On neighbourhood product of some Horn axiomatizable logics

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Abstract

The paper considers modal logics of products of neighbourhood frames. The n-product of modal logics is the logic of all products of neighbourhood frames of the corresponding logics. We find the n-product of any two pretransitive Horn axiomatizable logics. As a corollary, we find the d-logic of products of topological spaces from some classes of topological spaces.

Keywords: Neighbourhood semantics, product of modal logics, Horn sentences, topological semantics.

1 Introduction

The neighbourhood semantics is a generalization of Kripke semantics and topological semantics. It was introduced by Dana Scott in [12] and Richard Montague in [9] independently. In this paper, we will consider the product of neighbourhood frames introduced by Sano in [11]. It is a generalization of the product of topological spaces[1] presented in [17].

The product of neighbourhood frames is defined in the same manner as the product of Kripke frames (see [14] and [16]). But there are some differences. The axioms of commutativity and Church–Rosser property are valid in any product of Kripke frames. Whereas in [17] it was shown that the logic of the products of all topological spaces is the fusion[2] of logics $\mathsf{S4} * \mathsf{S4}$. Moreover, $\mathsf{S4} * \mathsf{S4}$ is complete w.r.t. the product $\mathbb{Q} \times_t \mathbb{Q}$ ($\times_t$ stands for the product of topological spaces defined in [17]).

In [7] it was proved that, for any pair L and L' of logics from $\{\mathsf{S4}, \mathsf{D4}, \mathsf{D}, \mathsf{T}\}$, the modal logic of the family of products of L-neighbourhood frames and L'-neighbourhood frames is the fusion of L and L'. But at that point it was unclear how to proceed in the case of logics that does not contain the seriality axiom $\Diamond\top$. In [8] it was shown that, for any variable-free and $\Box_2$-free formula B, formula $B \to \Box_2 B$ is valid in any product of neighbourhood frames (and the same holds for $B' \to \Box_1 B'$, where B' is variable-free and $\Box_1$-free). It was also proved that $\mathsf{K} * \mathsf{K}$ plus all these formulas is the logic of all products of neighbourhood frames. For any two modal logics L_1 and L_2 we can define $\langle \mathsf{L}_1, \mathsf{L}_2 \rangle$ as $\mathsf{L}_1 * \mathsf{L}_2$ plus all the formulas from the above. For details, see Definition 3.8.

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[1]'Product of topological spaces' is a well-known notion in Topology, but here we use a different definition (for details see [17]).

[2]Some authors (see [5, 17]) use $\oplus$ for the fusion.

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In this paper, we will find a sufficient conditions for two logics L_1 and L_2 to be n-product matching. Two logics are called n-product matching if $\mathsf{L}_1 \times_n \mathsf{L}_2 = \langle \mathsf{L}_1, \mathsf{L}_2 \rangle$, where $\mathsf{L}_1 \times_n \mathsf{L}_2$ is the logic of all products of neighbourhood frames $\mathfrak{X}_1 \times \mathfrak{X}_2$ such that $\mathfrak{X}_1 \models \mathsf{L}_1$ and $\mathfrak{X}_2 \models \mathsf{L}_2$.

Neighbourhood frames are often considered in the context of non-normal modal logics. Since many non-normal logics are complete w.r.t. neighbourhood semantics. Examples of Kripke incomplete normal modal logics that are complete w.r.t. neighbourhood semantics are rare and usually artificial. This paper, however, shows that in the case of the products neighbourhood frames give different results from Kripke frames even in case of normal modal logics. To be precise this paper (and others: [7, 11, 17]) shows that 'neighbourhood' product, in general, generates a weaker logic in comparison to 'Kripke' product. It also shows how the notion of the product of modal logics depends on the underlining semantics.

We also prove some corollaries for the derivational semantics of topological spaces. In particular the logic of all products of all T_1 spaces is $\langle \mathsf{K4}, \mathsf{K4} \rangle$. What is the logic of all products of all topological products is still unknown.

2 Language, logics and semantics

In this paper, we study propositional modal logics. A formula is defined recursively by using the Backus–Naur form as follows:

$$\phi ::= p \mid \bot \mid (\phi \to \phi) \mid \Box_i \phi,$$

where $p \in \mathrm{PROP}$ is a propositional letter and $\Box_i$ is a modal operator ($i = 1, \ldots, n$). Other connectives are introduced as abbreviations: classical connectives are expressed through $\bot$ and $\to$, and dual modal operators $\Diamond_i$ are expressed as $\neg\Box_i\neg$. The set of all modal formulas is denoted by $\mathcal{ML}_n$, and in order to specify the modalities used in the language we write them in subindex, e.g. $\mathcal{ML}_{\Box_1}$ or $\mathcal{ML}_{\Box_2}$.

DEFINITION 2.1
A *normal modal logic* (or *a logic,* for short) is a set of modal formulas closed under Substitution $\left(\frac{A(p)}{A(B)}\right)$, Modus Ponens $\left(\frac{A, A \to B}{B}\right)$ and Generalization rules $\left(\frac{A}{\Box_i A}\right)$, containing all the classical tautologies and the normality axioms:

$$\Box_i (p \to q) \to (\Box_i p \to \Box_i q).$$

$\mathsf{K_n}$ denotes *the minimal normal modal logic with n modalities* and $\mathsf{K} = \mathsf{K_1}$.

Let L be a logic and Γ a set of formulas, then $\mathsf{L} + \Gamma$ denotes the minimal logic containing L and Γ. If $\Gamma = \{A\}$, then we write $\mathsf{L} + A$ rather than $\mathsf{L} + \{A\}$.

DEFINITION 2.2
A formula B is called *closed* if it has no variables.

DEFINITION 2.3
Let $\mathsf{L_1}$ and $\mathsf{L_2}$ be two modal logics with one modality $\Box$ (unimodal logics), then the *fusion* of these logics is the following modal logic with 2 modalities:

$$\mathsf{L_1} * \mathsf{L_2} = \mathsf{K_2} + \mathsf{L}'_1 + \mathsf{L}'_2,$$

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where L_i' is the set of all formulas from L_i in which all instances of $\Box$ are replaced by $\Box_i$.

DEFINITION 2.4
Let $R \subseteq W \times W$ be a relation on $W \neq \varnothing$, then for $n \geq 1$ and $w \in W$ we define

$$R^0 = Id_W = \{(w, w) \mid w \in W\},$$
$$R^{n+1} = R^n \circ R,$$
$$R^* = \bigcup_{k=0}^{\infty} R^k,$$
$$R(w) = \{u \mid wRu\}.$$

Notice that R^* is the reflexive transitive closure of R.

A *Kripke frame* with n relations is a tuple $F = \big(W, R_1, \ldots, R_n\big)$, where W is a non-empty set and $R_i \subseteq W \times W$ is a relation on W for each $i \in \{1, \ldots, n\}$.

REMARK 2.5
We will sometimes write $w \in F$ as a shortcut for $w \in W$ and $F = \big(W, R_1, \ldots, R_n\big)$.

A frame F with a *valuation* $V : PROP \to 2^W$ is called *a model* $M = (F, V)$.

For a Kripke frame $F = (W, R_1, \ldots, R_n)$ we define the *subframe* generated by $w \in W$ as the frame $F^w = \big(W', R_1|_{W'}, \ldots, R_n|_{W'}\big)$, where $W' = \big(R_1 \cup \ldots, \cup R_n\big)^*(w)$ and $R_i|_{W'} = R_i \cap W' \times W'$. A frame F is called *rooted* if $F = F^w$ for some w.

The truth of a formula in a model M at a point $x \in W$ is defined, as usual, by induction on the length of the formula:

$$M, x \not\models \bot,$$
$$M, x \models p \iff x \in V(p);$$
$$M, x \models A \to B \iff M, x \not\models A \text{ or } M, x \models B;$$
$$M, x \models \Box_i A \iff \forall y \ \big(xR_i y \Rightarrow M, y \models A\big).$$

A formula is *true in a (Kripke) model* M if it is true at all points of M (notation $M \models A$). A formula is *valid on a (Kripke) frame* F if it is true in all models based on F (notation $F \models A$). We write $F \models \mathsf{L}$ if, for any $A \in \mathsf{L}$, $F \models A$. The *logic* of a class of Kripke frames $\mathcal{C}$ is $Log(\mathcal{C}) = \{A \mid F \models A \text{ for all } F \in \mathcal{C}\}$. For a logic L we also define $\mathcal{V}(\mathsf{L}) = \big\{F \,\big|\, F \text{ is a Kripke frame and } F \models \mathsf{L}\big\}$. Note that if there is no F such that $F \models \mathsf{L}$, then $\mathcal{V}(\mathsf{L}) = \varnothing$.

DEFINITION 2.6
Let $F = \big(W, R_1, \ldots, R_n\big)$ and $G = \big(U, S_1, \ldots, S_n\big)$ be Kripke frames. A function $f : W \to U$ is called a *p-morphism* (Notation: $f : F \twoheadrightarrow G$) if

1. f is surjective;
2. **[monotonisity]** for any $w, v \in W$ $wR_i v$ implies $f(w) S_i f(v)$;
3. **[lifting]** for any $w \in W$ and $v' \in U$ such that $f(w) S_i v'$ there exists $v \in W$ such that $wR_i v$ and $f(v) = v'$.

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The following p-morphism lemma is well known (see [2, Proposition 2.14]):

LEMMA 2.7
Let $f : F \twoheadrightarrow G$ and V be a valuation on G. We define a valuation on F by $\left[f^{-1}(V)\right](p) = f^{-1}\left(V(p)\right)$. Then for any $w \in F$ and any formula A

$$F, f^{-1}(V), w \models A \iff G, V, f(w) \models A.$$

The following is a straightforward corollary.

COROLLARY 2.8
If $f : F \twoheadrightarrow G$, then $Log(F) \subseteq Log(G)$.

For a consistent modal logic L with n modalities we define the canonical model (c.f. [2]) $\mathcal{M}_{\mathsf{L}} = \left(\mathcal{F}_{\mathsf{L}}, V_{\mathsf{L}}\right)$, where $\mathcal{F}_{\mathsf{L}} = \left(W, R_1, \ldots, R_n\right)$ such that

$$W = \{x \mid x \text{ is an } \mathsf{L}\text{-complete set of formulas}\},$$

$$xR_i y \iff \forall A\left(\Box_i A \in x \Rightarrow A \in y\right),$$

$$x \in V\left(p\right) \iff p \in x.$$

The classical result on canonical models is

LEMMA 2.9
For any formula A and any consistent logic L

$$\mathcal{M}_{\mathsf{L}}, x \models A \iff A \in x.$$

We also define 0-canonical frame $\mathcal{F}^0_{\mathsf{L}}$ being the counterparts of canonical frame in the modal language without variables. More precisely,

$$\mathcal{F}^0 = \left(W^0, R'_1, \ldots, R'_n\right),$$

$$W^0 = \{\bar{x} \mid \bar{x} \text{ is an } \mathsf{L}\text{-complete set of } \textit{closed} \text{ formulas}\},$$

$$\bar{x}R'_i\bar{y} \iff \forall A\left(\Box_i A \in \bar{x} \Rightarrow A \in \bar{y}\right).$$

Note that there are no 0-canonical models since there are no variables in closed formulas. So the lemma for canonical model transforms into

LEMMA 2.10
For any closed formula A and any logic L

$$\mathcal{F}^0_{\mathsf{L}}, \bar{x} \models A \iff A \in \bar{x}.$$

Here we describe a construction of continuum unravelling. It is similar to the construction in [4, Lemma 4.9].

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DEFINITION 2.11
Let $F = \big(W, R\big) = F^{w_0}$ be a rooted Kripke frame, S be a non-empty set and $x_0 \in S$ be a point in it (the starting point). Then

$$F \cdot S = \big(W \times S, R \cdot S\big);$$
$$(w, x) R \cdot S\big(v, y\big) \iff wRv;$$
$$F_S = \big(F \cdot S\big)^{\left(w_0, x_0\right)} = \big(W_S, R_S\big) \text{— a rooted subframe.}$$

F_S is called the *thickening* of F by S.

The proof of the following lemma is straightforward.

LEMMA 2.12
The first projection $p_1(w, x) = w$ is a p-morphism $p_1 : F_S \twoheadrightarrow F$.

The following construction is well known (c.f. [2]).

DEFINITION 2.13
Let $F = \big(W, R_1, \ldots, R_n\big) = F^{w_0}$ be a rooted Kripke frame.
We define the unravelling of it and a map π as follows

$$F^\sharp = \left(W^\sharp, R_1^\sharp, \ldots, R_n^\sharp\right)';$$
$$W^\sharp = \left\{w_0 R_{j_1} w_1 \ldots R_{j_m} w_m \,\middle|\, \forall i \in \{1, \ldots, m\} \left(w_{i-1} R_{j_i} w_i\right)\right\};$$
$$\pi\left(w_0 w_1 \ldots w_m\right) = w_m, \qquad \left(\pi : W^\sharp \to W\right);$$
$$\alpha R_j^\sharp \beta \iff \beta = \alpha w_{m+1} \text{ and } \pi(\alpha) R_j \pi\left(\beta\right).$$

LEMMA 2.14
The map π is a p-morphism: $\pi : F^\sharp \twoheadrightarrow F$.

The proof is straightforward.

DEFINITION 2.15
Let $F = F^{w_0}$ be a rooted frame. Then we define the *continuum unravelling* of it as $F^\sharp_{\mathbb{R}} = \big(F_{\mathbb{R}}\big)^\sharp$ (the unravelling of the thickening by $\mathbb{R}$ with 0 as the starting point).

Now we turn to neighbourhood frames (c.f. [13] and [3] or a recent book [10]).

DEFINITION 2.16
Let X be a non-empty set, then $\mathcal{F} \subseteq 2^X$ is a *filter* on X if

1 $X \in \mathcal{F}$;
2 if $U_1,\ U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$;
3 if $U_1 \in F$ and $U_1 \subseteq U_2$, then $U_2 \in \mathcal{F}$.

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It is usually required that $\varnothing \notin \mathcal{F}$ ($\mathcal{F}$ is a proper filter), but we will not require this in our paper.

Definition 2.17
A *(normal) neighbourhood frame* (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where X is a non-empty set and $\tau : X \to 2^{2^X}$ such that $\tau(x)$ is a filter on X for any x. The function τ is called the *neighbourhood function* of $\mathfrak{X}$, and sets from $\tau(x)$ are called *neighbourhoods of x*. *A neighbourhood model* (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is an n-frame and $V : PROP \to 2^X$ is a *valuation*. In a similar way, we define *neighbourhood 2-frame* (n-2-frame) as (X, τ_1, τ_2) such that $\tau_i(x)$ is a filter on X for any x, and an *n-2-model*.

Remark 2.18
Many authors consider neighbourhood semantics for non-normal and even non-monotone logics. In this case a set of neighbourhoods can be an arbitrary set of sets. Other authors consider *monotone* neighbourhood frames and require only item 3 from Definition 2.16 (a set of neighbourhoods is closed under supersets).

Definition 2.19
The valuation function in an n-model can be extended to all formulas by induction. For Boolean connectives the definition is as usual, so we omit it. For modalities the definition is as follows:

$$M, x \models \Box_i A \iff \exists U \forall y \big(y \in U \in \tau_i(x) \Rightarrow M, y \models A\big).$$

A formula is true in an n-model M if it is valid at all points of M (notation $M \models A$). A formula is valid on an n-frame $\mathfrak{X}$ if it is true in all models based on $\mathfrak{X}$ (notation $\mathfrak{X} \models A$). We write $\mathfrak{X} \models \mathsf{L}$ if for any $A \in \mathsf{L}$, $\mathfrak{X} \models A$. We define the logic of a class of n-frames $\mathcal{C}$ as $Log(\mathcal{C}) = \{A \mid \mathfrak{X} \models A \text{ for all } \mathfrak{X} \in \mathcal{C}\}$. For a logic L we also define $\mathcal{V}n(\mathsf{L}) = \{\mathfrak{X} \mid \mathfrak{X} \text{ is an n-frame and } \mathfrak{X} \models \mathsf{L}\}$. Note that if there is no $\mathfrak{X}$ such that $\mathfrak{X} \models \mathsf{L}$, then $\mathcal{V}n(\mathsf{L}) = \varnothing$.

Definition 2.20
Let $F = (W, R)$ be a Kripke frame. We define the n-frame $\mathcal{N}\big(F\big) = (W, \tau)$ in the following way

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}.$$

Lemma 2.21
Let $F = \big(W, R\big)$ be a Kripke frame. Then

$$Log\left(\mathcal{N}\big(F\big)\right) = Log\left(F\right).$$

The proof is straightforward (see [3] or [10]).

Definition 2.22
Let $\mathfrak{X} = (X, \tau_1, \ldots)$ and $\mathcal{Y} = \big(Y, \sigma_1, \ldots\big)$ be n-frames. Then a function $f : X \to Y$ is a *p-morphism* (notation $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$) if

1. f is surjective;
2. for any $x \in X$ and $U \in \tau_i(x)$, we have $f\big(U\big) \in \sigma_i\big(f(x)\big)$;
3. for any $x \in X$ and $V \in \sigma_i\big(f(x)\big)$, we have $f^{-1}(V) \in \tau_i(x)$.

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REMARK 2.23
According to Lemma 2.21, a Kripke frame is a special case of a neighbourhood frame. It is easy to check that for any two Kripke frames F and G a function f is a p-morphism (Definition 2.6) from F to G iff f is a p-morphism (Definition 2.22) from $\mathcal{N}(F)$ to $\mathcal{N}(G)$. So, a p-morphism for n-frames is a natural generalization of the notion of p-morphism for Kripke frames. This is why we use the same name for these two formally different notions.

LEMMA 2.24
Let $\mathfrak{X} = (X, \tau_1, \dots), \mathcal{Y} = (Y, \sigma_1, \dots)$ be n-frames and $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$. Let V be a valuation on $\mathcal{Y}$. We define $[f^{-1}(V)](p) = f^{-1}(V(p))$. Then

$$\mathfrak{X}, f^{-1}(V), x \models A \iff \mathcal{Y}, V, f(x) \models A.$$

The proof is by induction on the length of formula A (c.f. [10]). The following is a straightforward corollary.

COROLLARY 2.25
If $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$, then $Log(\mathfrak{X}) \subseteq Log(\mathcal{Y})$.

3 Products: from Kripke to neighbourhood frames

DEFINITION 3.1
Let $F_i = (W_i, R_i)$ $(i = 1, 2)$ be two Kripke frames. We define their *product* (see [4]) as a frame with two relations: $F_1 \times F_2 = (W_1 \times W_2, R_1^h, R_2^v)$, where

$$(x, y)R_1^h(z, t) \iff xR_1z \;\&\; y = t;$$
$$(x, y)R_2^v(z, t) \iff x = z \;\&\; yR_2t.$$

DEFINITION 3.2
Let $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$ be two n-frames. Then *the product* of these n-frames is the n-2-frame defined as follows:

$$\mathfrak{X}_1 \times \mathfrak{X}_2 = \left(X_1 \times X_2, \tau_1^h, \tau_2^v\right);$$
$$\tau_1^h(x_1, x_2) = \left\{U \subseteq X_1 \times X_2 \mid \exists V\left(V \in \tau_1(x_1) \;\&\; V \times \{x_2\} \subseteq U\right)\right\};$$
$$\tau_2^v(x_1, x_2) = \left\{U \subseteq X_1 \times X_2 \mid \exists V\left(V \in \tau_2(x_2) \;\&\; \{x_1\} \times V \subseteq U\right)\right\}.$$

REMARK 3.3
Note that $\tau_1^h(x_1, x_2)$ and $\tau_2^v(x_1, x_2)$ are closed under supersets. So it is possible to generalize the definition of the product to monotone neighbourhood frames. This was done in [11]. However, we consider only normal n-frames in this paper, because we use Kripke semantics in the proof of completeness.

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Definition 3.4
For two unimodal logics L_1 and L_2 such that $\mathcal{V}n(\mathsf{L}_1) \neq \varnothing$ and $\mathcal{V}n(\mathsf{L}_2) \neq \varnothing$, we define the *n-product* of them as follows:

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = Log\left(\{\mathfrak{X}_1 \times \mathfrak{X}_2 \mid \mathfrak{X}_1 \in \mathcal{V}n(\mathsf{L}_1) \;\&\; \mathfrak{X}_2 \in \mathcal{V}n(\mathsf{L}_2)\}\right).$$

If we forget about one of its neighbourhood functions, say τ_2^v, then $\mathfrak{X}_1 \times \mathfrak{X}_2$ will be the disjoint union of L_1 n-frames.[3] Hence,

Proposition 3.5 ([11])
For two unimodal normal logics L_1 and L_2

$$\mathsf{L}_1 * \mathsf{L}_2 \subseteq \mathsf{L}_1 \times_n \mathsf{L}_2.$$

In Section 8 we will show that for many logics with the seriality axiom (including $\mathsf{S4}$, $\mathsf{D4}$, T, D) their n-product coincides with the fusion. But this is not the case for logic K:

Proposition 3.6
$\mathsf{K} \times_n \mathsf{K} \neq \mathsf{K} * \mathsf{K}$.

Proof. Let $\mathfrak{X}_1 = (X_1, \tau_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2)$ be two n-frames and $\mathfrak{X}_1 \times \mathfrak{X}_2 = \left(X_1 \times X_2, \tau_1^h, \tau_2^v\right)$. Consider formula $\Box_1\bot \to \Box_2\Box_1\bot$. It is non-derivable from K_2 but valid in any product of two n-frames. Indeed, consider a Kripke frame $F = \left(W, R_1, R_2\right)$, where

$$W = \{x_1, x_2, y\},$$
$$x_1 R_2 x_2,$$
$$R_1(x_1) = \varnothing,$$
$$x_2 R_1 y.$$

Then $x_2 \models \neg\Box_1\bot$, $x_1 \models \Box_1\bot$ and $x_1 \models \neg\Box_2\Box_1\bot$.

Since this formula has no variables, the truth of this formula does not depend on the valuation. So

$$\mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_1\bot \iff \varnothing \in \tau_1^h(x,y) \iff$$
$$\varnothing \in \tau_1(x) \iff \forall y' \in X_2 \left(\varnothing \in \tau_1^h(x,y')\right) \iff$$
$$\forall y' \in X_2 \left(\mathfrak{X}_1 \times \mathfrak{X}_2, (x,y') \models \Box_1\bot\right) \implies \mathfrak{X}_1 \times \mathfrak{X}_2, (x,y) \models \Box_2\Box_1\bot.$$

Hence, $\mathfrak{X}_1 \times \mathfrak{X}_2 \models \Box_1\bot \to \Box_2\Box_1\bot$. $\Box$

Moreover,

[3] Indeed an n-frame $\left(\mathfrak{X}_1 \times \mathfrak{X}_2, \tau_1^h\right)$ is isomorphic to the disjoint union of copies of $\left(\mathfrak{X}_1, \tau_1\right)$. Neighbourhoods in the union are all supersets of all neighbourhoods of its components.

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LEMMA 3.7
For any two n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$ (i) if B is a closed formula without $\Box_2$, then for any two n-frames $\mathfrak{X}_1$ and $\mathfrak{X}_2$

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models B \to \Box_2 B,$$

(ii) if B is a closed formula without $\Box_1$, then

$$\mathfrak{X}_1 \times \mathfrak{X}_2 \models B \to \Box_1 B.$$

PROOF. We prove only (i) because (ii) can be proved analogously. Since B contains neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathfrak{X}_1 \times \mathfrak{X}_2$. So if $F, (x, y) \models B$, then $\forall y' \left(F, \left(x, y'\right) \models B\right)$, hence, $F, \left(x, y\right) \models \Box_2 B$. □

DEFINITION 3.8
We put

$$\Delta_1 = \{B_1 \to \Box_2 B_1 \mid B_1 \text{ is closed and } \Box_2\text{-free}\};$$

$$\Delta_2 = \{B_2 \to \Box_1 B_2 \mid B_2 \text{ is closed and } \Box_1\text{-free}\};$$

$$\Delta = \Delta_1 \cup \Delta_2.$$

For two unimodal logics L_1 and L_2, we define the *weak commutator* of them as

$$\left\langle L_1, L_2 \right\rangle = L_1 * L_2 + \Delta.$$

From Lemma 3.7 and Proposition 3.5 follows

LEMMA 3.9
For any two normal modal logics L_1 and $L_2 \left\langle L_1, L_2 \right\rangle \subseteq L_1 \times_n L_2$.

COROLLARY 3.10
$\left\langle \mathsf{K}, \mathsf{K} \right\rangle \subseteq \mathsf{K} \times_n \mathsf{K}$.

The converse inclusion also holds but the proof requires some work.

4 Dense neighbourhood frames

To prove the completeness of a logic w.r.t. neighbourhood frames we rely on Kripke completeness. It is possible that using purely neighbourhood semantics constructions one can prove more general results for non-normal logics. But we leave this for the future.

Starting from a Kripke frame we will construct a neighbourhood frame so that the neighbourhood frame will be *dense*. An n-frame is called *dense* if no point in it has a minimal neighbourhood. This is important, because otherwise n-frames will be equivalent to Kripke frames, and any product of Kripke frames satisfies the commutativity axioms and the Church–Rosser axiom.

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DEFINITION 4.1
Let Σ be a non-empty finite set (*alphabet*). A finite sequence of elements from Σ is called a *word*; by ϵ we denote the empty word. Let Σ^* be the set of all words. We will write words without brackets or commas, e.g. $a_1a_2...a_n \in \Sigma^*$. The length of a word is the number of elements in it:

$$len\left(a_1a_2\dots a_n\right) = n, \quad len(\epsilon) = 0.$$

We also define the concatenation of words:

$$a_1a_2\dots a_n \cdot b_1b_2\dots b_m = a_1a_2\dots a_nb_1b_2\dots b_m.$$

DEFINITION 4.2
For a frame $F = \left(W, R\right)$ with a fixed root a_0 we define *a (rooted) path with stops* as a word in alphabet $W \cup \{0\}$: $a_1...a_n$, so that $a_i \in W$ or $a_i = 0$, and after dropping zeros, each point is related to the next one by relation R, and the first one is a successor of the root. The empty word ϵ is allowed and it corresponds to the root a_0. In other words, a path with stops is a word of the following type:

$$0^{i_1}b_10^{i_2}\dots 0^{i_m}b_m, \text{ where } b_j \in W,\ i_j \geq 0,\ 0^i = \underbrace{00\dots 0}_{i\,\text{times}};$$

$$\text{and } f_0\left(0^{i_1}b_10^{i_2}\dots 0^{i_m}b_m\right) = a_0Rb_1R\dots Rb_m \in W^\sharp.$$

Let us consider some examples. $f_0(\epsilon) = a_0$. If the root is reflexive, then a_0 is a path with stops, and $f_0(a_0) = a_0Ra_0$. In the frame $\left(W, R\right)$, $W = \{a_0, b\}$, $R = \left\{\left(a_0, b\right)\right\}$ the following are paths with stops: ϵ, 000, 00b, 00b00.

We also consider infinite paths with stops that end with infinitely many zeros. We call these sequences *pseudo-infinite paths (with stops)*. A pseudo-infinite path can be presented uniquely in the following way:

$$\alpha = 0^{i_1}b_10^{i_2}\dots 0^{i_m}b_m0^\omega, \text{where } b_j \in W,\ i_j \geq 0.$$

Let W_ω be the set of all pseudo-infinite paths in W.

The function of dropping zeros can be extended to W_ω as $f_0 : W_\omega \to W^\sharp$ in the following way: for a pseudo-infinite path $\alpha = a_1...a_n...$ we define

$$st(\alpha) = \min\left\{N \,\middle|\, \forall k > N\left(a_k = 0\right)\right\};$$

$$\alpha|_k = a_1\dots a_k;\ \ \alpha|_0 = \epsilon;$$

$$f_0(\alpha) = f_0\left(\alpha|_{st(\alpha)}\right), \text{ i.e. } f_0\left(0^{i_1}b_10^{i_2}\dots 0^{i_m}b_m0^\omega\right) = a_0Rb_1R\dots Rb_m.$$

In order to introduce a neighbourhood function on W_ω, we define

$$U_k(\alpha) = \left\{\beta \in W_\omega \,\middle|\, \alpha|_m = \beta|_m \;\&\; f_0(\alpha)R^\sharp f_0\left(\beta\right),\ m = \max(k, st(\alpha))\right\}.$$

LEMMA 4.3
$U_k(\alpha) \subseteq U_m(\alpha)$ whenever $k \geq m$.

PROOF. Let $\beta \in U_k(\alpha)$. Since $\alpha\,|_k = \beta|_k$ and $k \geq m$, $\alpha\,|_m = \beta|_m$. Hence, $\beta \in U_m(\alpha)$. □

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DEFINITION 4.4
Due to Lemma 4.3, sets $U_n(\alpha)$ form a filter base. So we can define

$$\tau(\alpha) \text{— the filter with the base } \{U_n(\alpha) \mid n \in \mathbb{N}\};$$

$$\mathcal{N}_\omega(F) = (W_\omega, \tau) \text{—is } \textit{a dense n-frame based on } F.$$

The frame $\mathcal{N}_\omega(F)$ is dense unlike $\mathcal{N}(F)$. Indeed,

$$\bigcap_n U_n(\alpha) = \varnothing \notin \tau(\alpha).$$

LEMMA 4.5
For any α and any k we have that $f_0\big(U_k(\alpha)\big) = R^\sharp\big(f_0(\alpha)\big)$.

PROOF.
By definition for $k < st(\alpha)$ $U_k(\alpha) = U_{st(\alpha)}(\alpha)$, hence we can assume that $k \geq st(\alpha)$.
Let $x \in f_0\big(U_k(\alpha)\big)$. Then

$$\exists \beta \left(\alpha|_k = \beta|_k \;\&\; f_0(\alpha) R^\sharp f_0(\beta) = x\right),$$

so $x \in R^\sharp\big(f_0(\alpha)\big)$.
Let $x \in R^\sharp\big(f_0(\alpha)\big)$, then for some b' $x = f_0(\alpha) R b'$. We put

$$\beta = \alpha 0^{k - st(\alpha)} b'.$$

It is easy to show that $\beta \in U_k(\alpha)$ and $f_0(\beta) = x$. □

LEMMA 4.6
Let $F = (W, R)$ be a Kripke frame with root a_0, then

$$f_0 : \mathcal{N}_\omega(F) \twoheadrightarrow \mathcal{N}(F^\sharp).$$

PROOF. From now on in this proof we will omit the subindex in f_0. Since for any $b \in W$ there is a path $a_0 a_1 \ldots a_{n-1} b$ and hence for a pseudo-infinite path $\alpha = a_1 \ldots b 0^\omega \in X$, $f(\alpha) = b$ and f is surjective.

Assume that $\alpha \in W_\omega$ and $U \in \tau(\alpha)$. We need to prove that $R^\sharp\big(f(\alpha)\big) \subseteq f(U)$. There exists an m such that $U_m(\alpha) \subseteq U$, and since $f\big(U_m(\alpha)\big) = R^\sharp\big(f(\alpha)\big)$, we have

$$R^\sharp\big(f(\alpha)\big) = f\big(U_m(\alpha)\big) \subseteq f(U).$$

Assume that $\alpha \in W_\omega$ and V is a neighbourhood of $f(\alpha)$, i.e. $R^\sharp\big(f(\alpha)\big) \subseteq V$. We need to prove that there exists $U \in \tau(\alpha)$ such that $f(U) \subseteq V$. For U, we take $U_m(\alpha)$ for some $m \geq st(\alpha)$, then

$$f\big(U_m(\alpha)\big) = R^\sharp\big(f(\alpha)\big) \subseteq V.$$

□

COROLLARY 4.7
For any frame F, $Log\left(\mathcal{N}_\omega(F)\right) \subseteq Log\left(F\right)$.

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PROOF. It follows from Lemmas 2.21, 4.6, 2.14 and Corollary 2.25 that

$$Log\left(\mathcal{N}_\omega(F)\right) \subseteq Log\left(\mathcal{N}\left(F^\sharp\right)\right) = Log\left(F^\sharp\right) \subseteq Log\left(F\right).$$

□

Let us remark that it is possible that $Log\left(\mathcal{N}_\omega(F)\right) \neq Log\left(F\right)$. For example, let us consider the natural numbers with the 'next' relation. It is convenient here to regard a number as a word in a one-letter alphabet:

$$G = \left(\{1\}^*, S\right),\ 1^n S 1^m \iff m = n + 1.$$

Obviously $G \models \Diamond p \to \Box p$.

Since in G every point, except for the root point, has only one predecessor, we can identify a point and a path from the root to this point, i.e. $G^\sharp = G$. Therefore, points in $\mathcal{N}_\omega(G)$ can be presented as infinite sequences of 0 and 1 with only zeros at the end.

PROPOSITION 4.8
$\mathcal{N}_\omega(G) \not\models \Diamond p \to \Box p$

PROOF. Consider valuation $V(p) = \left\{0^{2n}10^\omega \mid n \in \mathbb{N}\right\}$. In every neighbourhood of point 0^ω there are points, where p is true and there are points where p is false. For example, if k is even, then in $U_k\left(0^\omega\right)$ there is a point 0^k10^ω where p is true, and a point $0^{k+1}10^\omega$ where p is false. Hence,

$$\mathcal{N}_\omega(G) \models \Diamond p \wedge \Diamond \neg p.$$

□

This formula does not preserved under the $\mathcal{N}_\omega$ operation and probably any formula that restricts branching does not preserved under the $\mathcal{N}_\omega$ operation. In Section 7 we define some formulas that are preserved.

5 Weak product of Kripke frames

In order to prove the completeness w.r.t. n-frames, we first establish completeness w.r.t. a special kind of Kripke frames. The *weak products* of Kripke frames were introduced in [8] for this purpose. Here we modify the definition. Nevertheless frames from the new construction are isomorphic to frames from the old one, but their description is better in some respect.

DEFINITION 5.1
Let $F_1 = \left(W_1, R_1\right)$ and $F_2 = \left(W_2, R_2\right)$ be two Kripke frames with the roots x_0 and y_0, respectively, and $W_1 \cap W_2 = \varnothing$. Let $\Sigma = W_1 \cup W_2$. Then we define the functions $p_1, p_2 : \Sigma^* \to \Sigma^*$ and $\pi : \Sigma^* \setminus \{\epsilon\} \to \Sigma$ by induction

$$\begin{aligned} p_1(\epsilon) &= \epsilon, \\ p_2(\epsilon) &= \epsilon, \\ p_1(\boldsymbol{a}u) &= p_1(\boldsymbol{a}) \cdot u \text{ for } \boldsymbol{a} \in \Sigma^*,\ u \in W_1; \\ p_1(\boldsymbol{a}u) &= p_1(\boldsymbol{a}) \text{ for } \boldsymbol{a} \in \Sigma^*,\ u \in W_2; \\ p_2(\boldsymbol{a}u) &= p_2(\boldsymbol{a}) \text{ for } \boldsymbol{a} \in \Sigma^*,\ u \in W_1; \\ p_2(\boldsymbol{a}u) &= p_2(\boldsymbol{a}) \cdot u \text{ for } \boldsymbol{a} \in \Sigma^*,\ u \in W_2; \\ \pi(\boldsymbol{a}u) &= u \text{ for } \boldsymbol{a} \in \Sigma^*,\ u \in \Sigma. \end{aligned}$$

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Intuitively p_1 is dropping all symbols not belonging to W_1 (the same for p_2) and π maps a non-empty word to its final symbol.

Since F_1 and F_2 are frames with roots and have only one relation, we will assume that paths in them do not contain relations and start from the roots:

$$W_1^\sharp = \{x_1 \dots x_n \mid x_0 R_1 x_1 R_1 \dots R_1 x_n \text{ is a path in the usual sense}\};$$
$$W_2^\sharp = \{y_1 \dots y_n \mid y_0 R_2 y_1 R_2 \dots R_2 y_n \text{ is a path in the usual sense}\}.$$

We define the *entanglement* of F_1 and F_2 as follows:

$$F_1 \between F_2 = \left\{ \boldsymbol{a} \in \Sigma^* \,\middle|\, p_1(\boldsymbol{a}) \in W_1^\sharp \text{ and } p_2(\boldsymbol{a}) \in W_2^\sharp \right\}.$$

We define the *weak product of frames* F_1 and F_2 as follows:

$$\begin{aligned} \langle F_1, F_2 \rangle &= \left(F_1 \between F_2, R_1^<, R_2^<\right), \\ \boldsymbol{a} R_1^< \boldsymbol{b} &\iff \exists u \in W_1 \big(\boldsymbol{b} = \boldsymbol{a}u\big); \\ \boldsymbol{a} R_2^< \boldsymbol{b} &\iff \exists v \in W_2 \big(\boldsymbol{b} = \boldsymbol{a}v\big). \end{aligned}$$

PROPOSITION 5.2

For any two rooted frames F_1 and $F_2 \langle F_1, F_2 \rangle \models \Delta$.

PROOF. Let B be a closed $\Box_2$-free formula and $\langle F_1, F_2 \rangle, \boldsymbol{a} \models B$, then for any $v \in W_2$ we need to show that $\langle F_1, F_2 \rangle, \boldsymbol{a}v \models B$. Indeed, the frames

$$\left(\left(R_1^<\right)^* (\boldsymbol{a}), R_1^< |_{\left(R_1^<\right)^* (\boldsymbol{a})} \right)$$

and

$$\left(\left(R_1^<\right)^* (\boldsymbol{a}v), R_1^< |_{\left(R_1^<\right)^* (\boldsymbol{a}v)} \right)$$

are isomorphic to $\left(F_1^\sharp\right)^{p_1(\boldsymbol{a})}$. Then, since B is closed and does not contain $\Box_2$, $\langle F_1, F_2 \rangle, \boldsymbol{a}v \models B$.

For Δ_2 the proof is similar. $\Box$

The aim of this section is to prove the following theorem:

THEOREM 5.3

The logic $\langle \mathsf{K}, \mathsf{K} \rangle$ is complete with respect to the class of all weak products of Kripke frames.

Let $\mathcal{F} = \mathcal{F}^{x_0}$ be a rooted subframe of the canonical frame of a logic L with two modalities.

By Υ we define all closed (variable-free) modal formulas of the modal language. For a point $x \in \mathcal{F}$ we define $\bar{x} = x \cap \Upsilon$. Then let $\mathcal{F}_0^{\bar{x}_0}$ be a rooted subframe of the 0-canonical frame.

We define Υ_i as the set of all closed formulas in the language with only $\Box_i$ modality.

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LEMMA 5.4
Let $\mathcal{F}_0 = (\bar{W}, \bar{R}_1, \bar{R}_2)$ be the 0-canonical frame for logic L, such that $\Delta \subset \mathsf{L}$. Then

$$\bar{x}\bar{R}_1\bar{y} \Rightarrow \bar{x} \cap \Upsilon_2 = \bar{y} \cap \Upsilon_2,$$

$$\bar{x}\bar{R}_2\bar{y} \Rightarrow \bar{x} \cap \Upsilon_1 = \bar{y} \cap \Upsilon_1.$$

PROOF.
We prove only one half, since the other half is similar.

For any $A \in \Upsilon_2$

$$A \to \Box_1 A \in \Delta \text{ and } \neg A \to \Box_1 \neg A \in \Delta.$$

So

$$A \in \Upsilon_2 \cap \bar{x} \Rightarrow A \to \Box_1 A \in \bar{x} \Rightarrow A \in \bar{y},$$

$$A \in \Upsilon_2 \text{ and } A \notin \bar{x} \Rightarrow \neg A \to \Box_1 \neg A \in \bar{x} \Rightarrow \neg A \in \bar{y} \Rightarrow A \notin \bar{y}.$$

□

By a straightforward induction we get

COROLLARY 5.5
Let $\mathcal{F}_0 = (\bar{W}, \bar{R}_1, \bar{R}_2)$ be the 0-canonical frame for logic L, such that $\Delta \subset \mathsf{L}$. Then

$$\bar{x}\left(\bar{R}_1 \cup \bar{R}_1^{-1}\right)^* \bar{y} \Rightarrow \bar{x} \cap \Upsilon_2 = \bar{y} \cap \Upsilon_2,$$

$$\bar{x}\left(\bar{R}_2 \cup \bar{R}_2^{-1}\right)^* \bar{y} \Rightarrow \bar{x} \cap \Upsilon_1 = \bar{y} \cap \Upsilon_1.$$

Since any closed formula is canonical, the following holds:

LEMMA 5.6
Let L_1 and L_2 be two canonical logics. Then $\langle \mathsf{L}_1, \mathsf{L}_2 \rangle$ is also canonical.

LEMMA 5.7
Let L be a 2-modal logic, $\mathsf{L}_i = \{A \mid A \in \mathsf{L} \cap \mathcal{ML}_{\Box_i}\}$ $(i = 1, 2)$ be the 1-modal fragments of it, $\mathcal{F}_{\mathsf{L}} = (W, R_1, R_2)$ be the canonical frame of L and $a \in \mathcal{F}_{\mathsf{L}}$. Let $F_i = \left(\left(\mathcal{F}^0_{\mathsf{L}_i}\right)^{a \cap \Upsilon_i}\right)^\sharp_{\mathbb{R}}$ be the continuum unravelling of the rooted subframe of the 0-canonical model of logic L_i with the root $a \cap \Upsilon_i$ $(i = 1, 2)$. Then for any $\boldsymbol{a}_0 \in F_1$ such that $\pi(\boldsymbol{a}_0) = (a \cap \Upsilon_1, r)$ for some $r \in \mathbb{R}$ there exist a p-morphism of 1-Kripke frames $f : F_1^{\boldsymbol{a}_0} \twoheadrightarrow \left(R_1^*(a), R_1|_{R_1^*(a)}\right)$ with the following property.

$$\forall \boldsymbol{b} \in F_1^{\boldsymbol{a}_0} \forall b \in R_1^*(a) \left(f(\boldsymbol{b}) = b \;\Rightarrow\; \exists l \in \mathbb{R}\big(\pi(\boldsymbol{b}) = (b \cap \Upsilon_1, l)\big)\right).$$

The same holds for F_2.

PROOF. We will describe the construction only for F_1 because for F_2 it is similar.

To simplify formulas we assume that $G = \left(R_1^*(a), R_1|_{R_1^*(a)}\right) = (W, R)$ and $F_1 = (W', R')$.

Since F_1 and G are rooted we can define a map $f : F_1 \to G$ recursively.

Base: $f(\epsilon) = x_0$.

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Step: We assume that $f(\boldsymbol{b}) = x$, $\pi(\boldsymbol{b}) = (x \cap \Upsilon_1, r)$ and $\boldsymbol{c} \in R'(\boldsymbol{b})$. We need to choose the image for $\boldsymbol{c}$ from $R(x)$.

For $y, z \in R(x)$ we define a relation

$$y \sim z \iff y \cap \Upsilon_1 = z \cap \Upsilon_1.$$

It is obviously an equivalence relation. Let $U = R(x)/\sim$ be the quotient set of $R(x)$ by $\sim$.

Since the cardinality of each equivalence class $[y] \in U$ is no greater than the cardinality of the canonical frame which is no greater than continuum, then there exists a partition of $\mathbb{R}$ indexed by elements of $[y]$ into sets of continuum cardinality:

$$\mathbb{R} = \bigsqcup_{z \in [y]} V_z^{[y]} \text{ and for each } z \in [y] \ \left|V_z^{[y]}\right| = |\mathbb{R}|.$$

This is due to the standard result of Set Theory: $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$.

For a fixed $\boldsymbol{c} \in R'(\boldsymbol{b})$ there exists $y \in R(x)$ and r' such that

$$\pi(\boldsymbol{c}) = (y \cap \Upsilon_1, r'),\ r' \in V_y^{[y]}.$$

We define

$$f(\boldsymbol{c}) = y.$$

Each point in $(\mathcal{F}^0)^\sharp_{\mathbb{R}}$ is reachable from $(\bar{x}_0, 0)$ in finitely many steps. A point reachable in m steps will appear on m-th iteration. So the function f is defined correctly.

Let us check that f is a p-morphism.

Monotonicity. It is obvious from the construction.

Lifting. Let $f(\boldsymbol{a}) = x$ and xRy then for any $r' \in V_y^{[y]}$ and $\boldsymbol{b} = \boldsymbol{a}R(\bar{y}, r')$ we have $\boldsymbol{a}R'\boldsymbol{b}$ and $f(\boldsymbol{b}) = y$.

Surjectivity. Since F_1 and G are rooted, and the root maps to the root, surjectivity follows from the lifting property. □

LEMMA 5.8

Let L_1 and L_2 be two unimodal logics and $\mathcal{F} = \mathcal{F}^{a_0}$ be a rooted subframe of the canonical frame for logic $\langle \mathsf{L}_1, \mathsf{L}_2 \rangle$; then there exist two rooted frames F_1, F_2 and a p-morphism $f : \langle F_1, F_2 \rangle \twoheadrightarrow \mathcal{F}$.

PROOF. We take $\left(\left(\mathcal{F}^0_{\mathsf{L}_1}\right)^{\bar{x}_0}\right)^\sharp_{\mathbb{R}}$ and $\left(\left(\mathcal{F}^0_{\mathsf{L}_2}\right)^{\bar{y}_0}\right)^\sharp_{\mathbb{R}}$ as F_1 and F_2, respectively, where $\bar{x}_0 = a_0 \cap \Upsilon_1$ and $\bar{y}_0 = a_0 \cap \Upsilon_2$. Let $F_1 = (W_1, R'_1)$, $F_2 = (W_2, R'_2)$ and $\mathcal{F} = (W, R_1, R_2)$.

Using Lemma 5.7 for each $\boldsymbol{a} \in \langle F_1, F_2 \rangle$ we fix two p-morphisms:

$$g_1^{\boldsymbol{a}} : F_1^{p_1(\boldsymbol{a})} \twoheadrightarrow \left(R_1^*(a), R_1|_{R_1(a)}\right),$$

$$g_2^{\boldsymbol{a}} : F_2^{p_2(\boldsymbol{a})} \twoheadrightarrow \left(R_2^*(a), R_2|_{R_2(a)}\right).$$

We also make sure that they are coordinated in the following way

$$\boldsymbol{a}R_i^< \boldsymbol{b}R_i^< \boldsymbol{c} \Longrightarrow g_i^{\boldsymbol{a}}(\boldsymbol{c}) = g_i^{\boldsymbol{b}}(\boldsymbol{c}),$$

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where $R_1^<$ and $R_2^<$ are the first and the second relations in $\langle F_1, F_2\rangle$.

We can do this because the restriction of a p-morphism to a rooted submodel is a p-morphism.

Let us define a map $f : \langle F_1, F_2\rangle \to \mathcal{F}$ recursively. The root of $\langle F_1, F_2\rangle$ maps to the root of $\mathcal{F}$:

$$f(\epsilon) = a_0.$$

We assume that for $\boldsymbol{a} \in F_1 \between F_2$ the map is defined. Let $\boldsymbol{b} = \boldsymbol{a}u$. If $u \in W_i$, then

$$f(\boldsymbol{b}) = g_i^{\boldsymbol{a}}(\boldsymbol{b}).$$

Let us check that f is a p-morphism. The monotonicity is due to the monotonicity of $g_i{}^{\boldsymbol{a}}$. To check the lifting, we assume that $f(\boldsymbol{a}) = a$ and aR_ib. Then $b \in R_i(a)$ and due to the surjectivity of $g_i{}^{\boldsymbol{a}}$ there exists $\boldsymbol{b}$ such that $\boldsymbol{a}R_i{}'\boldsymbol{b}$ and $f(\boldsymbol{b}) = b$. The surjectivity follows from the rootedness of frames and the lifting property. □

To prove Theorem 5.3, we assume that formula A is not in the logic $\langle \mathsf{K}, \mathsf{K}\rangle$. Then it is refutable in a rooted subframe of the canonical frame $\mathcal{F}_{\langle \mathsf{K},\mathsf{K}\rangle}$. By Lemma 5.8 there exist F_1 and F_2 such that $\langle F_1, F_2\rangle$ is a p-morphic preimage of the subframe. Hence by p-morphism lemma A is refutable in $\langle F_1, F_2\rangle$.

6 N-product completeness theorem for $\langle \mathsf{K}, \mathsf{K}\rangle$

Let $F_1 = (W_1, R_1) = F_1^{r_1}$ and $F_2 = (W_2, R_2) = F_2^{r_2}$ be two rooted frames. Assume that $W_1 \cap W_2 = \varnothing$. Consider the product of n-frames $\mathfrak{X}_1 = (X_1, \tau_1) = \mathcal{N}_\omega(F_1)$ and $\mathfrak{X}_2 = (X_2, \tau_2) = \mathcal{N}_\omega(F_2)$

$$\mathfrak{X} = \left(X_1 \times X_2, \tau_1^h, \tau_2^v\right) = \mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2).$$

We define a function $g : \mathfrak{X}_1 \times \mathfrak{X}_2 \to \langle F_1, F_2\rangle$ by induction, as follows.

Let $(\alpha, \beta) \in \mathfrak{X}_1 \times \mathfrak{X}_2$, so that $\alpha = x_1x_2\ldots$ and $\beta = y_1y_2\ldots$, $x_i \in W_1 \cup \{0\}$, $y_j \in W_2 \cup \{0\}$. We define $g(\alpha, \beta)$ to be the finite sequence that we obtain after dropping all zeros from the infinite sequence $x_1y_1x_2y_2\ldots$.

LEMMA 6.1

The function g defined above is a p-morphism:

$$g : \mathfrak{X}_1 \times \mathfrak{X}_2 \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2\rangle\right).$$

PROOF. First, we need to check that for any $\alpha \in \mathcal{N}_\omega(F_1)$ and any $\beta \in \mathcal{N}_\omega(F_2)$ we have that $g(\alpha, \beta) \in F_1 \between F_2$. This follows from the equalities:

$$p_1\left(g(\alpha, \beta)\right) = p_1\left(f_0(\alpha)\right),\ p_2\left(g(\alpha, \beta)\right) = p_2\left(f_0(\beta)\right).$$

To prove surjectivity, we take $\boldsymbol{z} = z_1\ldots z_n \in F_1 \between F_2$. For $i \le n$ we define

$$x_i = \begin{cases} z_i, & \text{if } z_i \in W_1; \\ 0, & \text{if } z_i \in W_2; \end{cases} \qquad y_i = \begin{cases} 0, & \text{if } z_i \in W_1; \\ z_i, & \text{if } z_i \in W_2. \end{cases}$$

Let $\alpha = x_1x_2\ldots x_n0^\omega$ and $\beta = y_1y_2\ldots y_n0^\omega$, then $g(\alpha, \beta) = \boldsymbol{z}$. Hence, g is surjective.

We check the next two conditions only for τ_1, since for τ_2 it is similar. We assume that $(\alpha, \beta) \in X_1 \times X_2$ and $U \in \tau_1(\alpha, \beta)$. We need to prove that $R_1^<(g(\alpha, \beta)) \subseteq g(U)$. There exists

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$m > \max\{st(\alpha), st(\beta)\}$ such that $U_m(\alpha) \times \{\beta\} \subseteq U$ and, since $g(U_m(\alpha) \times \{\beta\}) = R_1^{<}(g(\alpha,\beta))$, then

$$R_1^{<}(g(\alpha,\beta)) = g(U_m(\alpha) \times \{\beta\}) \subseteq g(U),$$

where $U_m(\alpha)$ is the corresponding neighbourhood from $\mathfrak{X}_1$.

We assume that $(\alpha,\beta) \in X_1 \times X_2$ and $R_1^{<}(g(\alpha,\beta)) \subseteq V$. We need to prove that there exists $U \in \tau_1(\alpha,\beta)$ such that $g(U) \subseteq V$. For U, we take $U_m(\alpha) \times \{\beta\}$ for some $m > \max\{st(\alpha), st(\beta)\}$, then

$$g(U_m(\alpha) \times \{\beta\}) = R_1^{<}(g(\alpha,\beta)) \subseteq V.$$

□

COROLLARY 6.2
Let $F_1 = (W_1, R_1)$ and $F_2 = (W_2, R_2)$, then $Log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \subseteq Log(\langle F_1, F_2\rangle)$.

This immediately follows from Lemma 6.1 and Corollary 2.25.

THEOREM 6.3
The logic $\langle \mathsf{K}, \mathsf{K}\rangle$ is complete with respect to products of normal neighbourhood frames, i.e.

$$\langle \mathsf{K}, \mathsf{K}\rangle = \mathsf{K} \times_n \mathsf{K}. \tag{1}$$

PROOF. The inclusion from left to right of (1) was proved in Corollary 3.10.

The converse inclusion follows from Theorem 5.3 and Corollary 6.2. Indeed,

$$\begin{aligned}
\mathsf{K} \times_n \mathsf{K} &= \bigcap_{\mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{V}n(\mathsf{K})} Log(\mathfrak{X}_1 \times \mathfrak{X}_2) \\
&\subseteq \bigcap_{F_1, F_2 \text{ — Kripke frames}} Log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \\
&\subseteq \bigcap_{F_1, F_2 \text{ — Kripke frames}} Log(\langle F_1, F_2\rangle) \subseteq \langle \mathsf{K}, \mathsf{K}\rangle.
\end{aligned}$$

□

7 Horn axioms

DEFINITION 7.1
Following [4], we define *a universal strict Horn sentence* as a first-order closed formula of the form

$$\forall x \forall y \forall z_1 \ldots \forall z_n \left(\phi(x, y, z_1, \ldots, z_n) \to \psi(x,y)\right),$$

where $\phi(x, y, z_1, \ldots, z_n)$ is quantifier-free positive (i.e. it is built from atomic formulas by using $\wedge$ and $\vee$) and $\psi(x,y)$ is an atomic formula in the signature $\Omega = \left\langle R_1^{(2)}, \ldots, R_m^{(2)}\right\rangle$, where $R_i^{(2)}$ is the propositional letter that corresponds to the relation R_i.

DEFINITION 7.2
A logic L is called an *HTC-logic* (from Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type $\Box p \to \Box^n p$, $n \geq 0$. These formulas correspond to universal strict Horn sentences (see [4]).

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Let Γ be a set of universal strict Horn formulas and F be a Kripke frame. By F^Γ we define the Γ-closure of F, that is the minimal (in terms of inclusion of relations) frame such that all formulas from Γ are valid in it. Such a frame exists due to [4]:

LEMMA 7.3 ([4, Prop 7.9])
For any Kripke frame $F = \left(W, R_1, \ldots, R_n\right)$ and a set of universal strict Horn formulas Γ, there exists $F^\Gamma = \left(W, R_1^\Gamma, \ldots, R_n^\Gamma\right)$ such that

- $R_i \subseteq R_i^\Gamma$ for all $i \in \{1, \ldots, n\}$;
- $F^\Gamma \vDash \Gamma$;
- if $G \vDash \Gamma$ and $f : F \twoheadrightarrow G$ then $f : F^\Gamma \twoheadrightarrow G$.

DEFINITION 7.4
Let Γ be a set of universal strict Horn formulas, $F = (W, R)$ be a rooted frame, $\alpha \in W_\omega$ and $f_0 : W_\omega \to W^\sharp$ be the 'zero-dropping' function. Then we define

$$\begin{aligned}
U_k^\Gamma(\alpha) &= \left\{\beta \in W_\omega \,\middle|\, \alpha|_m = \beta|_m \;\&\; f_0(\alpha) \left(R^\sharp\right)^\Gamma f_0\!\left(\beta\right),\ m = \max\left(k, st(\alpha)\right)\right\}, \\
\tau^\Gamma(\alpha) &= \left\{V \,\middle|\, \exists k \left(U_k^\Gamma(\alpha) \subseteq V\right)\right\}, \\
\mathcal{N}_\omega^\Gamma\left(F\right) &= \left(W_\omega, \tau^\Gamma\right).
\end{aligned}$$

We also need the following obvious lemmas:

LEMMA 7.5
For any closed modal formula A and a p-morphism of Kripke frames $f : F \twoheadrightarrow G$

$$F, x \vDash A \iff G, f(x) \vDash A.$$

And its neighbourhood analogue:

LEMMA 7.6
For any closed modal formula A and a p-morphism of n-frames $f : \mathfrak{X} \twoheadrightarrow \mathcal{Y}$

$$\mathfrak{X}, x \vDash A \iff \mathcal{Y}, f(x) \vDash A.$$

In [4], the product matching was proved for a large class of Horn axiomatizable logics, including $\mathsf{S5}$. But in our case, $\mathsf{S5} \times_n \mathsf{S5} \neq \langle \mathsf{S5}, \mathsf{S5} \rangle$. In fact, since neighbourhood frames correspond to topological spaces in case of transitive and reflexive logics, and due to [6],

$$\mathsf{S5} \times_n \mathsf{S5} = \mathsf{S5} \times \mathsf{S5} = \left[\mathsf{S5}, \mathsf{S5}\right] = \mathsf{S5} * \mathsf{S5} + \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p + \Diamond_1\Box_2 p \to \Box_2\Diamond_1 p.$$

LEMMA 7.7
Let L be an HTC-logic, Γ be the corresponding set of Horn formulas and $F \vDash \mathsf{L}$. If $\Box p \to \Box^n p \in \mathsf{L}$, then

$$\mathcal{N}_\omega^\Gamma\left(F\right) \vDash \Box p \to \Box^n p.$$

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PROOF. Let $M = \left(\mathcal{N}^{\Gamma}_{\omega}(F), V\right)$ be a neighbourhood model. We assume that $M, \alpha \not\models \Box^n p$, and then we prove that $M, \alpha \not\models \Box p$, i.e.

$$\forall m \exists \beta \in U^{\Gamma}_{m}(\alpha)\left(\beta \not\models p\right).$$

Let us fix m. Then

$$\begin{aligned} &\exists \alpha_1 \in U^{\Gamma}_{m}(\alpha)\left(\alpha_1 \not\models \Box^{n-1} p\right) \Rightarrow \\ \Rightarrow &\exists \alpha_2 \in U^{\Gamma}_{m}(\alpha_1)\left(\alpha_2 \not\models \Box^{n-2} p\right) \Rightarrow \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \Rightarrow &\exists \alpha_n \in U^{\Gamma}_{m}(\alpha_{n-1})\left(\alpha_n \not\models p\right). \end{aligned}$$

By the definition of $U^{\Gamma}_{m}(\alpha)$

$$f_0(\alpha)\left(R^{\sharp}\right)^{\Gamma} f_0(\alpha_1)\left(R^{\sharp}\right)^{\Gamma} \ldots \left(R^{\sharp}\right)^{\Gamma} f_0(\alpha_n)$$

and

$$\alpha\big|_m = \alpha_1\big|_m = \ldots = \alpha_n\big|_m.$$

Since $\left(W^{\sharp}, \left(R^{\sharp}\right)^{\Gamma}\right) \models \Box p \to \Box^n p$,

$$f_0(\alpha)\left(R^{\sharp}\right)^{\Gamma} f_0(\alpha_n).$$

It follows that $\alpha_n \in U^{\Gamma}_{m}(\alpha)$. $\square$

LEMMA 7.8
Let L be an HTC-logic, Γ be the corresponding set of Horn formulas and $F \models \mathsf{L}$. Then

$$f_0 : \mathcal{N}^{\Gamma}_{\omega}(F) \twoheadrightarrow \mathcal{N}\left(F^{\sharp\Gamma}\right).$$

PROOF. From now on in this proof we will omit the subindex in f_0. The surjectivity was established in Lemma 4.6.

Assume that $\alpha \in W_{\omega}$ and $U \in \tau^{\Gamma}(\alpha)$. We need to prove that $R^{\sharp\Gamma}\left(f(\alpha)\right) \subseteq f(U)$. There exists m such that $U^{\Gamma}_{m}(\alpha) \subseteq U$, and since $f\left(U^{\Gamma}_{m}(\alpha)\right) = R^{\sharp\Gamma}\left(f(\alpha)\right)$, then

$$R^{\sharp\Gamma}\left(f(\alpha)\right) = f\left(U^{\Gamma}_{m}(\alpha)\right) \subseteq f(U).$$

Assume that $\alpha \in W_{\omega}$ and V is a neighbourhood of $f(\alpha)$, i.e. $R^{\sharp\Gamma}\left(f(\alpha)\right) \subseteq V$. We need to prove that there exists $U \in \tau^{\Gamma}(\alpha)$ such that $f(U) \subseteq V$. For U, we take $U^{\Gamma}_{m}(\alpha)$ for some $m \geq st(\alpha)$, then

$$f\left(U^{\Gamma}_{m}(\alpha)\right) = R^{\sharp\Gamma}\left(f(\alpha)\right) \subseteq V.$$

$\square$

COROLLARY 7.9
Let L be an HTC-logic and $F \models \mathsf{L}$; then $\mathcal{N}^{\Gamma}_{\omega}(F) \models \mathsf{L}$.

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LEMMA 7.10
Let F_1 and F_2 be two frames, Γ_1 and Γ_2 be two sets of Horn sentences that correspond to HTC-logics, then

$$\mathcal{N}_\omega^{\Gamma_1}(F_1) \times \mathcal{N}_\omega^{\Gamma_2}(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2\rangle^{\Gamma_1 \cup \Gamma_2}\right).$$

The proof is similar to Lemma 6.1. The underlining sets are the same and we can take the same function g. So, surjectivity follows. Monotonicity and lifting are proved similarly.

THEOREM 7.11
Let L_1 and L_2 be two HTC-logics then

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \langle \mathsf{L}_1, \mathsf{L}_2\rangle.$$

PROOF. By Lemma 3.9 $\langle \mathsf{L}_1, \mathsf{L}_2\rangle \subseteq \mathsf{L}_1 \times_n \mathsf{L}_2$.

Let Γ_1 and Γ_2 be the sets of Horn sentences corresponding to L_1 and L_2. Let $A \notin \langle \mathsf{L}_1, \mathsf{L}_2\rangle$; then there is a rooted subframe $\mathcal{F}$ of the canonical frame of logic $\langle \mathsf{L}_1, \mathsf{L}_2\rangle$ such that $\mathcal{F} \not\models A$. Then by Lemma 5.8 there are frames F_1 and F_2 such that

$$\langle F_1, F_2\rangle \twoheadrightarrow \mathcal{F}.$$

Since L_1, L_2 and $\langle \mathsf{L}_1, \mathsf{L}_2\rangle$ are canonical then

$$\langle F_1, F_2\rangle^{\Gamma_1 \cup \Gamma_2} \twoheadrightarrow \mathcal{F}.$$

By Lemma 7.10

$$\mathcal{N}_\omega^{\Gamma_1}(F_1) \times \mathcal{N}_\omega^{\Gamma_2}(F_2) \twoheadrightarrow \mathcal{N}\left(\langle F_1, F_2\rangle^{\Gamma_1 \cup \Gamma_2}\right).$$

By Corollary 7.9

$$\mathcal{N}_\omega^{\Gamma_1}(F_1) \models \mathsf{L}_1 \text{ and } \mathcal{N}_\omega^{\Gamma_2}(F_2) \models \mathsf{L}_2.$$

At the same time

$$\mathcal{N}_\omega^{\Gamma_1}(F_1) \times \mathcal{N}_\omega^{\Gamma_2}(F_2) \not\models A.$$

So $\mathsf{L}_1 \times_n \mathsf{L}_2 \subseteq \langle \mathsf{L}_1, \mathsf{L}_2\rangle$. ∎

8 Seriality axiom

Consider the seriality axiom $\neg\Box\bot$. By induction on the length of a formula, one can easily prove

LEMMA 8.1
If $\neg\Box\bot \in \mathsf{L}$ then any closed formula is L-equivalent to $\bot$ or $\top$.

The base is obvious and the inductive step follows from

$$\vdash \Box\top \leftrightarrow \top, \qquad \neg\Box\bot \vdash \Box\bot \leftrightarrow \bot.$$

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LEMMA 8.2
For a bimodal logic L if $\mathsf{L} \vdash \neg\Box_1\bot$ then $\mathsf{L} \vdash B \to \Box_2 B$ for any closed formula $B \in \mathcal{ML}_{\Box_1}$.

This is a simple exercise.

COROLLARY 8.3
If L_1 and L_2 are HTC-logics and $\neg\Box\bot \in \mathsf{L}_1$, $\neg\Box\bot \in \mathsf{L}_2$ then

$$\langle \mathsf{L}_1, \mathsf{L}_2 \rangle = \mathsf{L}_1 * \mathsf{L}_2.$$

From Corollary 8.3 and Theorem 7.11 it follows.

THEOREM 8.4
Let L_1 and L_2 be HTC-logics with seriality then

$$\mathsf{L}_1 \times_n \mathsf{L}_2 = \mathsf{L}_1 * \mathsf{L}_2.$$

Note that this theorem covers the results from [7], since the logics D, T, $\mathsf{D4}$ and $\mathsf{S4}$ are all HTC-logics with seriality.

PROPOSITION 8.5
If L_1 and L_2 are finitely axiomatizable, and have only finitely many non-equivalent closed formulas then $\langle \mathsf{L}_1, \mathsf{L}_2 \rangle$ is finitely axiomatizable.

To prove this proposition it is enough to show that the set of formulas Δ has only finitely many non-equivalent formulas.

9 Derivational semantics

The derivational semantics studied by many authors (see, e.g. [1, 14]) can be equivalently defined as follows.

Let $\mathfrak{X} = (X, T)$ be a topological space. We define

$$\tau_d^{\mathfrak{X}}(x) = \left\{ U \;\middle|\; U' \setminus \{x\} \subseteq U, x \in U' \in T \right\}.$$

Then for any valuation V on X the following holds

$$\mathfrak{X}, V, x \models_d A \iff \left(X, \tau_d^{\mathfrak{X}}\right), V, x \models_n A,$$

where $\models_d$ corresponds to the derivational semantics, and $\models_n$ corresponds to the neighbourhood semantics. We define $\mathcal{N}_d(\mathfrak{X}) = (X, \tau_d^{\mathfrak{X}})$.

For a class of topological spaces $\mathcal{C}$ and logics L_1 and L_2 we put

$$Log_d(\mathcal{C}) = \left\{ A \;\middle|\; \forall \mathfrak{X} \in \mathcal{C}(\mathfrak{X} \models_d A) \right\},$$
$$\mathsf{L}_1 \times_d \mathsf{L}_2 = Log_d\left(\left\{ \mathfrak{X}_1 \times_t \mathfrak{X}_2 \mid \mathfrak{X}_1, \mathfrak{X}_2 \text{ — topological spaces}, \mathfrak{X}_1 \models_d \mathsf{L}_1, \mathfrak{X}_2 \models_d \mathsf{L}_2 \right\}\right).$$

Here $\times_t$ is the bitopological product defined in [17].

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We say that $Log_d\left(\mathcal{C}\right)$ is the *d-logic* of $\mathcal{C}$.

THEOREM 9.1

1. $\mathsf{K4} \times_d \mathsf{K4} = \left\langle \mathsf{K4}, \mathsf{K4} \right\rangle$,
2. $\mathsf{K4} \times_d \mathsf{D4} = \left\langle \mathsf{K4}, \mathsf{D4} \right\rangle$,
3. $\mathsf{D4} \times_d \mathsf{K4} = \left\langle \mathsf{D4}, \mathsf{K4} \right\rangle$,
4. $\mathsf{D4} \times_d \mathsf{D4} = \mathsf{D4} * \mathsf{D4}$.

PROOF. This follows from Theorems 7.11 and 8.4. But, it is not a straightforward corollary because for a logic L the set of L-n-frames and the set of all n-frames that correspond to L-topological spaces do not coincide. Indeed, in a topological space $\mathfrak{X} = \left(X, T\right)$ the family of $\tau_d^{\mathfrak{X}}$-neighbourhoods of a point x always contains set $X \setminus \{x\}$ and it is not the case for n-frames.

So to prove this theorem it is sufficient to say that all the logics mentioned in this theorem are not reflexive and the unravellings are irreflexive. So let $F = \left(W, R\right)$, then $F^\sharp$ is irreflexive, and $\mathcal{N}_\omega^\Gamma\left(F^\sharp\right)$ can be obtained as $\mathcal{N}_d\left(\mathfrak{X}\right)$, where $\mathfrak{X} = \left(W_\omega, T\right)$, Γ is the Horn sentence expressing transitivity, and sets U_n^Γ form the base for topology T. $\square$

THEOREM 9.2
The d-logic of the class of all products of T_1 spaces is $\left\langle \mathsf{K4}, \mathsf{K4} \right\rangle$.

It is enough to check that the topological space corresponding to $\mathcal{N}_\omega^\Gamma\left(F\right)$ is a T_1 space, whenever F is the unravelling of a rooted $\mathsf{S4}$-frame and Γ corresponds to transitivity. This can be easily checked.

10 Conclusions

We are still in the beginning of the road of studying products of neighbourhood frames.

This topic can be interesting from different points of view. It is interesting in itself because it is a natural way to combine modal logics, and the result is weaker then the product of logics based on Kripke semantics. It is also interesting because using the products we can express new properties, e.g. $\mathbb{Q}$ and $\mathbb{R}$ are indistinguishable in the unimodal language with topological semantics, whereas the logics of $\mathbb{Q} \times_t \mathbb{Q}$ and $\mathbb{R} \times_t \mathbb{R}$ are different (see [5]). It is also possible that this construction will be useful for epistemic modal logic as semantics for multi-agents systems.

There are a lot of open questions in this area, to name a few:

- find other sufficient conditions for product matching;
- investigate products of type $\mathsf{L} \times_n \mathsf{S5}$; we give a partial answer to this question in a forthcoming paper: for any HTC-logic L (this result is announced at Advances in Modal Logic '16 conference);
- find the logics of $\mathbb{R} \times_t \mathbb{R}$ and $\mathcal{C} \times_t \mathcal{C}$, where $\mathcal{C}$ is the Cantor space;
- find the n-products of well-known logics like $\mathsf{S4.1}$, $\mathsf{S4.2}$, $\mathsf{S4.3}$, GL, Grz, DL and others.

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