On planes through points off the twisted cubic in $\mathrm{PG}(3,q)$ and multiple covering codes

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Abstract. Let $\mathrm{PG}(3,q)$ be the projective space of dimension three over the finite field with q elements. Consider a twisted cubic in $\mathrm{PG}(3,q)$. The structure of the point-plane incidence matrix in $\mathrm{PG}(3,q)$ with respect to the orbits of points and planes under the action of the stabilizer group of the twisted cubic is described. This information is used to view generalized doubly-extended Reed-Solomon codes of codimension four as asymptotically optimal multiple covering codes.

Keywords: Twisted cubic, projective space, incidence matrix, multiple coverings, Reed-Solomon codes

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1 Introduction

Let $\mathbb{F}_q$ be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $\mathbb{F}_q^+ = \mathbb{F}_q \cup \{\infty\}$. Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over $\mathbb{F}_q$; it contains $\theta_{N,q} = (q^{N+1}-1)/(q-1)$ points. We denote by $[n,k,d]_qR$ an $\mathbb{F}_q$-linear code of length n, dimension k, minimum distance d, and covering radius R. For an introduction to projective spaces over finite fields and connections between projective geometry and coding theory see [11, 13, 15, 16].

An n-arc in $\mathrm{PG}(N,q)$, with $n \geq N+1 \geq 3$, is a set of n points such that no $N+1$ points belong to the same hyperplane of $\mathrm{PG}(N,q)$. An n-arc is complete if it is not contained in an $(n+1)$-arc. Arcs and linear maximum distance separable (MDS) $[n,k,n-k+1]_q$ codes are equivalent objects.

In $\mathrm{PG}(N,q)$, $2 \leq N \leq q-2$, a normal rational curve is any $(q+1)$-arc projectively equivalent to the arc $\{(t^N, t^{N-1}, \ldots, t^2, t, 1) : t \in \mathbb{F}_q\} \cup \{(1, 0, \ldots, 0)\}$. The points (in homogeneous coordinates) of a normal rational curve in $\mathrm{PG}(N,q)$ treated as columns define a parity check matrix of a $[q+1, q-N, N+2]_q$ generalized doubly-extended Reed-Solomon (GDRS) code [11, 22]. Clearly, a GDRS code is MDS. In $\mathrm{PG}(3,q)$, the normal rational curve is called a *twisted cubic* [14,16]. Twisted cubics have important connections with a number of other objects, see e.g. [4, 6, 7, 9, 12, 14–16, 19] and the references therein.

Twisted cubics in $\mathrm{PG}(3,q)$ have been widely studied; see [14] and the references therein. In particular, in [14], the orbits of planes and points under the group of the projectivities fixing a cubic are considered.

In this paper we investigate the intersection multiplicities of planes and twisted cubics, determining the structure of the point-plane incidence matrix in $\mathrm{PG}(3,q)$. As a by-product, we give also a number useful relations regarding these numbers.

As an application, we show that twisted cubics can be treated as multiple ρ-saturating sets with $\rho = 2$ which, in turn, give rise to asymptotically optimal non-binary linear multiple covering $[q+1, q-3, 5]_q3$ codes of radius $R = 3$. Thereby, we show that the $[q+1, q-3, 5]_q3$ GDRS code associated with the twisted cubic can be viewed as an asymptotically optimal multiple covering. Note that in the literature, see e.g. [2, 3, 8, 21], several examples of multiple coverings with $R = 2$ and $\rho = 1$ are given whereas asymptotically optimal multiple coverings with $R = 3$ and $\rho = 2$ are not considered.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3, the main results of the paper are presented. Section 4 provides a number useful relations. In Sections 5 and 6 we compute the spectrum of the intersections between planes and twisted cubics, and the structure of the point-plane incidence matrix in $\mathrm{PG}(3,q)$ is described. Covering properties of the codes associated with twisted cubics are considered in Section 7.

2 Preliminaries

For the convenience of readers, in this section we summarize known results on twisted cubics [14, Chapter 21] and on multiple covering codes [2,3,8,21].

2.1 Twisted cubic

Let $\mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $\mathrm{PG}(3, q)$ with the homogeneous coordinates $x_i \in \mathbb{F}_q$; the rightmost nonzero coordinate is equal to 1.

Let $\mathscr{C} \subset \mathrm{PG}(3, q)$ be the *twisted cubic* consisting of $q + 1$ points $P_1, \ldots, P_{q+1}$ no four of which are coplanar. We consider $\mathscr{C}$ in the canonical form

$$\mathscr{C} = \{P_1, P_2, \ldots, P_{q+1}\} = \{P(t) = \mathbf{P}(t^3, t^2, t, 1) | t \in \mathbb{F}_q^+, P(\infty) = \mathbf{P}(1, 0, 0, 0)\}. \tag{2.1}$$

Let $\boldsymbol{\pi}(c_0, c_1, c_2, c_3) \subset \mathrm{PG}(3, q)$, $c_i \in \mathbb{F}_q$, be the plane with equation $c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0$. The plane through three points $P(t_1), P(t_2), P(t_3)$ of $\mathscr{C}$ is

$$\boldsymbol{\pi}(1, -(t_1 + t_2 + t_3), t_1t_2 + t_1t_3 + t_2t_3, -t_1t_2t_3) \supset \{P(t_1), P(t_2), P(t_3)\}. \tag{2.2}$$

When three points coincide with each other and $t_1 = t_2 = t_3 = t$, we have, in the point $P(t) = \mathbf{P}(t^3, t^2, t, 1) \in \mathscr{C}$, an *osculating plane* $\pi_{\mathrm{osc}}(t)$ such that

$$\pi_{\mathrm{osc}}(t) = \boldsymbol{\pi}(1, -3t, 3t^2, -t^3), \quad P(t) = \mathbf{P}(t^3, t^2, t, 1) \in \pi_{\mathrm{osc}}(t); \tag{2.3}$$

$$\pi_{\mathrm{osc}}(\infty) = \boldsymbol{\pi}(0, 0, 0, 1), \quad P(\infty) = \mathbf{P}(1, 0, 0, 0) \in \pi_{\mathrm{osc}}(\infty). \tag{2.4}$$

The osculating plane $\pi_{\mathrm{osc}}(t)$ meets $\mathscr{C}$ only in $P(t)$. The osculating planes form the *osculating developable* to $\mathscr{C}$, that is, a pencil of planes for $q \equiv 0 \pmod 3$ or a cubic developable otherwise.

A *chord* of $\mathscr{C}$ is a line through a pair of real points of $\mathscr{C}$ or a pair of complex conjugate points. In the last case it is an *imaginary chord*. If the real points are distinct, it is a *real chord*. If the real points coincide with each other, it is a *tangent*. No two chords of $\mathscr{C}$ meet off $\mathscr{C}$. Every point off $\mathscr{C}$ lies on exactly one chord of $\mathscr{C}$.

Notation 2.1. The following notation is used:

G_q	the group of projectivities in $\mathrm{PG}(3, q)$ fixing $\mathscr{C}$;
$\mathbf{Z}_n$	cyclic group of order n;
$\mathbf{S}_n$	symmetric group of degree n;
Γ	the osculating developable to $\mathscr{C}$;
$\mathfrak{A}$	the null polarity [13, Chapter 2.1.5], [14, Theorem 21.1.2];
Γ-plane	an osculating plane of Γ;

$d_{\mathscr{C}}$-plane	a plane containing *exactly* d distinct points of $\mathscr{C}$, $d = 0, 1, 2, 3$;
$1_{\mathscr{C}} \setminus \Gamma$-plane	a $1_{\mathscr{C}}$-plane not in Γ;
$\mathscr{C}$-point	a point of $\mathscr{C}$;
μ_Γ-point	a point off $\mathscr{C}$ lying on *exactly* μ osculating planes, $\mu_\Gamma = 0, 1, 3, q+1$;
T-point	a point off $\mathscr{C}$ on a tangent to $\mathscr{C}$ for $q \not\equiv 0 \pmod 3$;
TO-point	a point off $\mathscr{C}$ on a tangent and one osculating plane for $q \equiv 0 \pmod 3$;
RC-point	a point off $\mathscr{C}$ on a real chord;
IC-point	a point on an imaginary chord;
A^{tr}	the transposed matrix A.

The following theorem summarizes known results from [14].

Theorem 2.2. [14, Chapter 21] *The following properties of the twisted cubic $\mathscr{C}$ of* (2.1) *hold:*

A. *The group G_q acts triply transitively on $\mathscr{C}$. Also,*

$$\begin{array}{ll} G_q \cong PGL(2,q), & \text{for } q \geq 5; \\ G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2,4), & \#G_4 = 2 \cdot \#PGL(2,4) = 120; \\ G_3 \cong \mathbf{S}_4\mathbf{Z}_2^3, & \#G_3 = 8 \cdot \#PGL(2,3) = 192; \\ G_2 \cong \mathbf{S}_3\mathbf{Z}_2^3, & \#G_2 = 8 \cdot \#PGL(2,2) = 48. \end{array}$$

B. *Let $q \geq 5$. Under G_q, there are five orbits $\mathscr{N}_i$ of planes and five orbits $\mathscr{M}_j$ of points. These orbits have the following properties:*

(i) *For all q, the orbits $\mathscr{N}_i$ of planes are as follows:*

$$\mathscr{N}_1 = \{\Gamma\text{-}planes\},\ \#\mathscr{N}_1 = q+1;\ \mathscr{N}_2 = \{2_{\mathscr{C}}\text{-}planes\},\ \#\mathscr{N}_2 = q(q+1); \tag{2.5}$$
$$\mathscr{N}_3 = \{3_{\mathscr{C}}\text{-}planes\},\ \#\mathscr{N}_3 = \frac{q(q^2-1)}{6};$$
$$\mathscr{N}_4 = \{1_{\mathscr{C}} \setminus \Gamma\text{-}planes\},\ \#\mathscr{N}_4 = \frac{q(q^2-1)}{2};$$
$$\mathscr{N}_5 = \{0_{\mathscr{C}}\text{-}planes\},\ \#\mathscr{N}_5 = \frac{q(q^2-1)}{3}.$$

(ii) *For $q \not\equiv 0 \pmod 3$, the orbits $\mathscr{M}_j$ of points are as follows:*

$$\mathscr{M}_1 = \mathscr{C},\ \#\mathscr{M}_1 = q+1;\ \mathscr{M}_2 = \{\text{T-}points\},\ \#\mathscr{M}_2 = q(q+1); \tag{2.6}$$
$$\mathscr{M}_3 = \{3_\Gamma\text{-}points\},\ \#\mathscr{M}_3 = \frac{q(q^2-1)}{6};$$

$$\mathscr{M}_4 = \{1_\Gamma\text{-}points\},\ \#\mathscr{M}_4 = \frac{q(q^2-1)}{2};$$
$$\mathscr{M}_5 = \{0_\Gamma\text{-}points\},\ \#\mathscr{M}_5 = \frac{q(q^2-1)}{3}.$$

Also,

$$\text{if } q \equiv 1 \pmod 3 \text{ then } \mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{RC-}points\},\ \mathscr{M}_4 = \{\text{IC-}points\}; \tag{2.7}$$
$$\text{if } q \equiv -1 \pmod 3 \text{ then } \mathscr{M}_3 \cup \mathscr{M}_5 = \{\text{IC-}points\},\ \mathscr{M}_4 = \{\text{RC-}points\}. \tag{2.8}$$

(iii) *For $q \equiv 0 \pmod 3$, the orbits $\mathscr{M}_k$ of points are as follows:*

$$\mathscr{M}_1 = \mathscr{C},\ \#\mathscr{M}_1 = q+1;\ \mathscr{M}_2 = \{(q+1)_\Gamma\text{-}points\},\ \#\mathscr{M}_2 = q+1; \tag{2.9}$$
$$\mathscr{M}_3 = \{\text{TO-}points\},\ \#\mathscr{M}_3 = q^2-1;$$
$$\mathscr{M}_4 = \{\text{RC-}points\},\ \#\mathscr{M}_4 = \frac{q(q^2-1)}{2};$$
$$\mathscr{M}_5 = \{\text{IC-}points\},\ \#\mathscr{M}_5 = \frac{q(q^2-1)}{2}.$$

C. *In total, there are $\binom{q+1}{2}$ real chords of $\mathscr{C}$, $q+1$ tangents to $\mathscr{C}$, and $\binom{q}{2}$ imaginary chords of $\mathscr{C}$.*

D. *For $q \not\equiv 0 \pmod 3$, the null polarity $\mathfrak{A}$ interchanges $\mathscr{C}$ and Γ; also,*

$$\mathscr{M}_i\mathfrak{A} = \mathscr{N}_i,\ \#\mathscr{M}_i = \#\mathscr{N}_i. \tag{2.10}$$

Remark 2.3. For $q \equiv 0 \pmod 3$, Γ is a pencil of $q+1$ planes, see [14, Theorem 21.1.2(i)]. Points lying on all these planes (the orbit $\mathscr{M}_2$) form a line external to $\mathscr{C}$. All $d_{\mathscr{C}}$-planes with $d = 0, 1, 2, 3$ intersect this line.

2.2 The point-plane incidence matrix of $\mathrm{PG}(3, q)$

Let $\mathcal{I}$ be the $\theta_{3,q} \times \theta_{3,q}$ point-plane incidence matrix of $\mathrm{PG}(3, q)$ in which columns correspond to points, rows correspond to planes, and there an entry is "1" if the corresponding point belongs to the corresponding plane. Every column and every row of $\mathcal{I}$ contains exactly $\theta_{2,q}$ ones, i.e. $\mathcal{I}$ is a tactical configuration [13, Chapter 2.3]. Moreover, $\mathcal{I}$ gives a symmetric 2-$(\theta_{3,q}, \theta_{2,q}, q+1)$ design as there are exactly $q+1$ planes through any two points of $\mathrm{PG}(3, q)$.

For $q \geq 5$, orbits $\mathscr{N}_i$ and $\mathscr{M}_j$ partition $\mathcal{I}$ in 25 submatrices $\mathcal{I}_{ij}$, with $i, j = 1, \ldots, 5$, where $\mathcal{I}_{ij}$ has size $\#\mathscr{N}_i \times \#\mathscr{M}_j$.

It is clear (see Lemma 4.12) that every plane of $\mathscr{N}_i$ contains the same number of points from $\mathscr{M}_j$; we denote this number as k_{ij}. And vice versa, through every point of $\mathscr{M}_j$ we have the same number of planes from $\mathscr{N}_i$; we denote this number as r_{ij}. This means that $\mathcal{I}_{ij}$ contains k_{ij} ones in each row and r_{ij} ones in each column, i.e. $\mathcal{I}_{ij}$ *is a tactical configuration*.

Tactical configurations are useful in distinct areas as, in particular, to construct bipartite graph codes, see e.g. [1, 10, 17] and the references therein.

2.3 Linear multiple covering codes and multiple saturating sets

Let $\mathbb{F}_q^n$ be the space of n-dimensional vectors over $\mathbb{F}_q$. Consider a linear code $C \subseteq \mathbb{F}_q^n$ and denote by $A_w(C)$ the number of its codewords of weight w. Let $d(x, c)$ be the Hamming distance between vectors x and c of $\mathbb{F}_q^n$ and denote by $d(x, C) = \min_{c \in C} d(x, c)$ the distance between x and C.

Definition 2.4. [2, 8, 21] An $[n, k, d]_q R$ code C is an (R, μ) *multiple covering of the farthest-off points* ((R, μ)-MCF code for short) if for all $x \in \mathbb{F}_q^n$ such that $d(x, C) = R$ the number of codewords c such that $d(x, c) = R$ is at least μ.

In the literature, MCF codes are also called multiple coverings of deep holes.

The covering quality of an $[n, k, d(C)]_q R$ MCF code C is characterized by its μ-*density* $\gamma_\mu(C, R) \geq 1$ so that

$$\gamma_\mu(C, R) = \frac{\binom{n}{R}(q-1)^R - \binom{2R-1}{R-1} A_{2R-1}(C)}{\mu \left(q^{n-k} - \sum\limits_{i=0}^{R-1} \binom{n}{i}(q-1)^i \right)} \text{ if } d(C) \geq 2R - 1; \tag{2.11}$$

see [2, Proposition 2.3], [3, Proposition 1]. From the covering problem point of view, the best codes are those with small μ-density. If $\gamma_\mu(C, R) = 1$ then C is called perfect MCF code. We call *asymptotically optimal code* an MCF code whose μ-density tends to 1 when q tends to infinity.

Definition 2.5. [2, 21] Let S be an n-subset of points of $\mathrm{PG}(N, q)$. Then S is said to be (ρ, μ)-saturating if:

(M1) S generates $\mathrm{PG}(N, q)$;

(M2) there exists a point Q in $\mathrm{PG}(N, q)$ which does not belong to any subspace of dimension $\rho - 1$ generated by the points of S;

(M3) every point Q in $\mathrm{PG}(N, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S is such that the number of subspaces of dimension ρ generated by the points of S and containing Q is at least μ.

Here we slightly simplified the corresponding definition of [2, 21].

Definition 2.6. [2] A (ρ,μ)-saturating n-set in $\mathrm{PG}(N,q)$ is called *minimal* if it does not contain a (ρ,μ)-saturating $(n-1)$-set in $\mathrm{PG}(N,q)$.

Proposition 2.7. [2, Proposition 3.6] *Let S be a (ρ,μ)-saturating n-set in* $\mathrm{PG}(n-k-1,q)$. *Let a linear $[n,k]_qR$ code C admit a parity-check matrix whose columns are homogeneous coordinates of the points in S. Then C is a $(\rho+1,\mu)$-MCF code.*

Proposition 2.7 allows us to consider (ρ,μ)-saturating sets as linear $(\rho+1,\mu)$-MCF codes and vice versa.

3 Main results

From now on we consider $q\ge 5$ apart from Theorems 3.1(B) and 6.6.

Tables 1 and 2 and Theorem 3.1 summarize the results of Sections 4–6.

In particular, for the point-plane incidence matrix, Tables 1 and 2 show values k_{ij} (top entry) and r_{ij} (bottom entry) for each possible pair $(\mathscr{N}_i,\mathscr{M}_j)$, where k_{ij} is the number of points from $\mathscr{M}_j$ in every plane of $\mathscr{N}_i$, whereas r_{ij} is the number of planes from $\mathscr{N}_i$ through every point of $\mathscr{M}_j$. In other words, k_{ij} (resp. r_{ij}) is the number of ones in every row (resp. column) of the $\#\mathscr{N}_i\times\#\mathscr{M}_j$ submatrix $\mathcal{I}_{ij}$ of the point-plane incidence matrix.

Table 1: Values k_{ij} (the number of ones in every row, top entry) and r_{ij} (the number of ones in every column, bottom entry) for the $\#\mathscr{N}_i\times\#\mathscr{M}_j$ submatrices $\mathcal{I}_{ij}$ of the point-plane incidence matrix of $\mathrm{PG}(3,q)$, $q\equiv\xi \pmod 3$, $\xi=-1,1$, $q\ge 5$

$\mathscr{N}_i \downarrow$	$\mathscr{M}_j \rightarrow$		$\mathscr{M}_1$ $\mathscr{C}$-points $q+1$	$\mathscr{M}_2$ T-points q^2+q	$\mathscr{M}_3$ 3_Γ-points $\frac{1}{6}(q^3-q)$	$\mathscr{M}_4$ 1_Γ-points $\frac{1}{2}(q^3-q)$	$\mathscr{M}_5$ 0_Γ-points $\frac{1}{3}(q^3-q)$
$\mathscr{N}_1$	Γ-planes	k_{1j}	1	$2q$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$	0
	$q+1$	r_{1j}	1	2	3	1	0
$\mathscr{N}_2$	$2_\mathscr{C}$-planes	k_{2j}	2	$2q-1$	$\frac{1}{6}(q^2-3q+2)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{3}(q^2-1)$
	q^2+q	r_{2j}	$2q$	$2q-1$	$q-2$	q	$q+1$
$\mathscr{N}_3$	$3_\mathscr{C}$-planes	k_{3j}	3	$q-2$	$\frac{1}{6}(q^2+\xi q+4)$	$\frac{1}{2}(q^2-\xi q)$	$\frac{1}{3}(q^2+\xi q-2)$
	$\frac{1}{6}(q^3-q)$	r_{3j}	$\frac{1}{2}(q^2-q)$	$\frac{q^2-3q+2}{6}$	$\frac{1}{6}(q^2+\xi q+4)$	$\frac{1}{6}(q^2-\xi q)$	$\frac{1}{6}(q^2+\xi q-2)$
$\mathscr{N}_4$	$1_\mathscr{C}\setminus\Gamma$-planes	k_{4j}	1	q	$\frac{1}{6}(q^2-\xi q)$	$\frac{1}{2}(q^2+\xi q)$	$\frac{1}{3}(q^2-\xi q)$
	$\frac{1}{2}(q^3-q)$	r_{4j}	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-\xi q)$	$\frac{1}{2}(q^2+\xi q)$	$\frac{1}{2}(q^2-\xi q)$
$\mathscr{N}_5$	$0_\mathscr{C}$-planes	k_{5j}	0	$q+1$	$\frac{1}{6}(q^2+\xi q-2)$	$\frac{1}{2}(q^2-\xi q)$	$\frac{1}{3}(q^2+\xi q+1)$
	$\frac{1}{3}(q^3-q)$	r_{5j}	0	$\frac{1}{3}(q^2-1)$	$\frac{1}{3}(q^2+\xi q-2)$	$\frac{1}{3}(q^2-\xi q)$	$\frac{1}{3}(q^2+\xi q+1)$

Table 2: Values k_{ij} (the number of ones in every row, top entry) and r_{ij} (the number of ones in every column, bottom entry) for the $\#\mathscr{N}_i\times\#\mathscr{M}_j$ submatrices $\mathcal{I}_{ij}$ of the point-plane incidence matrix of $\mathrm{PG}(3,q)$, $q\equiv 0 \pmod 3$, $q\geq 5$

$\mathscr{N}_i$ $\downarrow$	$\mathscr{M}_j \rightarrow$		$\mathscr{M}_1$ $\mathscr{C}$-points $q+1$	$\mathscr{M}_2$ $(q+1)_\Gamma$ -points $q+1$	$\mathscr{M}_3$ TO-points q^2-1	$\mathscr{M}_4$ RC-points $\frac{1}{2}(q^3-q)$	$\mathscr{M}_5$ IC-points $\frac{1}{2}(q^3-q)$
$\mathscr{N}_1$	Γ-planes	k_{1j}	1	$q+1$	$q-1$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$
	$q+1$	r_{1j}	1	$q+1$	1	1	1
$\mathscr{N}_2$	$2_{\mathscr{C}}$-planes	k_{2j}	2	1	$2q-2$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$
	q^2+q	r_{2j}	$2q$	q	$2q$	q	q
$\mathscr{N}_3$	$3_{\mathscr{C}}$-planes	k_{3j}	3	1	$q-3$	$\frac{1}{2}(q^2+q)$	$\frac{1}{2}(q^2-q)$
	$\frac{1}{6}(q^3-q)$	r_{3j}	$\frac{1}{2}(q^2-q)$	$\frac{1}{6}(q^2-q)$	$\frac{1}{6}(q^2-3q)$	$\frac{1}{6}(q^2+q)$	$\frac{1}{6}(q^2-q)$
$\mathscr{N}_4$	$1_{\mathscr{C}}\setminus\Gamma$-planes	k_{4j}	1	1	$q-1$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2+q)$
	$\frac{1}{2}(q^3-q)$	r_{4j}	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}(q^2+q)$
$\mathscr{N}_5$	$0_{\mathscr{C}}$-planes	k_{5j}	0	1	q	$\frac{1}{2}(q^2+q)$	$\frac{1}{2}(q^2-q)$
	$\frac{1}{3}(q^3-q)$	r_{5j}	0	$\frac{1}{3}(q^2-q)$	$\frac{1}{3}q^2$	$\frac{1}{3}(q^2+q)$	$\frac{1}{3}(q^2-q)$

Theorem 3.1. A. *Let $q\geq 5$. Let $q\equiv\xi \pmod 3$. The following hold:*

(i) *In $\mathrm{PG}(3,q)$, let notations of planes, points, and incidence submatrices be as in Sections 2.1 and 2.2. Then, for the point-plane incidence matrix, the values k_{ij} (i.e. the number of distinct points in distinct planes) and r_{ij} (i.e. the number of distinct planes through distinct points) are given by Tables 1 and 2.*

(ii) *Up to rearrangement of rows and columns, we have*

$$\mathcal{I}_{ij}^{tr}=\mathcal{I}_{ji},\ k_{ij}=r_{ji},\ r_{ij}=k_{ji},\ i,j=1,\dots,5,\ \textit{for } \xi\neq 0;$$
$$\mathcal{I}_{41}^{tr}=\mathcal{I}_{14},\ \mathcal{I}_{41}^{tr}=\mathcal{I}_{15},\ \mathcal{I}_{42}^{tr}=\mathcal{I}_{14},\ \mathcal{I}_{42}^{tr}=\mathcal{I}_{15},\ \textit{for } \xi=0;$$
$$\mathcal{I}_{i4} \textit{ for } \xi=1 \textit{ is the same as } \mathcal{I}_{i5} \textit{ for } \xi=0,\ i=1,\dots,5;$$
$$\mathcal{I}_{i4} \textit{ for } \xi=-1 \textit{ and for } \xi=0 \textit{ is the same, } i=1,\dots,5.$$

(iii) *Let $\xi\in\{-1,1\}$. Then the submatrix $\mathcal{I}_{21}$ gives a 2-$(q+1,2,2)$ design and the submatrix $\mathcal{I}_{31}$ defines 3-$(q+1,3,1)$ and 2-$(q+1,3,q-1)$ designs.*

B. *Let $q=2,3,4$. Then the point-plane incidence matrix can be represented as in Tables 1 and 2 if $\mathscr{N}_i,\mathscr{M}_j$ are orbits under a group isomorphic to $\mathbf{S}_{q+1}$, where $\mathbf{S}_{q+1}$ is isomorphic to a subgroup of G_q for $q=2,3$, whereas $\mathbf{S}_{4+1}\cong G_4$, cf. Theorem 6.6.*

Theorem 3.2 summarizes the results of Section 7.

Theorem 3.2. *Let*

$$\mu = \begin{cases} \frac{q^2-3q+2}{6} & \text{if} \quad q \not\equiv 0 \pmod 3 \\ \frac{q^2-3q}{6} & \text{if} \quad q \equiv 0 \pmod 3 \end{cases}. \tag{3.1}$$

(i) *The twisted cubic* $\mathscr{C}$ *of* (2.1) *is a minimal* $(2,\mu)$*-saturating* $(q+1)$*-set.*

(ii) *The generalized doubly-extended Reed-Solomon code associated with* $\mathscr{C}$ *is a* $(3,\mu)$ *multiple covering of the farthest-off points, i.e.* $(3,\mu)$*-MCF code, with parameters* $[q+1, q-3, 5]_q3$*. Its* μ*-density tends to* 1 *from above when* q *tends to infinity; the code is asymptotical optimal.*

4 Some useful relations

Notation 4.1. Let $d \in \{0,1,2,3\}$. The following notation is used:

n_d^{Σ}	the total number of $d_{\mathscr{C}}$-planes;
$n_{d,\mathscr{C}}$	the number of $d_{\mathscr{C}}$-planes through a $\mathscr{C}$-point;
$n_d(A)$	the number of $d_{\mathscr{C}}$-planes through a point A;
$n_{d,\mu_\Gamma}^{(\xi)}$	the number of $d_{\mathscr{C}}$-planes through a μ_Γ-point for $q \equiv \xi \pmod 3$ where $\mu_\Gamma \in \{0,1,3\}$ if $\xi \neq 0$ and $\mu_\Gamma = q+1$ if $\xi = 0$;
$n_{d,\mathrm{T}}^{(\neq 0)}$	the number of $d_{\mathscr{C}}$-planes through a T-point for $q \not\equiv 0 \pmod 3$;
$n_{d,\mathrm{TO}}^{(0)}$	the number of $d_{\mathscr{C}}$-planes through a TO-point for $q \equiv 0 \pmod 3$;
$n_{d,\mathrm{RC}}^{(0)}$	the number of $d_{\mathscr{C}}$-planes through an RC-point for $q \equiv 0 \pmod 3$;
$n_{d,\mathrm{IC}}^{(0)}$	the number of $d_{\mathscr{C}}$-planes through an IC-point for $q \equiv 0 \pmod 3$.

Remark 4.2. In Notation 4.1, the values $n_{d,\bullet}^{(\star)}$ are equal to the parameters r_{ij} of the submatrices $\mathcal{I}_{ij}$. Using numbers of orbits in Theorem 2.2(B) and Tables 1 and 2, one can easy set the correspondence between $n_{d,\bullet}^{(\star)}$ and r_{ij}. For example,

$$n_{0,\mathscr{C}} = r_{5,1},\ n_{1,\mathscr{C}} = r_{1,1} + r_{4,1},\ n_{2,\mathscr{C}} = r_{2,1},\ n_{3,\mathscr{C}} = r_{3,1};$$
$$n_{0,0_\Gamma}^{(\xi)} = r_{5,5};\ n_{1,0_\Gamma}^{(\xi)} = r_{1,5} + r_{4,5};\ n_{2,0_\Gamma}^{(\xi)} = r_{2,5};\ n_{3,0_\Gamma}^{(\xi)} = r_{3,5},\ q \equiv \xi \pmod 3,\ \xi \neq 0.$$

Lemma 4.3. *For all* q*, the number of* $3_{\mathscr{C}}$*-planes and* $2_{\mathscr{C}}$*-planes through a real chord of* $\mathscr{C}$ *is equal to* $q-1$ *and* 2*, respectively.*

Proof. We consider the real chord through points K, Q of $\mathscr{C}$. Every plane through a real chord is either a $2_{\mathscr{C}}$-plane or a $3_{\mathscr{C}}$-plane. Every of $q-1$ points R of $\mathscr{C}\setminus\{K,Q\}$ gives rise to the $3_{\mathscr{C}}$-plane through K, Q, R. Therefore, the number of the $3_{\mathscr{C}}$-planes through a real chord is equal to $q-1$. In total, we have $q+1$ planes through a line in $\mathrm{PG}(3,q)$. Thus, the number of the $2_{\mathscr{C}}$-planes through a real chord is $q+1-(q-1)=2$. □

Proposition 4.4. *For all q, we have*

$$n_0^{\Sigma}=\frac{q(q^2-1)}{3},\ n_1^{\Sigma}=\frac{q^3+q+2}{2},\ n_2^{\Sigma}=q(q+1),\ n_3^{\Sigma}=\frac{q(q^2-1)}{6}. \tag{4.1}$$

Proof. By Theorem 2.2(Bi), $n_0^{\Sigma}=\#\mathscr{N}_5$, $n_1^{\Sigma}=\#\mathscr{N}_1+\#\mathscr{N}_4$, $n_2^{\Sigma}=\#\mathscr{N}_2$, $n_3^{\Sigma}=\#\mathscr{N}_3$. □

Proposition 4.5. *The following hold:*

(i) *Let $q\not\equiv 0 \pmod 3$ and $q\equiv\xi \pmod 3$. Then for $\xi\neq 0$ we have*

$$n_{d,\mathrm{T}}^{(\xi)}+\frac{q-1}{3}n_{d,0_\Gamma}^{(\xi)}+\frac{q-1}{2}n_{d,1_\Gamma}^{(\xi)}+\frac{q-1}{6}n_{d,3_\Gamma}^{(\xi)}=\begin{cases}\frac{1}{3}(q^3-1) & \text{if } d=0\\ \frac{1}{2}(q^3+q+2) & \text{if } d=1\\ q^2+q-1 & \text{if } d=2\\ \frac{1}{6}(q-1)^2(q+2) & \text{if } d=3\end{cases}.$$

(ii) *Let $q\equiv 0 \pmod 3$. Then*

$$(q-1)n_{d,\mathrm{TO}}^{(0)}+n_{d,q+1_\Gamma}^{(0)}+\frac{q(q-1)}{2}n_{d,\mathrm{RC}}^{(0)}+\frac{q(q-1)}{2}n_{d,\mathrm{IC}}^{(0)}$$
$$=\begin{cases}\frac{1}{3}q(q^3-1) & \text{if } d=0\\ \frac{1}{2}q(q^3+q+2) & \text{if } d=1\\ q(q^2+q-1) & \text{if } d=2\\ \frac{1}{6}q(q-1)^2(q+2) & \text{if } d=3\end{cases}.$$

Proof. Every $d_{\mathscr{C}}$-plane contains $q^2+q+1-d$ points outside $\mathscr{C}$. Therefore,

(i) $\#\mathscr{M}_2 n_{d,\mathrm{T}}^{(\xi)}+\#\mathscr{M}_5 n_{d,0_\Gamma}^{(\xi)}+\#\mathscr{M}_4 n_{d,1_\Gamma}^{(\xi)}+\#\mathscr{M}_3 n_{d,3_\Gamma}^{(\xi)}=n_d^{\Sigma}(q^2+q+1-d)$.

(ii) $\#\mathscr{M}_3 n_{d,\mathrm{TO}}^{(0)}+\#\mathscr{M}_2 n_{d,q+1_\Gamma}^{(0)}+\#\mathscr{M}_4 n_{d,\mathrm{RC}}^{(0)}+\#\mathscr{M}_5 n_{d,\mathrm{IC}}^{(0)}$
$=n_d^{\Sigma}(q^2+q+1-d)$.

Now, we use the values of $\#\mathscr{M}_j$ and n_d^{Σ} from (2.6), (2.9), and (4.1). □

Proposition 4.6. *Let $q\equiv\xi \pmod 3$. Then*

$$\sum_{d=0}^{3}n_{d,\mathrm{T}}^{(\xi)}=\sum_{d=0}^{3}n_{d,0_\Gamma}^{(\xi)}=\sum_{d=0}^{3}n_{d,1_\Gamma}^{(\xi)}=\sum_{d=0}^{3}n_{d,3_\Gamma}^{(\xi)}=q^2+q+1,\ \xi\neq 0;$$
$$\sum_{d=0}^{3}n_{d,\mathrm{TO}}^{(0)}=\sum_{d=0}^{3}n_{d,q+1_\Gamma}^{(0)}=\sum_{d=0}^{3}n_{d,\mathrm{RC}}^{(0)}=\sum_{d=0}^{3}n_{d,\mathrm{IC}}^{(0)}=q^2+q+1.$$

Proof. There are q^2+q+1 planes through every point of $\mathrm{PG}(3,q)$. $\square$

Lemma 4.7. *For all q, for a point A off $\mathscr{C}$,*

$$n_2(A)+3n_3(A)=\begin{cases}\binom{q+1}{2} & \text{if } A \text{ does not lie on any real chord}\\ \frac{q^2+3q}{2} & \text{if } A \text{ lies on a real chord}\end{cases}.$$

Proof. Let A not lie on any real chord. There are $\binom{\#\mathscr{C}}{2}=\binom{q+1}{2}$ real chords. Every chord together with A defines a plane which is either a $2_{\mathscr{C}}$-plane or a $3_{\mathscr{C}}$-plane. All the $2_{\mathscr{C}}$-planes are distinct whereas every $3_{\mathscr{C}}$-plane contains 3 real chords and is repeated 3 times.

Let A lie on a real chord. Let $S(A)$ be the set of $\binom{q+1}{2}-1$ real chords not containing A. For $d=2,3$, let $n_d^*(A)$ be the number of $d_{\mathscr{C}}$-planes through A and a chord of $S(A)$. Every such $3_{\mathscr{C}}$-plane contains 3 real chords of $S(A)$ and is repeated 3 times while all the $2_{\mathscr{C}}$-planes are distinct.

Denote by $\mathcal{RC}$ the real chord containing A. By Lemma 4.3, in total there are $q-1$ $3_{\mathscr{C}}$-planes and two $2_{\mathscr{C}}$-planes through $\mathcal{RC}$. All these planes contain A and they do not contain any chord from $S(A)$. Therefore, $n_3(A)=n_3^*(A)+q-1,\ n_2(A)=n_2^*(A)+2$. Every of the $q-1$ $3_{\mathscr{C}}$-planes through $\mathcal{RC}$ contains 2 real chords of $S(A)$. Thus,

$$3n_3^*(A)+2(q-1)+n_2^*(A)=\binom{q+1}{2}-1$$

whence the assertion follows. $\square$

Corollary 4.8. *The following hold:*

$$n^{(1)}_{2,\mathrm{T}}+3n^{(1)}_{3,\mathrm{T}}=n^{(-1)}_{2,\mathrm{T}}+3n^{(-1)}_{3,\mathrm{T}}=n^{(1)}_{2,1_\Gamma}+3n^{(1)}_{3,1_\Gamma}=n^{(-1)}_{2,0_\Gamma}+3n^{(-1)}_{3,0_\Gamma} \tag{4.2}$$
$$=n^{(-1)}_{2,3_\Gamma}+3n^{(-1)}_{3,3_\Gamma}=n^{(0)}_{2,\mathrm{TO}}+3n^{(0)}_{3,\mathrm{TO}}=n^{(0)}_{2,q+1_\Gamma}+3n^{(0)}_{3,q+1_\Gamma}=n^{(0)}_{2,\mathrm{IC}}+3n^{(0)}_{3,\mathrm{IC}}=\binom{q+1}{2}.$$

$$n^{(1)}_{2,0_\Gamma}+3n^{(1)}_{3,0_\Gamma}=n^{(1)}_{2,3_\Gamma}+3n^{(1)}_{3,3_\Gamma}=n^{(-1)}_{2,1_\Gamma}+3n^{(-1)}_{3,1_\Gamma}=n^{(0)}_{2,\mathrm{RC}}+3n^{(0)}_{3,\mathrm{RC}}=\frac{q^2+3q}{2}. \tag{4.3}$$

Proof. Due to Theorem 2.2(Bii),(Biii), (4.2) holds for points off $\mathscr{C}$ not on a real chord whereas (4.3) concerns points lying on a real chord. $\square$

Lemma 4.9. *For all q, for a point A off $\mathscr{C}$ the following holds:*

$$n_1(A)+2n_2(A)+3n_3(A)=(q+1)^2.$$

Proof. We consider the line $\overline{AP_i}$ through points $A\notin\mathscr{C}$ and $P_i\in\mathscr{C}$, $i\in\{1,2,\dots,q+1\}$. Each of the $q+1$ planes through $\overline{AP_i}$ is a $d_{\mathscr{C}}$-plane with $d\in\{1,2,3\}$. Let $n_d(P_i)$ be the number of $d_{\mathscr{C}}$-planes through $\overline{AP_i}$. Clearly, $n_1(P_i)+n_2(P_i)+n_3(P_i)=q+1$. Moreover,

$$n_1(A)+2n_2(A)+3n_3(A)=\sum_{i=1}^{q+1}(n_1(P_i)+n_2(P_i)+n_3(P_i))=\sum_{i=1}^{q+1}q+1=(q+1)^2.$$

Here we take into account that in the sum $\sum_{i=1}^{q+1}(n_1(P_i)+n_2(P_i)+n_3(P_i))$ every $d_{\mathscr{C}}$-plane appears d times. $\square$

Corollary 4.10. *For all q, the following hold:*

$$n_{1,\mathrm{T}}^{(\xi)}+2n_{2,\mathrm{T}}^{(\xi)}+3n_{3,\mathrm{T}}^{(\xi)}=n_{1,\mu_\Gamma}^{(\xi)}+2n_{2,\mu_\Gamma}^{(\xi)}+3n_{3,\mu_\Gamma}^{(\xi)}=(q+1)^2,\ \ \mu_\Gamma=0,1,3,\ \ \xi\neq 0;$$
$$n_{1,\mathrm{TO}}^{(0)}+2n_{2,\mathrm{TO}}^{(0)}+3n_{3,\mathrm{TO}}^{(0)}=n_{1,q+1_\Gamma}^{(0)}+2n_{2,q+1_\Gamma}^{(0)}+3n_{3,q+1_\Gamma}^{(0)}$$
$$=n_{1,\mathrm{RC}}^{(0)}+2n_{2,\mathrm{RC}}^{(0)}+3n_{3,\mathrm{RC}}^{(0)}=n_{1,\mathrm{IC}}^{(0)}+2n_{2,\mathrm{IC}}^{(0)}+3n_{3,\mathrm{IC}}^{(0)}=(q+1)^2.$$

Lemma 4.11. *All $d_{\mathscr{C}}$-planes with $d=0,2,3$ and all osculating planes contain no imaginary chord. All $q+1$ planes through an imaginary chord are $1_{\mathscr{C}}\setminus\Gamma$-planes; these $q+1$ planes form a pencil.*

Proof. Any $2_{\mathscr{C}}$-plane and $3_{\mathscr{C}}$-plane contains a real chord. An osculating plane contains a tangent. If a $2_{\mathscr{C}}$,- or a $3_{\mathscr{C}}$-, or a Γ-plane contains an imaginary chord then it intersects the real chord or the tangent, contradiction. Thus, we have a $1_{\mathscr{C}}\setminus\Gamma$-plane through an imaginary chord and any point of $\mathscr{C}$. In total, there are $\#\mathscr{C}=q+1$ such $1_{\mathscr{C}}\setminus\Gamma$-planes for every imaginary chord. $\square$

The following lemma is obvious.

Lemma 4.12. *In $\mathrm{PG}(3,q)$, let $\mathscr{N}$ and $\mathscr{M}$ be, respectively, an orbit of planes and an orbit of points under some group G of projectivities.*

(i) *The number of planes from $\mathscr{N}$ through a point of $\mathscr{M}$ is the same for all points of $\mathscr{M}$.*

(ii) *The number of points from $\mathscr{M}$ in a plane of $\mathscr{N}$ is the same for all planes of $\mathscr{N}$.*

Proof. **(i)** Consider points P and Q of $\mathscr{M}$. Denote by π a plane of $\mathscr{N}$. Let $S(P)$ and $S(Q)$ be subsets of $\mathscr{N}$ such that $S(P)=\{\pi\in\mathscr{N}|P\in\pi\}$, $S(Q)=\{\pi\in\mathscr{N}|Q\in\pi\}$. There exists $\varphi\in G$ such that $Q=\varphi(P)$. Clearly, φ embeds $S(P)$ in $S(Q)$, i.e. $\varphi(S(P))\subseteq S(Q)$ and $\#S(P)\le\#S(Q)$. In the same way, φ^{-1} embeds $S(Q)$ in $S(P)$, i.e. $\#S(Q)\le\#S(P)$. Thus, $\#S(Q)=\#S(P)$.

(ii) The proof is similar to the point (i).

$\square$

5 The number r_{ij} of distinct planes through distinct points of $\mathrm{PG}(3,q)$

In this section we obtain all values r_{ij}, $i,j=1,\dots,5$.

Theorem 5.1. *The following hold:*

$$n_{0,\mathscr{C}}=0,\ n_{1,\mathscr{C}}=\frac{q^2-q+2}{2},\ n_{2,\mathscr{C}}=2q,\ n_{3,\mathscr{C}}=\frac{q^2-q}{2}.$$

Proof. By definition, $n_{0,\mathscr{C}}=0$. Obviously, $n_{1,\mathscr{C}}=\frac{n_1^\Sigma}{\#\mathscr{C}}$, see (4.1).

We consider a point $A\in\mathscr{C}$. There are q real chords through A. By Lemma 4.3, we have two $2_{\mathscr{C}}$-planes through every such chord. Finally, every pair of points of $\mathscr{C}\setminus\{A\}$ generates a $3_{\mathscr{C}}$-plane through A. $\square$

Theorem 5.2. *The following hold:*

$$n^{(1)}_{0,1_\Gamma}=n^{(0)}_{0,q+1_\Gamma}=n^{(0)}_{0,\mathrm{IC}}=\frac{q^2-q}{3},\ n^{(1)}_{1,1_\Gamma}=n^{(0)}_{1,q+1_\Gamma}=n^{(0)}_{1,\mathrm{IC}}=\frac{q^2+q+2}{2},$$
$$n^{(1)}_{2,1_\Gamma}=n^{(0)}_{2,q+1_\Gamma}=n^{(0)}_{2,\mathrm{IC}}=q,\ n^{(1)}_{3,1_\Gamma}=n^{(0)}_{3,q+1_\Gamma}=n^{(0)}_{3,\mathrm{IC}}=\frac{q^2-q}{6}.$$

Proof. By Theorem 2.2(Bii), for $q\equiv 1 \pmod 3$, 1_Γ-points are points on imaginary chords. We take an imaginary chord $\mathcal{IC}$. Clearly, $\#\mathcal{IC}=q+1$. By Lemma 4.11, all n_0^Σ $0_{\mathscr{C}}$-planes intersect $\mathcal{IC}$. By Theorem 2.2(Bii), for $q\not\equiv 0 \pmod 3$, all 1_Γ-points belong to the same orbit of the group G_q. Therefore, the number of $d_{\mathscr{C}}$-planes intersecting every 1_Γ-point is the same. Thus, see also Proposition 4.4, we have

$$n^{(1)}_{0,1_\Gamma}=\frac{n_0^\Sigma}{\#\mathcal{IC}}=\frac{q^2-q}{3}\,.$$

By Proposition 4.6,

$$\sum_{d=1}^{3} n^{(1)}_{d,1_\Gamma}=q^2+q+1-\frac{q^2-q}{3}\,.$$

This equation together with Corollaries 4.8 and 4.10 yields $n^{(1)}_{d,1_\Gamma}$, $d=1,2,3$.

A similar argument holds for $n^{(0)}_{d,\mathrm{IC}}$ and for $n^{(0)}_{d,q+1_\Gamma}$ (together with Remark 2.3). $\square$

Theorem 5.3. *Let $q\not\equiv 0 \pmod 3$. Then*

$$n^{(\neq 0)}_{0,\mathrm{T}}=\frac{q^2-1}{3},\ n^{(\neq 0)}_{1,\mathrm{T}}=\frac{q^2-q+4}{2},\ n^{(\neq 0)}_{2,\mathrm{T}}=2q-1,\ n^{(\neq 0)}_{3,\mathrm{T}}=\frac{q^2-3q+2}{6}.$$

Proof. We proceed as in Theorem 5.2.

We consider a tangent line $\mathcal{T}$ to $\mathscr{C}$ at a point $Q\in\mathscr{C}$. We denote $\widehat{\mathcal{T}}=\mathcal{T}\setminus\{Q\}$. Clearly, $\widehat{\mathcal{T}}$ consists of T-points and $\#\widehat{\mathcal{T}}=q$. All n_0^Σ $0_{\mathscr{C}}$-planes intersect $\widehat{\mathcal{T}}$. By Theorem 2.2(Bii), for $q \not\equiv 0 \pmod 3$, all T-points belong to the same orbit of the group G_q; the number of $d_{\mathscr{C}}$-planes intersecting every T-point is the same. Therefore,

$$n_{0,\mathrm{T}}^{(\neq 0)} = \frac{n_0^{\Sigma}}{\#\widehat{\mathcal{T}}} = \frac{q^2-1}{3}.$$

By Proposition 4.6 and Corollaries 4.8 and 4.10, the claim follows. $\square$

Theorem 5.4. *The following hold:*

$$n_{0,1_\Gamma}^{(-1)} = n_{0,\mathrm{RC}}^{(0)} = \frac{q^2+q}{3},\ n_{1,1_\Gamma}^{(-1)} = n_{1,\mathrm{RC}}^{(0)} = \frac{q^2-q+2}{2},$$
$$n_{2,1_\Gamma}^{(-1)} = n_{2,\mathrm{RC}}^{(0)} = q,\ n_{3,1_\Gamma}^{(-1)} = n_{3,\mathrm{RC}}^{(0)} = \frac{q^2+q}{6}.$$

Proof. We proceed as in Theorems 5.2 and 5.3.

By Theorem 2.2(Bii), for $q \equiv -1 \pmod 3$, 1_Γ-points are points on real chords. We take a real chord $\mathcal{RC}$ through points Q, K of $\mathscr{C}$. We denote $\widehat{\mathcal{RC}} = \mathcal{RC} \setminus \{Q, K\}$. Clearly, $\widehat{\mathcal{RC}}$ consists of 1_Γ-points and $\#\widehat{\mathcal{RC}} = q-1$. All n_0^{Σ} $0_{\mathscr{C}}$-planes intersect $\widehat{\mathcal{RC}}$. Also, by Theorem 2.2(Bii), for $q \equiv -1 \pmod 3$, all 1_Γ-points belong to the same orbit of the group G_q; the number of $d_{\mathscr{C}}$-planes intersecting every 1_Γ-point is the same. Therefore,

$$n_{0,1_\Gamma}^{(-1)} = \frac{n_0^{\Sigma}}{\#\widehat{\mathcal{RC}}} = \frac{q^2+q}{3}.$$

The claim follows using Proposition 4.6 and Corollaries 4.8 and 4.10. The argument for $n_{d,\mathrm{RC}}^{(0)}$ is the same. $\square$

Lemma 5.5. *Let $q \equiv 1 \pmod 3$. Let $\mathbb{T}$ be the $\binom{q-1}{3}$-multiset of all possible products of three distinct elements of $\mathbb{F}_q^*$. Then in $\mathbb{T}$, cubes (resp. non-cubes) of $\mathbb{F}_q^*$ appear m_c (resp. m_{nc}) times, where*

$$m_c = \frac{q-1}{3} \cdot \frac{q^2-5q+10}{6},\ m_{nc} = \frac{2(q-1)}{3} \cdot \frac{q^2-5q+4}{6}.$$

Proof. Let α be a primitive element of $\mathbb{F}_q$. We partition $\mathbb{F}_q^*$ in three $\frac{q-1}{3}$-subsets with elements of the form α^{3v}, α^{3v+1}, and α^{3v+2}, respectively. A product of three distinct elements of $\mathbb{F}_q^*$ is a cube if and only if all three elements belong to the same subset or to distinct subsets. So,

$$3\binom{(q-1)/3}{3} + \left(\frac{q-1}{3}\right)^3 = m_c.$$

Finally, $m_{nc} = \binom{q-1}{3} - m_c$. $\square$

Theorem 5.6. *Let $q \equiv 1 \pmod 3$. Then*

$$n^{(1)}_{3,0_\Gamma} = \frac{q^2+q-2}{6},\ n^{(1)}_{3,3_\Gamma} = \frac{q^2+q+4}{6}.$$

Proof. We consider the real chord $\mathcal{RC}_{0,\infty}$ through $P(0) = \mathbf{P}(0,0,0,1)$ and $P(\infty) = \mathbf{P}(1,0,0,0)$. We denote $\widehat{\mathcal{RC}}_{0,\infty} = \mathcal{RC}_{0,\infty} \setminus \{P(0), P(\infty)\}$. Points in $\widehat{\mathcal{RC}}_{0,\infty}$ have the form $(c,0,0,1)$, $c \in \mathbb{F}_q$. By (2.3), $\pi_\Gamma(t) = \boldsymbol{\pi}(1,-3t,3t^2,-t^3)$. Therefore, in $\widehat{\mathcal{RC}}_{0,\infty}$, we have 3_Γ-points of the form $\mathbf{P}(a^3,0,0,1)$, $a \in \mathbb{F}_q$, and 0_Γ-points of the form $\mathbf{P}(a^v,0,0,1)$, $a \in \mathbb{F}_q$, $v \not\equiv 0 \pmod 3$. In $\widehat{\mathcal{RC}}_{0,\infty}$, the number of 3_Γ-points and 0_Γ-points is $\frac{q-1}{3}$ and $\frac{2(q-1)}{3}$, respectively.

By (2.2), a 3_Γ-point $\mathbf{P}(a^3,0,0,1)$ and a 0_Γ-point $\mathbf{P}(a^v,0,0,1)$ lie on the plane through three points $P(t_1), P(t_2), P(t_3)$ of $\mathscr{C}$ if $a^3 = t_1t_2t_3$ and $a^v = t_1t_2t_3$, respectively. Now, by Lemma 5.5, one sees that through 3_Γ-points of $\widehat{\mathcal{RC}}_{0,\infty}$, in total, there are m_c $3_\mathscr{C}$-planes not containing the points $P(0), P(\infty)$. Also, by Lemma 4.3, through every 3_Γ-point of $\widehat{\mathcal{RC}}_{0,\infty}$ we have $q-1$ $3_\mathscr{C}$-planes containing $\mathcal{RC}_{0,\infty}$. Thus, through 3_Γ-points on $\mathcal{RC}_{0,\infty}$ we have, in total, $m_c + \frac{q-1}{3}(q-1)$ $3_\mathscr{C}$-planes. All 3_Γ-points belong to the same orbit $\mathscr{M}_3$ under G_q. Therefore, the number of $3_\mathscr{C}$-planes through a 3_Γ-point on $\mathcal{RC}_{0,\infty}$ is equal to

$$\left(m_c + \frac{q-1}{3}(q-1)\right)\left(\frac{q-1}{3}\right)^{-1} = \frac{q^2+q+4}{6}$$

Similarly, the number of $3_\mathscr{C}$-planes through a 0_Γ-point on $\mathcal{RC}_{0,\infty}$ is

$$\left(m_{nc} + \frac{2(q-1)}{3}(q-1)\right)\left(\frac{2(q-1)}{3}\right)^{-1} = \frac{q^2+q-2}{6}.$$

Finally, note that the number of intersecting $d_\mathscr{C}$-planes is the same for all points of an orbit under G_q. □

Theorem 5.7. *Let $q \equiv 1 \pmod 3$. Then*

$$n^{(1)}_{0,0_\Gamma} = \frac{q^2+q+1}{3},\ n^{(1)}_{1,0_\Gamma} = \frac{q^2-q}{2},\ n^{(1)}_{2,0_\Gamma} = q+1;$$
$$n^{(1)}_{0,3_\Gamma} = \frac{q^2+q-2}{3},\ n^{(1)}_{1,3_\Gamma} = \frac{q^2-q+6}{2},\ n^{(1)}_{2,3_\Gamma} = q-2.$$

Proof. By Corollary 4.8 and Theorem 5.6, we obtain $n^{(1)}_{2,0_\Gamma}$ and $n^{(1)}_{2,3_\Gamma}$. Then by Corollary 4.10 we get $n^{(1)}_{1,0_\Gamma}$ and $n^{(1)}_{1,3_\Gamma}$. Finally, we use Proposition 4.6 for $n^{(1)}_{0,0_\Gamma}$ and $n^{(1)}_{0,3_\Gamma}$. □

Theorem 5.8. *Let $q \equiv 0 \pmod 3$. Then*

$$n^{(0)}_{0,\mathrm{TO}} = \frac{q^2}{3},\ n^{(0)}_{1,\mathrm{TO}} = \frac{q^2-q+2}{2},\ n^{(0)}_{2,\mathrm{TO}} = 2q,\ n^{(0)}_{3,\mathrm{TO}} = \frac{q^2-3q}{6}.$$

Proof. We consider a tangent line $\mathcal{T}$ to $\mathscr{C}$ at a point $Q \in \mathscr{C}$. Let S be the $(q+1)_\Gamma$-point on $\mathcal{T}$. We denote $\widetilde{\mathcal{T}} = \mathcal{T} \setminus \{Q, S\}$. Clearly, $\widetilde{\mathcal{T}}$ consists of TO-points and $\#\widetilde{\mathcal{T}} = q-1$. All n_0^Σ $0_\mathscr{C}$-planes intersect $\mathcal{T} \setminus \{Q\}$. Therefore, the total number of $0_\mathscr{C}$-planes intersecting $\widetilde{\mathcal{T}}$ is $n_0^\Sigma - n_{0,q+1_\Gamma}^{(0)}$ where we subtract $0_\mathscr{C}$-planes through S. By Theorem 2.2(Bii), for $q \equiv 0 \pmod 3$, all TO-points belong to the same orbit of the group G_q; the number of $d_\mathscr{C}$-planes intersecting every TO-point is the same. Therefore, see also Theorem 5.2,

$$n_{0,\mathrm{TO}}^{(0)} = \frac{n_0^\Sigma - n_{0,q+1_\Gamma}^{(0)}}{\#\widetilde{\mathcal{T}}} = \frac{q^2}{3}.$$

The claim follows from Proposition 4.6 and Corollaries 4.8 and 4.10. □

Proposition 5.9. *Let $q \equiv -1 \pmod 3$. Then*

$$2n_{0,0_\Gamma}^{(-1)} + n_{0,3_\Gamma}^{(-1)} = q^2 - q,\ 2n_{1,0_\Gamma}^{(-1)} + n_{1,3_\Gamma}^{(-1)} = \frac{3(q^2+q+2)}{2},$$
$$2n_{2,0_\Gamma}^{(-1)} + n_{2,3_\Gamma}^{(-1)} = 3q,\ 2n_{3,0_\Gamma}^{(-1)} + n_{3,3_\Gamma}^{(-1)} = \frac{q^2-q}{2}.$$

Proof. By Theorem 2.2(Bii), for $\mu_\Gamma = 0, 3$, all μ_Γ-points belong to the same orbit under G_q. By Theorem 2.2(Bii), for $q \equiv -1 \pmod 3$, we have that 0_Γ-points and 3_Γ-points are points on imaginary chords. By Lemma 4.11, for $d = 0, 2, 3$, all n_d^Σ $d_\mathscr{C}$-planes intersect all $\binom{q}{2}$ imaginary chords. Thus, the total number of intersections of imaginary chords with $d_\mathscr{C}$-planes is $\binom{q}{2} n_d^\Sigma$. So,

$$\#\mathscr{M}_5 n_{d,0_\Gamma}^{(-1)} + \#\mathscr{M}_3 n_{d,3_\Gamma}^{(-1)} = \binom{q}{2} n_d^\Sigma,\ d = 0, 2, 3.$$

The assertions for $d = 0, 2, 3$ follow from (2.6), (4.1).

Finally, by Proposition 4.6, we obtain

$$2\sum_{d=0}^{3} n_{d,0_\Gamma}^{(-1)} + \sum_{d=0}^{3} n_{d,3_\Gamma}^{(-1)} = 3(q^2+q+1).$$

□

Lemma 5.10. *Let $q \equiv -1 \pmod 3$ be odd. Let $f(a) = a^2 + a + 1$. Let $V = \{a \in \mathbb{F}_q | f(a)$ is a square in $\mathbb{F}_q\}$. Then $\#V = \frac{q-1}{2}$.*

Proof. By [18, Theorem 5.18], $\sum_{a \in \mathbb{F}_q} \eta(f(a)) = -\eta(1) = -1$ where η is the quadratic character of $\mathbb{F}_q$. Also, $f(a) \neq 0, \forall a \in \mathbb{F}_q$. So, $\#V - (q - \#V) = -1$. □

Lemma 5.11. *Let $q \equiv -1 \pmod 3$. Then the point $W = \mathbf{P}(0, 1, -1, 0)$ off $\mathscr{C}$ lies on three osculating planes. Moreover, the number of $3_\mathscr{C}$-planes through W is equal to $(q^2 - q + 4)/6$.*

Proof. By (2.3), W belongs to $\pi_\Gamma(t)$ with $-3t-3t^2=0$ whence $t=0,1$. Also, by (2.4), W lies on $\pi_\Gamma(\infty)$.

(1) The $3_{\mathscr{C}}$-plane π' through points $P(t_1),P(t_2),P(\infty)$ of $\mathscr{C}$ has the form

$$\pi'=\boldsymbol{\pi}(0,-1,t_1+t_2,-t_1t_2)\supset\{P(t_1),P(t_2),P(\infty)\}.$$

This means that W belongs to π' if $-1-t_1-t_2=0$. So, under the condition $t_1\neq t_2$, there are n' distinct $3_{\mathscr{C}}$-planes π' through W where

$$n'=\begin{cases}\frac{q}{2} & \text{if } \ q \text{ even}\\ \frac{q-1}{2} & \text{if } \ q \text{ odd}\end{cases}.$$

(2) By (2.2), the $3_{\mathscr{C}}$-plane π'' through points $P(t_1),P(t_2),P(t_3)$ with $t_i\neq\infty$, $i=1,2,3$, contains W under the condition

$$(t_1+t_2+t_3)+(t_1t_2+t_1t_3+t_2t_3)=0,\ t_i\in\mathbb{F}_q,\ t_i\neq t_j,\ i,j\in\{1,2,3\}. \tag{5.1}$$

We now compute the number n'' of distinct triples t_1,t_2,t_3 satisfying (5.1).

(2.1) Let q be even, i.e. $q=2^{2v+1}\equiv-1 \pmod 3$.

In this case, by (5.1), we have

$$t_3=\frac{t_1+t_2+t_1t_2}{1+t_1+t_2}. \tag{5.2}$$

We fix $t_1\in\mathbb{F}_q$. By (5.1) and (5.2), there are the following restrictions on t_2:
(a) $t_2\neq t_1$;
(b) $t_2\neq t_1+1$ otherwise $1+t_1+t_2=0$;
(c) $t_2\neq t_3$ whence $t_2(1+t_1+t_2)\neq t_1+t_2+t_1t_2$ and $t_2\neq\sqrt{t_1}$;
(d) $t_1\neq t_3$ whence $t_1(1+t_1+t_2)\neq t_1+t_2+t_1t_2$ and $t_2\neq t_1^2$.

Suppose (a) and (c) or (a) and (d) coincide, i.e. $t_1=t_1^2$ or $t_1=\sqrt{t_1}$. This implies $t_1=0,1$.

Suppose (b) and (c) or (b) and (d) coincide, i.e. $t_1+1=t_1^2$ or $t_1+1=\sqrt{t_1}$. This yields $t_1^2+t_1+1=0$. As $q=2^{2v+1}$, the trace $\mathrm{Tr}_{\mathbb{F}_q}(1)\neq0$ [18, Cor. 3.79], a contradiction.

Finally, if (c) and (d) coincide then $\sqrt{t_1}=t_1^2$, $t_1=t_1^4$ and therefore $t_1=0,1$.

Thus, for $t_1\in\mathbb{F}_q$, $t_1\neq0,1$, (a)–(d) are distinct. Here we have $q-2$ possibilities for t_1 and $q-4$ possibilities for t_2 for every t_1. Also, there are $q-2$ possibilities of t_2 if $t_1=0,1$.

The number of distinct triples t_1,t_2,t_3 satisfying (5.1) is therefore $(q-2)(q-4)+2(q-2)=q^2-4q+4$. Because of symmetry, each plane is generated by 6 triples, so $n''=(q^2-4q+4)/6$.

Now $n'+n''$ gives the needed result for even q.

(2.2) Let q be odd, i.e. $q=p^{2v+1}$, $p>3$ prime, $p\equiv-1 \pmod 3$.

First we count the number of triples satisfying

$$(t_1 + t_2 + t_3) + (t_1t_2 + t_1t_3 + t_2t_3) = 0, \ t_i \in \mathbb{F}_q, \tag{5.3}$$

without the condition $t_i \neq t_j, \ i, j \in \{1, 2, 3\}$.

Relation (5.3) can be rewritten as the set of q conditions

$$\begin{cases} t_1 + t_2 + t_3 = k \\ t_1t_2 + t_1t_3 + t_2t_3 = -k \end{cases} \tag{5.4}$$

where $k \in \mathbb{F}_q$.
The triples satisfying (5.4) can be seen as the affine coordinates of the points of the 3-dimensional affine space $\mathrm{AG}(3, q)$ belonging to a plane conic defined by

$$\begin{cases} t_1 + t_2 + t_3 = k \\ t_2^2 + t_3^2 + t_2t_3 - kt_2 - kt_3 - k = 0 \end{cases} . \tag{5.5}$$

For $k = 0$ and $k = -3$, the conic is degenerate and, as $\sqrt{-3}$ is not a square in $\mathbb{F}_q$, the unique triples satisfying (5.5) are $(0, 0, 0)$ and $(-1, -1, -1)$.

For each $k \in \mathbb{F}_q \setminus \{0, -3\}$, there are exactly $q + 1$ triples (t_1, t_2, t_3) satisfying (5.5).

Therefore $2 + (q - 2)(q + 1) = q(q - 1)$ triples satisfy (5.3).
To count the triples satisfying (5.1), we exclude the triples satisfying (5.3) having at least two equal elements.

(2.2.1) $t_1 = t_2 = t_3$.
Equation (5.3) reads $3t_1 + 3t_1^2 = 0$, so $t_1 = 0, -1$.

(2.2.2) $t_i = t_j \neq t_k \ i, j, k \in \{1, 2, 3\}$.
Equation (5.3) reads

$$t_i^2 + 2(t_k + 1)t_i + t_k = 0. \tag{5.6}$$

Discriminant of (5.6) is $4(t_k^2 + t_k + 1)$. Let $V = \{t_k \in \mathbb{F}_q | t_k^2 + t_k + 1 \text{ is a square in } \mathbb{F}_q\}$. By Lemma 5.10, $\#V = \frac{q-1}{2}$. As $q \equiv -1 \pmod 3$, q odd, by [13, Ch. 1] $t_k^2 + t_k + 1 \neq 0, \ \forall t_k \in \mathbb{F}_q$. Then $\forall t_k \in V$ we obtain two distinct values of t_i. On the other hand, when $t_k = 0, -1$, one of the values of t_i we obtain is equal to t_k. Therefore the number of triples satisfying (5.3) such that $t_i = t_j \neq t_k, \ i, j \in \{1, 2, 3\}$ is $3(2(\frac{q-1}{2} - 2) + 2) = 3(q - 3)$.

So, the number of distinct triples t_1, t_2, t_3 satisfying (5.1) is $q(q-1)-2-3(q-3) = q^2 - 4q + 7$. Because of symmetry, each plane is generated by 6 triples, so $n'' = (q^2 - 4q + 7)/6$. Now $n' + n''$ gives the needed result for odd q. $\square$

Theorem 5.12. *Let* $q \equiv -1 \pmod 3$. *Then*

$$n_{0,0_\Gamma}^{(-1)} = \frac{q^2 - q + 1}{3}, \ n_{1,0_\Gamma}^{(-1)} = \frac{q^2 + q}{2}, \ n_{2,0_\Gamma}^{(-1)} = q + 1, \ n_{3,0_\Gamma}^{(-1)} = \frac{q^2 - q - 2}{6};$$

$$n^{(-1)}_{0,3_\Gamma} = \frac{q^2-q-2}{3},\ n^{(-1)}_{1,3_\Gamma} = \frac{q^2+q+6}{2},\ n^{(-1)}_{2,3_\Gamma} = q-2,\ n^{(-1)}_{3,3_\Gamma} = \frac{q^2-q+4}{6}.$$

Proof. As all points of the orbit $\mathscr{M}_3$ have the same number of intersecting $d_{\mathscr{C}}$-planes, we have by Lemma 5.11 that $n^{(-1)}_{3,3_\Gamma} = \frac{q^2-q+4}{6}$. Then we obtain the value $n^{(-1)}_{3,0_\Gamma}$ by Proposition 5.9. By Lemma 4.7 and Corollary 4.8, see (4.2), we obtain $n^{(-1)}_{2,0_\Gamma}$ and $n^{(-1)}_{2,3_\Gamma}$. Then by Lemma 4.9 and Corollary 4.10, we get $n^{(-1)}_{1,0_\Gamma}$ and $n^{(-1)}_{1,3_\Gamma}$. Finally, we use Proposition 4.6 for $n^{(-1)}_{0,0_\Gamma}$ and $n^{(-1)}_{0,3_\Gamma}$. □

Theorem 5.13. *For $q \equiv \xi \pmod 3$, the following hold:*

(i) $\xi = -1, 1.$

$$r_{11} = r_{14} = 1,\ r_{12} = 2,\ r_{13} = 3,\ r_{15} = 0,$$
$$r_{41} = r_{42} = \frac{1}{2}(q^2-q),\ r_{43} = r_{45} = \frac{1}{2}(q^2-\xi q),\ r_{44} = \frac{1}{2}(q^2+\xi q).$$

(ii) $\xi = 0.$

$$r_{11} = r_{13} = r_{14} = r_{15} = 1,\ r_{12} = q+1,$$
$$r_{41} = r_{42} = r_{43} = r_{44} = \frac{1}{2}(q^2-q),\ r_{45} = \frac{1}{2}(q^2+q).$$

Proof. **(i)** By definition, $r_{11} = r_{14} = 1,\ r_{13} = 3,\ r_{15} = 0$.

We consider a tangent $\mathcal{T}$ to $\mathscr{C}$ at a point Q of $\mathscr{C}$. We denote $\widehat{\mathcal{T}} = \mathcal{T} \setminus \{Q\}$. Clearly, $\widehat{\mathcal{T}}$ consists of T-points and lies in a Γ-plane. The rest q osculating planes intersect $\widehat{\mathcal{T}}$. As all q points of $\widehat{\mathcal{T}}$ belong to the same orbit under G_q, every point corresponds to $\frac{q}{\#\widehat{\mathcal{T}}} = \frac{q}{q} = 1$ intersection. Thus, $r_{12} = 2$.

We note, see Table 1 and Notation 4.1, that $r_{41} = n_{1,\mathscr{C}} - r_{11}$, $r_{42} = n^{(\neq 0)}_{1,\mathrm{T}} - r_{12}$, $r_{43} = n^{(\xi)}_{1,3_\Gamma} - r_{13}$, $r_{44} = n^{(\xi)}_{1,1_\Gamma} - r_{14}$, $r_{45} = n^{(\xi)}_{1,0_\Gamma} - r_{15}$. Finally, we take the values $n_{1,\mathscr{C}}, n^{(\neq 0)}_{1,\mathrm{T}}, n^{(\xi)}_{1,\mu_\Gamma}$ from Theorems 5.1–5.4, 5.7, and 5.12.

(ii) By definition, $r_{11} = 1$, $r_{12} = q+1$.

We consider a tangent line $\mathcal{T}$ to $\mathscr{C}$ at a point $Q \in \mathscr{C}$. Let K be the $(q+1)_\Gamma$-point in $\mathcal{T}$. We denote $\widehat{\mathcal{T}} = \mathcal{T} \setminus \{Q, K\}$. Clearly, $\widehat{\mathcal{T}}$ consists of OT-points. All Γ-planes form a pencil of planes; their common line passes through K. Therefore, no Γ-plane intersects $\widehat{\mathcal{T}}$. On the other hand, $\widehat{\mathcal{T}}$ lies in the Γ-plane through Q. So, $r_{13} = 1$.

We consider a real chord $\mathcal{RC}$ through points Q, K of $\mathscr{C}$. We denote $\widehat{\mathcal{RC}} = \mathcal{RC} \setminus \{Q, K\}$. Apart from the osculating planes in Q and K, all the other $q-1$ such planes intersect $\widehat{\mathcal{RC}}$. All $q-1$ points of $\widehat{\mathcal{RC}}$ belong to the same orbit under G_q. Therefore, the number of the osculating planes through every point of $\widehat{\mathcal{RC}}$ is the same and $r_{14} = \frac{q-1}{q-1} = 1$.

We take an imaginary chord $\mathcal{IC}$. By Lemma 4.11, all $q+1$ osculating planes intersect $\mathcal{IC}$. As all $q+1$ points of $\mathcal{IC}$ belong to the same orbit under G_q, the number of the osculating planes through every point of $\mathcal{IC}$ is the same and $r_{15} = \frac{q+1}{q+1} = 1$.

We note, see Table 2 and Notation 4.1, that $r_{41} = n_{1,\mathscr{C}} - r_{11}$, $r_{42} = n^{(0)}_{1,q+1_\Gamma} - r_{12}$, $r_{43} = n^{(0)}_{1,\mathrm{TO}} - r_{13}$, $r_{44} = n^{(0)}_{1,\mathrm{RC}} - r_{14}$, $r_{45} = n^{(0)}_{1,\mathrm{IC}} - r_{15}$. Finally, Theorems 5.1, 5.2, 5.4, and 5.8 provide $n_{1,\mathscr{C}}, n^{(0)}_{1,q+1_\Gamma}, n^{(0)}_{1,\mathrm{TO}}, n^{(0)}_{1,\mathrm{RC}}, n^{(0)}_{1,\mathrm{IC}}$.

□

6 The number k_{ij} of distinct points in distinct planes of $\mathrm{PG}(3,q)$. Structure of the point-plane incidence matrix

Recall that, by Lemma 4.12, we have the same number r_{ij} of planes from an orbit $\mathscr{N}_i$ through every point of an orbit $\mathscr{M}_j$, and vice versa, the number k_{ij} of points from $\mathscr{M}_j$ in a plane of $\mathscr{N}_i$ is the same for all planes of $\mathscr{N}_i$.

Theorem 6.1. *For $i,j = 1,\dots,5$, the following hold:*

$$k_{ij} \cdot \#\mathscr{N}_i = r_{ij} \cdot \#\mathscr{M}_j\,; \tag{6.1}$$

$$\sum_{j=1}^{5} r_{ij} = \sum_{i=1}^{5} k_{ij} = q^2+q+1. \tag{6.2}$$

Proof. The cardinality of the multiset consisting of points of $\mathscr{M}_j$ in all planes of $\mathscr{N}_i$ is equal to $r_{ij} \cdot \#\mathscr{M}_j$. By Lemma 4.12, every plane of $\mathscr{N}_i$ contains the same number of points of $\mathscr{M}_j$. Thus, $k_{ij} = \frac{r_{ij}\cdot\#\mathscr{M}_j}{\#\mathscr{N}_i}$.

Relation (6.2) holds as $\mathrm{PG}(3,q)$ is partitioned under G_q in 5 orbits $\mathscr{M}_j$ and $\mathscr{N}_i$. □

The values r_{ij} and k_{ij} are collected in Tables 1 and 2.

Recall that the point-plane incidence matrix of the $\mathrm{PG}(3,q)$ consists of 25 submatrices $\mathcal{I}_{ij}$. The submatrix $\mathcal{I}_{ij}$ has size $\#\mathscr{N}_i \times \#\mathscr{M}_j$; it contains k_{ij} ones in every row and r_{ij} ones in every column, see (6.1).

Proposition 6.2. *For $q \not\equiv 0 \pmod 3$, $\mathcal{I}_{ij}^{tr} = \mathcal{I}_{ji}$ up to rearrangement of rows and columns. Also,*

$$\#\mathscr{N}_i = \#\mathscr{M}_i,\ \#\mathscr{M}_j = \#\mathscr{N}_j,\ k_{ij} = r_{ji},\ r_{ij} = k_{ji},\ i,j \in \{1,\dots,5\}.$$

Proof. The assertion follows from Theorem 2.2(D), see (2.10). □

Proposition 6.3. *Let $q \not\equiv 0 \pmod 3$. Then the submatrix $\mathcal{I}_{21}$ gives a 2-$(q+1,2,2)$ design and the submatrix $\mathcal{I}_{31}$ defines 3-$(q+1,3,1)$ and 2-$(q+1,3,q-1)$ designs.*

Proof. For 2-designs we use Lemma 4.3. For the 3-design note that there is one and only one $3_{\mathscr{C}}$-plane through any three points of $\mathscr{C}$. □

Corollary 6.4. *From Tables 1 and 2 the following hold:*

(i) *For $q \equiv 0 \pmod 3$, up to rearrangement of rows and columns, we have*

$$\mathcal{I}_{41}^{tr} = \mathcal{I}_{14},\ \mathcal{I}_{41}^{tr} = \mathcal{I}_{15},\ \mathcal{I}_{42}^{tr} = \mathcal{I}_{14},\ \mathcal{I}_{42}^{tr} = \mathcal{I}_{15}.$$

(ii) *If $\#\mathscr{N}_i = \#\mathscr{M}_j$, then the submatrix $\mathcal{I}_{ij}$ gives rise to a symmetric tactical configuration with $k_{ij} = r_{ij}$. This holds for $\mathcal{I}_{ii}$, $i = 1,\ldots,5$, when $q \not\equiv 0 \pmod 3$ and for $\mathcal{I}_{44},\mathcal{I}_{45}$ when $q \equiv 0 \pmod 3$.*

Proposition 6.5. *Let $q \equiv \xi \pmod 3$. Let $i = 1,\ldots,5$. Up to rearrangement of rows and columns, the following hold:*

(i) *The submatrix $\mathcal{I}_{i1}$ for $\xi = -1, 1$ and for $\xi = 0$ is the same;*

(ii) *The submatrix $\mathcal{I}_{i4}$ for $\xi = 1$ is the same as the submatrix $\mathcal{I}_{i5}$ for $\xi = 0$;*

(iii) *The submatrix $\mathcal{I}_{i4}$ for $\xi = -1$ and for $\xi = 0$ is the same.*

Proof. The assertion (i) is clear. Regarding (ii) and (iii), by Theorem 2.2(B), we have $\mathscr{M}_4 = \{\text{IC-points}\}$ for $\xi = 1$ and $\mathscr{M}_5 = \{\text{IC-points}\}$ for $\xi = 0$. Also, $\mathscr{M}_4 = \{\text{RC-points}\}$ for $\xi = -1$ as well as for $\xi = 0$. Finally, see Theorems 5.2 and 5.4. □

Theorem 6.6. *Let the orbits $\mathscr{N}_i$ and $\mathscr{M}_j$ be as in Theorem 2.2(B), see (2.5)–(2.9). For the twisted cubic $\mathscr{C}$ of (2.1) the following hold:*

(i) *Let $q = 2$. Under the action of the group $G_2 \cong \mathbf{S}_3\mathbf{Z}_2^3$ fixing $\mathscr{C}$, there are four orbits $\widehat{\mathscr{N}_i}$ of planes and four orbits $\widehat{\mathscr{M}_j}$ of points where*

$$\widehat{\mathscr{N}_1} = \mathscr{N}_1 \cup \mathscr{N}_4,\ \widehat{\mathscr{N}_2} = \mathscr{N}_2,\ \widehat{\mathscr{N}_3} = \mathscr{N}_3,\ \widehat{\mathscr{N}_4} = \mathscr{N}_5; \tag{6.3}$$

$$\widehat{\mathscr{M}_1} = \mathscr{M}_1,\ \widehat{\mathscr{M}_2} = \mathscr{M}_2 \cup \mathscr{M}_5,\ \widehat{\mathscr{M}_3} = \mathscr{M}_3, \widehat{\mathscr{M}_4} = \mathscr{M}_4.$$

The subgroup $\mathbf{S_3} \cong PGL(2,2)$ *of* G_2 *partitions* PG(3, 2) *to the orbits* $\mathscr{N}_i$ *and* $\mathscr{M}_j$ *as in Theorem* 2.2(B) *for* $q \not\equiv 0 \pmod 3$. *In this case, the point-plane incidence matrix has the form of Table* 1.

(ii) *Let* $q = 3$. *Under the action of the group* $G_3 \cong \mathbf{S}_4\mathbf{Z}_2^3$ *fixing* $\mathscr{C}$, *there are orbits* $\widehat{\mathscr{N}_i}$ *and* $\widehat{\mathscr{M}_j}$ *as in* (6.3). *The subgroup* $\mathbf{S_4} \cong PGL(2,3)$ *of* G_3 *partitions* PG(3, 3) *to the orbits* $\mathscr{N}_i$ *and* $\mathscr{M}_j$ *as in Theorem* 2.2(B) *for* $q \equiv 0 \pmod 3$; *the point-plane incidence matrix has the form of Table* 2.

(iii) *Let* $q = 4$. *Under the action of the group* $G_4 \cong \mathbf{S}_5 \cong P\Gamma L(2,4)$ *fixing* $\mathscr{C}$, *there are orbits* $\mathscr{N}_i$ *and* $\mathscr{M}_j$ *as in Theorem* 2.2(B) *for* $q \not\equiv 0 \pmod 3$. *In this case, the point-plane incidence matrix has the form of Table* 1.

Proof. The groups G_i are given in Theorem 2.2(A). The rest of the assertions are obtained by computer search using the MAGMA computational algebra system [5]. □

7 The twisted cubic as a multiple covering code and a multiple 2-saturating set

For $\rho = 2$ and $N = 3$, Definition 2.5 can be viewed as follows.

Definition 7.1. Let S be a subset of points of $PG(3, q)$. Then S is said to be $(2, \mu)$-saturating if:

(M1) S generates $\mathrm{PG}(3, q)$;

(M2) there exists a point Q in $\mathrm{PG}(3, q)$ which does not belong to any bisecant line of S;

(M3) every point Q in $\mathrm{PG}(3, q)$ not belonging to any bisecant line of S is such that the number of planes through three points of S containing Q is at least μ.

Theorem 7.2. *The twisted cubic* $\mathscr{C}$ *of* (2.1) *is a minimal* $(2, \mu)$*-saturating* $(q + 1)$*-set with* μ *as in* (3.1).

Proof. **(M1)** Any 4 points of $\mathscr{C}$ generate $\mathrm{PG}(3, q)$.

(M2) Apart from RC-points, all points off $\mathscr{C}$ do not belong to any bisecant line of $\mathscr{C}$.

(M3) Recall that $n^{(\xi)}_{3,\bullet}$ is the number of $3_{\mathscr{C}}$-planes through a point of the type $\bullet$. By Theorem 3.1 and Tables 1 and 2, among points not lying on real chords the smallest value of $n^{(\xi)}_{3,\bullet}$ is $n^{(\neq 0)}_{3,\mathrm{T}} = (q^2 - 3q + 2)/6$ if $q \not\equiv 0 \pmod 3$ or $n^{(0)}_{3,\mathrm{TO}} = (q^2 - 3q)/6$ if $q \equiv 0 \pmod 3$.

It can be easily seen that $\mathscr{C}$ is a *minimal* $(2, \mu)$-saturating set. □

Theorem 7.3. *Let μ be as in* (3.1). *Let C be the code associated with the twisted cubic $\mathscr{C}$ of* (2.1). *Then*

(i) *The code C is a $[q+1, q-3, 5]_q3$ quasi-perfect GDRS code of covering radius $R = 3$ and, moreover, C is a $(3, \mu)$-MCF code.*

(ii) *The μ-density $\gamma_\mu(C, 3)$ of the code C tends to 1 from above when q tends to infinity, i.e.*

$$\lim_{q\to\infty} \gamma_\mu(C, 3) = 1, \ \ \gamma_\mu(C, 3) > 1, \tag{7.1}$$

and the code is asymptotical optimal.

Proof. **(i)** The twisted cubic is a normal rational curve. It is well known that a normal rational curve in $\mathrm{PG}(N, q)$ gives rise to a $[q+1, q-N, N+2]_q$ GDRS code. Also, by Proposition 2.7 and Theorem 7.2, C is a $(3, \mu)$-MCF code.

(ii) Since $d(C) = 2R - 1$, we have, by (2.11),

$$\gamma_\mu(C, 3) = \frac{\binom{q+1}{3}(q-1)^R - \binom{5}{2}(q-1)\binom{q+1}{5}}{\mu\left(q^4 - 1 - (q^2-1) - \binom{q+1}{2}(q-1)^2\right)},$$

where $A_{2R-1}(C) = A_d(C) = (q-1)\binom{n}{d}$ as C is an MDS code [20, 22]. After simple transformations, for $\mu = \frac{q^2-3q+2}{6}$, we obtain

$$\gamma_\mu(C, 3) = \frac{\frac{1}{12}q^6 - \frac{1}{2}q^4 + \frac{1}{3}q^3 + \frac{5}{12}q^2 - \frac{1}{3}q}{\frac{1}{12}q^6 - \frac{3}{4}q^4 + \frac{2}{3}q^3 + \frac{2}{3}q^2 - \frac{2}{3}q}$$

whence (7.1) immediately follows. For $\mu = \frac{q^2-3q}{6}$ the proof is the same.

□

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