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HOMOGENIZATION OF TRAJECTORY ATTRACTORS OF GINZBURG-LANDAU EQUATIONS WITH RANDOMLY OSCILLATING TERMS

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To the blessed memory of I. D. Chueshov

Abstract. We consider complex Ginzburg-Landau (GL) type equations of the form:

$$\partial_t u = (1+\alpha i)\Delta u + R\,u + (1+\beta i)\,|u|^2 u + g,$$

where R, β, and g are random rapidly oscillating real functions. Assuming that the random functions are ergodic and statistically homogeneous in space variables, we prove that the trajectory attractors of these systems tend to the trajectory attractors of the homogenized equations whose terms are the average of the corresponding terms of the initial systems.

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1. **Introduction.** The phenomenological theory of superconductivity was formulated at the mid of 1950-th by Ginzburg V.L., Landau L.D., Abrikosov A.A., and Gor'kov L.P. Nowadays it is known as the *GLAG-theory*. Varying the free energy functional and taking into account the special form of the Lagrangian, one comes to the stationary Ginzburg–Landau (GL) equation:

$$\gamma u+\delta|u|^2u+\frac{1}{2m}\big(-i\hbar\nabla-2e\,\mathbf{A}\big)^2u=0,$$

where u is the complex field of electronic Cooper pairs; γ,δ are empirical constants; m is the mass of an electron; $i=\sqrt{-1}$, $\hbar$ is the Plank constant; e is the elementary charge (an electron charge); $\mathbf{A}$ is the electro-magnetic potential.

In this paper we consider the evolution complex Ginzburg–Landau equation with inhomogeneous terms of the form:

$$\partial_t u=(1+\alpha i)\Delta u+R\,u+(1+\beta i)\,|u|^2u+g. \tag{1}$$

Assuming α to be a fixed real number and $R=R\left(x,\frac{x}{\varepsilon},\omega\right)$, $\beta=\beta\left(x,\frac{x}{\varepsilon},\omega\right)$, and $g=g\left(x,\frac{x}{\varepsilon},\omega\right)$ to be random rapidly oscillating real functions, we construct the homogenized equation and prove the convergence of trajectory attractors of the initial equation to the trajectory attractors of the homogenized one. We refer here to homogenization methods (cf., for example, [4, 7, 14, 38, 42, 45]), which are basic for studying mathematical problems in micro–nonhomogeneous media. These methods enable to consider media with periodic microstructure as well as with random one (for random case cf., for instance, [1, 15, 16, 17, 18, 19]).

Notice that the GL equation (1) is a particular case of the reaction-diffusion equations of the form:

$$\partial_t u=A\Delta u+B\left(x,\frac{x}{\varepsilon},\omega\right)f(u)+g\left(x,\frac{x}{\varepsilon},\omega\right), \tag{2}$$

where $u=(u^1,\dots,u^N)$, $f=(f^1,\dots,f^N)$, and $g=(g^1,\dots,g^N)$ are real vector functions. Here A is a fixed $N\times N$ matrix with positive symmetric part, $B\left(x,\frac{x}{\varepsilon},\omega\right)$ and $g\left(x,\frac{x}{\varepsilon},\omega\right)$ are random rapidly oscillating functions. Usually one assumes that the vector function $f(v)\in C(\mathbb{R}^N;\mathbb{R}^N)$ satisfies the following inequalities:

$$f(v)\cdot v\geqslant\gamma|v|^p-C,\ |f(v)|\leqslant C_1\left(|v|^{p-1}+1\right)\quad(p\geqslant 2). \tag{3}$$

This condition can be replaced by the following more general inequalities with different degrees $\mathbf{p}=(p_1,\dots,p_2)$ of the form

$$f(v)\cdot v\geqslant\gamma\sum_{k=1}^N|v^k|^{p_k}-C,$$
$$\sum_{k=1}^N|f^k(v)|^{\frac{p_k}{p_k-1}}\leqslant C_1\left(\sum_{k=1}^N|v^k|^{p_k}+1\right)\quad(p_k\geqslant 2,\forall v\in\mathbb{R}^N).$$

Let us also notice that the first results on homogenization of Ginzburg–Landau and Ginzburg–Landau heat flow equations were obtained in [39, 44] by Khruslov's method of local energy characteristics (see, e.g., [42]). In particular, in paper [39], the well known Josephson effect was rigorously justified.

In this paper we study the attractors (see Fig. 1 [1] for example) of differential equations that model the long time behavior of superconductive media with complicated microstructure.

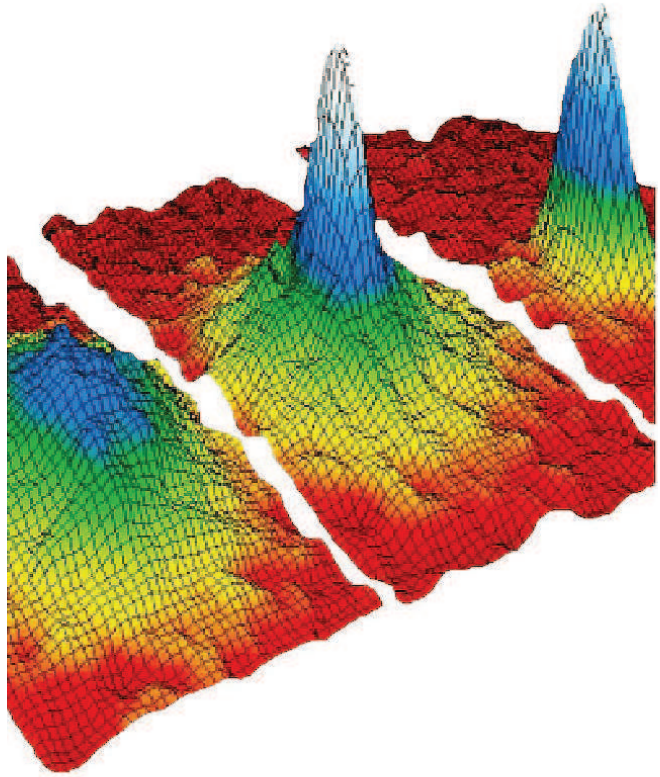

FIGURE 1. Attractors of the Ginzburg–Landau Equations

The attractors describe the behaviour of dissipative dynamical systems corresponding to nonlinear evolution partial differential equations when time goes to infinity. Informally speaking, the attractor is the smallest set of an infinite dimensional phase space of dissipative PDE able to attract in a suitable sense the trajectories arising from bounded regions of initial data. The attractor characterizes the dynamical system in a whole (see, e.g., the monographes [3, 24, 26, 35, 46] and the references therein). Using the attractors, it is also convenient to study global perturbations of trajectories (solutions) of evolution equations.

More precisely, our interest is the asymptotic behavior of the trajectory attractors of the Ginzburg–Landau equations (see Fig. 1) with randomly oscillating terms.

The first results related to the attractors of evolution equations with rapidly but non-randomly oscillating terms of periodic or almost periodic type (see [36, 37]) have been obtained by using the Bogolyubov averaging principle [9]. The averaging of global attractors of parabolic equations with oscillating parameters have been studied in [10, 11, 12, 24, 27, 32, 33]. Several problems related to the homogenization

[1]The image of the attractors of the Ginzburg-Landau equations was taken from the internet http://neurocomp.ru/c/kvantovaya-mexanika/reakcionno-diffuznye-sredy/

of uniform global attractors for dissipative wave equations were considered in [20, 21, 37] in the case of time oscillations and in [24, 28, 47, 50] for the case of oscillations in space. For 2D Navier–Stokes system, similar problems have been considered in [24, 47]. Averaging of the global attractors of non-autonomous Ginzburg-Landau equations with singularly oscillating terms has been studied in [25].

In the paper [29] the authors used the averaging method to study random or non-autonomous systems on a fast time scale. They applied this method to a random abstract evolution equation on a fast time scale whose long time behavior can be characterized by a random attractor or a random inertial manifold. The main purpose of the paper was to show that the long-time behavior of such a system can be described by a deterministic evolution equation with averaged coefficients (the respective mathematical expectation). The main result deals with a global averaging procedure. Under some spectral gap condition the authors showed that inertial manifolds of the fast time scale system tend to an inertial manifold of the averaged system when the scaling parameter goes to zero.

The theory of trajectory attractors for dissipative partial differential equations were developed in [23, 24] (see also [22] and the review [49]) with an emphasis on equations for which the uniqueness of a solution of the corresponding Cauchy problem is not known (e.g., for the inhomogeneous 3D Navier-Stokes system) or does not hold (e.g., for elliptic equations). For such equations, the traditional theory of global attractors (see [3, 46]) is not directly applicable. The trajectory attractors were constructed for a number of important equations and systems of mathematical physics, e.g. for the 3D N.S. system, various reaction-diffusion systems, the dissipative hyperbolic equation with arbitrary polynomial growth of the nonlinear term, and for other equations (see [24, 49]). Trajectory attractors for inhomogeneous Ginzburg-Landau equations have been constructed in [48]. Some averaging problem for trajectory attractors of evolution equation with rapidly (non-randomly) oscillating terms were studied in [24, 47]. Paper [5] deals with homogenization of trajectory attractors for autonomous and non-autonomous 3D Navier–Stokes systems with randomly oscillating external forces (see also [6] for random homogenization of reaction–diffusion systems).

In present paper we consider the Ginzburg-Landau equations under the assumption that the coefficients and the right-hand sides $R\left(x, \frac{x}{\varepsilon}, \omega\right)$, $\beta\left(x, \frac{x}{\varepsilon}, \omega\right)$, and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ of the systems are random functions which oscillate rapidly with respect to the space variables. Here ω is an element of a standard probability space $(\Omega, \mathcal{B}, \mu)$. The parameter $\varepsilon > 0$ characterizes the oscillation frequency in space variable. Along with such systems we also consider the corresponding homogenized Ginzburg-Landau equations with terms $R^{\mathrm{hom}}(x)$, $\beta^{\mathrm{hom}}(x)$, and $g^{\mathrm{hom}}(x)$ that are the mathematical expectations of $R\left(x, \frac{x}{\varepsilon}, \omega\right)$, $\beta\left(x, \frac{x}{\varepsilon}, \omega\right)$, and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$. We prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the equation with randomly oscillating term converges almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in an appropriate functional space. Under the assumption that the random function $R\left(x, \frac{x}{\varepsilon}, \omega\right)$, $\beta\left(x, \frac{x}{\varepsilon}, \omega\right)$, and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ are statistically homogeneous and ergodic with smooth realizations (for detailed definitions see below), we prove that the mathematical expectation coincides with deterministic spatial mean.

The rest of the paper is organized as follows. In Section 2 we formulate the problem and give the necessary definitions of the random spaces. In Section 3 we consider global and trajectory attractors for Ginzburg–Landau equation and formulate and generalize known theorem on these attractors. Section 4 is devoted to the homogenization of trajectory and global attractors of the Ginzburg–Landau equations with randomly rapidly oscillating terms.

2. **Preliminaries.**

We consider the Ginzburg–Landau equation in the torus $\mathbb{T}^n := (\mathbb{R} \mod 2\pi)^n$:

$$\partial_t u_\varepsilon = (1+\alpha i)\Delta u_\varepsilon + R_\varepsilon\, u_\varepsilon + (1+\beta_\varepsilon i)\, |u_\varepsilon|^2 u_\varepsilon + g_\varepsilon, \tag{4}$$

where $\varepsilon > 0$ is a small parameter; $u_\varepsilon = u_\varepsilon(x,t) := (u_\varepsilon^1, u_\varepsilon^2, \ldots, u_\varepsilon^N),\ x \in \mathbb{T}^n, t \geqslant 0$, is an unknown complex vector-field of electronic Cooper pairs; $g_\varepsilon = g_\varepsilon(x) := (g_\varepsilon^1, g_\varepsilon^2, \ldots, g_\varepsilon^N)$ is a known complex function; α is a real number; β_ε and R_ε are real scalar functions.

We study the case of autonomous equation, i.e. equation (4) with rapid space oscillations having the external force $g_\varepsilon = g\left(x, \frac{x}{\varepsilon}, \omega\right)$ and the coefficients $R_\varepsilon = R(x, \frac{x}{\varepsilon}, \omega)$, $\beta_\varepsilon = \beta(x, \frac{x}{\varepsilon}, \omega)$, where $g = g(x,\xi,\omega)$, $R = R(x,\xi,\omega)$, $\beta = \beta(x,\xi,\omega)$, $x \in \mathbb{T}^n$, $\xi \in \mathbb{R}^n$.

In what follows, for the sake of brevity, we omit the subindex ε.

We assume that R, β and g are random statistically homogeneous ergodic functions with smooth realizations, ω is an element of a standard probability space $(\Omega, \mathcal{B}, \mu)$ (for more details see the definitions below).

Randomness. Notational convention. Now we turn to the definitions and results that will be used in the paper. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, i.e., the set Ω is endowed with a σ-algebra $\mathcal{B}$ of its subsets and a σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

Definition 2.1. A family of measurable maps $\mathcal{T}_\xi : \Omega \to \Omega$, $\xi \in \mathbb{R}^n$ is called a *dynamical system* if the following properties hold:

1) *the group property*: $\mathcal{T}_{\xi_1+\xi_2} = \mathcal{T}_{\xi_1}\mathcal{T}_{\xi_2}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^n$; $\mathcal{T}_0 = Id$, where Id is the identity mapping on Ω;

2) *the isometry property* (the mappings $\mathcal{T}_\xi$ preserve the measure μ on Ω): $\mathcal{T}_\xi B \in \mathcal{B}$, $\mu(\mathcal{T}_\xi B) = \mu(B)$, $\forall \xi \in \mathbb{R}^n$, $\forall B \in \mathcal{B}$;

3) *the measurability*: for any measurable function $\psi(\omega)$ on Ω, the function $\psi(\mathcal{T}_\xi \omega)$ is measurable on $\Omega \times \mathbb{R}^n$ and continuous in ξ.

Let $L_q(\Omega, \mu)$ $(q \geqslant 1)$ be the space of measurable functions on Ω whose absolute value at the power q is integrable with respect to the measure μ. If $\mathcal{T}_\xi : \Omega \to \Omega$ is a dynamical system, then on the space $L_q(\Omega, \mu)$ we define a group of operators $\{\mathcal{T}_\xi\}$ depending on the parameter $\xi \in \mathbb{R}^n$ by the formula $(\mathcal{T}_\xi \psi)(\omega) := \psi(\mathcal{T}_\xi \omega)$, $\psi \in L_q(\Omega, \mu)$.

Note that Condition 3) in Definition 2.1 implies that the group $\mathcal{T}_\xi$ is strongly continuous, i.e.,

$$\lim_{\xi \to 0} \|\mathcal{T}_\xi \psi - \psi\|_{L_q(\Omega,\mu)} = 0$$

for any $\psi \in L_q(\Omega, \mu)$.

Definition 2.2. Suppose that $\psi(\omega)$ is a measurable function on Ω. The function $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$ $(\xi \in \mathbb{R}^n,\ \psi \in \mathbb{R})$ for a fixed $\omega \in \Omega$ is called the *realization of the function* ψ.

The following assertion is proved, for instance, in [38] (see also [14]).

Proposition 1. *If $\psi \in L_q(\Omega,\mu)$, then ω-almost all realizations $\xi \mapsto \psi(\mathcal{T}_\xi\omega)$ belong to $L_q^{loc}(\mathbb{R}^n)$. If the sequence $\{\psi_k\} \subset L_q(\Omega,\mu)$ converges in $L_q(\Omega,\mu)$ to the function ψ, then there exists a subsequence $\{\psi_{k'}\}$ such that ω-almost all realizations $\xi \mapsto \psi_{k'}(\mathcal{T}_\xi\omega)$ converge in $L_q^{loc}(\mathbb{R}^n)$ to the realization $\xi \mapsto \psi(\mathcal{T}_\xi\omega)$.*

Definition 2.3. A measurable function $\psi(\omega)$ on Ω is called *invariant*, if $\psi(\mathcal{T}_\xi\omega) = \psi(\omega)$ for any $\xi \in \mathbb{R}^n$ and almost all $\omega \in \Omega$.

Definition 2.4. The dynamical system $\mathcal{T}_\xi$ is called *ergodic*, if any invariant function is ω-almost everywhere constant.

In what follows, $\mathcal{R}$ stands for the natural Borel σ-algebra of subsets of $\mathbb{R}^n$.

Definition 2.5. Let $\digamma(\xi)$ be an arbitrary function from the space $L_1^{loc}(\mathbb{R}^n)$. We say that $\digamma(\xi)$ has a *space average*, if the limit

$$M(\digamma) := \lim_{\varepsilon\to 0} \frac{1}{|R|} \int\limits_R \cdots \int \digamma\left(\frac{x}{\varepsilon}\right)\, dx$$

exists for any bounded Borel set $R \in \mathcal{R}$ and does not depend on the choice of R. The number $M(\digamma)$ is called the SPACE MEAN VALUE of the function $\digamma$.

Equivalently, the space average is defined by

$$M(\digamma) := \lim_{s\to+\infty} \frac{1}{|R_s|} \int\limits_{R_s} \cdots \int \digamma(\xi)\, d\xi, \quad \text{where} \quad R_s = \left\{\xi \in \mathbb{R}^n \;\middle|\; \frac{\xi}{s} \in R\right\}.$$

The following statement holds true (the proof is step by step repetition of the proof from [14]).

Proposition 2. *Let P be a measurable subset of $\mathbb{R}^n$. Let $p \geqslant 1$ or $p = \infty$. Suppose that a measurable function $\digamma(x,\xi)$, $x \in P$, $\xi \in \mathbb{R}^n$, has a space mean value $M(\digamma)(x)$ in $\mathbb{R}^n_\xi$ for every $x \in P$ and that the family $\{\digamma\left(x, \frac{x}{\varepsilon}\right),\ 0 < \varepsilon \leqslant 1\}, x \in \mathcal{K}$, is bounded in $L_p(\mathcal{K})$, where $\mathcal{K}$ is an arbitrary compact subset in P. Then $M(\digamma)(\cdot) \in L_p^{loc}(P)$ and, for $p \geqslant 1$, we have $\digamma\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\digamma)(x)$ weakly in $L_p^{loc}(P)$ as $\varepsilon \to 0$ and, for $p = \infty$, we have $\digamma\left(x, \frac{x}{\varepsilon}\right) \rightharpoonup M(\digamma)(x)$ $*$-weakly in $L_\infty^{loc}(P)$ as $\varepsilon \to 0$.*

Remark 1. We note that if a function $\digamma(x,\xi)$ is periodic or quasi-periodic in x, then the corresponding space mean value function $M(\digamma)(x)$ is also periodic or quasi-periodic.

In the next sections considering the GL system, we shall apply Proposition 2 for the case $P = \mathbb{T}^n$.

From now on we make use of a generalization of the well known Birkhoff theorem (see [8], [2] and also [38] and [14]). Namely, following the lines of [31, Ch. VIII, §7] it can be obtained in the form (see, e.g., [52]).

Theorem 2.6. ***(Birkhoff ergodic theorem)*** *Let $P \subset \mathbb{R}^n$. Let the dynamical system $\mathcal{T}_\xi$ ($\xi \in \mathbb{R}^n$) satisfy Definition 2.1. Consider a measurable real function $\psi = \psi(x,\omega)$, $x \in P$, $\omega \in \Omega$, such that, for every $x \in P$, the function $\psi(x,\cdot) \in L_q(\Omega,\mu)$ with $q \geqslant 1$. Then, for every $x \in P$ and for almost all $\omega \in \Omega$, the realization $\psi(x,\mathcal{T}_\xi\omega)$ has the space mean value $M(\psi(x,\mathcal{T}_\xi\omega))$. Moreover, the space mean value $M(\psi(x,\mathcal{T}_\xi\omega))$ is a conditional mathematical expectation of the function $\psi(x,\omega)$ with* *respect to the σ-algebra of invariant subsets. Hence, $M(\psi(x, \mathcal{T}_\xi\omega))$ is an invariant function and*

$$\mathbb{E}(\psi)(x) \equiv \int_\Omega \psi(x,\omega)\ d\mu = \int_\Omega M(\psi(x,\mathcal{T}_\xi\omega))\ d\mu.$$

In particular, if the dynamical system $\mathcal{T}_\xi$ is ergodic then, for almost all $\omega \in \Omega$, we have the identity

$$\mathbb{E}(\psi)(x) = M(\psi)(x).$$

Note that in the formulation of Theorem 2.6 the variable $x \in P$ plays the role of the parameter. In the next sections, we consider $P = \mathbb{T}^n$.

Definition 2.7. Let $P \subset \mathbb{R}^n$. A random function $\psi(x, \xi, \omega)$, $x \in P$, $\xi \in \mathbb{R}^n$, $\omega \in \Omega$, is called *statistically homogeneous* for any x, if the representation $\psi(x, \xi, \omega) = \Psi(x, \mathcal{T}_\xi\omega)$ is valid for some measurable function $\Psi : P \times \Omega \to \mathbb{R}$, where $\mathcal{T}_\xi$ is a dynamical system in Ω.

2.1. **Some examples.**

2.1.1. *Periodic case.* Let Ω be the unit cube $\{\omega \in \mathbb{R}^d, 0 \le \omega_j \le 1, j = 1, \dots, d\}$. On Ω we have a dynamical system $\mathcal{T}_\xi\omega = \omega + \xi \pmod 1$. The Lebesque measure is invariant and ergodic due to this dynamical system. The realization of the function $f(\omega) \in L_q(\Omega)$ has the form $f(\xi + \omega)$.

2.1.2. *Quasi-periodic case.* Let Ω be a unit cube in $\mathbb{R}^d$, μ be a Lebesque measure on it. For $\xi \in \mathbb{R}^m$ we set $\mathcal{T}_\xi\omega = \omega + \lambda\xi \pmod 1$, where $\lambda = \{\lambda_{ij}\}_{i,j=1}^d$ is a matrix $m \times d$. Obviously the mapping $\mathcal{T}_\xi$ preserve the measure μ on Ω. The dynamical system is ergodic if and only if $\lambda_{ij}k_j \neq 0$ for any integer vector $k \neq 0$.

Thus, $L_q(\Omega)$ is the space of periodic functions of d variables, and the realizations have the form $f(\omega + \lambda\xi)$. These realizations are called *quasi-periodic functions*, if $f(\omega)$ is continuous on Ω.

3. **Attractors for Ginzburg-Landau equations.**

In this section we discuss attractors for Ginzburg–Landau (GL) equations with space-dependent coefficients recalling and generalizing some known facts concerning these equations for $N = 1$ with constant coefficients. Consider the following system on the n-dimensional torus $\mathbb{T}^n$:

$$\partial_t u = (1 + i\alpha)\Delta u + R(x)u - (1 + i\beta(x))|u|^2 u + g(x),\ x \in \mathbb{T}^n,\ t \geqslant 0, \tag{5}$$

where $u = u(x,t)$, $u = u_1 + iu_2 = (u_1^1, \dots, u_1^N) + i(u_2^1, \dots, u_2^N)$ is a periodic in x unknown complex vector function and $|u|^2 = |u_1|^2 + |u_2|^2 = \sum_{k=1}^N |u_1^k|^2 + |u_1^k|^2$. In equation (5), α is a fixed real number, $\beta(x)$ and $R(x)$ are real function. The quantity α and the function $\beta(x)$ are called the *dispersion coefficients* and the function $R(x)$ is called the *instability coefficient*.

We assume that functions $\beta(x)$, $R(x)$ are measurable and for almost all $x \in \mathbb{T}^n$

$$-R_0 \leqslant R(x) \leqslant R_1, \tag{6}$$

$$-\beta_0 \leqslant \beta(x) \leqslant \beta_1 \quad (R_0, R_1, \beta_0, \beta_1 > 0). \tag{7}$$

Remark 2. In a similar way, we can consider more general systems (5) where α is a real $N \times N$-matrix and $\beta(x)$ and $R(x)$ are real $N \times N$-matrix functions with coefficients from $L_\infty(\mathbb{T}^n)$ that satisfy inequalities (6) and (7).

Let $\mathbf{H}^s, s \in \mathbb{R}$, denote the scale of Sobolev spaces with $\mathbf{H}^0 = \mathbf{H} := L_2(\mathbb{T}^n; \mathbb{C}^N)$. In particular, $\mathbf{H}^1 = H^1(\mathbb{T}^n; \mathbb{C}^N)$. We shall use also the short notation $\mathbf{L}_p := L_p(\mathbb{T}^n; \mathbb{C}^N)$.

Assume that the complex vector function $g(x) = g_1(x) + ig_2(x) = (g_1^1(x), \ldots, g_1^N(x)) + i(g_2^1(x), \ldots, g_2^N(x))$ in (5) belongs to $\mathbf{H}^{-1}$, which is the dual space for $\mathbf{H}^1$.

For $t = 0$ we consider the initial conditions

$$u|_{t=0} = u_0(x) = u_{01}(x) + iu_{02}(x),\ u_0 \in \mathbf{H}. \tag{8}$$

GL equation (5) can be written as a reaction-diffusion system with respect to the unknown real vector function $\mathbf{u} = (u_1, u_2)^\top$ in the following form:

$$\partial_t \mathbf{u} = A\Delta\mathbf{u} + R(x)\mathbf{u} - B(x)f(\mathbf{u}) + \mathbf{g}(x), \tag{9}$$

where the matrices $A = \begin{pmatrix} I_N & -\alpha I_N \\ \alpha I_N & I_N \end{pmatrix}$, $B(x) = \begin{pmatrix} I_N & -\beta(x) I_N \\ \beta(x) I_N & I_N \end{pmatrix}$, the nonlinear real function $f(\mathbf{u}) = |\mathbf{u}|^2\mathbf{u}$, and the external force $\mathbf{g}(x) = (g_1(x), g_2(x))^\top$.

It is well-known (see the case with constant coefficients R and β in [24], [43]) that under the condition

$$|\beta(x)| \leqslant \sqrt{3}, \tag{10}$$

the Cauchy problem (5), (8) for each $u_0 \in \mathbf{H}$ has a unique weak solution $u(x,t), x \in \mathbb{T}^n, t \geq 0$ that belongs to the space

$$L_2(0, M; \mathbf{H}^1) \cap L_4(0, M; \mathbf{L}_4) \cap C(0, M; \mathbf{H})$$

for each $M > 0$. Therefore, this solution can be presented in the form

$$u(\cdot, t) = S(t)u_0(\cdot),\ S(t) : \mathbf{H} \to \mathbf{H},\ t \geq 0,$$

where $\{S(t)\} := \{S(t), t \geq 0\}$ denotes the dynamical semigroup in $\mathbf{H}$ corresponding to the problem (5), (8).

The semigroup $\{S(t)\}$ is continuous in the space $\mathbf{H}$ and if $g \in \mathbf{H}$, then the semigroup $\{S(t)\}$ has an absorbing set B_0 that is compact in $\mathbf{H}$. This implies (see, e.g., [3, 46]) that the semigroup $\{S(t)\}$ has the *global attractor* $\mathcal{A}$ in the space $\mathbf{H}$, that is, the set $\mathcal{A}$ is compact in $\mathbf{H}$, strictly invariant with respect to the action of the semigroup $\{S(t)\}$:

$$S(t)\mathcal{A} = \mathcal{A},\ \forall t \geq 0,$$

and $\mathcal{A}$ attracts bounded trajectories of equation (5), i.e., for any bounded set $B \subset \mathbf{H}$:

$$\mathrm{dist}_{\mathbf{H}}(S(t)B, \mathcal{A}) \to 0\ (t \to +\infty).$$

Here and in the sequel

$$\mathrm{dist}_{\mathcal{M}}(X, Y) := \sup_{x \in X} \mathrm{dist}_{\mathcal{M}}(x, Y) = \sup_{x \in X} \inf_{y \in Y} \rho_{\mathcal{M}}(x, y)$$

denotes the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Remark 3. We note that in dimension $n = 1, 2$, the uniqueness theorem for the GL equation is proved for any values of dispersion parameters α and β (see, e.g., [34, 46]). Besides, there are deep results concerning the attractors of GL equations for $n \geq 3$ in the phase spaces $\mathbf{L}_p$ under some restrictions on the parameters α, β, and R (the degree p depends on α, β, and R (see [43, 51])). However, to the best of our knowledge, the uniqueness of a weak solution of the GL equation is not yet proved for arbitrary parameters α, β, and R and in space dimension $n \geq 3$.

Let now assume that the dispersion coefficient $\beta(x)$ does not satisfy condition (10). In that case, since the uniqueness of a solution to problem (5), (8) is not proved for dimensions $n \geq 3$, we can not construct the global attractor for this equation following the above standard scheme. To study the behaviour of solutions of GL equation (5) as $t \to +\infty$ we shall use an alternative approach and we shall construct the trajectory attractor of this equation.

Let $g \in \mathbf{H}^{-1}$. First of all, we observe that, for each $u_0(\cdot) \in \mathbf{H}$, the Cauchy problem (5), (8) has at least one weak solution $u(\cdot, t), t \geq 0$, that can be constructed by the Galerkin approximation method (see, e.g., [22, 23, 24]).

Recall that a function $u(x,t)$ is called a weak solution of equation (5) for $t \geq 0$ if $u(t) := u(\cdot,t) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1) \cap L_4^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_4)$ and $u(t)$ satisfies equation (5) in the distribution sense of the space $\mathcal{D}'(0,M;\mathbf{H}^{-r})$ for every $M > 0$. Here $\mathcal{D}'(0,M;\mathbf{H}^{-r})$ is the space of distributions with values in the Banach space $\mathbf{H}^{-r},\ r = \max\{1, n/4\}$ (see [40, 41]).

Let $u(t)$ be a weak solution of equation (5). Considering the corresponding vector-function $\mathbf{u}(t) = (u_1(t), u_2(t))^\top$ that satisfy the system (9), we have

$$\begin{aligned} \partial_t \mathbf{u}(t) &= A\Delta\mathbf{u} + R(x)\mathbf{u} - B(x)|\mathbf{u}(t)|^2\mathbf{u}(t) + \mathbf{g}(x) \\ &= \mathbf{w}(t) + \mathbf{h}(t), \end{aligned} \tag{11}$$

where

$$\mathbf{w}(t) = A\Delta u(t) + \mathbf{g},\ \mathbf{h}(t) = R(x)\mathbf{u}(t) - B(x)|\mathbf{u}(t)|^2\mathbf{u}(t), \tag{12}$$

and clearly

$$\mathbf{w}(\cdot) \in L_2(0,M;\mathbf{H}^{-1}),\ \mathbf{h}(\cdot) \in L_{4/3}(0,M;\mathbf{L}_{4/3}) \tag{13}$$

for all $M > 0$. The Sobolev embedding theorem implies that $H^s(\mathbb{T}^n) \subset L_\varrho(\mathbb{T}^n)$ for $s \geq n(1/2 - 1/\varrho)$. In particular, for $\varrho = 4$, we have $H^{n/4}(\mathbb{T}^n) \subset L_4(\mathbb{T}^n)$, and, hence, for the dual spaces we obtain the embedding $L_{4/3}(\mathbb{T}^n) \subset H^{-n/4}(\mathbb{T}^n)$. Thus,

$$L_2(0,M;\mathbf{H}^{-1}) \subset L_{4/3}(0,M;\mathbf{H}^{-1}),\ L_{4/3}(0,M;\mathbf{L}_{4/3}) \subset L_{4/3}(0,M;\mathbf{H}^{-n/4}). \tag{14}$$

From (13) and (14), we conclude that

$$\partial_t \mathbf{u}(\cdot) \in L_{4/3}(0,M;\mathbf{H}^{-r}) \subset \mathcal{D}'(0,M;\mathbf{H}^{-r}),\ r = \max\{1, n/4\}. \tag{15}$$

We have the following

Proposition 3. *Let $u(t)$ be a weak solution of GL equation (5). Then*

(a): *the function $u(\cdot) \in C(\mathbb{R}_+;\mathbf{H})$;*

(b): *the function $\|u(t)\|_{\mathbf{H}}^2$ is absolutely continuous on $\mathbb{R}_+$ and the following identity holds for almost every $t \geq 0$:*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + \|\nabla u(t)\|_{\mathbf{H}}^2 + \|u(t)\|_{\mathbf{L}_4}^4 - \int\limits_{\mathbb{T}^n} R(x)|u(x,t)|^2 dx = \langle u(t), g\rangle\,. \tag{16}$$

Here

$$\langle u, v\rangle := \int\limits_{\mathbb{T}^n} \left[\sum_{k=1}^N u_1^k(x)v_1^k(x) + u_2^k(x)v_2^k(x)\right] dx$$

denotes the real scalar product of complex vector functions $u = u_1 + iu_2$ and $v = v_1 + iv_2$.

This proposition is proved in [48] for the case of constant coefficients β and R. The x-dependent case can be proved in the similar way. We shortly explain the formal derivation of the equality (16). Taking the scalar product of (11) and $\mathbf{u}$, we integrate the result over $\mathbb{T}^n$ and obtain the identity

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_{\mathbf{H}}^2 = \langle \mathbf{w}(t), \mathbf{u}(t)\rangle + \langle \mathbf{h}(t), \mathbf{u}(t)\rangle . \tag{17}$$

Since matrices A and $B(x)$ are skew-symmetric, we clearly have that

$$\langle \mathbf{w}(t), \mathbf{u}(t)\rangle = \langle A\Delta\mathbf{u}(t) + \mathbf{g}, \mathbf{u}\rangle = -\|\nabla\mathbf{u}(t)\|_{\mathbf{H}}^2 + \langle \mathbf{g}, \mathbf{u}(t)\rangle , \tag{18}$$

$$\langle \mathbf{h}(t), \mathbf{u}(t)\rangle = \big\langle R(x)\mathbf{u}(t) - B(x)|\mathbf{u}(t)|^2\mathbf{u}(t), \mathbf{u}(t)\big\rangle = \int\limits_{\mathbb{T}^n} R(x)|\mathbf{u}(x,t)|^2 dx - \int\limits_{\mathbb{T}^n} |\mathbf{u}(x,t)|^4 dx, \tag{19}$$

where in the second equality of (18) we have integrated by part in $x \in \mathbb{T}^n$ (recall that α is a real constant). Finally, we apply (18) and (19) in (17) and obtain (16). The rigorous proof of the identity (17) uses the properties (13) (see, e.g., Theorem II.1.8 in [24] or Theorem 3.2 in [48]).

Identity (16) implies that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + \|\nabla u(t)\|_{\mathbf{H}}^2 + \|u(t)\|_{\mathbf{L}_4}^4 - \int\limits_{\mathbb{T}^n} R(x)|u(x,t)|^2 dx = \langle u(t), g\rangle \leqslant \|u(t)\|_{\mathbf{H}^1}\|g\|_{\mathbf{H}^{-1}} \leqslant \frac{1}{2}\|u(t)\|_{\mathbf{H}^1}^2 + \frac{1}{2}\|g\|_{\mathbf{H}^{-1}}^2.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + \frac{1}{2}\|u(t)\|_{\mathbf{H}^1}^2 + \|u(t)\|_{\mathbf{L}_4}^4 - \int\limits_{\mathbb{T}^n} [R(x)+1]\, |u(x,t)|^2 dx \leqslant \frac{1}{2}\|g\|_{\mathbf{H}^{-1}}^2$$

since $\|\nabla u(t)\|_{\mathbf{H}}^2 = \|u(t)\|_{\mathbf{H}^1}^2 - \|u(t)\|_{\mathbf{H}}^2$ and, due to (6), we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + \frac{1}{2}\|u(t)\|_{\mathbf{H}^1}^2 + \|u(t)\|_{\mathbf{L}_4}^4 - \rho\|u(t)\|_{\mathbf{H}}^2 \leqslant \frac{1}{2}\|g\|_{\mathbf{H}^{-1}}^2, \tag{20}$$

where $\rho = R_1 + 1$. Elementary inequality $z^2 - \rho z \geq \rho z - \rho^2$ implies that

$$\|u(t)\|_{\mathbf{L}_4}^4 - \rho\|u(t)\|_{\mathbf{H}}^2 = \int\limits_{\mathbb{T}^n} \big(|u(x,t)|^4 - \rho|u(x,t)|^2\big)\, dx \geqslant \int\limits_{\mathbb{T}^n} \big(\rho|u(x,t)|^2 - \rho^2\big)\, dx = \rho\|u(t)\|_{\mathbf{H}}^2 + \rho^2(2\pi)^n, \tag{21}$$

Combining (20) and (21) we obtain the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + \frac{1}{2}\|u(t)\|_{\mathbf{H}^1}^2 + \rho\|u(t)\|_{\mathbf{H}}^2 \leqslant \frac{1}{2}\|g\|_{\mathbf{H}^{-1}}^2 + \rho^2\mu(\mathbb{T}^n).$$

Therefore,

$$\frac{d}{dt}\|u(t)\|_{\mathbf{H}}^2 + 2\rho\|u(t)\|_{\mathbf{H}}^2 \leqslant D, \tag{22}$$

where $D = \|g\|_{\mathbf{H}^{-1}}^2 + 2\rho^2\mu(\mathbb{T}^n)$. Inequality (22) implies that

$$\|u(t)\|_{\mathbf{H}}^2 \leqslant \|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + \frac{D}{2\rho},\ \forall t \geqslant 0. \tag{23}$$

Integrating (22) over $[t, t+1]$, we have

$$\|u(t+1)\|_{\mathbf{H}}^2 + 2\rho \int\limits_t^{t+1} \|u(s)\|_{\mathbf{H}}^2 ds \leqslant \|u(t)\|_{\mathbf{H}}^2 + D \leqslant \|u(0)\|_{\mathbf{H}}^2 e^{-2R_1 t} + \frac{D}{2\rho} + D.$$

Hence,

$$2\rho \int\limits_t^{t+1} \|u(s)\|_{\mathbf{H}}^2 ds \leqslant \|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + D\left(\frac{1}{2\rho} + 1\right). \tag{24}$$

Finally, integrating (20) over $[t, t+1]$ and using (23) and (24), we obtain

$$\int\limits_t^{t+1} \|u(s)\|_{\mathbf{H}^1}^2 ds + 2\int\limits_t^{t+1} \|u(s)\|_{\mathbf{L}_4}^4 ds \leqslant \|u(t)\|_{\mathbf{H}}^2 + 2\rho \int\limits_t^{t+1} \|u(s)\|_{\mathbf{H}}^2 ds + \|g\|_{\mathbf{H}^{-1}}^2$$

$$\leqslant \|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + \frac{D}{2\rho} + \|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + D\left(\frac{1}{2\rho} + 1\right) + \|g\|_{\mathbf{H}^{-1}}^2$$

$$= 2\|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + D_1,$$

where $D_1 = D\left(\frac{1}{\rho} + 1\right) + \|g\|_{\mathbf{H}^{-1}}^2$. Thus, we have proved

Proposition 4. *Any weak solution $u(t)$ of GL equation (5) satisfies the inequalities*

$$\|u(t)\|_{\mathbf{H}}^2 \leqslant \|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + \frac{D}{2\rho}, \tag{25}$$

$$\int\limits_t^{t+1} \|u(s)\|_{\mathbf{H}^1}^2 ds + 2\int\limits_t^{t+1} \|u(s)\|_{\mathbf{L}_4}^4 ds \leqslant 2\|u(0)\|_{\mathbf{H}}^2 e^{-2\rho t} + D_1,\ \forall t \geqslant 0, \tag{26}$$

where constants $\rho = R_1 + 1$, $D = \|g\|_{\mathbf{H}^{-1}}^2 + 2\rho^2 \mu(\mathbb{T}^n)$, and $D_1 = D\left(\frac{1}{\rho} + 1\right) + \|g\|_{\mathbf{H}^{-1}}^2$ are independent of the dispersion coefficients α and $\beta(x)$.

We note that the proof of an analogous result is given in [22, 24] for more general reaction-diffusion systems (see also [23]).

We now construct the trajectory attractor for GL equation (5). We consider the trajectory space $\mathcal{K}^+$ consisting of all the weak solutions $u(t), t \geqslant 0$, of equation (5), i.e. all the functions $u(\cdot) \in L_\infty^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}) \cap L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}^1) \cap L_4^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_4)$ that satisfy (5) in the space of distributions $\mathcal{D}'(0, M; \mathbf{H}^{-r})$.

Proposition 5. *Let the sequence $u_m \in \mathcal{K}^+$ and $u_m \rightharpoonup u$ $(m \to \infty)$ weakly in the space $L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}^1)$, weakly in the space $L_4^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_4)$, and $*$-weakly in the space $L_\infty^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H})$. Then $u \in \mathcal{K}^+$.*

Proof. It is clear that $\{u_m\}$ is bounded in the spaces $L_2(0, M; \mathbf{H})$, $L_4(0, M; \mathbf{L}_4)$, and $L_\infty(0, M; \mathbf{H})$ for any fixed $M > 0$. The functions $u_m(t)$ satisfies the GL equations

$$\partial_t u_m = (1 + i\alpha)\Delta u_m + Ru_m - (1 + i\beta)|u_m|^2 u_m + g(x). \tag{27}$$

Therefore, the sequence of derivatives $\{\partial_t u_m\}$ is bounded in $L_{4/3}(0, M; \mathbf{H}^{-r})$. Using the known compactness theorem (see [30, 40]), we conclude that the sequence $\{u_m\}$ converges strongly in $L_2(0, M; \mathbf{H})$ and, hence, $u_m(x, t)$ converges to $u(x, t)$ as $m \to \infty$ for almost every $(x, t) \in \mathbb{T}^n \times [0, M]$. It follows from the assumptions of Proposition 5 that the sum $(1 + i\alpha)\Delta u_m + Ru_m$ converges to $(1 + i\alpha)\Delta u + Ru$ weakly as $m \to \infty$ in the space $L_2(0, M; \mathbf{H}^{-1})$ and the sequence $(1 + i\beta)|u_m|^2 u_m$

converges to $(1+i\beta)|u|^2u$ weakly in $L_{4/3}(0,M;\mathbf{L}_{4/3})$ for an arbitrary fixed $M>0$. The last assertion follows from the fact that $u_m(x,t)\to u(x,t)$ $(m\to\infty)$ for almost every $(x,t)\in\mathbb{T}^n\times[0,M]$. Therefore, the right-hand side of equation (27) converges to $(1+i\alpha)\Delta u+Ru-(1+i\beta)|u|^2u+g(x)$ as $m\to\infty$ in the space $\mathcal{D}'(0,M;\mathbf{H}^{-r})$. The operator ∂_t is continuous in the space of distributions $\mathcal{D}'(0,M;\mathbf{H}^{-r})$, that is, $\partial_t u_m\to\partial_t u$ $(m\to\infty)$ in $\mathcal{D}'(0,M;\mathbf{H}^{-r})$. Consequently, passing to the limit as $m\to\infty$ in (27), we obtain the equation for u

$$\partial_t u=(1+i\alpha)\Delta u+Ru-(1+i\beta)|u|^2u+g(x). \tag{28}$$

It is clear that $u(\cdot)\in L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H})\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1)\cap L_4^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_4)$. Hence, $u\in\mathcal{K}^+$. $\square$

Corollary 1. *The space $\mathcal{K}^+$ of weak solutions of the GL equation (5) is sequentially closed in the weak topology of the space $L_{2,\mathrm{w}}^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1)\cap L_{4,\mathrm{w}}^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_4)\cap L_{\infty,*\mathrm{w}}^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H})$.*

We now introduce the spaces $\mathcal{F}_+^{\mathrm{loc}}$, $\mathcal{F}_+^{\mathrm{b}}$, and Θ_+^{loc} for equation (5) that we need to define the trajectory attractor.

Definition 3.1. The spaces

$$\mathcal{F}_+^{\mathrm{loc}}:=L_\infty^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H})\cap L_2^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^1)\cap L_4^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{L}_4)\cap\left\{v\mid\partial_t v\in L_{4/3}^{\mathrm{loc}}(\mathbb{R}_+;\mathbf{H}^{-r})\right\} \tag{29}$$

$$\mathcal{F}_+^{\mathrm{b}}:=L_\infty(\mathbb{R}_+;\mathbf{H})\cap L_2^{\mathrm{b}}(\mathbb{R}_+;\mathbf{H}^1)\cap L_4^{\mathrm{b}}(\mathbb{R}_+;\mathbf{L}_4)\cap\left\{v\mid\partial_t v\in L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;\mathbf{H}^{-r})\right\}, \tag{30}$$

where the norm in the space $L_p^{\mathrm{b}}(\mathbb{R}_+;E)$ is

$$\|\varphi\|_{L_p^{\mathrm{b}}(\mathbb{R}_+;E)}:=\left(\sup_{t\geqslant 0}\int_t^{t+1}\|\varphi(\tau)\|_E^p d\tau\right)^{1/p}.$$

Obviously $\mathcal{F}_+^{\mathrm{b}}$ is a Banach space with norm

$$\|v\|_{\mathcal{F}_+^{\mathrm{b}}}:=\|v\|_{L_\infty(\mathbb{R}_+;\mathbf{H})}+\|v\|_{L_2^{\mathrm{b}}(\mathbb{R}_+;\mathbf{H}^1)}+\|v\|_{L_4^{\mathrm{b}}(\mathbb{R}_+;\mathbf{L}_4)}+\|\partial_t v\|_{L_{4/3}^{\mathrm{b}}(\mathbb{R}_+;\mathbf{H}^{-r})}. \tag{31}$$

We denote by Θ_+^{loc} the space $\mathcal{F}_+^{\mathrm{loc}}$ equipped with the following local convergence topology in $\mathcal{F}_+^{\mathrm{loc}}$. By definition, a sequence of functions $\{v_m\}\subset\mathcal{F}_+^{\mathrm{loc}}$ converges to a function $v\in\mathcal{F}_+^{\mathrm{loc}}$ as $m\to\infty$ in Θ_+^{loc}, if, for each $M>0$, the sequence $v_m\rightharpoonup v$ $(m\to\infty)$ weakly in $L_2(0,M;\mathbf{H}^1)$, weakly in $L_4(0,M;\mathbf{L}_4)$ and $*$-weakly in $L_\infty(0,M;\mathbf{H})$, and moreover, $\partial_t v_m\rightharpoonup\partial_t v$ $(m\to\infty)$ weakly in $L_{4/3}(0,M;\mathbf{H}^{-r})$. It is also possible to define the topological space Θ_+^{loc} in terms of the suitable neighbourhoods (see [24]).

We note that $\mathcal{F}_+^{\mathrm{b}}\subset\Theta_+^{\mathrm{loc}}$ and a ball $B_d=\left\{w(\cdot)\in\mathcal{F}_+^{\mathrm{b}}\mid\|w\|_{\mathcal{F}_+^{\mathrm{b}}}\leqslant d\right\}$ in $\mathcal{F}_+^{\mathrm{b}}$ is a compact set in the topology Θ_+^{loc}. Therefore, the corresponding topological subspace $B_d\subset\Theta_+^{\mathrm{loc}}$ is metrizable and complete (see, e.g., [24]).

We consider the translation semigroup $\{T(h)\}:=\{T(h),h\geqslant 0\}$ acting in the space Θ_+^{loc} by the formula $T(h)w(t)=w(t+h),h\geqslant 0$. It is easy to see that the semigroup $\{T(h)\}$ maps the trajectory space $\mathcal{K}^+$ to itself, i.e. $T(h)\mathcal{K}^+\subset\mathcal{K}^+$ for any $h\geqslant 0$.

Proposition 6. *Let, in equation (5), $g\in\mathbf{H}^{-1}$. Then*

(a): *the trajectory space $\mathcal{K}^+\subset\mathcal{F}_+^{\mathrm{b}}$;*

(b): *for an arbitrary trajectory $u \in \mathcal{K}^+$, the following inequality holds:*

$$\|T(h)u(\cdot)\|_{\mathcal{F}^{\mathrm{b}}_+} \leqslant C_0\|u(0)\|^2_{\mathbf{H}} e^{-2\rho h} + D_2,\ \forall t \geqslant 0, \tag{32}$$

where the constants D_2 and $\rho = R_1 + 1$ are independent of $u(\cdot)$.

Proof. Estimate (26) yields

$$\|T(h)u(\cdot)\|^2_{L_2(0,1;\mathbf{H}^1)} + \|T(h)u(\cdot)\|^4_{L_4(0,1;\mathbf{L}_4)} = \\ \int_h^{h+1} \left(\|u(s)\|^2_{\mathbf{H}^1} + \|u(s)\|^4_{\mathbf{L}_4}\right) ds \leqslant 2\|u(0)\|^2_{\mathbf{H}} e^{-2\rho t} + D_1,\ \forall h \geqslant 0. \tag{33}$$

Thus, since $u(t) \in \mathcal{K}^+$ the function $T(h)u(t)$ is a weak solution of equation (5). Hence, we conclude that

$$\|T(h)\partial_t u(\cdot)\|_{L_{4/3}(0,1;\mathbf{H}^{-r})} \leq \|T(h)\left((1+i\alpha)\Delta u + R(\cdot)u + g(x)\right)\|_{L_{4/3}(0,1;\mathbf{H}^{-r})} + \\ \|T(h)\left((1+i\beta(\cdot))|u|^2 u\right)\|_{L_{4/3}(0,1;\mathbf{H}^{-r})} \leqslant \\ C_1\left(\|T(h)\Delta u\|_{L_{4/3}(0,1;\mathbf{H}^1)} + \|T(h)\left(|u|^2 u\right)\|_{L_{4/3}(0,1;\mathbf{H}^{-r})} + 1\right) \leqslant \\ C_2\left(\|T(h)u\|^2_{L_2(0,1;\mathbf{H}^1)} + \|T(h)u\|^4_{L_4(0,1;\mathbf{L}_4)} + 1\right) \leqslant C_4\|u(0)\|^2_{\mathbf{H}} e^{-2\rho t} + D_2'. \tag{34}$$

Inequalities (25), (33), and (34) yield estimate (32). Point **(b)** of Proposition 6 is proved. Inequality (32) evidently provides Point **(a)**. □

Recall the notion of the trajectory attractor of equation (5) (see, e.g. [23, 24, 49]).

We consider the translation semigroup $\{T(h)\}$ on the trajectory space $\mathcal{K}^+$ of GL equation (5). We have already proved that $\mathcal{K}^+ \subset \mathcal{F}^{\mathrm{b}}_+ \subset \Theta^{\mathrm{loc}}_+$. It is easy to see that $T(h)\mathcal{K}^+ \subset \mathcal{K}^+$ for $h \geqslant 0$ and, for every $u \in \mathcal{K}^+$, the function $T(h)u(\cdot)$ satisfies the estimate (32). Moreover, $\mathcal{K}^+$ is sequentially closed in the topology Θ^{loc}_+ (see Corollary 1).

Definition 3.2. A set $P \subseteq \mathcal{K}^+$ is called an *attracting set* for the semigroup $\{T(h)\}$ in the topology Θ^{loc}_+ if, for each set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}^{\mathrm{b}}_+$, the set P attracts $T(h)B$ as $h \to +\infty$ in the topology Θ^{loc}_+, that is, for each neighbourhood $\mathcal{O}(P)$ of P in Θ^{loc}_+, there exists a quantity $h_1 = h_1(B, \mathcal{O})$ such that $T(h)B \subset \mathcal{O}(P)$ for all $h \geqslant h_1$.

Definition 3.3. A set $\mathfrak{A} \subset \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{T(h)\}$ on $\mathcal{K}^+$ in the topology Θ^{loc}_+ if $\mathfrak{A}$ is bounded in $\mathcal{F}^{\mathrm{b}}_+$, compact in Θ^{loc}_+, strictly invariant with respect to $\{T(h)\}$:

$$T(h)\mathfrak{A} = \mathfrak{A},\ \forall h \geqslant 0,$$

and $\mathfrak{A}$ is an attracting set of the semigroup $\{T(h)\}$ on $\mathcal{K}^+$ in the topology Θ^{loc}_+.

It follows from estimate (32) that the set

$$P = \left\{u(\cdot) \in \mathcal{K}^+ \mid \|u(\cdot)\|_{\mathcal{F}^{\mathrm{b}}_+} \leqslant 2D_2\right\}$$

is absorbing and attracting for the semigroup $\{T(h)\}|_{\mathcal{K}^+}$. The set P is bounded in $\mathcal{F}^{\mathrm{b}}_+$ and compact in Θ^{loc}_+. Then, P is a complete metric space (recall the topology Θ^{loc}_+ is metrizable on any ball in $\mathcal{F}^{\mathrm{b}}_+$). Besides, the semigroup $\{T(h)\}|_{\mathcal{K}^+}$ is continuous in the topology Θ^{loc}_+. It is known that the above facts imply the following

Theorem 3.4. *The semigroup $\{T(h)\}|_{\mathcal{K}^+}$ has the trajectory attractor $\mathfrak{A} \subset \mathcal{K}^+ \cap P$ in the topology $\Theta_+^{\rm loc}$ that attracts all bounded sets B of $\mathcal{K}^+ \cap \mathcal{F}_+^{\rm b}$.*

Proof. It follows from the above-listed properties of the semigroup $\{T(h)\}|_{\mathcal{K}^+}$ that the ω-limit set

$$\bigcap_{h\geqslant 0}\left[\bigcup_{s\geqslant h} T(s)P\right]_{\Theta_+^{\rm loc}} = \mathfrak{A}$$

is the global $\left(\mathcal{F}_+^{\rm b}, \Theta_+^{\rm loc}\right)$-attractor of the semigroup $\{T(h)\}|_{\mathcal{K}^+}$. Here, we use the terminology from the well-known theorem on the existence of global attractors of semigroups (see, e.g., [3, 24, 35, 46]). The so-constructed global $\left(\mathcal{F}_+^{\rm b}, \Theta_+^{\rm loc}\right)$-attractor is just the trajectory attractor of the equation (5). □

We now describe the structure of the attractor $\mathfrak{A}$ in terms of the complete trajectories $\{u(t), t \in \mathbb{R}\}$ of equation (5), that is, of weak solutions of this equation that are defined on the entire time axis.

A function $\{u(t), t \in \mathbb{R}\}$ belonging to $L_\infty^{\rm loc}(\mathbb{R}; \mathbf{H}) \cap L_2^{\rm loc}(\mathbb{R}; \mathbf{H}^1) \cap L_4^{\rm loc}(\mathbb{R}; \mathbf{L}_4)$ is called the *complete trajectory* of GL equation (5) if $u(t)$ satisfies equation (5) in the sense of the distribution space $\mathcal{D}'(-M, M; \mathbf{H}^{-r})$ for each $M > 0$.

The spaces $\mathcal{F}^{\rm loc}$ and $\mathcal{F}^{\rm b}$ are defined similarly to the spaces $\mathcal{F}_+^{\rm loc}$ and $\mathcal{F}_+^{\rm b}$ except that one must replace in formulae (29), (30), and (31) the half-axis $\mathbb{R}_+$ by the entire axis $\mathbb{R}$, and the topological space $\Theta^{\rm loc}$ is the space $\mathcal{F}^{\rm loc}$ equipped with the following topology of sequential convergence: by definition, a sequence $\{v_m\} \subset \mathcal{F}^{\rm loc}$ converges to $v \in \mathcal{F}^{\rm loc}$ as $m \to \infty$ in $\Theta^{\rm loc}$ if, for each $M > 0$, the sequence $v_m \rightharpoonup v$ converges as $m \to \infty$ weakly in $L_2(-M, M; \mathbf{H}^1)$, weakly in $L_4(-M, M; \mathbf{L}_4)$, and $*$-weakly in $L_\infty(-M, M; \mathbf{H})$, and moreover $\partial_t v_m \rightharpoonup \partial_t v$ as $m \to \infty$ weakly in $L_{4/3}^{\rm loc}(-M, M; \mathbf{H}^{-1})$.

Definition 3.5. By the *kernel* $\mathcal{K}$ of GL equation (5) in the space $\mathcal{F}^{\rm b}$ we mean the union of all complete trajectories $\{u(t), t \in \mathbb{R}\}$ of this equation that belong to $\mathcal{F}^{\rm b}$ (that is, with bounded norm $\|u\|_{\mathcal{F}^{\rm b}}$).

We denote by Π_+ the restriction operator to the positive half-axis that maps a function $\{v(t), t \in \mathbb{R}\}$ to the function $\{v(t), t \in \mathbb{R}_+\}$.

Theorem 3.6. *The kernel $\mathcal{K}$ of GL equation (5) is bounded in the space $\mathcal{F}^{\rm b}$ and compact in $\Theta^{\rm loc}$. The trajectory attractor $\mathfrak{A} \subset \mathcal{K}^+$ of equation (5) coincides with restriction of the kernel $\mathcal{K}$ of this equation on the positive half-axis:*

$$\mathfrak{A} = \Pi_+\mathcal{K}.$$

The proof of this theorem in a more general case is presented in [24, p.224].

Remark 4. If, in GL equation (5), $|\beta(x)| \leqslant \sqrt{3}$ and $g \in \mathbf{H}$, then the uniqueness theorem holds for the Cauchy problem (5), (8) and GL equation has the global attractor $\mathcal{A}$ and the trajectory attractor $\mathfrak{A}$. The following relation holds for these attractors:

$$\mathcal{A} = \mathfrak{A}(0) := \{u(0) \mid u(\cdot) \in \mathfrak{A}\}. \tag{35}$$

The proof of this fact can be found in [24, p.227].

We note that the considered in $\mathcal{K}^+$ local weak topology $\Theta_+^{\rm loc}$ is stronger that the local strong topologies $L_2^{\rm loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ and $C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$ for each $\delta > 0$, that is, if a sequence $\{v_m\} \subset \mathcal{F}_+^{\rm loc}$ converges to a function $v \in \mathcal{F}_+^{\rm loc}$ as $m \to \infty$ in $\Theta_+^{\rm loc}$, then $v_m \rightharpoonup v$ $(m \to \infty)$ strongly in $L_2(0, M; \mathbf{H}^{1-\delta})$ and strongly in $C([0, M]; \mathbf{H}^{-\delta})$ for every $M > 0$. This assertion follows from Aubin-Lions-Simon theorem (see, e.g., [5], [13], [24]). We have

Corollary 2. *For any set $B \subset \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ we have*

$$\mathrm{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})} (\Pi_{0,M} S(t)B, \Pi_{0,M}\mathcal{K}) \to 0 \ (t \to \infty),$$
$$\mathrm{dist}_{C([0,M];\mathbf{H}^{-\delta})} (\Pi_{0,M} S(t)B, \Pi_{0,M}\mathcal{K}) \to 0 \ (t \to \infty), \ \forall M > 0, \ (\delta > 0),$$

where $\Pi_{0,M}$ denotes the restriction operator onto the interval $[0, M]$.

In the next section we study GL equations and their global and trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 3.7. **(a):** We say that the global attractors $\mathcal{A}_\varepsilon$ converge to the trajectory attractor $\mathcal{A}_0$ as $\varepsilon \to 0$ in the space $\mathbf{H}^s$ if

$$\mathrm{dist}_{\mathbf{H}^s}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0 \ (\varepsilon \to 0+).$$

(b): We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\mathfrak{A}_0$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A}_0) \subset \Theta_+^{loc}$ there is an $\varepsilon_1 \geqslant 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\mathfrak{A}_0)$ for any $\varepsilon < \varepsilon_1$.

4. **Ginzburg-Landau equations with randomly rapidly oscillating coefficients.** Consider the GL equation in the torus $\mathbb{T}^n$

$$\partial_t u = (1 + i\alpha)\Delta u + R\left(x, \frac{x}{\varepsilon}, \omega\right) u + \left(1 + i\beta\big(x, \frac{x}{\varepsilon}, \omega\big)\right) |u|^2 u + g\left(x, \frac{x}{\varepsilon}, \omega\right). \quad (36)$$

Let $\mathcal{T}_\xi, \xi \in \mathbb{R}^n$, be an ergodic dynamical system.

Assume that the functions $R(x, \xi, \omega)$ and $\beta(x, \xi, \omega)$ are statistically homogeneous for any x, that is, $R(x, \xi, \omega) = \mathbf{R}(x, \mathcal{T}_\xi\omega)$, $\beta(x, \xi, \omega) = \mathbf{B}(x, \mathcal{T}_\xi\omega)$, where $\mathbf{R}, \mathbf{B} : \mathbb{T}^n \times \Omega \to \mathbb{R}$ are measurable functions.

We also assume that $\mathbf{R}(x, \omega), \mathbf{B}(x, \omega) \in L_\infty(\mathbb{T}^n \times \Omega)$ and

$$\begin{aligned} -R_0 &\leqslant \mathbf{R}(x, \omega) \leqslant R_1, \\ -\beta_0 &\leqslant \mathbf{B}(x, \omega) \leqslant \beta_1, \quad \forall x \in \mathbb{T}^n, \ \omega \in \Omega. \end{aligned} \quad (37)$$

Due to the Birkhoff ergodic theorem the functions $R(x, \xi, \omega)$ and $\beta(x, \xi, \omega)$ have the average $R^{\mathrm{hom}}(x) = \mathbb{E}(\mathbf{R})(x)$, $\beta^{\mathrm{hom}}(x) = \mathbb{E}(\mathbf{B})(x)$ for every $x \in \mathbb{T}^n$. It is apparent that the functions $R^{\mathrm{hom}}(x)$ and $\beta^{\mathrm{hom}}(x)$ also satisfies inequalities

$$-R_0 \leqslant R^{\mathrm{hom}}(x) \leqslant R_1, \quad -\beta_0 \leqslant \beta^{\mathrm{hom}}(x) \leqslant \beta_1, \ \forall x \in \mathbb{T}^n.$$

It follows from Proposition 2 (see also [38, Ch. VII, §1]), the case $p = \infty$, that almost surely in $\omega \in \Omega$

$$\int_{\mathbb{T}^n} R\left(x, \frac{x}{\varepsilon}, \omega\right) \varphi(x) dx \longrightarrow \int_{\mathbb{T}^n} R^{\mathrm{hom}}(x) \varphi(x) dx, \quad (38)$$

$$\int_{\mathbb{T}^n} \beta\left(x, \frac{x}{\varepsilon}, \omega\right) \varphi(x) dx \longrightarrow \int_{\mathbb{T}^n} \beta^{\mathrm{hom}}(x) \varphi(x) dx \quad (\varepsilon \to 0+) \quad (39)$$

for any $\varphi \in L_1(\mathbb{T}^n)$.

For the random vector function $g(x, \frac{x}{\varepsilon}, \omega)$, we assume that

$$g\left(x, \frac{x}{\varepsilon}, \omega\right) = g_0\left(x, \frac{x}{\varepsilon}, \omega\right) + \sum_{i=1}^{n} \partial_{x_i} g_i\left(x, \frac{x}{\varepsilon}, \omega\right),$$

where the function $g_i(x,\xi,\omega)$ are statistically homogeneous, i.e.

$$g_i(x,\xi,\omega) = \mathbf{G}_i(x,\mathcal{T}_\xi\omega),\ i=0,\dots,n,$$

$\mathbf{G}_i : \mathbb{T}^n \times \Omega \to \mathbb{C}^N$ are measurable functions and the following inequalities hold for almost all $x \in \mathbb{T}^n$:

$$|\mathbf{G}_i\,(x,\omega)\,| \leqslant \phi_i(x),\ \forall \omega \in \Omega, i=0,\dots,n, \tag{40}$$

where the positive majorant $\phi_i(\cdot) \in L_2(\mathbb{T}^n)$, for $i=0,\dots,n$.

Birkhoff theorem implies that the functions $g_i(x,\xi,\omega)$, $i=0,\dots,n$, have the averages $g_i^{\mathrm{hom}}(x) = \mathbb{E}(\mathbf{G_i})(x)$ for every $x \in \mathbb{T}^n$. It follows from (40) that

$$|g_i^{\mathrm{hom}}(x)| \leqslant \phi_i(x),\ i=0,\dots,n,$$

for almost all $x \in \mathbb{T}^n$ and therefore $g_i^{\mathrm{hom}}(\cdot) \in \mathbf{H}$.

Inequalities (40) imply that the vector functions $g_i\left(x,\frac{x}{\varepsilon},\omega\right)$ belong to the space $\mathbf{H}$ and they are uniformly (w.r.t. $\varepsilon \in (0,1]$) bounded in this space. Therefore, Proposition 2 is applicable with $P = \mathbb{T}^n$ and $p=2$ and we obtain that, almost surely in $\omega \in \Omega$,

$$g_i\left(x,\frac{x}{\varepsilon},\omega\right) \rightharpoonup g_i^{\mathrm{hom}}(x)\ (\varepsilon \to 0+) \text{ weakly in } \mathbf{H}, i=0,\dots,n, \tag{41}$$

that is,

$$\int\limits_{\mathbb{T}^n} \left\langle g_i\left(x,\frac{x}{\varepsilon},\omega\right), \varphi(x)\right\rangle dx \longrightarrow \int\limits_{\mathbb{T}^n} \left\langle g_i^{\mathrm{hom}}\,(x)\,, \varphi(x)\right\rangle dx \quad (\varepsilon \to 0+)$$

for any $\varphi \in \mathbf{H}$.

That is, the vector function $g(x,\frac{x}{\varepsilon},\omega) = g_0\left(x,\frac{x}{\varepsilon},\omega\right) + \sum_{i=1}^n \partial_{x_i} g_i\left(x,\frac{x}{\varepsilon},\omega\right)$ belongs to the space $\mathbf{H}^{-1}$, is uniformly bounded (w.r.t. $\varepsilon \in (0,1]$) and almost surely

$$g\left(x,\frac{x}{\varepsilon},\omega\right) \rightharpoonup g^{\mathrm{hom}}(x)\ (\varepsilon \to 0+) \text{ weakly in } \mathbf{H}^{-1}, \tag{42}$$

where

$$g^{\mathrm{hom}}(x) = g_0^{\mathrm{hom}}(x) + \sum_{i=1}^n \partial_{x_i} g_i^{\mathrm{hom}}(x).$$

This function clearly also belongs to $\mathbf{H}^{-1}$.

We note that the $\mathbf{H}$-norms of the functions

$$\partial_{x_i} g_i\left(x,\frac{x}{\varepsilon},\omega\right) = g_{i\,x_i}\left(x,\frac{x}{\varepsilon},\omega\right) + \frac{1}{\varepsilon} g_{i\,\xi_i}\left(x,\frac{x}{\varepsilon},\omega\right)$$

may tend to infinity as $\varepsilon \to 0+$. These functions are bounded in the space $\mathbf{H}^{-1}$ only.

Remark 5. As examples, we can consider the functions R, β and g to be periodic or quasiperiodic in $\omega \in \Omega$, where the probability space Ω is a unit cube in $\mathbb{R}^d$ with Lebesgue measure and with corresponding translation dynamical system on it (see Subsection 2.1).

We study the weak solutions to system (2), that is, the functions

$$u(x,t) \in L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{H}^1) \cap L_p^{loc}(\mathbb{R}_+;\mathbf{L}_p)$$

which satisfy (36) in the sense of distributions of the space $D'(\mathbb{R}_+;\mathbf{H}^{-r})$, (see Section 2). For every $u_0 \in \mathbf{H}$ there exists at least one weak solution $u(x,t)$ of the equation (36) such that $u(0) = u_0$. This solution is not necessarily unique because we do not assume that the function β satisfies inequality (10).

We denote by $\mathcal{K}_\varepsilon^+$ the set of all weak solutions of (36).

For every fixed $\varepsilon > 0$, the GL equation (36) satisfies assumptions of Theorems 3.4 and 3.6. Therefore, there exist the trajectory attractor $\mathfrak{A}_\varepsilon$ in topological space Θ_+^{loc}.

Moreover, the sets $\mathfrak{A}_\varepsilon$ are uniformly (in $\varepsilon \in (0,1]$) bounded in $\mathcal{F}_+^b$ since the constant D_2 in (32) depends only on R_1 and $\|g\|^2_{\mathbf{H}^{-1}}$ which are independent of ε due to (37) and (40). We obtain the following

Proposition 7. *Under the hypotheses (37), (38), (39), (40), and (42) the GL equation (36) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is almost surely uniformly (in $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon.$$

The kernel $\mathcal{K}_\varepsilon$ is non-empty, uniformly (in $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$ and compact in Θ^{loc}.

Along with the system (36) we consider the homogenized GL equation

$$\partial_t u_0 = (1+\alpha i)\Delta u_0 + R^{\text{hom}}(x) u_0 + \left(1+\beta^{\text{hom}}(x) i\right) |u_0|^2 u_0 + g^{\text{hom}}(x). \tag{43}$$

It is clear that the system (43) also has the trajectory attractor $\mathfrak{A}^{\text{hom}}$ in the trajectory space $\mathcal{K}^{\text{hom}+}$ corresponding to the equation (43) and

$$\mathfrak{A}^{\text{hom}} = \Pi_+ \mathcal{K}^{\text{hom}},$$

where $\mathcal{K}^{\text{hom}}$ is the kernel of (43) in $\mathcal{F}^b$ (see Theorems 3.4 and 3.6).

Let us formulate the main result of the paper concerning the trajectory attractors of Ginzburg-Landau equations.

Theorem 4.1. *The following limit holds almost surely in the topological space Θ_+^{loc}*

$$\mathfrak{A}_\varepsilon \longrightarrow \mathfrak{A}^{\text{hom}} \quad \text{as } \varepsilon \to 0+\,. \tag{44}$$

Moreover, almost surely

$$\mathcal{K}_\varepsilon \longrightarrow \mathcal{K}^{\text{hom}} \text{ as } \varepsilon \to 0+ \text{ in } \Theta^{loc}. \tag{45}$$

Proof of Theorem 4.1. It is clear that (45) implies (44). Therefore, it is sufficient to prove (45), that is, for every neighborhood $\mathcal{O}(\mathcal{K}^{\text{hom}}) \subset \Theta^{loc}$ there exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that, almost surely,

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\mathcal{K}^{\text{hom}}) \text{ for } \varepsilon < \varepsilon_1. \tag{46}$$

Suppose that (45) is not true. Consider the corresponding subset $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and (45) does not hold for all $\omega \in \Omega'$. Then, for each $\omega \in \Omega'$, there exists a neighborhood $\mathcal{O}'(\mathcal{K}^{\text{hom}}) \subset \Theta^{loc}$, a sequence $\varepsilon_n \to 0+$ $(n \to \infty)$, and a sequence $u_{\varepsilon_n}(\cdot) = u_{\varepsilon_n}(\omega, t) \in \mathcal{K}_{\varepsilon_n}$ such that

$$u_{\varepsilon_n} \notin \mathcal{O}'(\mathcal{K}^{\text{hom}}) \text{ for all } n \in \mathbb{N}, \omega \in \Omega'. \tag{47}$$

For each $\omega \in \Omega'$, the function $u_{\varepsilon_n}(t), t \in \mathbb{R}$ is the solution to the equation

$$\begin{aligned}\partial_t u_{\varepsilon_n} = [1+\alpha i]\, \Delta u_{\varepsilon_n} + R\left(x, \frac{x}{\varepsilon_n}, \omega\right) u_{\varepsilon_n} + \\ \left[1 + \beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) i\right] |u_{\varepsilon_n}|^2 u_{\varepsilon_n} + g\left(x, \frac{x}{\varepsilon_n}, \omega\right), \quad t \in \mathbb{R}.\end{aligned} \tag{48}$$

Moreover, the sequence $\{u_{\varepsilon_n}(t)\}$ is bounded in $\mathcal{F}^b$ for each $\omega \in \Omega'$, that is,

$$\begin{aligned}\|u_{\varepsilon_n}\|_{\mathcal{F}^b} = \ & \sup_{t\in\mathbb{R}} \|u_{\varepsilon_n}(t)\|_{\mathbf{H}} + \\ & \sup_{t\in\mathbb{R}} \left(\int\limits_t^{t+1} \|u_{\varepsilon_n}(t)\|^2_{\mathbf{H}^1}\, ds \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int\limits_t^{t+1} \|u_{\varepsilon_n}(t)\|^4_{\mathbf{L}_4}\, ds \right)^{1/4} + \\ & \sup_{t\in\mathbb{R}} \left(\int\limits_t^{t+1} \|\partial_t u_{\varepsilon_n}(t)\|^{4/3}_{\mathbf{H}^{-r}}\, ds \right)^{3/4} \leqslant C \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{49}$$

Hence there exists a subsequence, $\{u_{\varepsilon_n'}(t)\} \subset \{u_{\varepsilon_n}(t)\}$, for which we keep the same notation such that

$$u_{\varepsilon_n}(t) \longrightarrow u_0(t) \text{ as } n \to \infty \ \text{ in } \Theta^{loc}, \tag{50}$$

where $u_0(t) \in \mathcal{F}^b$ and $u_0(t)$ satisfies (49) with the same constant C. In detail, we have that $u_{\varepsilon_n}(t) \rightharpoonup u_0(t)$ $(n \to \infty)$ weakly in $L^{loc}_{2,w}(\mathbb{R}; \mathbf{H}^1)$, weakly in $L^{loc}_{4,w}(\mathbb{R}; \mathbf{L}_4)$, $*$-weakly in $L^{loc}_{\infty,*w}(\mathbb{R}_+; \mathbf{H})$ and $\partial_t u_{\varepsilon_n}(t) \rightharpoonup \partial_t u_0(t)$ weakly in $L^{loc}_{4/3,w}(\mathbb{R}; \mathbf{H}^{-r})$. We claim that $u_0(t) \in \mathcal{K}^{\text{hom}}$. We have already proved that $\|u_0\|_{\mathcal{F}^b} \leqslant C$. Then we have to establish that $u_0(t)$ is a weak solution of (43). Since the derivative operators are continuous in the space of distributions, then using (49) and (42) we obtain that

$$\partial_t u_{\varepsilon_n} - (1+\alpha i)\Delta u_{\varepsilon_n} - g\left(x, \frac{x}{\varepsilon_n}, \omega\right) \quad \to \quad \partial_t u_0 - (1+\alpha i)\Delta u_0 - g^{\text{hom}}(x) \tag{51}$$

as $n \to \infty$ in the space $D'(\mathbb{R}; \mathbf{H}^{-r})$. Let us prove that

$$R\left(x, \frac{x}{\varepsilon_n}, \omega\right) u_{\varepsilon_n} \rightharpoonup R^{\text{hom}}(x)\, u_0 \quad \text{and} \quad \beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) |u_{\varepsilon_n}|^2 u_{\varepsilon_n} \rightharpoonup \beta^{\text{hom}}(x)\, |u_0|^2 u_0 \tag{52}$$

as $n \to \infty$ weakly in $L^{loc}_{4/3,w}(\mathbb{R}; \mathbf{L}_{4/3})$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_n}(t)\}$ is bounded in $L_4(-M, M; \mathbf{L}_4)$ (see (49)). Hence the sequence $\{|u_{\varepsilon_n}(t)|^2 u_{\varepsilon_n}(t)\}$ is bounded in $L_{4/3}\left(-M, M; \mathbf{L}_{4/3}\right)$. Since $\{u_{\varepsilon_n}(t)\}$ is bounded in $L_2\left(-M, M; \mathbf{H}^1\right)$ and $\{\partial_t u_{\varepsilon_n}(t)\}$ is bounded in $L_{4/3}(-M, M; \mathbf{H}^{-r})$, then we can assume that $u_{\varepsilon_n}(t) \to u_0(t)$ as $n \to \infty$ strongly in $L_2(-M, M; \mathbf{L}_2) = L_2(\mathbb{T}^n \times (-M, M))^2$ and therefore

$$u_{\varepsilon_n}(x,t) \longrightarrow u_0(x,t) \text{ as } n \to \infty \text{ a.e. in } (x,t) \in \mathbb{T}^n \times (-M, M).$$

We also conclude that

$$|u_{\varepsilon_n}(x,t)|^2 u_{\varepsilon_n}(x,t) \to |u_0(x,t)|^2 u_0(x,t) \text{ as } n \to \infty \text{ a.e. in } (x,t) \in \mathbb{T}^n \times (-M, M). \tag{53}$$

Then we have

$$\begin{aligned}&\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) |u_{\varepsilon_n}|^2 u_{\varepsilon_n} - \beta^{\text{hom}}(x)\, |u_0|^2 u_0 = \\ &\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) \left(|u_{\varepsilon_n}|^2 u_{\varepsilon_n} - |u_0|^2 u_0\right) + \left(\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) - \beta^{\text{hom}}(x)\right) |u_0|^2 u_0.\end{aligned} \tag{54}$$

Let us show that both terms on the right-hand side of (54) converge to zero as $n \to \infty$ weakly in $L_{4/3}\left((-M, M); \mathbf{L}_{4/3}\right) = \left(L_{4/3}(\mathbb{T}^n \times (-M, M))\right)^2$. The sequence

$$\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) \left(|u_{\varepsilon_n}|^2 u_{\varepsilon_n} - |u_0|^2 u_0\right)$$

tends to zero as $n \to \infty$ almost everywhere in $(x,t) \in \mathbb{T}^n \times (-M,M)$ (see (53)) and is bounded in $\left(L_{4/3}\left(\mathbb{T}^n \times (-M,M)\right)\right)^2$ (see (37)). Therefore Lemma 1.3 from [40, Chapter 1, Section 1] implies that

$$\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right)\left(|u_{\varepsilon_n}|^2 u_{\varepsilon_n} - |u_0|^2 u_0\right) \rightharpoonup 0 \text{ as } n \to \infty$$

weakly in $\left(L_{4/3}\left(\mathbb{T}^n \times (-M,M)\right)\right)^2$. The sequence $\left(\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) - \beta^{\mathrm{hom}}(x)\right)|u_0|^2 u_0$ also goes to zero as $n \to \infty$ weakly in $\left(L_{4/3}\left(\mathbb{T}^n \times (-M,M)\right)\right)^2$ since by the assumption, $\beta\left(x, \frac{x}{\varepsilon_n}, \omega\right) \rightharpoonup \beta^{\mathrm{hom}}(x)$ as $n \to \infty$ $*$-weakly in $L_{\infty,*w}\left((-M,M); \mathbf{L}_\infty\right)$ and $|u_0|^2 u_0 \in \left(L_{4/3}\left(\mathbb{T}^n \times (-M,M)\right)\right)^2$. We have proved the second convergence result in (52). In order to prove the first convergence result in (52) we proceed in a similar way. Using (51) and (52) we pass to the limit in the equation (48) as $n \to \infty$ in the space $D'\left(\mathbb{R}_+; \mathbf{H}^{-r}\right)$ and we obtain that the function $u_0(x,t)$ satisfies the equation

$$\partial_t u_0 = (1 + \alpha i)\Delta u_0 + R^{\mathrm{hom}}(x) u_0 + \left(1 + \beta^{\mathrm{hom}}(x) i\right)|u_0|^2 u_0 + g^{\mathrm{hom}}(x), \ t \in \mathbb{R}.$$

Consequently, $u_0 \in \mathcal{K}^{\mathrm{hom}}$. It was proved above that $u_{\varepsilon_n}(t) \longrightarrow u_0(t)$ as $n \to \infty$ in Θ^{loc} for each $\omega \in \Omega'$. The hypotheses $u_{\varepsilon_n}(t) \notin \mathcal{O}'(\mathcal{K}^{\mathrm{hom}})$ implies that $u_0 \notin \mathcal{O}'(\mathcal{K}^{\mathrm{hom}})$ and moreover $u_0 \notin \mathcal{K}^{\mathrm{hom}}$ for all $\omega \in \Omega'$. This contradiction completes the proof of the theorem. □

Since the topology Θ_+^{loc} is stronger that the local strong topologies $L_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ and $C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$ for each $\delta > 0$, similar to Corollary 2 we obtain the following

Corollary 3. *For every* $0 < \delta \leqslant 1$ *and any* $M > 0$*, almost surely*

$$\mathrm{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\mathfrak{A}^{\mathrm{hom}}\right) \to 0,$$

$$\mathrm{dist}_{C([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\mathfrak{A}^{\mathrm{hom}}\right) \to 0 \quad (\varepsilon \to 0+) \tag{55}$$

Finally, we consider the case of GL equation (36) for which the dispersion coefficient $\beta(x,\xi,\omega)$ satisfies almost surely the inequality

$$|\beta(x,\xi,\omega)| \leqslant \sqrt{3}, \ \forall x \in \mathbb{T}^n, \ \xi \in \mathbb{R}^n, \ \omega \in \Omega.$$

that is, the corresponding Cauchy problem has a unique solution and generates the semigroup $\{S_\varepsilon(t)\}$ acting in $\mathbf{H}$. The averaged equation (43) also generates the semigroup $\{S^{\mathrm{hom}}(t)\}$ acting in $\mathbf{H}$. Moreover, these semigroups possess the global attractors $\mathcal{A}_\varepsilon$ and $\mathcal{A}^{\mathrm{hom}}$ (see Section 3). Using formula (33) and limit relation (55) we obtain

Corollary 4. *The following limit holds almost surely:*

$$\mathrm{dist}_{H^{-\delta}}\left(\mathcal{A}_\varepsilon, \mathcal{A}^{\mathrm{hom}}\right) \to 0 \ (\varepsilon \to 0+) \ \forall \delta > 0.$$

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