Homogenization of random trajectory attractors for reaction–diffusion systems

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In this work we study asymptotic behavior of trajectory attractors of autonomous reaction–diffusion systems with randomly oscillating terms. To study such phenomenon we apply the homogenization method (cf., for example, [1], [2], [3], for random case cf., for instance, [4]) as well as a delicate analysis of attractors for dissipative partial differential equations (see, for example, [5], [6], [7] and the references therein).

Assume that $(\Omega, \mathcal{B}, \mu)$ is a probability space, i.e., the set Ω is endowed with a σ-algebra $\mathcal{B}$ of its subsets and a σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

We consider reaction-diffusion systems with randomly oscillating terms of the form

$$\partial_t u = a\Delta u - b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u) + g\left(x, \frac{x}{\varepsilon}, \omega\right),\ u|_{\partial D} = 0, \tag{1}$$

where $x \in D \Subset \mathbb{R}^n,\ u = (u^1, \ldots, u^N),\ f = (f^1, \ldots, f^N)$, and $g = (g^1, \ldots, g^N)$. Here a is an $N \times N$ matrix with positive symmetric part and $b(x, z, \omega) \in C(D \times \mathbb{R}^N \times \Omega)$ is a real positive function. For the simplicity we assume that the vector function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfies the following inequalities:

$$f(v) \cdot v \geq \gamma |v|^p - C,\ |f(v)| \leq C_1 \left(|v|^{p-1} + 1\right),\ p \geq 2. \tag{2}$$

Notice that we *do not assume* that the function $f(v)$ satisfies Lipschitz-type condition with respect to v. This means that the uniqueness theorem for the Cauchy problem of system (1) may not hold.

Assume that $T_\xi, \xi \in \mathbb{R}^n$, is an *ergodic dynamical system* on Ω. Let the function $b(x, \frac{x}{\varepsilon}, \omega)$ and vector function $g(x, \frac{x}{\varepsilon}, \omega)$ be *statistically homogeneous*, i.e. $b(x, \xi, \omega) =$

$\mathbf{B}(x,T_\xi\omega)$ and $g(x,\xi,\omega)=\mathbf{G}(x,T_\xi\omega)$, where $\mathbf{B}:D\times\Omega\to\mathbb{R}$ and $\mathbf{G}:D\times\Omega\to\mathbb{R}^N$ is measurable. We also assume that $b(x,z,\omega)\in C_b(\overline{D}\times\mathbb{R}\times\Omega)$ and

$$\beta_1\geq b(x,z,\omega)\geq\beta_0>0,\ \forall x\in D,\ z\in\mathbb{R}^n,\ \omega\in\Omega, \tag{3}$$

and the function $b\left(x,\frac{x}{\varepsilon},\omega\right)$ has the average $b^{hom}(x)=\mathbb{E}(\mathbf{B})(x)$ as $\varepsilon\to 0+$ in $L_{\infty,*w}(D)$, that is, almost surely

$$\int\limits_D b\left(x,\frac{x}{\varepsilon},\omega\right)\varphi(x)dx\to\int\limits_D b^{hom}\left(x\right)\varphi(x)dx\ (\varepsilon\to 0+),\ \forall\ \varphi\in L_1(D). \tag{4}$$

For the vector function $g\left(x,\frac{x}{\varepsilon},\omega\right)$ we also assume that it has the average $g^{hom}(x)=\mathbb{E}(\mathbf{G})(x)$ in the space $V'=\left(H^{-1}(D)\right)^N$, that is, almost surely

$$\left\langle g\left(x,\frac{x}{\varepsilon},\omega\right),\varphi(x)\right\rangle\to\left\langle g^{hom}\left(x\right),\varphi(x)\right\rangle\ (\varepsilon\to 0+),\ \forall\ \varphi\in V=\left(H_0^1(D)\right)^N. \tag{5}$$

Denote the space $H=\left(L_2(D)\right)^N$. We consider weak solutions (trajectories) of the system (1), that is, the functions $u(x,t)\in L_\infty^{loc}(\mathbb{R}_+;H)\cap L_2^{loc}(\mathbb{R}_+;V)\cap L_p^{loc}\left(\mathbb{R}_+;(L_p(D))^N\right)$ which satisfy (1) in the sense of distributions. We denote by $\mathcal{K}_\varepsilon^+$ the set of all weak solutions of the system (1). Consider the *translation semigroup* $\{T(h)\}$ acting on the *trajectory space* $\mathcal{K}_\varepsilon^+$ by the formula $T(h)u(x,t)=u(x,t+h)$ for $h\geq 0$.

We study the (strong) *trajectory attractor* $\mathfrak{A}_\varepsilon$ of the system (1) that, by definition, coincides with the global $\left(\mathcal{F}_+^b,\Theta_+^{s,loc}\right)$-attractor of the translation semigroup $\{T(h)\}$ acting on $\mathcal{K}_\varepsilon^+$ (see [5], [6], [7]). Here, $\Theta_+^{s,loc}$ denotes the local *STRONG* topology, which is determined by the local strong convergence of sequences $\{v_m\}$ and $\{\partial_t v_m\}$ in the corresponding spaces. The trajectory space $\mathcal{K}_\varepsilon^+$ is supplied with topology $\Theta_+^{s,loc}$. The Banach space $\mathcal{F}_+^b$ is used to define bounded sets in $\mathcal{K}_\varepsilon^+$.

Along with the random system (1) we consider the averaged deterministic system

$$\partial_t\bar{u}=a\Delta\bar{u}-b^{hom}\left(x\right)f(\bar{u})+g^{hom}\left(x\right),\ \bar{u}|_{\partial D}=0. \tag{6}$$

System (6) also has the strong trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the system (6).

Theorem. *The following limit holds almost surely in the local strong topology* $\Theta_+^{s,loc}$

$$\mathfrak{A}_\varepsilon\to\overline{\mathfrak{A}}\ \text{ as } \varepsilon\to 0+\,. \tag{7}$$

Let $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y)$ denote the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Corollary. *For any $M > 0$ we have almost surelly in Ω*

$$\begin{aligned}
\mathrm{dist}_{L_2([0,M];H^1)} \left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) &\to 0 \ (\varepsilon \to 0+), \\
\mathrm{dist}_{C([0,M];H)} \left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) &\to 0 \ (\varepsilon \to 0+).
\end{aligned}$$

Remark. Analogous theorem holds for random non-autonomous reaction-diffusion systems of the form (1) which contain the terms $b\left(x, \frac{t}{\varepsilon}, t, \omega\right)$ *and* $g\left(x, \frac{t}{\varepsilon}, t, \omega\right)$ *having the uniform averages in time as* $\varepsilon \to 0+$.

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