi_{t_1,t_2} f) \to 0$ as $n \to \infty$ for each $[t_1,t_2] \subset \mathbb{R}_+$. It is not hard to prove that the topology Θ_+^{loc} is metrizable using, for example, the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{7}$$

If it is known that all metric spaces Θ_{t_1,t_2} are complete, then clearly the space Θ_+^{loc} is also complete.

We claim that the translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This assertion follows directly from the definition of the topological space Θ_+^{loc}.

We also consider the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\},$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{8}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$, then the space $\mathcal{F}_+^b = C^b(\mathbb{R}_+; E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0} \|f(h)\|_E$ and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+; E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0} \int_h^{h+1} \|f(s)\|_E^p ds\right)^{1/p}$.

Recall that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We require the Banach space $\mathcal{F}_+^b$ only to define bounded subsets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we do not consider the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$. Instead, we utilize the local convergence topology Θ_+^{loc} which is much weaker.

We suppose that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e., every trajectory $u(s) \in \mathcal{K}^+$ of equation (2) has a finite norm (8). Let us define a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 3.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set

$\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ε-neighbourhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $t \geq t_1$.

It is clear that the attracting property of $\mathcal{P}$ can be formulated in the following equivalent form: for any set $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where $\operatorname{dist}_{\mathcal{M}}(X, Y) := \sup_{x\in X} \operatorname{dist}_{\mathcal{M}}(x, Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x, y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Definition 3.2. (see [20]) A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}, **(ii)** the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 2. Using the terminology from [3] we can say that the trajectory attractor $\mathfrak{A}$ is the *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$*-attractor* of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$, that is, $\mathfrak{A}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc} for any bounded (in $\mathcal{F}_+^b$) set $\mathcal{B}$ from $\mathcal{K}^+$:

$$\operatorname{dist}_{\Theta_+^{loc}}(S(t)\mathcal{B}, \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We now formulate the central result on the trajectory attractor for equation (2).

Theorem 3.3. *Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation (2) is contained in $\mathcal{F}_+^b$ and (6) holds. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.*

Proof. Indeed, the semigroup $\{S(t)\}$ is continuous on $\mathcal{K}^+$ in the metric space Θ_+^{loc}. The set $\mathcal{P}$ is $(\mathcal{F}_+^b, \Theta_+^{loc})$-attracting, compact in the space Θ_+^{loc}, and bounded in $\mathcal{F}_+^b$. Then the semigroup $\{S(t)\}$ has the global $(\mathcal{F}_+^b, \Theta_+^{loc})$-attractor $\mathfrak{A}$ which is evidently the trajectory attractor (see [3, 19, 20] for the complete proof). This attractor can be constructed from the set $\mathcal{P}$ by the standard formula $\mathfrak{A} = \omega(\mathcal{P}) := \cap_{h\geq 0} \left[\cup_{t\geq h} S(t)\mathcal{P}\right]_{\Theta_+^{loc}}$. □

We now describe the structure of the trajectory attractor $\mathfrak{A}$ of equation (2) in terms of complete trajectories of this equation.

Consider the equation (2) on the entire time axis

$$\partial_t u = A(u),\ t \in \mathbb{R}. \tag{9}$$

We have defined the trajectory space $\mathcal{K}^+$ of equation (9) on $\mathbb{R}_+$. We now extend this definition on the entire $\mathbb{R}$. If a function $f(s),\ s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s + h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of equation (9) if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to the semiaxis $\mathbb{R}_+$.

We have introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. We now define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}\ \forall\, [t_1,t_2] \subseteq \mathbb{R}\};$$
$$\mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h\in\mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ in Θ_{t_1,t_2} for each $[t_1,t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 3.4. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (9) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (9) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \le C_u, \quad \forall h \in \mathbb{R}. \tag{11}$$

Theorem 3.5. *Assume that the hypotheses of Theorem 3.3 holds. Then*

$$\mathfrak{A} = \Pi_+ \mathcal{K}, \tag{12}$$

the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

The complete proof can be found in [19, 20].

We now describe briefly the construction of the trajectory attractors for non-autonomous evolution equations. Non-autonomous equations contain terms that explicitly depend on time. For example, such terms can be the external forces, the interaction functions or other time dependent coefficients of the equation. Thus, it is convenient to introduce a function $\sigma(t), t \in \mathbb{R}$, which is called the *time symbol* of the considering equation and consists of all time dependent terms of the equation. The values of the function $\sigma(t)$ belong to the corresponding function (Banach or metric) space Ψ.

A non-autonomous equation can be written in the form

$$\partial_t u = A_{\sigma(t)}(u). \tag{13}$$

Here similarly to equation (2), for every $t \in \mathbb{R}$, we are given an operator $A_{\sigma(t)}(\cdot) : E_1 \to E_0$. For example, $A_{\sigma(t)}(u) = a\Delta u - f(u) + g(x,t)$, where $\sigma(t) = g(x,t)$ (see also Section 4).

The *time symbol* $\sigma(s)$, as a function of time, belongs to a topological space Ξ. We assume for simplicity that Ξ is the space $C^{loc}(\mathbb{R};\Psi)$ or $\Xi = L_p^{loc}(\mathbb{R};\Psi)$, where Ψ is a Banach space, with corresponding topology of the local strong convergence on each finite interval of the time axis. To construct the trajectory attractor for equation (13) we start from the following requirement: the attractor must not change when the symbol $\sigma(s)$ is replaced by any shifted symbol $\sigma(s+h)$, $h \in \mathbb{R}$. This is why we consider the entire family of equations (13) with various symbols $\sigma(s) \in \Sigma$, where Σ is a translation invariant set:

$$S(h)\Sigma = \Sigma, \quad \forall h \in \mathbb{R}. \tag{14}$$

Usually in applications Σ coincides with the hull $\mathcal{H}(\sigma_0)$ (in Ξ) of some initial symbol $\sigma_0(s)$ of the equation:

$$\Sigma = \mathcal{H}(\sigma_0) := [\{\sigma_0(t+h) \mid h \in \mathbb{R}\}]_{\Xi}\,.$$

Recall that a function $\sigma(s)$ is called *translation compact* in the space Ξ if the hull $\mathcal{H}(\sigma_0)$ is compact in the topological space Ξ (see [20]).

We now assume that the initial symbol $\sigma_0(s)$ of the equation (13) is a translation compact function in Ξ. Consequently the set Σ is compact in Ξ. For example, $\sigma_0(t)$ can be a periodic, quasiperiodic or almost periodic function. Then its hull $\mathcal{H}(\sigma_0)$ is compact in the space $C_b(\mathbb{R};\Psi)$, and therefore the symbol space $\Sigma = \mathcal{H}(\sigma_0)$ is compact in $\Xi = C^{loc}(\mathbb{R};\Psi)$.

Similarly to the autonomous case we assume that corresponding to each symbol $\sigma \in \Sigma$ there is the trajectory space $\mathcal{K}_\sigma^+ \subset \mathcal{F}_+^b$, that consists of the solutions $u(s), s \geq 0$ of equation (13). The function $u(t)$ satisfies the equation in the distributional sense of the space $D'((t_1,t_2);E_0)$ for each $(t_1,t_2) \subset \mathbb{R}_+$. The spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc} are defined exactly as in the autonomous case.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\sigma^+$. It is clear that in the general case the trajectory space $\mathcal{K}_\sigma^+$ is not invariant with respect to $\{S(t)\}$, i.e., $S(t)\mathcal{K}_\sigma^+ \nsubseteq \mathcal{K}_\sigma^+$. Nevertheless, we have the inclusion

$$S(t)\mathcal{K}_\sigma^+ \subseteq \mathcal{K}_{S(t)\sigma}^+, \quad \forall t \geq 0. \tag{15}$$

Speaking informally, this means that if $u(s) \in \mathcal{K}_\sigma^+$ is a solution of equation (13) with symbol $\sigma(s)$, then $u(h+s) = S(h)u(s) \in \mathcal{K}_{S(h)\sigma}^+$ is a solution of the h-shifted equation (13) with symbol $\sigma(h+s) = S(h)\sigma(s)$.

Consider now the *aggregate trajectory space*

$$\mathcal{K}_\Sigma^+ = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma^+,$$

which is already invariant with respect to $\{S(t)\}$:

$$S(t)\mathcal{K}_\Sigma^+ \subseteq \mathcal{K}_\Sigma^+, \quad \forall t \geq 0.$$

Similarly to autonomous case we define the *uniform (w.r.t. $\sigma \in \Sigma$) attractor* of the translation semigroup on $\mathcal{K}_\Sigma^+$ in the topology Θ_+^{loc}.

Definition 3.6. The global $(\mathcal{F}_+^b, \Theta_+^{loc})$-attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\Sigma^+$, is called the *uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor* $\mathfrak{A}_\Sigma$ of equation (13), that is, **(i)** the set $\mathfrak{A}_\Sigma \subseteq \mathcal{K}_\Sigma^+$ is compact in Θ_+^{loc}, bounded in $\mathcal{F}_+^b$, **(ii)** strictly invariant, $S(t)\mathfrak{A}_\Sigma = \mathfrak{A}_\Sigma$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}_\Sigma$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc} each bounded (in $\mathcal{F}_+^b$) set $\mathcal{B} \subset \mathcal{K}_\Sigma^+$, that is, for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}_\Sigma) \to 0 \quad (t \to +\infty).$$

Similarly to the autonomous case we define the *kernel* $\mathcal{K}_\sigma$ of the non-autonomous equation (13) that consists of all trajectories $u(s), s \in \mathbb{R}$, of equation (13) defined on the entire axis and bounded with respect to the norm of $\mathcal{F}^b$ (see (10)).

Theorem 3.7. *Let the symbol space of equation (2) be the hull $\Sigma = \mathcal{H}(\sigma_0)$ of a translation compact function $\sigma_0(s)$. Assume that the aggregated trajectory space $\mathcal{K}_\Sigma^+$ corresponding to the equation (2) belongs to $\mathcal{F}_+^b$. Assume further that there exists an attracting set $\mathcal{P} \subset \mathcal{K}_\Sigma^+$ of $\{S(t)\}$ in the topology Θ_+^{loc} such that $\mathcal{P}$ is compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}_\Sigma^+$ has the uniform (w.r.t. $\sigma \in \Sigma$) trajectory attractor $\mathfrak{A}_\Sigma \subseteq \mathcal{P}$. Moreover,*

$$\mathfrak{A}_\Sigma = \Pi_+ \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma, \tag{16}$$

the kernel $\mathcal{K}_\sigma$ is non-empty for every $\sigma \in \Sigma$ and the set $\mathfrak{A}_\Sigma$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

Theorem 3.7 is proved in [19, 20] for more general symbol spaces Σ.

Theorems 3.3 and 3.7 show that for the construction of the trajectory attractor we require an attracting set $\mathcal{P}$ compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$. Usually in application, a large ball $B_R = \{\|f\|_{\mathcal{F}_+^b} \le R\}$ in $\mathcal{F}_+^b$ $(R \gg 1)$ can be taken as such an attracting (or even absorbing) set and the existence of such ball B_R follows from the inequality of the form

$$\|S(t)u\|_{\mathcal{F}_+^b} \le C(\|u\|_{\mathcal{F}_+^b})e^{-\gamma t} + R_0, \quad \forall t \ge 0, \quad (\gamma > 0) \tag{17}$$

holding for each trajectory $u(\cdot)$ of autonomous equation (2) or non-autonomous equation (13). Here, $C(\xi)$ depends on ξ and R_0 does not depend on a trajectory u. Inequality (17) follows usually from a priori estimates for solutions of equations (2) or (13).

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$\begin{aligned} W_{p_1,p_0}([0,M];E_1,E_0) = \{\psi(s),\ s\in[0,M] \mid \psi(\cdot) \in L_{p_1}([0,M];E_1),\\ \psi'(\cdot) \in L_{p_0}([0,M];E_0)\},\\ W_{\infty,p_0}([0,M];E_1,E_0) = \{\psi(s),\ s\in[0,M] \mid \psi(\cdot) \in L_{\infty}([0,M];E_1),\\ \psi'(\cdot) \in L_{p_0}([0,M];E_0)\}, \end{aligned}$$

(where $p_1 \ge 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s \in [0,M]\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0}.$$

Lemma 3.8. ***(Aubin-Lions-Simon)*** [9] *Assume that $E_1 \Subset E \subset E_0$.Then the following embeddings are compact:*

$$W_{p_1,p_0}([0,T];E_1,E_0) \Subset L_{p_1}([0,T];E), \tag{18}$$

$$W_{\infty,p_0}([0,T];E_1,E_0) \Subset C([0,T];E). \tag{19}$$

In the next section we study evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 3.9. We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\varepsilon_1 \ge 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0).$$

4. Homogenization of 3D Navier–Stokes system .

Consider the autonomous 3D Navier-Stokes system:

$$\partial_t u + \nu L u + B(u) = g\left(x, \frac{x}{\varepsilon}, \omega\right), \quad \operatorname{div} u = 0, \quad u|_{\partial D} = 0. \tag{20}$$

Assume that $T_\xi, \xi \in \mathbb{R}^3$, is an ergodic dynamical system. The function $g(x, \frac{x}{\varepsilon}, \omega)$ is statistically homogeneous, i.e. $g(x,\xi,\omega) = \mathbf{G}(x, T_\xi\omega)$, where $\mathbf{G} : D \times \Omega \to \mathbb{R}^3$ is measurable.

We suppose that ω-almost all realizations $\mathbf{G}(x, T_{\frac{x}{\varepsilon}}\omega)$ are taken from the space H for every $\varepsilon > 0$ and the function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ has the average $g^{hom}(x) = \mathbb{E}(\mathbf{G})(x)$ as $\varepsilon \to 0$ in the space H_w, that is, almost surely

$$\left(g\left(x, \frac{x}{\varepsilon}, \omega\right), \varphi(x)\right) \to \left(g^{hom}(x), \varphi(x)\right) \quad \text{as } \varepsilon \to 0, \quad \forall \varphi \in H. \tag{21}$$

Remark 3. Here we refer to the Birkhoff Theorem 2.6.

Remark 4. We can construct many examples of function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ that satisfy (21). For instance, $g\left(x, \frac{x}{\varepsilon}, \omega\right) = Pg_1(x)\, g_2(\frac{x}{\varepsilon}, \omega)$, where $g_1 \in H$ and $g_2(\xi, \omega)$ is statistically homogeneous ergodic function with smooth realization. More examples can be constructed in the form $g\left(x, \frac{x}{\varepsilon}, \omega\right) = P\sum_{k=1}^{N} g_1^k(x)\, g_2^k(\frac{x}{\varepsilon}, \omega)$, where $g_1^k \in H$ and $g_2^k(\xi, \omega)$ have the above properties.

Applying to equation (20) the general scheme from Section 3, we set $E_1 = H^1,\ E_0 = H^{-1},\ E = H$, where H^{-1} is the dual space of H^1.

To describe the trajectory space $\mathcal{K}_\varepsilon^+$ of equation (20) we consider weak solutions (trajectories) of this equation in the spaces $L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H)$. If $u(s) \in L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H)$, then equation (20) makes sense in the space of distributions $D'(\mathbb{R}_+; H^{-1})$, (see [33]).

Definition 4.1. The trajectory space $\mathcal{K}_\varepsilon^+$ is the union of all weak solutions (trajectories) $u(\cdot) \in L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H)$ of equation (20) that satisfy almost surely (for almost all ω or with probability one) the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu\|u(t)\|_1^2 \le (g, u(t)), \quad t \in \mathbb{R}_+. \tag{22}$$

Here $g = g\left(x, \frac{x}{\varepsilon}, \omega\right)$. Inequality (22) means the following: for any test function $\psi(s) \in C_0^\infty(]0, +\infty[), \psi \ge 0$, we have almost surely

$$-\frac{1}{2}\int_0^{+\infty} \|u(s)\|^2 \psi'(s)ds + \nu \int_0^{+\infty} \|u(s)\|_1^2 \psi(s)ds \le \int_0^{+\infty} (g, u(s))\, \psi(s)ds. \tag{23}$$

If $u_0 \in H$, then there exists a weak solution $u(s)$ of equation (20) belonging to the space $L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H)$ such that $u(0) = u_0$ and $u(s)$ satisfies inequality (23). For the proof see [19, 20, 33].

Remark 5. For the 3D Navier–Stokes system the problem of the uniqueness of the weak solution is still open. Neither it is known whether an arbitrary weak solution of (20) satisfies inequality (22). Nevertheless, weak solutions $u(t), t \ge 0$ provided by the Galerkin approximation method satisfy (22).

It is well known that for any weak solution $u(s) \in L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H)$ of equation (20) the derivative $\partial_t u \in L_{4/3}^{loc}(\mathbb{R}_+; H^{-1})$ (see [20, 33]).

Following the general scheme of Section 3, we define the Banach spaces

$$\mathcal{F}_{t_1,t_2} := L_2([t_1, t_2]; H^1) \cap L_\infty([t_1, t_2]; H) \cap \{v \mid \partial_t v \in L_{4/3}([t_1, t_2]; H^{-1})\}.$$

It is clear that equalities (3) hold for the spaces $\mathcal{F}_{t_1,t_2}$ and the translation semigroup $\{S(h)\}$ satisfies (4).

It is obvious that

$$\mathcal{F}_+^{loc} = L_2^{loc}(\mathbb{R}_+; H^1) \cap L_\infty^{loc}(\mathbb{R}_+; H) \cap \{v \mid \partial_t v \in L_{4/3}^{loc}(\mathbb{R}_+; H^{-1})\}.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathcal{F}_{t_1,t_2}$ using the norms of the spaces $L_2([t_1,t_2];H)$, that is,

$$\rho_{t_1,t_2}(u,v) = \left(\int_{t_1}^{t_2} \|u(s)-v(s)\|^2 ds\right)^{1/2}, \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

These metrics generates the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$. Recall that a sequence $\{v_n\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $n \to \infty$ if $\|v_n(\cdot) - v(\cdot)\|_{L_2([0,M];H)} \to 0$ $(n \to \infty)$ for each $M > 0$. Recall that the topology Θ_+^{loc} is metrizable (see (7)) and the corresponding metric space is complete.

To define bounded sets in $\mathcal{K}_\varepsilon^+$, we shall use the Banach space

$$\mathcal{F}_+^b = L_2^b(\mathbb{R}_+; H^1) \cap L_\infty(\mathbb{R}_+; H) \cap \{v \mid \partial_t v \in L_{4/3}^b(\mathbb{R}_+; H^{-1})\} \tag{24}$$

which is clearly a subspace of $\mathcal{F}_+^{loc}$. Recall that

$$\|v\|_{L_p^b(\mathbb{R}_+;E)} = \sup_{h\in\mathbb{R}_+} \|v\|_{L_p([h,h+1];E)}.$$

We observe that the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, that is if $u(s) \in \mathcal{K}_\varepsilon^+$, then $u(h+s) \in \mathcal{K}_\varepsilon^+$ for all $h \geq 0$. Therefore,

$$S(t)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+, \quad \forall t \geq 0.$$

Proposition 3. *For every $u(s) \in \mathcal{K}_\varepsilon^+$ the following inequality holds:*

$$\|S(t)u(\,\cdot\,)\|_{\mathcal{F}_+^b} \leq C\|u(\,\cdot\,)\|^2_{L_\infty([0,1];H)} \exp(-\lambda t) + R_0, \quad \forall t \geq 0, \tag{25}$$

where λ is the first eigenvalue of the operator νL; C depends on λ and R_0 depends on λ and $\|g\|_H^2$ (see [19, 20]).

It follows from (25) that the ball $B_0 = \|v\|_{\mathcal{F}_+^b} \leq 2R_0$ is an absorbing set of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\varepsilon^+$. The set B_0 is bounded in $\mathcal{F}_+^b$. Consider the set $\mathcal{P}_\varepsilon = B_0 \cap \mathcal{K}_\varepsilon^+$. It is clear that $\mathcal{P}_\varepsilon \subset \mathcal{K}_\varepsilon^+$ is also absorbing and

$$S(t)\mathcal{P}_\varepsilon \subseteq \mathcal{P}_\varepsilon, \quad \forall t \geq 0. \tag{26}$$

Using Lemma 3.8 and reasoning from [20] we obtain

Proposition 4. *The set $\mathcal{P}_\varepsilon \subset \mathcal{K}_\varepsilon^+$ is compact in the topology Θ_+^{loc} (see also [19, 20]).*

The kernel $\mathcal{K}_\varepsilon$ of equation (20) consists of all weak solutions $u(s), s \in \mathbb{R}$, that satisfy inequality (23) for every $\psi(s) \in C_0^\infty(\mathbb{R}), \psi \geq 0$ and that are bounded in the space

$$\mathcal{F}^b = L_2^b(\mathbb{R}; H^1) \cap L_\infty(\mathbb{R}; H) \cap \{v \mid \partial_t v \in L_{4/3}^b(\mathbb{R}; H^{-1})\}.$$

By Propositions 3 and 4, Theorems 3.3 and 3.5 are applicable.

Theorem 4.2. *The system (20) has the trajectory attractor $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_+^{loc} = L_2^{loc}(\mathbb{R}_+; H)$. The set $\mathfrak{A}_\varepsilon$ is almost surely uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon, \tag{27}$$

the kernel $\mathcal{K}_\varepsilon$ is non-empty, uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$, and compact in Θ^{loc}.

We note that Lemma 3.8 implies that

$$B_0 \Subset L_2^{loc}(\mathbb{R}_+; H^{1-\delta}), \tag{28}$$

$$B_0 \Subset C^{loc}(\mathbb{R}_+; H^{-\delta}), \quad 0 < \delta \leq 1. \tag{29}$$

Inclusion (28) follows from (18) where we set $E_0 = H^{-1}$, $E = H^{1-\delta}$, $E_1 = H^1$, and $p_1 = 2$, $p_0 = 4/3$. Inclusion (29) follows from (19) and from the embeddings $H \Subset H^{-\delta} \subset H^{-1}$, if we set $E_0 = H^{-1}$, $E = H^{-\delta}$, $E_1 = H$, and $p_0 = 4/3$.

Using compact inclusions (28) and (29), we can strengthen the attraction to the constructed trajectory attractor (27).

Corollary 1. *For any set $B \subset \mathcal{K}_\varepsilon^+$ bounded in $\mathcal{F}_+^b$ we have almost surely*

$$\operatorname{dist}_{L_2([0,M];H^{1-\delta})}\left(\Pi_{0,M}S(t)B, \Pi_{0,M}\mathcal{K}_\varepsilon\right) \to 0 \ (t \to \infty),$$
$$\operatorname{dist}_{C([0,M];H^{-\delta})}\left(\Pi_{0,M}S(t)B, \Pi_{0,M}\mathcal{K}_\varepsilon\right) \to 0 \ (t \to \infty),$$

where M is an arbitrary positive number.

Along with equation (20) we consider the averaged equation

$$\partial_t u_0 + \nu L u_0 + B(u_0) = g^{hom}(x), \quad \operatorname{div} u_0 = 0, \quad u_0|_{\partial D} = 0. \tag{30}$$

Clearly equation (30) also has the trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the equation (30) (see Definition 4.1) and

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}, \tag{31}$$

where $\overline{\mathcal{K}}$ is the kernel of equation (30) in $\mathcal{F}^b$. Let us formulate the first main theorem concerning the autonomous 3D Navier-Stokes system.

Theorem 4.3. *The following limit holds almost surely in the topological space Θ_+^{loc} :*

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \ \text{ as } \varepsilon \to 0. \tag{32}$$

Moreover, almost surely

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \ \text{ as } \varepsilon \to 0 \text{ in } \Theta^{loc}. \tag{33}$$

Proof. It is clear that (33) implies (32). Therefore it is sufficient to prove (33), that is, for every neighborhood $\mathcal{O}\left(\overline{\mathcal{K}}\right)$ in Θ^{loc} there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that almost surely

$$\mathcal{K}_\varepsilon \subset \mathcal{O}\left(\overline{\mathcal{K}}\right) \quad \text{for } \varepsilon < \varepsilon_1. \tag{34}$$

Suppose that (34) is not true. Consider the corresponding subset $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and (34) does not hold for all $\omega \in \Omega'$. Then, for each $\omega \in \Omega'$, there exists a neighborhood $\mathcal{O}'\left(\overline{\mathcal{K}}\right)$ in Θ^{loc}, a sequence $\varepsilon_n \to 0$ $(n \to \infty)$, and a sequence $u_{\varepsilon_n}(\cdot) = u_{\varepsilon_n}(\omega, s) \in \mathcal{K}_{\varepsilon_n}^+$ such that

$$u_{\varepsilon_n} \notin \mathcal{O}'\left(\overline{\mathcal{K}}\right), \quad \forall n \in \mathbb{N}, \ \omega \in \Omega'. \tag{35}$$

It follows from the condition (21) that the sequence $\left\{g\left(x, \frac{x}{\varepsilon_n}, \omega\right)\right\}$ is almost surely bounded in H. Therefore from Proposition 3 we conclude that the sequence $\{u_{\varepsilon_n}\}$ is also bounded in $\mathcal{F}^b$ for each $\omega \in \Omega'$. Passing to a subsequence we can assume that

$$u_{\varepsilon_n} \to u_0 \ (n \to \infty) \ \text{ in } \Theta^{loc},$$

where $u_0 = u_0(\omega)$. We claim that $u_0 \in \overline{\mathcal{K}}$. For each $\omega \in \Omega'$, the functions $u_{\varepsilon_n}(x,s)$ satisfy the equation

$$\partial_t u_{\varepsilon_n} + \nu L u_{\varepsilon_n} + B(u_{\varepsilon_n}) = g\left(x, \frac{x}{\varepsilon_n}, \omega\right), \quad t \in \mathbb{R} \tag{36}$$

and the inequality

$$\begin{aligned} &-\frac{1}{2}\int_{-M}^{M} \|u_{\varepsilon_n}(s)\|^2 \psi'(s)ds + \nu \int_{-M}^{M} \|u_{\varepsilon_n}(s)\|_1^2 \psi(s)ds \\ &\leq \int_{-M}^{M} \left(g\left(x, \frac{x}{\varepsilon_n}, \omega\right), u_{\varepsilon_n}(s)\right) \psi(s)ds \end{aligned} \tag{37}$$

for any $M > 0$ and for every function $\psi \in C_0^\infty(]-M,M[)$, $\psi \geq 0$. Moreover, $u_{\varepsilon_n}(s) \rightharpoonup u_0(s)$ $(n \to \infty)$ weakly in $L_2([-M,M];H^1)$, $*$-weakly in $L_\infty([-M,M];H)$, and $\partial_t u_{\varepsilon_n}(s) \rightharpoonup \partial_t u_0(s)$ $(n \to \infty)$ weakly in $L_{4/3}([-M,M];H^{-1})$. By Lemma 3.8 (see inclusion (28) with $\delta = 0$) we can assume that $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $L_2([-M,M];H)$ and $u_{\varepsilon_n}(x,s) \to \bar{u}(x,s)$ $(n \to \infty)$ for almost all $(x,s) \in D \times]-M,M[$. In particular, $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ strongly in $\Theta_+^{loc} = L_2^{loc}(\mathbb{R};H)$.

In view of (21) we have $g\left(x, \frac{x}{\varepsilon_n}, \omega\right) \rightharpoonup g^{hom}(x)$ $(n \to \infty)$ weakly in H_w and therefore weakly in $L_2([-M,M];H)$. We now pass to the limit in (36) and (37) using the standard reasoning from [33] (see the complete proof in [19, 20, 39]). Hence $u_0 \in \overline{\mathcal{K}}$, i.e., u_0 is the solution to (30) that satisfies the corresponding inequality (37) for the external force $g^{hom}(x)$. At the same time we have established that, for each $\omega \in \Omega'$, the sequence $u_{\varepsilon_n}(s) \to u_0(s)$ $(n \to \infty)$ in Θ_+^{loc} and therefore $u_{\varepsilon_n}(s) \in \mathcal{O}'(u_0(s)) \subset \mathcal{O}'\left(\overline{\mathcal{K}}\right)$ for $\varepsilon_n \ll 1$ and for all $\omega \in \Omega'$. This contradicts (35). The proof is complete. □

Applying the embeddings (28) and (29) we obtain

Corollary 2. *For every $0 < \delta \leq 1$ and for any $M > 0$ almost surely*

$$\begin{aligned} \operatorname{dist}_{L_2([0,M];H^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0), \\ \operatorname{dist}_{C([0,M];H^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{38}$$

We now briefly consider the 3D Navier–Stokes system with random time-dependent external force which oscillates rapidly in time. The system has the form

$$\partial_t u + \nu L u + B(u) = g\left(x, \frac{t}{\varepsilon}, \omega\right), \quad \operatorname{div} u = 0, \quad u|_{\partial D} = 0, \tag{39}$$

We assume that $g(x,\tau,\omega), x \in D, t \in \mathbb{R}$, is a statistically homogeneous ergodic function with smooth realizations, that is, $g(x,\tau,\omega) = \mathbf{G}(x, T_\tau \omega)$, where $\mathbf{G} : D \times \Omega$ is measurable. We also assume that, with probability one (in $\omega \in \Omega$) the function $g(\cdot,\tau,\omega)$ is translation compact in $L_2^{loc}(\mathbb{R}_\tau;H)$, that is, the hull

$$\mathcal{H}(g(\cdot,\cdot,\omega)) := [\{g(\cdot,\tau+h,\omega) \mid h \in \mathbb{R}\}]_{L_2^{loc}(\mathbb{R}_\tau;H)}$$

is compact in $L_2^{loc}(\mathbb{R}_\tau;H)$. It is easy to prove that the function $g\left(\cdot, \frac{t}{\varepsilon}, \omega\right)$ is almost surely translation compact in $L_2^{loc}(\mathbb{R}_t;H)$ for every $\varepsilon > 0$.

Further we assume that the function $g(\cdot,\tau,\omega)$ has almost surely the uniform mean value in time in $L_{2,w}^{loc}(\mathbb{R}_\tau;H)$ (the definition is similar to Definition 2.5). Precisely, we assume that $g(\tau,\omega)$ has almost surely the uniform average, that is,

$$\frac{1}{\mu}\int_h^{h+\mu} g(\cdot,\tau,\omega)d\tau = \frac{1}{\mu}\int_h^{h+\mu} \mathbf{G}(\cdot,T_\tau\omega)d\tau \longrightarrow \mathbb{E}(\mathbf{G}) := g^{hom}\ (\mu\to\infty) \text{ in } H \quad (40)$$

uniformly with respect to $h\in\mathbb{R}$. Here $g^{hom}(\cdot)\in H$. Then, we prove that the function $g^\varepsilon(t)\equiv g\left(\frac{t}{\varepsilon},\omega\right)$ has the uniform average g^{hom} in the local weak topology $L_{2,w}^{loc}(\mathbb{R}_t;H)$ as $\varepsilon\to 0$, that is for each $M>0$ and every function $\varphi\in L_2([0,M];H)$

$$\int_0^M \left\langle g\left(\frac{t+h}{\varepsilon},\omega\right),\varphi(t)\right\rangle dt \to \int_0^M \left\langle g^{hom},\varphi(t)\right\rangle dt \quad (\varepsilon\to 0+) \quad (41)$$

uniformly with respect to $h\in\mathbb{R}$.

Remark 6. Here we used the Birkhoff Theorem 2.6, bearing in mind Remark 1.

Consider the hull $\mathcal{H}(g^\varepsilon(\cdot))$ of the function $g^\varepsilon = g\left(\cdot,\frac{t}{\varepsilon},\omega\right)$ in $L_2^{loc}(\mathbb{R}_t;H)$. Then, using (41) we prove that almost surely

$$\mathcal{H}(g^\varepsilon(\cdot))\to g^{hom} \quad (\varepsilon\to 0+) \quad (42)$$

in the local weak topology $L_{2,w}^{loc}(\mathbb{R}_t;H)$ (see [20] for more details).

Equation (39) has the time symbol $g^\varepsilon = g\left(x,\frac{t}{\varepsilon},\omega\right)$. Along with equation (39) we consider the entire family of this equations

$$\partial_t u + \nu L u + B(u) = \hat{g}(x,t,\omega), \quad \operatorname{div} u = 0, \quad u|_{\partial D} = 0, \quad (43)$$

with symbols $\hat{g}(\cdot,\cdot,\omega)\in\Sigma^\varepsilon := \mathcal{H}(g^\varepsilon)$. Here Σ^ε is the symbol space of the non-autonomous equation (43).

Similar to autonomous case, we define the trajectory space $\mathcal{K}_{\hat{g}}^+$ of the equation (43) with symbol $\hat{g}\in\Sigma^\varepsilon$. Recall that $\mathcal{K}_{\hat{g}}^+$ consists of all weak solutions $u(s)$ of the non-autonomous equation (43) from the space $L_2^{loc}(\mathbb{R}_+;H^1)\cap L_\infty^{loc}(\mathbb{R}_+;H)$ that satisfies the energy inequality (22), where we replace g by $\hat{g}(s)$. It is clear that $\mathcal{K}_{\hat{g}}^+$ satisfy (15), that is

$$S(t)\mathcal{K}_{\hat{g}}^+ \subseteq \mathcal{K}_{S(t)\hat{g}}^+, \quad \forall t\ge 0.$$

Consider now the aggregate trajectory space for equation (39):

$$\mathcal{K}_{\Sigma^\varepsilon}^+ = \bigcup_{\hat{g}\in\Sigma^\varepsilon}\mathcal{K}_{\hat{g}}^+,$$

which is invariant with respect to the translation semigroup $\{S(t)\}$:

$$S(t)\mathcal{K}_{\Sigma^\varepsilon}^+ \subseteq \mathcal{K}_{\Sigma^\varepsilon}^+, \quad \forall t\ge 0.$$

Proposition 3 holds true for all $u\in\mathcal{K}_{\Sigma^\varepsilon}^+$. Moreover, the constants C and R_0 are independent of ε (see [19, 20]). Therefore, the ball $B_0 = \|v\|_{\mathcal{F}_+^b}\le 2R_0$ is an absorbing set of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_{\Sigma^\varepsilon}^+$. The set $\mathcal{P}_\varepsilon = B_0\cap\mathcal{K}_{\Sigma^\varepsilon}^+$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. It is clear that $\mathcal{P}_\varepsilon\subset\mathcal{K}_\varepsilon^+$ is also absorbing and

$$S(t)\mathcal{P}_\varepsilon\subseteq\mathcal{P}_\varepsilon, \quad \forall t\ge 0.$$

Similar to Proposition 4 we prove that the set $\mathcal{P}_\varepsilon\subset\mathcal{K}_{\Sigma^\varepsilon}^+$ is compact in the topology Θ_+^{loc} (see [19, 20]).

We also define the kernel of equation (43) with symbol $\hat{g}\in\Sigma^\varepsilon$ that consists of all weak solutions $u(s), s\in\mathbb{R}$ of this equation that satisfy inequality (22) (with $g=\hat{g}$) and that are bounded in the space $\mathcal{F}^b$ (see (24)).

We now apply Theorem 3.7 and obtain the following

Theorem 4.4. *The system* (39) *has the uniform (w.r.t. $\hat{g} \in \Sigma^\varepsilon = \mathcal{H}(g^\varepsilon)$) trajectory attractor $\mathfrak{A}_{\Sigma^\varepsilon}$ in the topological space $\Theta_+^{loc} = L_2^{loc}(\mathbb{R}_+; H)$. The set $\mathfrak{A}_{\Sigma^\varepsilon}$ is almost surely uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_{\Sigma^\varepsilon} = \Pi_+ \bigcup_{\hat{g} \in \Sigma^\varepsilon} \mathcal{K}_{\hat{g}},$$

the kernel $\mathcal{K}_{\hat{g}}$ is non-empty for every $\hat{g} \in \Sigma^\varepsilon$ and the set $\mathfrak{A}_{\Sigma^\varepsilon}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$ uniformly with respect to $\varepsilon \in (0,1]$.

Finally, we consider the averaged equation (30) with averaged external force g^{hom} (see (40) and (41)). This equation is autonomous. Consider its trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+ := \mathcal{K}^+_{g^{\text{hom}}}$ that we have constructed in the first, "autonomous", part of this section.

We have the second main result for the non-autonomous 3D NS system (39).

Theorem 4.5. *The uniform trajectory attractor $\mathfrak{A}_{\Sigma^\varepsilon}$ of the equation* (39) *converges almost surely to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized autonomous equation* (30) *as $\varepsilon \to 0$ in the space Θ_+^{loc}.*

The proof of this theorem is analogous to the proof of Theorem 4.3. We use a generalization of Propositions 3, 4 and apply the limit relation (42).

Using compact inclusions (28) and (29) we obtain

Corollary 3. *For every $\delta > 0$, almost surely*

$$\text{dist}_{L_2([0,M];H^{1-\delta})} \left(\Pi_{0,M}\mathfrak{A}_{\Sigma^\varepsilon}, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0),$$
$$\text{dist}_{C([0,M];H^{-\delta})} \left(\Pi_{0,M}\mathfrak{A}_{\Sigma^\varepsilon}, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0), \quad \forall M > 0.$$

Remark 7. We can also consider in analogous way rapidly oscillating functions random in time and locally periodic in space variables as well as locally periodic in time and random in space variables or random in both space and time variables.

Acknowledgments. The text partially was written in Lausanne during March 2015. GAC thanks the Bernoulli Center of the École Polytechnique Fédérale de Lausanne for wonderful conditions to work.

REFERENCES

[1] Y. Amirat, O. Bodart, G. A. Chechkin and A. L. Piatnitski, Boundary homogenization in domains with randomly oscillating boundary, *Stochastic Processes and their Applications*, **121** (2011), 1–23.

[2] V. I. Arnol'd and A. Avez, *Ergodic Problems of Classical Mechanics*, W.A. Benjamin Inc., New York – Amsterdam, 1968.

[3] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland Publishing Co., Amsterdam, 1992.

[4] N. S. Bakhvalov and G. P. Panasenko, *Averaging Processes in Periodic Media*, Mathematics and its Applications (Soviet Series), **36**. Kluwer Academic Publishers Group, Dordrecht, 1989.

[5] K. A. Bekmaganbetov, G. A. Chechkin and V. V. Chepyzhov, Homogenization of Random Attractors for Reaction–Diffusion Systems, *C R Mécanique*, **344** (2016), 753–758.

[6] A. Bensoussan, J.-L. Lions and G. Papanicolau, *Asymptotic Analysis for Periodic Structures*, Corrected reprint of the 1978 original [MR0503330]. AMS Chelsea Publishing, Providence, RI, 2011.

[7] G. D. Birkhoff, Proof of the ergodic theorem, *Proc Natl Acad Sci USA*, **17** (1931), 656–660.

[8] N. N. Bogolyubov and Ya. A. Mitropolski, *Asymptotic Methods in the Theory of Non-Linear Oscillations*, International Monographs on Advanced Mathematics and Physics, Hindustan Publishing Corp., Delhi, Gordon & Breach Science Publishers, New York, 1961.

[9] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, Applied Mathematical Sciences, **183**. Springer, New York, 2013.

[10] G. A. Chechkin, A. L. Piatnitski and A. S. Shamaev, *Homogenization. Methods and Applications*, Translated from the 2007 Russian original by Tamara Rozhkovskaya. Translations of Mathematical Monographs, **234**. American Mathematical Society, Providence, RI, 2007.

[11] G. A. Chechkin, T. P. Chechkina, C. D'Apice and U. De Maio, Homogenization in Domains Randomly Perforated Along the Boundary, *Discrete Continuous Dynam. Systems - B*, **12** (2009), 713–730.

[12] G. A. Chechkin, T. P. Chechkina, C. D'Apice, U. De Maio and T. A. Mel'nyk, Asymptotic Analysis of a Boundary Value Problem in a Cascade Thick Junction with a Random Transmission Zone, *Applicable Analysis*, **88** (2009), 1543–1562.

[13] G. A. Chechkin, T. P. Chechkina, C. D'Apice, U. De Maio and T. A. Mel'nyk, Homogenization of 3D Thick Cascade Junction with the Random Transmission Zone Periodic in One direction, *Russ. J. Math. Phys.*, **17** (2010), 35–55.

[14] G. A. Chechkin, C. D'Apice, U. De Maio and A. L. Piatnitski, On the Rate of Convergence of Solutions in Domain with Random Multilevel Oscillating Boundary, *Asymptotic Analysis*, **87** (2014), 1–28.

[15] G. A. Chechkin, T. P. Chechkina, T. S. Ratiu and M. S. Romanov, Nematodynamics and Random Homogenization, *Applicable Analysis*, **95** (2016), 2243–2253.

[16] V. V. Chepyzhov, A. Yu. Goritski and M. I. Vishik, Integral manifolds and attractors with exponential rate for nonautonomous hyperbolic equations with dissipation, *Russ. J. Math. Phys.*, **12** (2005), 17–39.

[17] V. V. Chepyzhov, V. Pata and M. I. Vishik, Averaging of nonautonomous damped wave equations with singularly oscillating external forces, *J. Math. Pures Appl.*, **90** (2008), 469—491.

[18] V. V. Chepyzhov, V. Pata and M. I. Vishik, Averaging of 2D Navier-Stokes equations with singularly oscillating forces, *Nonlinearity*, **22** (2009), 351–370.

[19] V. V. Chepyzhov and M. I. Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.*, **76** (1997), 913–964.

[20] [0-8218-2950-5] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications, 49. American Mathematical Society, Providence, RI, 2002.

[21] V. V. Chepyzhov and M. I. Vishik, Non-autonomous 2D Navier-Stokes system with a simple global attractor and some averaging problems, *ESAIM Control Optim. Calc. Var.*, **8** (2002), 467–487.

[22] V. V. Chepyzhov and M. I. Vishik, Global attractors for non-autonomous Ginzburg-Landau equation with singularly oscillating terms, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.*, **29** (2005), 123–148.

[23] V. V. Chepyzhov and M. I. Vishik, Non-autonomous 2D Navier-Stokes system with singularly oscillating external force and its global attractor, *J. Dynam. Differential Equations*, **19** (2007), 655–684.

[24] V. V. Chepyzhov, M. I. Vishik and W. L. Wendland, On non-autonomous sine-Gordon type equations with a simple global attractor and some averaging, *Disc. Contin. Dyn. Syst.*, **12** (2005), 27–38.

[25] M. Efendiev and S. Zelik, Attractors of the reaction-diffusion systems with rapidly oscillating coefficients and their homogenization, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **19** (2002), 961–989.

[26] M. Efendiev and S. Zelik, The regular attractor for the reaction-diffusion system with a nonlinearity rapidly oscillating in time and its averaging, *Adv. Diff. Eq.*, **8** (2003), 673–732.

[27] B. Fiedler and M. I. Vishik, Quantitative homogenization of analytic semigroups and reaction-diffusion equations with Diophantine spatial sequences, *Adv. Diff. Eq.*, **6** (2001), 1377–1408.

[28] B. Fiedler and M. I. Vishik, Quantitative homogenization of global attractors for reaction–diffusion systems with rapidly oscillating terms, *Asymptotic Anal.*, **34** (2003), 159–185.

[29] J. K. Hale and S. M. Verduyn Lunel, Averaging in infinite dimensions, *J. Int. Eq. Appl.*, **2** (1990), 463–494.

[30] A. A. Ilyin, Averaging principle for dissipative dynamical systems with rapidly oscillating right-hand sides, *Sb. Math.*, **187** (1996), 635—677.
[31] A. A. Ilyin, Global averaging of dissipative dynamical systems, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.*, **22** (1998), 165–191.
[32] V. V. Jikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer–Verlag, Berlin, 1994.
[33] J.-L. Lions, *Quelques Méthodes de Résolutions Des Problèmes Aux Limites non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
[34] V. A. Marchenko and E. Ya. Khruslov, *Kraevye Zadachi v Oblastyakh s Melkozernistoi Granitsei*, (Russian) [Boundary value problems in domains with a fine-grained boundary] Naukova Dumka, Kiev, 1974.
[35] V. A. Marchenko and E. Ya. Khruslov, *Homogenization of Partial Differential Equations*, Progress in Mathematical Physics, **46**. Birkhäuser Boston, Inc., Boston, MA, 2006.
[36] L. S. Pankratov and I. D. Cheushov, Averaging of attractors of nonlinear hyperbolic equations with asymptotically degenerate coefficients, *Sb. Math.*, **190** (1999), 1325–1352.
[37] E. Sánchez-Palencia, *Homogenization Techniques for Composite Media*, Lecture Notes in Physics, **272**. Springer-Verlag, Berlin, 1987.
[38] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2^{nd} edition. Applied Mathematical Sciences, **68**. Springer-Verlag, New York, 1997.
[39] M. I. Vishik and V. V. Chepyzhov, Averaging of trajectory attractors of evolution equations with rapidly oscillating terms, *Sb. Math.*, **192** (2001), 11–47.
[40] M. I. Vishik and V. V. Chepyzhov, Approximation of trajectories lying on a global attractor of a hyperbolic equation with an exterior force that oscillates rapidly over time, *Sb. Math.*, **194** (2003), 1273–1300.
[41] M. I. Vishik and V. V. Chepyzhov, Attractors of dissipative hyperbolic equations with singularly oscillating external forces, *Math. Notes*, **79** (2006), 483–504.
[42] M. I. Vishik and V. V. Chepyzhov, Trajectory attractors of equations of mathematical physics, *Russian Math. Surveys*, **66** (2011), 637–731.
[43] M. I. Vishik and B. Fiedler, Quantitative averaging of global attractors of hyperbolic wave equations with rapidly oscillating coefficients, *Russian Math. Surveys*, **57** (2002), 709–728.
[44] S. Zelik, Global averaging and parametric resonances in damped semilinear wave equations, *Proc. Roy. Soc. Edinburgh Sect.A*, **136** (2006), 1053–1097.

Received January 2016; revised December 2016.

E-mail address: bekmaganbetov-ka@yandex.kz
E-mail address: chechkin@mech.math.msu.su
E-mail address: chep@iitp.ru
E-mail address: goritsky@mech.math.msu.su