Shortest and Straightest geodesics in Sub-Riemannian Geometry

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Dedicated to Jubilee of Joseph Krasil'shchik

Abstract

There are several different, but equivalent definitions of geodesics in a Riemannian manifold, based on two characteristic properties: geodesics as shortest curves and geodesics as straightest curves. They are generalized to sub-Riemannian manifolds, but become non-equivalent. We give an overview of different approach to definition, study and generalisation of sub-Riemannian geodesics and interrelations between them. For Chaplygin transversally homogeneous sub-Riemannian manifold Q, we prove that straightest geodesics (defines as geodesics of the Schouten partial connection) coincides with shortest geodesics. defined as the projection to Q of integral curves (with trivial initial covector) of the sub-Riemannian Hamiltonian system.
This gives a hamiltonization of Chaplygin systems in non-holonomic mechanics.
We describe some classes of homogeneous sub-Riemannian manifolds, where straightest geodesics coincides with shortest geodesics. In particular, we give a description of sub-Riemannian symmetric spaces in terms of affine symmetric spaces.

Contents

*The work was partially supported by the grant no. 18-00496S of the Czech Science Foundation

1 Introduction

The important role of Riemannian geometry in applications is based on the fact that many important equations , arising in mechanics, mathematical physics , biology, economy, information theory etc., can be reduced to the geodesic equation. Moreover, Riemannian geometry gives an effective tool for investigation of geodesic equation and other equations associated with the metric (Laplace, wave, heat and Schrödinger equation, Einstein equation, Yang-Mills equation etc).
There are many equivalent definitions of geodesics in a Riemannian manifold. They are naturally generalised to sub-Riemannian manifold, but become non-equivalent. H.R. Herz remarked that there are two main approaches for definition of geodesics: geodesics as **shortest curves** based on Mopertrui's principle of least action (variational approach) and geodesics as **straightest curves** based on d'Alembert's principle of virtual work (which leads to geometric descriptions based on the notion of parallel transport).

We consider three variational definitions of sub-Riemannian geodesics as (locally) shortest curves (Euler-Lagrange (EL-geodesics), Pontyagin (P-geodesics) and Hamilton (H-geodesics)) and
three definitions of sub-Riemannian geodesics as straightest or autoparallel curves, used in nonholonomic mechanics. (d'Alembert (dA-geodesics), Levi-Civita-Schouten -Synge-Vrancheanu (S-geodesics) and Cartan-Tanaka-Morimoto (M-geodesics)) and discuss interrelations between them.

A. Vershik and L. Faddeev [V-F] had formulated the problem of characterisation of sub-Riemannian manifolds such that straightest S-geodesics "coincides"(more precisely, consistent) with shortest H-geodesics in the following sense.
It is known that an S-geodesic of a sub-Riemannian manifold (Q, D, g^D) is determined by the initial velocity $v \in D_q \subset T_qQ$, and the initial data for an H-geodesic is a pair $(v, \lambda) \in D_q \times D_q^0$ where $D^0 := \text{AnnD} \subset T^*Q$ is the codistribution ((the annihilator of the distribution D).
Due to this , following [V-F], we say that straightest geodesics coincides with shortest geodesics if the class of straightest geodesics coincides with the class of shortest geodesics with zero codistribution covector λ.

Vershik and Faddeev showed that for a generic sub-Riemaniian manifold all shortest geodesics are different from straightest geodesics. They gave first example (left invariant distribution on a compact Lie group with a bi-invariant metric) when shortest geodesics coincides with straightest geodesics (with zero initial covector λ).

We generalised this example and show that this is true for any G-invariant sub-Riemannian Chaplygin metric $(D = \ker(\varpi), g^D)$ on a principal bundle $\pi : Q \to M = Q/G$ with a principal connection $\varpi : TQ \to \mathfrak{g}$, associated to a Riemannian metric g^M on M. Moreover, sub-Riemannian H-geodesics of g^D are the horizontal lifts of the projection to M of geodesics of the standard extension of g^D to a metric g^Q of Q , defined by any left invariant metric of G. The (straightest) S-geodesic are horizontal lifts of geodesics of the Riemannian manifold (M, g).

This is a generalization of results by R. Montgomery [Mont], [Mont1] who considered the bi-invariant extension g^Q of g^D, associated to a bi-invariant metric on G.
We give a simple proof of Wong results on description of the evolution of charge

particle in a classical Yang-Mills field in terms of geodesics of the bi-invariant extension g^Q of the Chaplugin sub-Riemannian metric and clarify relations between sub-Riemannian geodesics and geodesics of g^Q.

We describe some classes of invariant sub-Riemannian structures on homogeneous manifolds, where straightest geodesics coincides be shortest ones.
In particular, we give a description of all bracket generated symmetric sub-Riemannian manifolds , introduced by R.S.Strichartz [Str], and indicate a construction of compact sub-Riemannian symmetric space associated to a graded complex semisimple Lie algebra.

Acknowledgment. I thank A.M. Vershik for his comments on non-holonomic geometry and explanation of his joint with L.D. Faddeev results and A. V. Bolsinov for information about his results on integrability of geodesic flows and useful discussions.

2 Sub-Riemannian geodesics as shortest curves

2.1 Euler-Lagrange variational definition of sub-Riemannian geodesics (EL-geodesics)

Recall that a distribution $D \subset TM$ on a connected n dimensional manifold M is called bracket generated (or totally nonholonomic) if the space ΓD of sections generates the Lie algebra $\mathfrak{X}(M)$ of vector fields.
According to Rashevsky-Chow theorem , any two points on such manifold can be joint by a horizontal (i.e. tangent to D) curve. Then any Lagrangian $L \in C^\infty(TQ)$ defines a nonholonomic variational problem:
Find a curve $q(t)$, $t \in [0, T]$ in the space $C^D(q_0, q_1)$ of horizontal curves,joints points $q_0, q_1 \in Q$, which delivers a minimum or, more generally, a critical point, for the action functional

$$A(\gamma) = \int_0^T L(q(t), \dot{q(t)})dt.$$

The Lagrangian $L(q, \dot{q})$ determines a horizontal 1-form

$$F_L = (\delta L)_i dq^i := (\frac{d}{dt} L_{\dot{q}_i} - L_{q_i})dq^i$$

on TQ, called the Lagrangian force [Sy] [Sy1], [V-F]. Denote by $(\omega^1, \cdots, \omega^m)$ a coframe of D^0 such that the 1-form

$$\omega_\lambda = \sum \lambda_a(q, \dot{q})\omega^a_i(q)dq^i$$

vanishes on D for any functions $\lambda(q, \dot{q}) = (\lambda_1, \cdots, \lambda_m)$ on TQ.

Then critical points $q(t)$ of $A(q(t))$ are solution of the Euler-Lagrange equations

$$(\delta L)_i \dot{q}^i(t) = (\frac{d}{dt}\omega_\lambda)_i \dot{q}^i \tag{1}$$

$$\omega_\lambda(\dot{q}) = 0$$

for unknown curve $q(t) \in C^L(q_0, q_1)$ and vector function $\lambda(q(t), \dot{q}(t))$. The r.h.s. of (1) can be written as

$$\frac{d}{dt}\omega_\lambda = \mathcal{L}_{\dot{q}(t)}\omega_\lambda = \dot{\lambda}_a \omega^a + \lambda_a \dot{q} \lrcorner d\omega^a.$$

An sub-Riemannian manifold (Q, D, g^D) is a manifold with a distribution D and a Riemannian metric g^D on D . We assume that D is bracket generated.
An Euler-Lagrange (shortly, EL) non-parametrized geodesic of an sub-Riemannian manifold is a critical point of the length functional ($L = \sqrt{g(\dot{q}, \dot{q})}$) and a naturally parametrized EL geodesic is a critical point of the energy functional ($L = \frac{1}{2}g(\dot{q}, \dot{q})$) in $C^D(q_0, q_1)$. It is known that any curve in $C^D(q_0, q_1)$ of minimal length is an EL geodesic.

2.2 Pontryagin optimal control definition of sub-Riemannian geodesics (P-geodesics)

Let (Q, D, g_D) be a bracket generated sub-Riemannian manifold as above. Denote by $(X_1, \cdots, X_m)$ a field of orthonormal frames in D. Then any horizontal curve $q(t) \in C^D(q_0, q_1)$ is a solution of the first order ODE

$$\dot{q}^(t) = \sum_{a=1}^{m} u^a(t) X_a(q(t)),\ q(0) = q_0. \tag{2}$$

where q^i are coordinates in Q and vector-function $u(t) = (u^1(t), \cdots, u^m(t))$ (called the control) consists of the coordinates of $\dot{q}$ with respect to the frame (X_a) .

The length $\ell(u) = \int_0^T \sqrt{\sum u^a(t)^2} dt$ of the curve $q(t)$ depends only on the control $u(t)$ and is called the **cost function.** It is the functional of the space of **admissible control** $u(t)$ such that the associated equation has solution $q^u(t) \in C^D(q_0, q_1)$.

Pontryagin (or P) geodesic (resp. minimal P-geodesic) is the integral curve $q^u(t) \in C^D(q_0, q_1)$ of ODE with a control $u(t)$ which is a critical point (resp., minimum) of the cost functional in the space of admissible control.

P-geodesics coincide with EL-geodesics.

For $D = TQ$, P-geodesics are geodesics of the Riemannian manifold (Q, g^Q).

2.3 Hamiltonian definition of sub-Riemannian geodesics (H-geodesics) of a sub-Riemannian manifold

We will denote by ξ^D the restriction of a covector $\xi \in T_q^*Q$ to D and define the **cometric** as the contravariant symmetric tensor $g_D^* \in \Gamma S^2(TQ)$, given by

$$g^*(\xi, \xi) = (g^D)-1(\xi^D, \xi^D) \in C^\infty(T^*Q).$$

The **sub-Riemannian Hamiltonian** is defined as

$$h = h_{g^D} := \frac{1}{2} g^*(\xi, \xi).$$

H-geodesics are projection to Q of orbits of Hamiltonian vector field $\vec{h} = \omega^{-1} dh \in \mathfrak{X}(T^*Q)$ with quadratic (degenerate) sub-Riemannian Hamiltonian $h = \frac{1}{2} g^*(\xi, \xi)$. Here $\omega = dp_i \wedge dq^i$ is the standard symplectic form of T^*Q.

2.4 Pontryagin Maximum Principle for bracket generated sub-Riemannian manifold

Recall that vector fields $X = X^i \partial_{q^i}$ bijectively corresponds to fiberwise linear functions

$$p_X : T^*Q \to \mathbb{R},\ \xi = p_i \partial_{q^i} \mapsto p(X) = X^i p_i$$

on the cotangent bundle T^*Q. The function p_X is the Hamiltonian of the Hamiltonian vector field

$$\vec{p}_X = X^i \partial_{q^i} - p_i \partial_{q^j} X^i \partial_{p_j},$$

which is the complete lift of X onto T^*Q. The map

$$\mathfrak{X}(Q) \ni X \to \vec{p}_X \in \mathfrak{X}(T^*Q)$$

is an isomorphism of the Lie algebra $\mathfrak{X}(Q)$ onto the Lie algebra $\mathfrak{X}(T^*Q)^1$ of fiberwise linear vector fields on T^*Q.

Theorem 1 *(Pontryagin Maximim Principle) Let $q(t) \in C^D(q_0, q_1)$ be a minimal P-geodesic on a sub-Riemannian manifold (Q, D, g_D) with natural parametrization (s.t. $|\dot{q}(t)| = const$), which corresponds to a control $u(t) = (u_a(t))$:*

$$\dot{q}(t) = u^a(t) X_a(q(t)). \tag{3}$$

Denote by φ_t the flow , generated by the non-autonomous vector field $X^u = u^a(t)X_a$. Then for some covector $\xi_0 \in T^*_{q_0}Q$ the curve

$$\xi(t) := \varphi^*_{-t}\xi_0 := \xi_0 \circ \varphi_{-t_*} \in T^*_{q(t)}$$

satisfies the equation

$$\dot{\xi}(t) = u^a(t) \vec{p}_a(q(t)) \tag{4}$$

where $p_a := p_{X_a}$ and one of the following conditions holds

$$\begin{array}{ll} (N) & u^a(t) \equiv < \xi(t), X_a(q(t)) > \\ (A) & \quad 0 \equiv < \xi(t), X_i(q(t)) > . \end{array}$$

Here the bracket $< \xi, X >$ denotes the pairing between covectors and vectors.

An extremal curve $\xi(t) \subset T^*Q$, which satisfy (N) (resp., (A)) is called **normal** (resp., **abnormal**) **extremal**, and its projection $q(t) \subset Q$ is called a **normal** (resp., **abnormal**) **P-geodesic**.

2.4.1 Normal P-geodesics as H-geodesics

Corollary 2 *Normal extremal $\xi(t) \subset T^*Q$ for (Q, D, g_D) is an integral curve of the Hamiltionian equation on T^*Q with the sub-Riemannian Hamiltonian $h_{g^D}(\xi) = \frac{1}{2} g^*_D(\xi, \xi) = \frac{1}{2} \sum (g^D)^{ij} p^D_i p^D_j$ where $p = p_i dx^i \in T^*Q$, $p^D_i = p_i|_D$.*

Proof: In the case of normal geodesic, the equation (4) take the form

$$\dot{\xi}(t) = \sum p_a(\xi(t)) \vec{p}_a(\xi(t)) = \omega^{-1} d \frac{1}{2} \sum p^2_a(\xi(t)) = \vec{h}_{g^D}.$$

□

Since the Hamiltonian vector field $\vec{H}$ preserves the Hamiltonian, the normal extremal $\xi(t)$ belongs to a level set $L_c = \{H = c\} \subset T^*Q$ of the Hamiltonian.

A curve $\xi(t) \subset L$ on a submanifold $L \subset T^*Q$ is called **characteristic** if its velocity $\dot{\xi}(t)$ belongs to the kernel ker $\omega|L$ of the restriction of the symplectic form to L.

Corollary 3 *Assume that an extremal $\xi(t) \subset L_c$ belongs to a regular level set of the Hamiltonian $h = h_{g^Q}$, i.e. $L_c \subset T^*Q$ is a smooth hypersurface. Then* ker $\omega|_{L_c}$ *is the 1-dimensional distribution generated by $\vec{H}$. In particular, the extremals $\xi(t)$ are the characteristic curves of L_c.*

Proof: The tangent space of the level set L_c is described as follows

$$T_\xi L_c = \{w \in T_\xi(T^*Q), 0 =< dh_\xi, w >=< \omega(\vec{h})_\xi, w >\}.$$

This shows that $T_\xi L_c$ consists of all ω-orthogonal to $\vec{h}_\xi$ vectors. Since the $\omega|T_\xi L_c$ has 1-dimensional kernel, it is generated by $\vec{h}_\xi$. □

2.4.2 Abnormal P-geodesics

Denote by $C^D(q_0) = \{\gamma : [0,T] \to Q,\, \gamma(0) = q_0, \dot{\gamma} \in D\}$ the space of horizontal curves, starting from q_0 , where $D \subset TQ$ is a bracket generated distribution. A curve $\gamma(t) \in C^D(q_0)$ is called **singular (regular)** if the end-point map

$$\varepsilon : C^D(q_0) \to Q,\, \gamma(t) \mapsto g(T)$$

is singular (resp., regular).
The following theorem by R. Montgomery shows that abnormal geodesics coincides with skingular curves and they are projection on Q of characteristic curves of the codistribution D^0, considered as a submanifold of the symplectic manifold (T^*Q, ω).

Theorem 4 *([Mont], [Mont1]) i) Abnormal geodesics of any* sub-Riemannian metric g^D on D are exactly singular horizontal curves in Q.
ii) A horizontal curve $\gamma \subset Q$ is singular if and only if it is a projection to Q of a characteristic curve of the submanifold $\tilde{D^0} := D^0 \setminus \{\text{zero section}\} \subset T^*Q$.

Now we give a description of characteristic curves in $\tilde{D^0}$, following [Mont]. Denote by $\tau : T^*Q \to Q$, $\rho : T^*(T^*Q) \to T^*Q$ the natural projections.

We fix a complementary to D distribution V such that $TQ = D{+}V$. Let X_i be an orthonormal frame in D , Y_a a frame in V and $\theta^i \in \Gamma D^* = \Gamma V^0$ and $\eta^a \in \Gamma V^* = \Gamma D^0$ the dual coframes. The Liouville tautological 1-form $\theta_\xi = \xi \circ \tau_*$ in T^*Q (where $\tau_* : T(T^*Q) \to TQ$ is the induces projection) at a point $\xi = h_i\theta^i + k_a\eta^a \in T^*Q$ can be written as

$$\theta_\xi = h_i\theta^i + k_a\eta^a.$$

Then the restrictions $\theta^0 := \theta|_{D^0}, \omega^0 = d\theta|_{D^0}$ of the forms $\theta, \omega = d\theta$ to D^0 are given by

$$\theta^0_\xi = \xi|_D = k_a\eta^a,\, \omega^0 = d\theta^0 = dk_a \wedge \eta^a + k_a d\eta^a.$$

By definition, characteristic curves are non-vanishing curves $\eta(t) \subset D^0$ tangent to the distribution (with singularities)

$$ch(D^0) := \ker \omega^0 = T(D^0)^{\perp} \cap T(D^0),$$

where the vector bundle $T(D^0)^{\perp} \subset T(T^*Q)$ is the ω-orthogonal complement to the tangent bundle $T(D^0) \subset T(T^*Q)|_{D^0}$.

We consider h_i, k_a as fiberwise coordinates in $\tau : T^*Q \to Q$.

Lemma 5 *The vector bundle* $T(D^0)^{\perp} = \mathrm{span}\{\vec{\eta}_i,\ i = 1, \cdots, m\}$ *and the projection* $\tau_* : T(T^*Q) \to TQ)$ *induces for any* $\eta \in D^0$ *the isomorphism*

$$\tau_* : T_\eta(D^0)^{\perp} \to D_{\tau(\eta)},$$

$$u^i\vec{h}_i \mapsto u^i X_i|_q,\ q = \tau(\eta).$$

Proof: The submanifold $D^0 = \{\eta = k_a\eta^a\}$ is defined by the equations $h_i = 0$, $i = 1, \cdots, m$. Hence, $T_\eta D^0 = \{v \in T_\eta(T^*Q), 0 =< dh_i, v >= \omega(\vec{h}_i, v)\}$. □

Since $D^0 = \{\eta = k_a\eta^a\}$, k_a are fiberwise coordinate in $D^0 \to Q$. Hence ∂_{k_a}, X_i, Y_a is a frame in the tangent bundle $T(D^0)$ and the tangent vector to a curve $\gamma(t) = k_a(t)\eta^a(t) \subset D^0$ can be written as

$$\dot{\gamma}(t) = \dot{k}_a\partial_{k_a} + \dot{\gamma}^i X_i(\gamma(t)) + \dot{\gamma}^a Y_a(\gamma(t)).$$

The restriction to $D^0 \subset T^*Q$ of the canonical form $\theta = h_i\theta^i + k_a\eta^a$ is given by $\theta^0 = k_a\eta^a$ and the form $\omega^0 = d\theta^0 = dk_a \wedge \eta^a + k_a d\eta^a$. The 2-form $d\eta^a$ can be written as

$$d\eta^a = c^a_{bd}\eta^b \wedge \eta^d + c^a_{ib}\theta^i \wedge \eta^b + c^a_{ij}\theta^i \wedge \theta^j$$

where $c^a_{ij} = -\eta^a([X_i, X_j])$, $c^a_{bi} = -\eta^a([Y_b, X_i])$, $c^a_{bd} = -\eta^a(Y_b, Y_d)$.

Calculating the contraction $i_{\dot{\gamma}}\omega^0 = A^a dk_a + B_a\eta^a + C_i\theta^i$ of the tangent vector

$$\dot{\gamma} = \dot{k}_\alpha\partial_{k_\alpha} + \dot{\gamma}^i X_i + \dot{\gamma}^a Y_a \in T(D^0)$$

with 2-form ω^0 , we obtain the following conditions that $\dot{\gamma}$ is in $ch(D^0)$:

$$\begin{array}{llll} i) & A^a = & \dot{\gamma}^a & = 0, \\ ii) & B^a = & \dot{k}_a + k_b c^b_{ia}\dot{\gamma}^i & = 0, \\ iii) & C^i = & k_a c^a_{ij}\dot{\gamma}^j & = 0. \end{array}$$

Denote by $\bar{d} : D_x \to \Lambda^2 D_x$ the linear map, defined by

$$\bar{d}\eta = d\tilde{\eta}|_{\Lambda^2 D_x},$$

where $\tilde{\eta}$ is an extensions of η . Then

$$\bar{d}\tilde{\eta}(X, X') = -\eta([\tilde{X}, \tilde{X}']).$$

This map does not depends on extensions $\tilde{\xi}, \tilde{X}, \tilde{X}'$ of the 1-form $\eta \in D^0_x \subset T^*_x Q$ and the vectors $X, X' \in D_x$. The above formulas implies the following

Theorem 6 *[Mont] Let $\eta(t) = k_a \eta^a \subset D^0$ be a characteristic curve. Then*
i) The projection $\gamma(t) := \tau(\eta(t))$ is a horizontal curve of the form $\dot{\gamma}(t) = \dot{\gamma}^i X_i$.
ii) The velocity vector

$$\dot{\eta}(t) = \dot{k}_a \partial_{k_a} + \dot{\gamma}^i X_i = k_b c^b_{ai} \dot{\gamma}^i + \dot{\gamma}^i X_i$$

is completely determines by the vector $\eta(t)$ and the velocity $\dot{\gamma}(t)$ of the projection .
iii) The velocity vector $\dot{\gamma}(t)$ belong to the kernel $\ker \bar{d}\eta(t) \subset D_{\gamma(t)}$ of the 2-form $\bar{d}\eta(t)$ on $D_{\gamma(t)}$.
iv) The characteristic distribution (with singularites) at a point $\eta \in D^0$ has rank $r = \dim \ker \bar{d}\eta$.

P-geodesics are exhausted by **normal geodesics** , which are exactly H-geodesics, and **abnormal geodesics**, which depend only on distribution D.

3 Sub-Riemannian geodesics as straightest curves

3.1 d'Alemebert's definition of sub-Riemannian geodesics (dA-geodesics)

To define dA-geodesics, we extend the sub-Riemannian metric g_D to a Riemannian metric g^Q.
We will see that in the case, when the sub-Riemannian structure is regular the deep results by N. Tanaka and T. Morimoto provide a canonical extension g^Q.

The d'Alembert's principle of virtual dispalacements for a mechanical system may be formulated as follows, see [V-F].
1) The evolution of a mechanical system with a (smooth) configuration space Q is described by projection to Q of integral curves of a special vector field $X \in \mathfrak{X}(TQ)$ on the tangent bundle TQ. The field is special if it corresponds to a second order equation , that is $\pi_* X_{(q,\dot{q})} = \dot{q}$ where $\pi : TQ \to Q$ is the projection.
2) The vector field X is determined by the Lagrangian force defined as the horizontal 1-form $F_L := (\delta L(q,\dot{q}))_i dq^i$ on TQ, associated with the Lagrangian $L(q,\dot{q})$, and the external force.
3) d'Alembert's principle states that the real trajectories of the system (defined by the special vector field X) is determined by the condition that the Lagrangian force equal to the external force.
Assume that
i)the Lagrangian of the system is quadratic in velocities and positively defined (that is can be written as $L = \frac{1}{2} g(\dot{q},\dot{q})$ where g is a Riemannian metric in Q) and that
ii) the only external force is the reaction of a non-holonomic constrain, defined by a distribution $D = \ker \eta^1 \cap \cdots \cap \ker \eta^k$, where $\eta^a = \eta^a_i dq^i$, $a = 1, cdots, k$ is a coframe of the codistribution D^0 . The reaction of the constrain is the horizontal 1-form $\phi_\lambda = \lambda_a(q,\dot{q})\eta^a$ defined by the condition that the equation $F_L - \phi_\lambda = 0$ corresponds to a vector field $X \in \mathfrak{X}(TQ)$ tangent to the distribution $D \subset TQ$.
In coordinates, this equation take form [V-G1]

$$\frac{d}{dt} L_{\dot{q}_i} - L_{q_i} = \lambda_a(q(t),\dot{q}(t))\eta^a_i$$

or

$$\frac{d}{dt}L_{\dot{q}_i} - L_{q_i} \equiv 0(\mathrm{mod}D^0).$$

The projection to Q of integral curves of this equation is called **dA-geodesic** of the sub-Riemannian metric (D, g^D) , associated with an extension of g^D to a Riemannian metric g on Q. In general, the equation of dA-geodesics is neither Lagrangian no Hamiltonian.

3.2 Levi-Civita-Schouten-Synge-Vranceanu definition of sub-Riemannian geodesics (S-geodesics)

3.2.1 Levi-Civita definition of Riemannian geodesics

Recall that Levi-Civita associated to a Riemannian manifold (Q, g) the canonical torsion free connection ∇^g, which preserves the metric (called the Levi-Civita connection) and defined a geodesic as an autoparallel curve , that is a curve $q(t)$ with satisfies the geodesic equation

$$\nabla^g_{\dot{\gamma}}\dot{\gamma} \equiv \ddot{q}^i(t) + \Gamma^i_{jk}(q^j(t))\dot{q}^j(t)\dot{q}^k(t) = 0$$

where $q^i(t)$ are coordinates of the curve $q(t)$ (with respect to local coordinate q^i) and Γ^i_{jk} the Christoffel symbols.
The extension of this notion to the sub-Riemannian manifold (Q, D, g_D) had been proposed independently by J.A. Schouten, J.L.Synge and G. Vranceanu.

3.2.2 Schouten partial connection of a sub-Riemannian manifold

Let $D \subset TQ$ be a distribution. A partial D-connection in D is a bilinear map

$$\nabla^D : \Gamma D \times \Gamma D \to \Gamma D,\ (X, Y) \mapsto \nabla^D_X Y$$

which is $C^\infty(Q)$ linear in X and satisfies the Leibnitz rule
$\nabla^D_X(fY) = f\nabla^D_X Y + X \cdot fY$ in $Y \in \Gamma D$.
Let e_a, $a = 1, \cdots, k$ be a frame of D defined in a neighborhood of a horizontal curve $q(t)$.
The Christoffel symbols are the local functions $\Gamma^a_{bc}(q)$ on Q defined by

$$(\nabla_{e_b} e_c)^a = \Gamma^a_{bc}(q)e_a.$$

The value of the functions $\Gamma^a_{bc}(t) := \Gamma^a_{bc}(q(t))$ on the curve $q(t)$ depends only of the frame $e_a(t) := e_a(q(t))$ along the curve $q(t)$. Due to this , the partial connection defines a parallel transport of a vector $Y_0 \in D_{q_0}$ along the curve $q(t)$ as the solution $Y(t) = Y^c(t)e_c(t) \in D_{q(t)}$ of the equation

$$0 = \nabla_{\dot{q}(t)}Y(t) = \nabla_{\dot{q}(t)}(Y^a(t)e_a(t)) = [\dot{Y}^a(t) + \Gamma^a_{bc}(t)q^b(t)Y^c(t)]e_a(t)$$

where $\dot{q} = \dot{q}^a e_a$.

I.A. Schouten showed that a complementary to D distribution V on a sub-Riemannian manifold $()Q, D, g_D)$ (called a **rigging**) defines a partial connection

∇^S in D with preserves the metric g_D and has zero torsion T . The torsion tensor is defined by

$$T(X,Y) = \nabla^S_X Y - \nabla^S_Y X - [X,Y]_D, \; X,Y \in \Gamma D, \; X,Y \in \Gamma D$$

where X_D is the horizontal part of the vector

$$X = X_D + X_V \in T_q Q = D_q + V_q.$$

In coordinate free way, the Schouten partial connection of (Q, D, g_D) , associated to a rigging V is defined by the Koszul formula

$$\begin{aligned} 2g(\nabla^S_X Y, Z) = \quad & X \cdot g(Y,Z) + Y \cdot g(X,Z) - Z \cdot g(X,Y) + \\ & g([X,Y]_D, Z) - g(Y,[X,Z]_D) - g(X,[Y,Z]_D), \\ & X,Y,Z \in \mathfrak{X}(Q). \end{aligned}$$

3.2.3 Schouten-Synge-Vranceasnu definition of sub-Riemannian geodesics (S-geodesics) and non-holonomic mechanics

Schouten-Synge-Vranceasnu geodesic (**S-geodesic**) of a sub-Riemannian manifold (Q, D, g^D) associated to a rigging V a is a horizontal curve $\gamma(t)$ with parallel (w.r.t. Schouten connection) tangent vector field $\dot{g}(t)$, i.e. a solution of the equation

$$\nabla^S_{\dot\gamma} \dot\gamma = 0.$$

Schouten defined also the curvature tensor $R \in \mathfrak{so}(D) \otimes \Lambda^2 T^* M$ of the Schouten connection by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - [[X,Y]_V, Z]_D, \; X,Y,Z \in \Gamma D.$$

V.V. Wagner generalized this notion and defined Wagner curvature tensor, such that the vanishing of the Wagner tensor is equivalent to the flatness of the Schouten connection (such the the associated parallel transport does not depend on the path, jointed two points), see [B], [D-G].

Assume that the sub-Riemannian metric g_D is extended to a Riemannian metric g on Q such that $g(D, V) = 0$. Then the Levi-Civita connection ∇^g induces a connection ∇^D in D . Using the projection $\mathrm{pr}_D : TQ \to D$, it is defined by

$$\nabla^D_X Y = \pi_D \nabla^g_X Y = (\nabla^g_X Y)_D, \; X \in TQ, Y \in \Gamma D.$$

The connection ∇^D in the distribution D is an extension of the partial Schouten connection ∇^S.

Theorem 7 *(Vershik-Faddeev)[V-F], [V-F1] Let* (Q, D, g_D) *be a* sub-Riemannian manifold, g is an extension of g_D to a metric in Q and $V = D^\perp$ the orthogonal complement to D. Then S-geodesics coincides with dA-geodesics and they describe evolution of the free mechanical system with kinetic energy g in configuration space Q with nonholonomic linear constrains D.

4 Cartan-Tanaka-Morimoto frame bundle definition of sub-Riemannian geodesics (M-geodesics)

4.1 Cartan definition of a Riemannian geodesics

An important definition of Riemannian geodesics as straightest curves had been proposed by E. Cartan. It is easily generalized to the case of G-structures of finite type and to sub-Riemannian manifolds (and other Tanaka structures).
A Riemannian metric g on a manifold Q can be consider as a $G = SO_n$-structure, i.e. a principal G-subbundle $\pi : P \to Q = P/G$ of orthonormal coframes (i.e. isometries $f : T_xQ \to \mathbb{R}^n = V$) with the tautological soldering form

$$\theta : TP \to \mathfrak{so}_n,\ \theta_f(X) := f(\pi_* X).$$

Remark 8 *If $f : T_qQ \to V$ is a coframe, then $f^{-1} : V \to T_qQ$ is a frame. Due to this a G-structure can be consider also as the principal G-bundle of frame with the right action of the linear group G.*

The total space P of a SO_n-structure admits a canonical SO_n-equivariant absolute parallelism (Cartan connection)

$$\kappa = \theta + \omega : TP \to V + \mathfrak{so}(V),$$

which is an extension of the vertical parallelism $i_p : T^v_pP \simeq \mathfrak{so}_n, \forall p \in P$ (defined by the free action of SO_n on P). Here $\omega : TP \to \mathfrak{so}_n$ is the connection form of the Levi-Civita connection.
Geodesics are projection to Q of constant horizontal vector fields $X \in \kappa^{-1}(V) \in \mathfrak{X}(P)$, see [K-N].

4.1.1 Cartan connection and generalized Cartan geodesics (C-geodesics)

We recall the definition of a Cartan connection and discuss definition of generalised geodesics associated with a Cartan connection.
Let $M_0 = L/G$ be a homogeneous n-dimensional manifold.
A **Cartan connection of type** $M_0 = L/G$ on n-dimensional manifold Q is a principal G-bundle $\pi : P \to Q = P/G$ together with a $\mathfrak{l}$-valued G-equivariant (s.t.$r^*_g\kappa = \mathrm{Ad}_g^{-1} \circ \kappa,\ g \in G$) kernel free 1-form

$$\kappa : TP \to \mathfrak{l}$$

which extends the vertical parallelism $T^v_pP \simeq \mathfrak{g}$.
The form κ defines an absolute parallelism $T_pP \simeq \mathfrak{l}$. Hence, tensor fields on P may be identified with tensor-valued functions.
In particular, the **curvature 2-form** $d\kappa + \frac{1}{2}[\kappa, \kappa]$ on P is horizontal and can be identified with a function $K : P \to C^2(V, \mathfrak{l}) := \mathfrak{l} \otimes \Lambda^2 V^*$ where V is complementary subspace to $\mathfrak{g}$ in $\mathfrak{l}$.
Assume that the homogeneous manifold $M_0 = L/G$ is **reductive**, i.e. there is a reductive decomposition $\mathfrak{l} = \mathfrak{g} + V$, where V is an Ad_G-invariant complement to $\mathfrak{g}$. Then the Cartan connection κ is decomposed as

$$\kappa = \theta + \omega := \mathrm{pr}_V \circ \kappa + \mathrm{pr}_{\mathfrak{g}} \circ \kappa$$

into the horizontal form θ with ker $\theta = T^v P$ and the vertical part ω with $\mathcal{H} :=$ ker $\omega = \mathrm{span}(\kappa^{-1}V)$.
Moreover, θ is a soldering form, which allows identify $\pi : P \to Q$ with a G-structure , i.e. a principal bundle of coframes, see definition below. The form ω is a connection form, which defines a principal connection with horizontal distribution $\mathcal{H}$. It indices a linear connection ∇ in Q with the holonomy group Hol $\subset G$, see [K-N].
We say that the G-structure $(\pi : P \to Q)$ with connection , defined by (θ, ω) is the G-structure, associated to a Cartan connection of reductive type L/G. We define **C-geodesics** for such Cartan connection as projection to Q of the orbits of constant (horizontal) vector fields from $\kappa^{-1}(V)$.
These geodesics are geodesics of the linear connections ∇.
This construction will be used for definition of Morimoto sub-Riemannian geodesics.

There is another important case, when the complementary distribution V can be canonically chosen and one can define C-geodesics as above.
Let $M_0 = L/G$ be a n-dimensional flag manifold , i.e. the coset space of a semisimple Lie group (real or complex) modulo parabolic subgroup G . The parabolic subalgebra $\mathfrak{g} \subset \mathfrak{l}$ defines the generalized Gauss decomposition $\mathfrak{l} = \mathfrak{l}^- + \mathfrak{l}^0 + \mathfrak{l}^+$ such that $\mathfrak{g} := \mathfrak{l}^0 + \mathfrak{l}^+$ and $\mathfrak{g}_- := \mathfrak{l}^- + \mathfrak{l}^0$ are the opposite parabolic subalgebras.
A **parabolic geometry** is the geometry of the Cartan connection $(\pi : P \to P/G = Q, \kappa : TP \to \mathfrak{l})$ of type L/G. We chose the complementary to $\mathfrak{g}$ subspace $V := \mathfrak{l}^-$ and defines C-geodesics as the projection to Q of constant vector fields from $\kappa^{-1}V$. (This definition is too general and has to be specified).

The principal bundle $\pi : P \to Q$ admits many Cartan connections. One of the problems is to define some normalization conditions of the curvature function $K : P \to C^2(V, \ell)$, which guaranties the existence of unique Cartan connection. The standard condition is that at any point $p \in P$ the 2-form $K_p \in C^2(V, \mathfrak{l})$ is coclosed $\delta^* K_p = 0$ with respect to an appropriate codifferential , dual to the Spencer differential $\delta : C^i(V, \mathfrak{l}) \to C^{i+1}(V, \mathfrak{l})$ for cohomology of a Lie algebra V with value in V-module, see [C-S], [A-D]. Existence and uniqueness of such **normal** Cartan connection for parabolic geometry is known [C-S]. A very general normalisation conditions for existence and uniqueness of normal Cartan connections is given in [M1] and [C].

4.1.2 Generalized geodesics for a G-structures of finite type

Given a linear group $G \subset GL(V)$, $V = \mathbb{R}^n$.
A **G -structure** can be defines as a G-principal bundle $\pi : P \to Q = P/G$ with a **soldering 1-form**

$$\theta : TP \to V$$

i.e. a horizontal G-equivariant form with ker $\theta = T^{vert}P$.

Such bundle is naturally identified with a principal G-bundle of coframes on TQ. Assume that the group G is of finite type k, that is its Lie algebra $\mathfrak{g}$ has non trivial k-th prolongation $\mathfrak{g}^{(k)}$ and $\mathfrak{g}^{(k+1)} = 0$. Then ([Stern]) one can prolong π to a bundle $P^{(k)} \to Q$ with absolute parallelism

$$\kappa : TP^{(k)} \to \mathfrak{g}^{\infty} = V + \mathfrak{g} + \mathfrak{g}^{(1)} + \cdots + \mathfrak{g}^{(k)}.$$

Generalised C-geodesics for the G-structure are defined as the projection of orbits of constant vector fields $\kappa^{-1}V$ to Q.

In the case $k = 0$ when the first prolongation

$$\mathfrak{g}^{(1)} = \mathfrak{g} \otimes V^* \cap V \otimes S^2 V^*$$

of the linear Lie algebra $\mathfrak{g}$ is trivial , the absolute parallelism

$$\kappa : TP \to V + \mathfrak{g}$$

is a Cartan connection. It defines a linear connection in Q with holonomy group $H \subset G$ and generalised C-geodesics are geodesics of this connection.
Let $\mathfrak{g} \subset \mathfrak{l}(V)$ be an irreducible linear Lie algebra of type 2. Then the full prolongation $\mathfrak{g}^\infty = V + \mathfrak{g} + \mathfrak{g}^{(1)} = \mathfrak{g}^{-1} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)}$ is a simple 3-graded Lie algebra. All such 3-graded Lie algebra are known. The corresponding G-structure $P \to M$ admits unique normal Cartan connection $\kappa : TP \to V + \mathfrak{g} + \mathfrak{g}^{(1)}$, see [C-S], and corresponding generalized C-geodesics form an interesting class of distinguished curves in Q.
For example, for the conformal structure , which can be considered as $\mathbb{R}^+ \cdot SO_n$-structure, generalized geodesics are conformal circles.

The theory of such generalised geodesics for different parabolic geometries had been developed by A. Čap, J. Slovak, V. Žadnik [C-S-Z], B. Doubrov and I. Zelenko [D-Z], M Herzlich [H] and others.

4.2 Cartan-Tanaka -Morimoto definition of sub-Riemannian geodesics on a regular sub-Riemannian manifold

4.2.1 The symbol algebra

A distribution $D \subset TQ$ on a manifold defines an decreasing filtration

$$0 \subset \mathcal{D}^{-1} = \Gamma D \subset \mathcal{D}^{-2} := \mathcal{D}^{-1} + [\mathcal{D}^{-1}, \mathcal{D}^{-1}] \subset \cdots$$

in the Lie algebra $\mathcal{X}(\mathcal{Q})$ of vector fields and for $q \in Q$ a flag

$$0 \subset \mathcal{D}_q^{-1} \subset \mathcal{D}_q^{-2} \subset \cdots$$

in $T_q Q$. The Lie bracket induces in the associated graded space

$$\mathfrak{m}_q = \mathfrak{m}_q^{-1} + \mathfrak{m}_q^{-2} + \cdots + \mathfrak{m}_q^{-k},\ \mathfrak{m}_q^{-i} := \mathcal{D}_q^{-i} / \mathcal{D}_q^{-i-1}$$

a structure of a negatively graded Lie algebra , called the **symbol algebra at a point q.**
By construction, the graded Lie algebra is **fundamental**, i.e. it is generated by $\mathfrak{m}^{-1}$.

4.3 Regular distributions and graded tangent bundle

The distribution $D \subset TQ$ is called **bracket generated or totally nonholonomic of depth** k if k is the minimum integer, s.t. $\mathcal{D}_q^{-k} = T_q Q$ for any $q \in Q$.
Then the negatively grades Lie algebra

$$T_q^{gr} Q = \mathfrak{m}_q = \mathfrak{m}_q^{-1} + \cdots + \mathfrak{m}_q^{-k} = \mathcal{D}_q^{-1} + \mathcal{D}_q^{-2} / \mathcal{D}_q^{-1} + \cdots + T_q Q / \mathcal{D}_q^{-(k-1)}$$

is called the **graded tangent space** at $q \in Q$.
If, moreover the symbol algebra $\mathfrak{m}_q$ at any point is isomorphic to a fixed negatively graded Lie algebra $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$, the distribution D is called a **regular distribution of type** $\mathfrak{m}$. For a regular distribution D, the associated filtration $\mathcal{D}^{-1} \subset \mathcal{D}^{-2} \subset \mathcal{D}^{-3}$ consists of the space of sections of distributions $D = D^{-1} \subset D^{-2} \subset \cdots$ that is $\mathcal{D}^{-i} = \Gamma D^{-i}$.

4.4 Regular sub-Riemannian manifolds

A sub-Riemannian manifold (Q, D, g_D) with bracket generated distribution D is called **bracket generated**. Then the graded tangent space $T_q^{gr}Q = \mathfrak{m}_q$ has the structure of a metric negative graded Lie algebra, i.e. a graded Lie algebra $\mathfrak{m}_q = \sum_{i=-1}^{-k} \mathfrak{m}_q^i$ with an Euclidean metric $g_q^{\mathfrak{m}}$ such that the graded spaces $\mathfrak{m}_q^i$ are mutually orthogonal.

The metric $g_q^{\mathfrak{m}}$ is a natural extension of the sub-Riemannian metric g_q^D in D_q, which is described in the following Lemma

Lemma 9 *Let* $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$ *be a negatively graded fundamental Lie algebra. Then an Euclidean metric* g *on* $\mathfrak{m}^{-1}$ *has a natural extension to an Euclidean metric* $g^{\mathfrak{m}}$ *in* $\mathfrak{m}$.

Proof: The construction of the extension is inductive. Denote by $\beta : \Lambda^2\mathfrak{m}^{-1} \to \mathfrak{m}^{-2}$ the linear map defined by the Lie bracket. Then $\mathfrak{m}^{-2} = \mathfrak{m}^{-1} + \beta(\Lambda^2(\mathfrak{m}^{-1}))$. The metric in $\mathfrak{m}^{-1}$ induces a metric
in $\Lambda^2(\mathfrak{m}^{-1})$ and on $\beta(\Lambda^2(\mathfrak{m}^{-1}))$. Now we can canonically construct a metric at $\mathfrak{m}^{-2}$, which is represented as a (not direct) sum of two Euclidean subspaces. Then we iterate this construction. □

A sub-Riemannian manifold (Q, D, g_D) with a regular distribution D of type $\mathfrak{m}$ is called a **regular** sub-Riemannian manifold of type $(\mathfrak{m}, g^{\mathfrak{m}})$ if all metric Lie algebras $(\mathfrak{m}_q, g_q^{\mathfrak{m}})$ are isomorphic to the metric graded Lie algebra $(\mathfrak{m}, g^{\mathfrak{m}})$ (called the metric symbol of the sub-Riemannian manifold).

Denote by $\mathfrak{g}^0 = \mathfrak{der}(\mathfrak{m}, g^{\mathfrak{m}})$ the Lie algebra of skew-symmetric (graded preserving) derivation of the metric graded Lie algebra. Then

$$\tilde{\mathfrak{m}} = \mathfrak{g}^0 + \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$$

is a non-positively graded Lie algebra. If $(\mathfrak{m}_q, g_q^{\mathfrak{m}})$ is a metric symbol, then $\tilde{\mathfrak{m}}_q$ is called the **extended symbol of a** sub-Riemannian manifold.
Note that a derivation $A \in \mathfrak{der}(\mathfrak{m})$ is skew-symmetric if its restriction to $\mathfrak{m}^{-1}$ is skew-symmetric.

4.4.1 Regular sub-Riemannian structure as Tanaka structure

Let $D \subset TQ$ be a regular rank m distribution of type $\mathfrak{m}$, and Aut$(\mathfrak{m})$ the group of graded preserving automorphisms of $\mathfrak{m}$.

An **admissible frame** of D is an isomorphism

$$f : T_q^{gr}Q = \mathfrak{m}_q \to \mathfrak{m}$$

of graded Lie algebras. The group $\mathrm{Aut}(\mathfrak{m})$ acts freely and properly on the manifold $\mathrm{Fr}(D)$ of admissible frames on D with the orbit space $\mathrm{Fr}(D)/\mathrm{Aut}(\mathfrak{m}) = Q$. Hence, $\mathrm{Fr}(D) \to Q$ is a principal bundle (called the bundle of admissible frames on D).
A **Tanaka structure** (or relative G-structure) is a G -principal subbundle $\pi : P \to Q = P/G$ of the bundle of admissible frames on D.

The classical identification of a Riemannian manifold with a G-structure with the orthogonal group $G = SO(n)$ is extended to the sub-Riemannian case:

Proposition 10 *A regular sub-Riemannian manifold of a type* $(\mathfrak{m}, g^{\mathfrak{m}})$ *is identified with a Tanaka G-structure with the structure group* $G = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$ *.*

Proof: Let D be a regular distribution of type $\mathfrak{m} = \sum_{i=-1}^{-k} \mathfrak{m}^i$ and

$g^{\mathfrak{m}}$ is the Euclidean metric in $\mathfrak{m}$ generated by an Euclidean metric g in $\mathfrak{m}^{-1}$. Let $\pi : P \to Q$ be a Tanaka G-structure with $G = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$. Then the associated sub-Riemannian metric is defined by the condition that for any admissible frame $f \in P$, its restriction

$$f_{\mathfrak{m}^{-1}} : \mathfrak{m}^{-1} \to D_q = \mathfrak{m}^{-1}_q$$

is an isometry.
The converse statement is obvious. □

4.4.2 Tanaka and Morimoto theorems

N. Tanaka generalised the theory of G-structures to Tanaks structures. In particular, he defined the full prolongation of a non positively graded Lie algebra $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{g}^{-1} + \cdots \mathfrak{g}^{-k}$ as a maximal graded algebra of the form

$$\mathfrak{g}^{\infty} = \mathfrak{g} + \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)} + \cdots$$

such that for any $X \in \mathfrak{g}^i$, $i > 0$ the condition $[X, \mathfrak{g}^{-1}] = 0$ implies $X = 0$.

Theorem 11 *(Tanaka) [T], ,see also [C-S], [Z],[A-D] . Let* $\pi : P \to Q$ *be a Tanaka* G*-structure on* (Q, D) *where* D *is a regular distribution of type* $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$*. Assume that the full prolongation* $\tilde{\mathfrak{m}}^{\infty}$ *of the extended symbol algebra* $\tilde{\mathfrak{m}} = \mathfrak{g} + \mathfrak{m}$ *is finite dimensional. Then there is a canonical bundle* $P^{\infty} \to Q$ *, constructed by successive prolongations, with an absolute parallelism* $\kappa : TP^{\infty} \to \tilde{\mathfrak{m}}^{\infty}$ *.*
If the first prolongation $\mathfrak{g}^{(1)} = 0$*, then the absolute parallelism* $\kappa : TP \to \mathfrak{m} + \mathfrak{g}$ *is a Cartan connection.*

Theorem 12 *(T. Morimoto) see [M], [C], [A-M-S]. The first prolongation of the graded Lie algebra* $\mathfrak{g} = \mathfrak{g}^0 + \mathfrak{m}$*, which is the extended symbol algebra of a regular sub-Riemannian manifold, is trivial.*
Moreover, let (Q, D, g_D) *be a regular sub-Riemannian manifold of type* $(\mathfrak{m}, g^{\mathfrak{m}})$ *and* $\pi : P \to Q$ *the associated* $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$ *Tanaka structure. Then* π *admits a unique normal Cartan connection*

$$\kappa : T(P) \to \tilde{\mathfrak{m}} = \mathfrak{g}^0 + \mathfrak{m}.$$

This remarkable theorem has important applications. According to subsection 4.1.1, the Tanaka G^0-structure $\pi : P \to Q$ has the canonical G^0-structure with the soldering form $\theta := \mathrm{pr}_{\mathfrak{m}}\kappa$,and connection, defined by the connection form $\omega := \mathrm{pr}_{\mathfrak{g}^0}\kappa$. Due to this, the group $A = \mathrm{Aut}(Q, D, g_D)$ of sub-Riemannian automorphisms naturally acts on P as the group of automorphisms of the Cartan connection. i.e. the group of automorphisms of the principal bundle π, which preserves the absolute parallelism κ. By Kobayashi theorem [Stern], A is a Lie group, which acts freely on P with closed orbits. Also it preserves the Riemannian metric on Q, which is the projection to Q of the G^0-invariant metric on the horizontal distribution $\mathcal{H} = \mathrm{span}\{\kappa^{-1}(\mathfrak{m})\}$, defined by $g^{\mathfrak{m}}$.
We get

Theorem 13 *Let (Q, D, g_D) be a regular sub-Riemannian manifold of types $(\mathfrak{m}, g^{\mathfrak{m}})$. Then*
i) The sub-Riemannian metric g_D of a regular sub-Riemannian manifold (Q, D, g_D) admits a canonical extension to a Riemannian metric on Q.
ii) The group $A = \mathrm{Aut}(Q, D, g_D)$ of automorphisms of type $\mathfrak{m}$ is a Lie group of dimension $\dim A \leq \dim \mathfrak{g}_0(\mathfrak{m}) + n$ and the stability subgroup A_q of a point $q \in Q$ is compact.

A sub-Riemannian manifold (Q, D, g^Q) is called **homogeneous**, if a Lie group A of automorphisms acts on Q transitively. Since any bracket generated homogeneous sub-Riemannian manifold is regular, we get

Corollary 14 *i) Let (Q, D, g^Q) be a bracket generated homogeneous sub-Riemannian manifold. Then Q is identified with a coset space $Q = A/H$ of a Lie group of automorphisms by a compact subgroup H.*
Let $\mathfrak{a} = \mathfrak{h} + \mathfrak{p}$ be a reductive decomposition such that $\mathfrak{p}$ is identified with the tangent space T_oQ. Then $D_o \subset \mathfrak{p}$ is Ad_H-invariant subspace with Ad_H-invariant metric g_o^D.
iii) Conversely, let $Q = A/H$ be a homogeneous manifold with compact stabilizer H and a reductive decomposition $\mathfrak{a} = \mathfrak{h} + \mathfrak{p}$. Then invariant sub-Riemannian structures bijectively correspond to Ad_H-invariant subspaces $D_0 \subset \mathfrak{p}$ with Ad_H-invariant Euclidean metric. Invariant sub-Riemannian metrics on D bijectively correspond to Ad_H-invariant metrics on D_0. The distribution D is bracket generated if and only if the subalgebra, generated by D_0, contains $\mathfrak{m}$.

4.4.3 Sub-Riemannian M-geodesics as projection of constant vector fields

Let (Q, D, g_D) be a regular sub-Riemannian manifold of type $(\mathfrak{m}, g^{\mathfrak{m}})$ and $\pi : P \to Q$ the asslociated Tanaka G^0-structure with the normal Cartan connection

$$\kappa : T(P) \to \tilde{\mathfrak{m}} = \mathfrak{g}^0 + \mathfrak{m}.$$

M-geodesics of a regular sub-Riemannian manifold (Q, D, g_D) are the projection to Q of the orbit of constant vector fields from $\kappa^{-1}(\mathfrak{m}^{-1}) \subset \mathfrak{X}(P)$.

Proposition 15 *The M-geodesics of a regular sub-Riemannian manifold (Q, D, g_D) are horizontal curves.*

The result follows from the explicit description of the first prolongation of a Tanaka structures, given in [A-D]. □

It is not easy to construct the normal Cartan connection for the Tanaka G_0-structure $\pi : P \to Q$ associated with a regular sub-Riemannian manifold. But there is a simple construction of a Cartan connection (see [A-M-S]), associated to a rigging V, which is consistent with the filtration $\mathcal{D}^{-i} = \Gamma D^{-i}$ defined by D .This means that V has the form

$$V = V_2 + V_3 + \cdots + \cdots V_k$$

where $D^{-i} = D^{-(i-1)} + V_i$ for $i = 2, 3, \cdots$.
Such rigging V defines an isomorphism of the graded tangent bundle $T^{gr}Q$ with the tangent bundle TQ, which defines a Riemannian metric in Q and induces an isomorphism of the Tanaka G^0-structure $\pi : P \to Q$ with a G^0-structure $\pi_0 : P_0 \to Q$ on Q.
The Cartan connection κ on the Tanaka structure is defined in terms of soldering form θ_0 and the connection form ω_0 of the G_0-structure π_0, which has trivial first prolongation since $G_0 \subset SO(\mathfrak{m})$. We still can define the sub-Riemannian geodesics as the projection to Q of the constant vector fields from $\kappa^{-1}(\mathfrak{m})$. It would be interesting to study these geodesics from point of vierw of non-holonomic mechanics.

5 Sub-Riemannian Chaplygin metric on principal bundle

5.1 Principal connection and its curvature

Let $\pi : Q \to M = Q/G$ be a G-principal bundle with a right action $R_g q = qg$ of a Lie group G.
For $a \in \mathfrak{g} = \mathrm{Lie(G)}$,we denote by $a^* : q \mapsto qa := \frac{d}{dt} q \exp(ta)|_{t=0} \in T_qQ$ the fundamental vector field (the velocity vector field of $R_{\exp ta}$).

Recall that the principal connection is an G-equivariant $\mathfrak{g}$-valued 1-form $\varpi : TQ \to \mathfrak{g}$, which is an extension of the vertical parallelism, defined by $\mathfrak{g} \ni a \mapsto a^*_q$. The equivariancy means that

$$R_g^* \varpi = \mathrm{Ad}_g^{-1} \circ \varpi, \; g \in G.$$

The connection ϖ is completely determined by the horizontal G-invariant distribution $D = \ker \varpi$.

The horizontal lift $X^D \in \Gamma D$ of a vector field $X \in \mathfrak{X}(M)$ is defined by the isomorphisms $\pi_* : D_q \to T_{\pi(q)}TM$.
Recall formulas, [K-N],

$$[a^*, b^*] = [a, b]^*, [a^*, X^D] = 0, \; [X^D, Y^D] = [X, Y]^D + 2A(X^D, Y^D). \qquad (5)$$

Here

$$A : \Gamma D \times \Gamma D \to \Gamma T^v Q, \; X \to A_X Y = A(X, Y) = \frac{1}{2}[X, Y]^v, \; X, Y \in \Gamma D$$

is the O'Neil tensor, which describes the deviation of the distribution from non-holonomicity. We denote by $X = X^v + X^h \in TQ = T^vQ + D$ the decomposition of $X \in TQ$ into vertical and horizontal parts.

The tensor A is related to the $\mathfrak{g}$-valued horizontal curvature 2-form $F = d\varpi + \frac{1}{2}[\varpi, \varpi]$ of the connection by

$$-2A(X^D, Y^D) = F(X^D, Y^D)^* := F(X^D, Y^D)^a e_a^*.$$

where e_a is a basis of $\mathfrak{g}$.
We denote by $A_X^* : \Gamma(T^vQ) \to \Gamma D$ the $C^\infty(Q)$ linear map, dual to the O'Neil map $A_X : \Gamma D \to \Gamma T^vQ$. Then

$$g^Q(A_{X^D} a^*, Y^D) = g^Q(a^*, A(X^D, Y^D)) = -\frac{1}{2} g^{\mathfrak{g}}(a, F(X, Y)).$$

Denote by $\lambda = g^Q \circ a^* \in \Gamma D^0$ the 1-form dual to fundamental vector field a^*. Then the horizontal vector field

$$A^*_{X^D} a^* = -\frac{1}{2} F^*_X \lambda$$

where $F^*_{X_q} : \mathfrak{g} \to D_q$ is the linear map, dual to the linear map $F_{X_q} : D_q \to \mathfrak{g}$, $Y_q \mapsto F_{X_q} Y_q = F(X_q, Y_q)$, $X_q, Y_Q \in D_q$.

To write formulas in coordinates , we fix a (local) section $s : M \to Q$. It defines a trivialization

$$M \times G = Q \ \ (x, g) = s(x)g$$

of the principal bundle , where the group G acts on $M \times G$ as

$$R_g(x, g_1) = (x, g_1 g).$$

A fundamental vector field a^* is identified with the left invariant vector fields

$$a^L : (x, g) \mapsto (x, ga).$$

Similarly, $a^R : (x, g) \mapsto (x, ag)$ stands for the right invariant vector field.

The connection ϖ is an extension of the left invariant Maurer-Cartan form μ^L and can be written as

$$\varpi = \mu^L + \eta = (e^a_L + \eta^a_i(x) dx^i) \otimes e_a,$$

where the potential η is a $\mathfrak{g}$-valued 1-form on M and e_a is a basis of $\mathfrak{g}$.
The horizontal lift of a vector field $X \in \mathfrak{X}(M)$ is given by

$$X^D = X^i(\partial_i^D) = X^i(\partial_i - \eta_i^a(x) e_a^L).$$

The commutator of X^D, Y^D is given by

$$\begin{array}{rcl} [X^D, Y^D] = & & [X^i(\partial_i - \eta_i^a(x) e_a^L, Y^j(\partial_j - \eta_j^b(x) e_b^L)] \\ = & & [X, Y]^D - (X \cdot \eta)(Y)^a + (Y \cdot \eta)(Y)^a + [\eta(X), \eta(Y)]^a e_a^L \\ = & & [X, Y]^D + 2A(X^D, Y^D). \end{array}$$

The curvature $F = F^a \otimes e_a = d\varpi + \frac{1}{2}[\varpi, \varpi]$ of ϖ is a $\mathfrak{g}$-valued horizontal 2-form. Its value on $X, Y \in \mathfrak{X}(M)$ is

$$F(X, Y) = F(X, Y)^a e_a = \left\{ X \cdot (\eta_i^a) Y^i - Y \cdot (\eta_i^a) X^i + \frac{1}{2} \eta_i^a [X, Y]_i \right\} e_a.$$

5.2 Chaplygin metric and its standard extension

The metric g^M on the base manifold M defines a canonical invariant sub-Riemannian metric g^D on D such that the projection $\pi_* : D_q \to T_{\pi(q)}M$ is an isometry.

The sub-Riemannian metric (D, g^D) is called a Chaplygin metric and the sub-Riemannian manifold (Q, D, g^D) is called a **Chaplygin system** or a **transversally homogeneous** sub-Riemannian manifold.

Let (Q, D, g_D) be a Chaplygin system associated to a principal bundle $(\pi : Q \to M, \varpi)$ as above and $g^{\mathfrak{g}}$ an Euclidean metric on $\mathfrak{g}$. It defines a degenerate metric

$$g^F(X, Y) = g^{\mathfrak{g}}(\varpi(X), \varpi(Y))$$

with kernel D, whose restriction to a fiber $F_x = \pi^{-1}(x)$ is a Riemannian metric. We will consider also g^D as a degenerate metric on Q with ker $g^D = T^vQ$.

Then

$$g^Q = g^F + g^D \tag{6}$$

is a Riemannian metric in Q which is called the **standard extension of the Chaplygin metric g^D.**

Note that the metric g^Q is G-invariant if and only if the degenerate metric g^F is invariant which means that the metric $g^{\mathfrak{g}}$ is Ad_G-invariant. In this case the metric g^Q is called the **bi-invariant extension** of the sub-Riemannian matric g^D.

Denote by ∇^M (resp., ∇^Q) the Levi-Civita connection of g^M (resp., g^Q), and by ∇^F the Levi-Civita connection of the induced metric g^F on a fiber, which is a totally geodesic submanifold.

The Koszul formula implies the following O'Neil formulas for the covariant derivative of fundamental vector field b^* and the horizontal lift X^D of a vector field $X \in \mathfrak{X}(M)$, see [Besse].

$$\begin{array}{lll} i) & \nabla^Q_{a^*} b^* = & \nabla^F_{a^*} b^*, \\ ii) & \nabla^Q_{a^*} X^D = & \nabla^Q_{X^D} a^* = (\nabla^Q_{a^*} X^D)^h = A^*_{X^D} a^*, \\ iii) & \nabla^Q_{X^D} Y^D = & (\nabla^M_X Y)^D + A_{X^D} Y^D. \end{array}$$

The connection ∇^F describes in terms of Lie brackets as follows

$$2g(\nabla^F_{a^*} b^*, c^*) = g^{\mathfrak{g}}([a,b],c) - g^{\mathfrak{g}}(b,[a,c]) - g^{\mathfrak{g}}(a,[b,c]), \; a,b,c \in \mathfrak{g}.$$

5.2.1 S-geodesics of a Chaplygin metric

The O'Neil formulas implies the following relations between S-geodesics of the Chaplygin sub-Riemannian metric and geodesics of the Riemannian metrics g^Q, g^M, see also [Besse].

Theorem 16 *i) The principal bundle $\pi : Q \to M$ with a standard metric g^Q associated to a Riemannian metric g^M is a Riemannian submersion with totally geodesic fibers.*
ii)A Riemannian geodesic $\gamma(s)$ of (Q, g^Q) which is horizontal at one point is horizontal and it projects onto a geodesic $\pi\gamma(t)$ of the base manifold (M, g^M).
iii) S-geodesics of the Chaplygin metric are precisely horizontal geodesics of (Q, g^Q) and they are horizontal lifts of geodesics of the base manifold.

Proof: Recall that S-geodesics are geodesics of the Schouten connection $\nabla^S_X Y = \mathrm{pr}_D \nabla^Q_X Y$, $X, Y \in \Gamma D$, that is horizontal curves $\gamma(s)$ which satisfy the equation

$$0 = \nabla^S_{\dot\gamma(s)} \dot\gamma(s) = \mathrm{pr}_D \nabla^Q_{\dot\gamma(s)} \dot\gamma(s) = (\nabla^M_{\dot x} \dot x)^D.$$

We used the O'Neil formula iii) and skew-symmetry of the O'Neul tensor A. □

5.2.2 H-geodesics of a Chaplygin metric

Now we consider H-geodesics of the Chaplygin metric g^D on Q and study their relation with geodesics of its standard extension g^Q and the metric g^M of the base manifold.

As above, we fix a trivialization $Q = M \times G$ of the principal bundle π, defined by a section s. Then any fiber $F_x = x \times G$ with the metric g^F is identified with the Lie group G with the left invariant metric (still denoted by g^F) defined by the metric $g^{\mathfrak{g}}$. The metric $g^Q = g^D + g^F$ is invariant with respect to vector fields a^R and the sub-Riemannian metric g^D is invariant with respect to a^L and a^R for $a \in \mathfrak{g}$.

The decomposition $g^Q = g^D + g^F$ of the metric in $TQ = D + T^vQ$, defines the decomposition $g_Q^{-1} = g_D^{-1} + g_F^{-1}$ of the contravariant metric g_Q^{-1} where the cometrics

$$g_D^{-1} = \sum_{i=1}^{m} X_i \otimes X_i,\; g_F^{-1} = \sum e_a^* \otimes e_a^*$$

are considered as functions on T^*Q. Here X_i is an orthonormal frame in D and e_a and orthonormal basis of $\mathfrak{g}$.

The Hamiltonian h_Q of the Riemannian metric Q is the sum $h_Q = h_D + h_F$ of the Hamiltonian $h_D(\xi) := \frac{1}{2} g_D^{-1}(\xi, \xi)$ of the sub-Riemannian metric and the Hamiltonian $h_F(\xi) = \frac{1}{2} g_F^{-1}(\xi, \xi)$ of the fiberwise metric g^F.

Now we describe the relations between sub-Riemannian H-geodesics and Riemannian geodesics of the metric g^M on the base M and left invariant metric g^F on the group G. In the case, when the extension g^Q in bi-invariant, i.e. the metric $g^{\mathfrak{g}}$ is Ad_G invariant, they had been proved by R. Montgomery [Mont], Theorem 11.2.5 (The main theorem).

Lemma 17 *Let $g^Q = g^D + g^F$ be a standard extension of a Chaplygin metric g^D. Then the Hamiltonians h_Q, h_F, h_D Poisson commute:*

$$\{h_F, h_D\} = \{h_F, h_Q\} = 0.$$

and associated Hamiltonian vector fields $\vec{h}_F, \vec{h}_D, \vec{h}_F$ commute.

Proof: A fundamental field a^* commutes with the horizontal lift X^D of basic vector field, see (5), Then

$$\begin{aligned}
&\mathcal{L}_{a^*} g^D)(X^D, Y^D) \\
&= a^* \cdot (g^D(X^D, Y^D)) - g^D([a^*, X^D], Y^D) + g^D(X^D, [a^*, Y^D]) \\
&= a^* \cdot g^M(X, Y) = 0
\end{aligned}$$

which shows that the field preserves the sub-Riemannian metric g^D. It preserves also the decomposition $TQ = D + T^vQ$ and the dual decomposition $T^*Q = D^* + D^0$.

Let X_i be an orthonormal frame in D and ξ^i the dual coframe in $D^* = (T^v Q)^0$. Then we mar write

$$\mathcal{L}_{a^*} X_j = A^k_j(q) X_k,\ \mathcal{L}_{a^*}\xi^i = B^i_m(q)\xi^m.$$

Since a^* preserves the metric $g^D = \sum \xi^i \otimes \xi^i$, the matrix B^m_i is skew-symmetric. But

$$0 = \mathcal{L}_{a^*} < \xi^i, X_j > = < \mathcal{L}_{a^*}\xi^i, X_j > + < \xi^i, \mathcal{L}_{a^*} X_j > = B^i_j + A^i_j.$$

This shows that the matrix A^i_j is also skew-symmetric and the field a^* preserves the contravariant metric $g_Q^{-1} = \sum X_i \otimes X_i$. In particular, the Poisson bracket $\{a^*, g^{-1}\} = \{a^*, \sum X_i \otimes X_i\} = 0$. Then the Leibnitz rule shows that the Hamiltonians $h_F = \frac{1}{2}\sum(e^*_i)\otimes(e^*_i)$ and $h_D = \frac{1}{2}g^D$ Poisson commute. □

As a corollary, we get

Theorem 18 *i)The sub-Riemannian geodesic flow of the sub-Riemannian metric g^D is a composition $\exp t\vec{h}_D = expt\vec{h}_Q \circ exp(-t\vec{h}_F)$ of the Riemannian geodesic flows of the metric g^Q and the fiberwise metric g^F.*
ii) Denote by $g_a(t) \subset G$ the geodesic of the group G with the left invariant metric g^F as above with initial conditions $g(0) = e, \dot{g}(0) = a \in \mathfrak{g}$ and by $\gamma_w(t)$ the geodesic of the standard metric g^Q with initial conditions $\gamma(0) = q \in Q$, $\dot{\gamma}(0) = w = w^v + w^h \in T_qQ$. Then the curve $q(t) = \gamma_w(t)g_a(t)$ is a sub-Riemannian geodesic if and only if it has horizontal velocity $\dot{q}(0) = w + a^*_q = w^h \in D_q$ that is $\varpi(w) = -a$.
iii) Horizontal geodesic of g^Q are sub-Riemannian geodesics and they projects to geodesics of (M, g^M).
iv) Sub-Riemannian geodesics are horizontal lifts of the projection of geodesics $\gamma_w(t)$ of g^Q to M.

Proof: i) is obvious. ii) The restriction $g^F|_{F_x}$ of g^F to any fiber $F_x = (x, G)$ is identified with the left invariant Riemannian metric (denoted again by g^F) on G , defined by the metric $g^{\mathfrak{g}}$.
The projection to Q of integral curves $\exp t\vec{h}_F$ are geodesics of the metric g^F , hence also g^Q, since the fibers are totally geodesics. The projection to Q of the integral curves of g^Q are geodesics of g^Q. The projection of the composed curve

$$\xi(t) = \exp t\vec{h}_F \circ \exp t\vec{h}_Q \circ (\xi),\ \xi \in T^*_q Q$$

are curves of the form $q(t) = R_{g(t)}\gamma(t) = \gamma(t)g(t)$ where $\gamma(t) = \tau(\exp)t\vec{h}(\xi)$ is a geodesic of g^Q and $g(t) \subset G$ is a geodesic of the metric g^F on G. We may assume that $g(0) = e$ and $\dot{g}(0) = a \in \mathfrak{g}$. If $\dot{\gamma}(0) = w$, then the curve $q(t)$ is a sub-Riemannian geodesic if and only if its velocity vector $\dot{q}(0) = w + a^*_q$ is horizontal , that is $\varpi(w + a^*_q) = \varpi(w) + a = 0$. This proves ii), which implies iii). iv) follows from the remark that the transformation $R_{g(t)}$ deforms the geodesic $\gamma(t)$ in vertical directions. Hence $q(t)$ and $\gamma(t)$ have the same projection to M (which are geodesics if and only if $\gamma(t)$ is a horizontal geodesic.) □

A sub-Riemannian geodesic $q(t) = \tau(\xi(t))$ is determined by the initial covector $\xi(t) = \xi(t)_D + \lambda$, where $\lambda \in D^0 = \mathrm{Ann}(D)_q$ and the first term determines the velocity vector $v \in D_q$. We call $\lambda \in D^0$ the codistribution part of the sub-Riemannian geodesic . The sub-Riemannian geodesics , which are horizontal geodesics of g^Q are characterized as geodesics with trivial codistribution part. As a corollary of Theorem 18 and theorem 16, we get

Theorem 19 *Let g^D be a Chaplygin sub-Riemannian metric in a principal bundle $(\pi : Q \to M, \varpi)$ and g^Q is the standard metric. Then sub-Riemannian S-geodesics coincide with H-geodesics with trivial codistribution part.*

5.2.3 Bi-invariant extension of Chaplygin metric and Yang-Mills dynamics

Assume now that g^Q is a bi-invariant extension of the Chaplygin sub-Riemannian metric g^D in the total space Q of a principal bundle $\pi : Q \to M$ with a connection ϖ. In this case , the geodesic Hamiltonian system with Hamiltonian h_Q has nice physical interpretation as dynamical system, which describe the evolution of a charged particle in the base manifold M in the presence of the Yang -Mills field , defined by the principal connection $\varpi : TQ \to \mathfrak{g}$, see [W],[Mont], [Mont1]. More precisely, using a trivialisation $Q = M \times G$ and local coordinates x^i in M, we can write the connection form as

$$\varpi = \mu + A = (e_L^a + A_i^a dx^i) \otimes e_a$$

where e_a is an orthonormal basis of $\mathfrak{g}$, e_a^L (resp. , e_L^a) corresponding left invariant field of frames (resp., coframes) on G and $A = A_i^a dx^i \otimes e_a$ the Yang-Mills potential. The horizontal lifts of the coordinates vector fields $\partial_i := \partial_{x^i}$ are given by $\partial_i^D := \partial_i - A_i^a e_a^L$. Together with the fundamental field e_a^L they form a frame in $Q = M \times G$, which is invariant with respect to the right invariant fields e_a^R. The sub-Riemannian metric is given by $g^D(\partial_i^D, p_j^D) = g^M(\partial_i, \partial_j) = g_{ij}$.
The vertical metric is given by $g^F(e_a^L, e_b^L) = g^{\mathfrak{g}}(e_a, e_b) = \delta_{ab}$. The metric $g^Q = g^F + g^D$ is R_G-invariant and the fundamental fields e_a^L and (locally defined) right invariant fields e_a^R are Killing vector fields. The contravariant metric is $g_Q^{-1} = g_F^{-1} + g_D^{-1}$ where

$$g_F^{-1} = \sum e_a^L \otimes e_a^L,$$

$$g_D^{-1} = g^{ij}(x)\partial_i^D \otimes \partial_j^D = g^{ij}(\partial_i - A_i^a e_a^L)(\partial_j - A_i^a e_a^L).$$

The cotangent bundle locally is identified with $T^*Q = T^*M \times T^*G$, where (x^i, p_i) are local coordinates in T^*M with $p = p_i dx^i$ and (g^a, λ_a) are local coordinates in T^*G, where g^a are local coordinates in G and $T_g^*G \ni \lambda = \lambda_a e_L^a$. Note that λ_a as linear forms on T_g^*G are identified with e_a^L. The left invariant vector fields

$$\partial_i, \partial_{p_i}, \partial_{\lambda_a}, e_L^a$$

form a frame on $T^*Q = T^*M \times T^*G$. The quadratic in momenta Hamiltonians $h_M, h_F, h_D, h_Q = h_F + h_D$ can be written as follows

$$\begin{aligned} h_M &= \tfrac{1}{2} g^{ij}(x) p_i p_j \\ h_F &= \tfrac{1}{2} \sum e_a^L e_a^L \\ h_D &= \tfrac{1}{2} g^{ij}(x)(p_i - A_i^a(x)\lambda_a)(p_j - A_j^b(x)\lambda_b) \end{aligned}$$

Using formula for the Poisson structure $\Lambda = \omega^{-1}$ on T^*G, one can easily calculate the Hamiltonian vector fields and the geodesic equation. But we prefer to use the O'Neil formulas.

Lemma 20 *The angle between the geodesic γ of g^Q and a fundamental field a^*, $a \in \mathfrak{g}$ is a constant. In particular, the orthogonal projection $\mathrm{pr}_{T^vQ}\dot{\gamma}(t)$ of the velocity vector field $\dot{\gamma}$ to vertical subbundle is the restriction to $\gamma(t)$ of some fundamental vector field a^* and the velocity vector filed can be written as*

$$\dot{\gamma}(t) = a^*(\gamma(t)) + \dot{x}^D(\gamma(t))$$

where $\dot{x}^D(\gamma(t))$ is the horizontal lift of the velocity vector filed $\dot{x}(t))$ of the projection $x(t)$ of $\gamma(t)$ to M.

Remark 21 *Physically, the angles α_k between $\dot{\gamma}$ and basic fundamental fields e_k^* characterise the charges of a particle with respect to components of the Yang-Mills field and the condition $\alpha_k = const$ is called the conservation of charges. In particular, the evolution of neutral particle is described by horizontal geodesics.*

Proof: Let $\gamma(t)$ be a geodesic and $x(t) = \mathrm{pr}_M\gamma(t)$ its projection to M. Then $\dot{x}(t) = \mathrm{pr}_{TM}(\dot{\gamma}(t)$ and is the horizontal part of the velocity vector field is $\dot{\gamma}(t)^h = \dot{x}(t)^D$. Hence, we can write

$$\dot{\gamma}(t) = \dot{x}(t)^D + u^a(t)e_a^*(\gamma(t)).$$

Then

$$\begin{aligned} \tfrac{d}{dt}g^Q(e_b, \dot{\gamma}(t)) &= \dot{u}^b(t) \\ &= \nabla_{\dot{\gamma}}g^Q(e_b^*, \dot{\gamma}(t)) \\ &= g^Q(\nabla_{\dot{\gamma}}e_b^*, \dot{\gamma}(t)) + g^Q(e_b^*, \nabla_{\dot{\gamma}}\dot{\gamma}(t)) \\ &= 0. \end{aligned}$$

since the covariant derivative $\nabla_{\cdot}e_b^*$ of a Killing vector field $e_b^* = e_b^L$ is a skew-symmetric operator. □

The following theorem describes the relation between geodesics of the Riemannian metric g^M and geodesics of its bi-invariant extension g^Q.

Theorem 22 *A curve $\gamma(t) \subset Q$ with projection $x(t) = \mathrm{pr}_M\gamma(t)$ and velocity vector field $\dot{g}(t) = a^*(\gamma(t)) + \dot{x}^D(\gamma(t))$ is a geodesic of g^Q if and only if it satisfy the equation*

$$\nabla^Q_{\dot{g}(t)}\dot{\gamma}(t) = (\nabla^M_{\dot{x}}\dot{x})^D + 2A^*_{\dot{x}^D}a^* = 0. \tag{7}$$

Proof: Using O'Neil formulas, we calculate the covariant derivative $\nabla^Q_{\dot{\gamma}}\dot{\gamma}$ of the velocity field $\dot{\gamma} = a^*(\gamma(t)) + \dot{x}^D(\dot{\gamma}(t))$ as follows

$$n^Q_{\dot{\gamma}}\dot{\gamma} = (\nabla^Q_{a^*}a^*)(\gamma(t)) + 2\nabla^Q_{X^D}a^* + \nabla^Q_{\dot{x}^D}\dot{x}^D = (\nabla^M_{\dot{x}}\dot{x})^D(\gamma(t)) + 2A^*_{\dot{x}^D}a^*(\gamma(t)).$$

We use the fact that the geodesics of the bi-invariant metric on a group G starting from e are 1-parameter subgroups, hence $\nabla^Q_{a^*}a^* = \nabla^F_{a^*}a^* = 0$ and denote by X an extension of $\dot{x}$ to a local vector field. Since the left hand side depends only on $\gamma(t)$, the result does not depend on such extension. □

Recall that $2A^*_{X^D}a^* = -F^*_X\lambda$, $\lambda = g^Q \circ a^* \in D^0$ where $F^*_X : \mathfrak{g}^* \to \Gamma D$ is the linear map, dual to the map $F_{X_q} : D_q \to \mathfrak{g}$, associated with the curvature 2-form F.

The equation (7) is equivalent to the equation

$$\nabla^M_{\dot{x}}\dot{x} = g_M^{-1}\lambda_b F_i^b(\dot{x}) \tag{8}$$

where the right hand side is the vector field metrically dual to the 1-form $\lambda_b F_i^b(\dot{x}$ and $\lambda \in \mathfrak{g}^*$ is a constant covector. □

The equation (8) describes the motion of a charged particle in the Yang-Mills field ϖ with the strength tensor F . In the case, when $\pi : Q \to M$ is a circle bundle , the connection ϖ defines the Maxwell field (if g^M has Lorentz signature) and the equation reduces to the Lorentz equation for a charge particle in the electro-magnetic field.

6 Homogeneous sub-Riemannian manifolds

In this section , we consider some classes of homogeneous sub-Riemannian manifolds, for which S-geodesics coincides with H-geodesics.

6.1 Chaplygin system on Lie group associated to a homogeneous Riemannian manifold

Let $\pi : G \to M = G/H$ be the principal bundle associated to a homogeneous Riemannian manifold $(M = G/H, g^M)$. A reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ defines a principal connection with connection form $\varpi = \mathrm{pr}_{\mathfrak{h}} \mu^L : TG \to \mathfrak{h}$, which is the projection to $\mathfrak{h}$ of the left invariant Maurer -Cartan form μ^L . The horizontal distribution $D := \ker \varpi$ is the left invariant distribution, obtained by the left translations of the subspace $\mathfrak{m} \subset \mathfrak{g}$. The metric $g^{\mathfrak{m}}$ extends to a left invariant metric g^D in D. So any Riemannian manifold defines a left invariant sub-Riemannian Chaplygin metric (D, g^D) in G. Since the stability subgroup H is compact, it admits a bi-invariant metric g^H and sub-Riemannian metric g^D admits the bi-invariant extension $g^G = g^H + g^D$. Note that the metric g^G is left invariant and invariant with respect to the right action of H.

Remark 23 *Note that the distribution $D \subset TG$ associated to $\mathfrak{m}$ is bracket generated if and only if the Lie subalgebra $\mathfrak{g}'$ generated by $\mathfrak{m}$ coincides with $\mathfrak{g}$. Since the subgroup $G' \subset G$ associated with the subalgebra $\mathfrak{g}'$ acts on M transitively, changing G to G' we may assume that $\mathfrak{m}$ generates the Lie algebra $\mathfrak{g}$. In this case, the distribution D is bracket generated.*

As a summary, we get

Proposition 24 *Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold. A reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ defines a sub-Riemannian left invariant Chaplygin metric (D, g^D) on the Lie group G. Moreover, an Ad_H-invariant metric $g^{\mathfrak{h}}$ on the stability subalgebra $\mathfrak{h}$ defines a bi-invariant extension g^G of the sub-Riemannian metric to a left invariant and H-right invariant metric on G.*

6.2 Sub-Riemaniian homogeneous manifold associated with a homogeneous Riemannian manifold with non-simple stabilizer

Now we consider a generalisation of the above construction.

Assume that the stabilizer H of a Riemannian homogeneous manifold $M = G/H$ is an almost direct product $H = K \cdot L$ of two compact normal subgroups. Then $\pi : Q = G/K \to M = G/H$ is an L-principal bundle with the right action of L. Again a reductive decomposition for the manifold M has the form $\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{k} + \mathfrak{l}) + \mathfrak{m}$. The projection

$$\varpi^G = \mathrm{pr}_{\mathfrak{l}} \mu^L : TG \to \mathfrak{l}$$

of the left invariant Maurer-Cartan form defines a left invariant $\mathfrak{l}$-valued 1-form . It is right K-invariant, since K acts trivially on $\mathfrak{l}$, π-horizontal and L-equivariant (i.e. $R_g^* \varpi = \mathrm{Ad}^{-1} \circ \varpi$). Hence it is projected to a G-invariant principal connection form $\varpi : TQ \to \mathfrak{l}$. The kernel $D = \ker \varpi$ is the invariant distribution associated with the subspace $\mathfrak{m} \subset \mathfrak{m} + \mathfrak{l} = T_o Q$. The metric $g^{\mathfrak{m}} = g^M|_{T_0 M}$ defines an invariant sub-Riemannian Chaplugin metric g^D in the distribution D. As above, it admits a bi-invariant extension to the metric g^Q on the manifold $Q = G/K$. We get

Proposition 25 *A homogeneous Riemannian manifold* $(M = G/H, g^M)$ *with non simple stabilizer* $H = K \cdot L$ *defines an invariant* sub-Riemannian Chaplygin metric (D, g^D) on the total space of the principal L- bundle $\pi : Q = G/K \to M = G/H$, which admits a bi-invariant extension to an invariant metric g^Q on Q.

6.2.1 Homogeneous contact sub-Riemannian manifolds

The above construction may be applied to homogeneous Sasaki manifolds For simplicity, consider case of regular compact homogeneous Sasaki manifolds. They are described as total spaces $\pi : Q = G/K \to M = G/H$ of a principal circle bundle over a flag manifold $M = G/H = \mathrm{Ad}_G Z \subset \mathfrak{g}$ (that is the adjoint orbit of a compact semisimple Lie group G), see [A-C-H-K] with Kähler structure (g^M, ω^M) such that the Kähler form ω^M is integer. Then there exists a homogeneous circle bundle $\pi : Q = G/K \to M = G/H = G/K \cdot T^1$ with a principal connection $\varpi : TQ \to \mathbb{R} = \mathrm{LieT}^1$ such that the $\omega^M = d\varpi$ is the curvature of ϖ. The Chaplygin sub-Riemannian metric on $D = \ker \varpi$, defined by the Kähler metric g^M admits a bi-invariant extension to the invariant metric g^Q on Q, which is the invariant Sasaki metric on Q (under appropriate normalization).
From physical point of view, the principal T^1-bundle $\pi : Q \to M$ with Sasaki metric corresponds to Kaluza-Klein description of electro-magnetic field (abelian Yang-Mills field) and projections to M of geodesics of Sasaki metric describe the evolution of charges in electro-magnetic field and satisfy the Lorentz equation.

6.3 Symmetric sub-Riemannian manifolds

Strichartz [Str] defined the notion of sub-Riemannian symmetric space as a homogeneous sub-Riemannian manifold $(Q = G/H, D, g_D)$ such that H contains an involutive element σ (called sub-Riemannian symmetry) which acts on the subspace D_o of the point $o = eH \in Q$ as $-\mathrm{id}$.
He considered some example of sub-Riemannian symmetric spaces and classified 3-dimensional sub-Riemannian symmetric spaces. He stated the problem of extension of this classification to higher dimensions.
E. Falbel and C. Gorodski classified symmetric sub-Riemannian manifolds of contact type (1995). W. Respondek and A. J. Maciejewski describes all integrable sub-Riemannian metrics on 3-dimensional Lie groups with integrable H-geodesic flow (2008). They are exhausted by sub-Riemannian symmetric spaces.

Below we give a description of bracket generated symmetric sub-Riemannian manifolds in terms of affine symmetric spaces.

6.3.1 Sub-Riemannian symmetric spaces associated with an affine symmetric space $M = G/H$

Let σ be an involutive automorphism of a Lie algebra with eigenspace (symmetric) decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, $\sigma|_{\mathfrak{g}_\pm = \pm \mathrm{id}}$. It is integrated to an involutive decomposition σ of the simply connected Lie group G associated to $\mathfrak{g}$. Let H be a closed subgroup such that $G^\sigma_{con} \subset H \subset G^0$, where G^σ is the fixed point subgroup of G and G^σ_{con} its connected component. Then $M = G/H$ is an affine symmetric space (associated with the above symmetric decomposition) and any symmetric space is obtained by this construction. The invariant (torsion free) linear connection ∇ in M is given by $\nabla_X Y^*|_o = -\frac{1}{2}[X,Y]_o$, where $X, Y \in \mathfrak{g}_-$, Y^* the vector field generated by Y and $o = eH \in M$. The involution is called the central symmetry at the point eH

Since the subalgebra $\mathfrak{g}_1 := [\mathfrak{g}_-, \mathfrak{g}_-] + \mathfrak{g}_-$ generates a transitive subgroup G_1 (called the group of transvections) such that $M = G_1/H_1$ we may assume that $G = G_1$ is the group of transvections of the symmetric space.

Let $S = G/H$ be an affine symmetric space with a symmetric decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}^-$. Chose a compact subgroup $K \subset H$. Then the homogeneous manifold $Q := G/K$ is the total space of a homogeneous fibration $Q = G/K = G \times_H (H/K) \to S = G/H$ is a homogeneous fibration with the typical fiber $F = H/K$ over the symmetric space G/H , defined by the right action of K on H.

A reductive decomposition $\mathfrak{h} = \mathfrak{k} + \mathfrak{m}^+$ of the fiber $F = H/K$ defines a reductive decomposition $\mathfrak{g} = \mathfrak{k} + (\mathfrak{m}^+ + \mathfrak{g}^-)$ of G/K. If G is the group of transvections of S, the subspace $D_o := \mathfrak{g}_-$ generates an invariant bracket generated distribution $D \subset TQ$. An Ad_K-invariant Euclidean metric g in D_0 defines an invariant sub-Riemannian structure (D, g^D) in Q. Moreover , the involution σ induces an involutive automorphism σ of $(Q = G/K, D, g_D)$, which is a sub-Riemannian symmetry. The sub-Riemannian symmetric manifold $(Q = G/K, D, g_D)$ is called sub-Riemannian symmetric space associated to an affine symmetric space $M = G/H$ and a subgroup $K \subset H$.

Conversely, let $(Q = G/K, D, g)$ be a sub-Riemannian symmetric space. The sub-Riemannian symmetry σ , which preserves the point $o = eK$ defines a symmetric decomposition $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ of the Lie algebra of G such that the stability subalgebra $\mathfrak{k} \subset \mathfrak{g}^+$ and the horizontal subspace at the point o is an Ad_K-invariant subspace $D_0 \subset \mathfrak{g}^-$. Denote by $S = G/G^+$ the affine symmetrkic space associated with the symmetric decomposition. Then $Q = G/K \to S = S = G/G^+$ is a homogeneous fibration over the affine symmetric space.

We get

Theorem 26 *Let $(S = G/H, \sigma)$ be an affine symmetric space associated to a symmetric decomposition $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ where G is the group of transvections, i.e. $\mathfrak{g}^-$ generates $\mathfrak{g}$. Let K be a compact subgroup of H and g a K-invariant Euclidean metric in $\mathfrak{g}^-$. Then the homogeneous manifold $Q = G/K$ admits a structure of bracket generated sub-Riemannian symmetric space (D, g^D, σ), where D is the invariant distribution defined by $D_0 = \mathfrak{g}^-$ with invariant metric , defined by g. The sub-Riemannian symmetry σ is the automorphism of (Q, D, g^D), defined by the involutive automorphism σ of G. This symmetry is consistent with the fibration $Q = G/K \to M = G/H$ and induces the central symmetry of M.*

Any sub-Riemannian symmetric space can be obtained by this construction.

6.3.2 Compact sub-Riemannian symmetric space associated to a graded complex semisimple Lie algebra

We describe the structure of bracket generated symmetric sub-Riemannian manifold on a flag manifold.
Let

$$\mathfrak{g} = \sum_{i=-d}^{d} \mathfrak{g}_i$$

be a graded complex semisimple Lie algebra of depth $d \geq 2$ and

$$\mathfrak{g} = \mathfrak{g}^{ev} + \mathfrak{g}^{odd}$$

associated symmetric decomposition.
Denote by τ the anti-linear involution which defines the compact real form s.t. $\mathfrak{g}^\tau = \mathfrak{g}_0^\tau + \sum_{i>0}(\mathfrak{g}_{-i} + \mathfrak{g}_i)^\tau$ We denote by G a complex Lie group with Lie$G = \mathfrak{g}$ and by $F = G/P$ the complex (compact) flag manifold , where P is the parabolic subgroup generated by $\mathfrak{p} = \sum_{i\geq 0} \mathfrak{g}_i$. We set $\mathfrak{h} = \mathfrak{g}_0^\tau$ and denote by H the associated subgroup of the compact Lie group G^τ, which acts transitively on F . Hence, $F = G^\tau/H$.

We choose an $ad_{\mathfrak{h}}$-invariant metric $g^{\mathfrak{m}}$ on the space $\mathfrak{m} = (\mathfrak{g}^{-1} + \mathfrak{g}^1)^\tau$. The pair $(\mathfrak{m}, g^{\mathfrak{m}})$ defines an invariant bracket generated sub-Riemannian structure (D, g_D) on the flag manifold $F = G^\tau/H$.

Theorem 27 *The flag manifold $F = G^\tau/H$ with invariant sub-Riemannian structure (D, g_D) defined by $(\mathfrak{m}, g^{\mathfrak{m}})$ is a* sub-Riemannian symmetric space.

Example Let

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_9 + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \dim \mathfrak{g}_{\pm 2} = 1$$

be the contact gradation of a complex simple Lie algebra $\mathfrak{g}$, i.e. the eigenspace decomposition of ad $_{H_\mu}$ where H_μ is the coroot associated to the maximal root μ of $\mathfrak{g}$. Then the symmetric space G^τ/G^τ_{ev} is the quaternionic Kähler symmetric space (the Wolf space) and the flag manifold $F = G^\tau/H$ is the associated twistor space. The distribution D is the holomorphic contact distribution and g_D is unique (up to scaling) invariant sub-Riemannian metric on D. It is the restriction of the invariant Kähler-Einstein metric on F (for $\mathfrak{g} \neq \mathfrak{sl}_n(\mathbb{C})$.)

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