On neighbourhood product of some Horn axiomatizable logics

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Abstract

The paper considers modal logics of products of neighbourhood frames. The n-product of modal logics is the logic of all products of neighbourhood frames of the corresponding logics. We find the n-product of any two pretransitive Horn axiomatizable logics. As a corollary, we find the d-logic of products of topological spaces from some classes of topological spaces.

Keywords: Neighbourhood semantics, product of modal logics, Horn sentences, topological semantics.

1 Introduction

The neighbourhood semantics is a generalization of Kripke semantics and topological semantics. It was introduced by Dana Scott in [12] and Richard Montague in [9] independently. In this paper, we will consider the product of neighbourhood frames introduced by Sano in [11]. It is a generalization of the product of topological spaces1 presented in [17].

The product of neighbourhood frames is defined in the same manner as the product of Kripke frames (see [14] and [16]). But there are some differences. The axioms of commutativity and Church–Rosser property are valid in any product of Kripke frames. Whereas in [17] it was shown that the logic of the products of all topological spaces is the fusion2 of logics $S_4 \ast S_4$. Moreover, $S_4 \ast S_4$ is complete w.r.t. the product $\mathbb{Q} \times \mathbb{Q}$ ($\times$ stands for the product of topological spaces defined in [17]).

In [7] it was proved that, for any pair $L$ and $L'$ of logics from $\{S_4, D_4, D, T\}$, the modal logic of the family of products of $L$-neighbourhood frames and $L'$-neighbourhood frames is the fusion of $L$ and $L'$. But at that point it was unclear how to proceed in the case of logics that does not contain the seriality axiom $\Diamond T$. In [8] it was shown that, for any variable-free and $\Box_2$-free formula $B$, formula $B \rightarrow \Box_2 B$ is valid in any product of neighbourhood frames (and the same holds for $B' \rightarrow \Box_1 B'$, where $B'$ is variable-free and $\Box_1$-free). It was also proved that $K \ast K$ plus all these formulas is the logic of all products of neighbourhood frames. For any two modal logics $L_1$ and $L_2$ we can define $\langle L_1, L_2 \rangle$ as $L_1 \ast L_2$ plus all the formulas from the above. For details, see Definition 3.8.

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1'Product of topological spaces' is a well-known notion in Topology, but here we use a different definition (for details see [17]).

2Some authors (see [5, 17]) use $\oplus$ for the fusion.
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In this paper, we will find a sufficient conditions for two logics $L_1$ and $L_2$ to be $n$-product matching. Two logics are called $n$-product matching if $L_1 \times_n L_2 = \langle L_1, L_2 \rangle$, where $L_1 \times_n L_2$ is the logic of all products of neighbourhood frames $X_1 \times X_2$ such that $X_1 \models L_1$ and $X_2 \models L_2$.

Neighbourhood frames are often considered in the context of non-normal modal logics. Since many non-normal logics are complete w.r.t. neighbourhood semantics. Examples of Kripke incomplete normal modal logics that are complete w.r.t. neighbourhood semantics are rare and usually artificial. This paper, however, shows that in the case of the products neighbourhood frames give different results from Kripke frames even in case of normal modal logics. To be precise this paper (and others: [7, 11, 17]) shows that ‘neighbourhood’ product, in general, generates a weaker logic in comparison to ‘Kripke’ product. It also shows how the notion of the product of modal logics depends on the underlining semantics.

We also prove some corollaries for the derivational semantics of topological spaces. In particular the logic of all products of all $T_1$ spaces is $\langle K4, K4 \rangle$. What is the logic of all products of all topological products is still unknown.

2 Language, logics and semantics

In this paper, we study propositional modal logics. A formula is defined recursively by using the Backus–Naur form as follows:

$$\phi ::= p \mid \bot \mid (\phi \rightarrow \phi) \mid \Box_i \phi,$$

where $p \in \text{PROP}$ is a propositional letter and $\Box_i$ is a modal operator ($i = 1, \ldots, n$). Other connectives are introduced as abbreviations: classical connectives are expressed through $\bot$ and $\rightarrow$, and dual modal operators $\Diamond_i$ are expressed as $\neg \Box_i \neg$. The set of all modal formulas is denoted by $\mathcal{ML}_n$, and in order to specify the modalities used in the language we write them in subindex, e.g. $\mathcal{ML}_{\Box_1}$ or $\mathcal{ML}_{\Box_2}$.

**Definition 2.1**

A normal modal logic (or a logic, for short) is a set of modal formulas closed under Substitution $\left( \frac{A}{A'} \right)$, Modus Ponens $\left( \frac{A \rightarrow B}{A, A \rightarrow B} \right)$ and Generalization rules $\left( \frac{A}{\Box_i A} \right)$, containing all the classical tautologies and the normality axioms:

$$\Box_i (p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q).$$

$K_n$ denotes the minimal normal modal logic with $n$ modalities and $K = K_1$.

Let $L$ be a logic and $\Gamma$ a set of formulas, then $L + \Gamma$ denotes the minimal logic containing $L$ and $\Gamma$. If $\Gamma = \{A\}$, then we write $L + A$ rather than $L + \{A\}$.

**Definition 2.2**

A formula $B$ is called closed if it has no variables.

**Definition 2.3**

Let $L_1$ and $L_2$ be two modal logics with one modality $\Box$ (unimodal logics), then the fusion of these logics is the following modal logic with 2 modalities:

$$L_1 \ast L_2 = K_2 + L'_1 + L'_2,$$
where $L'_i$ is the set of all formulas from $L_i$ in which all instances of $\Box$ are replaced by $\Box_i$.

**Definition 2.4**

Let $R \subseteq W \times W$ be a relation on $W \neq \emptyset$, then for $n \geq 1$ and $w \in W$ we define

$$R^0 = Id_w = \{(w, w) \mid w \in W\},$$

$$R^{n+1} = R^n \circ R,$$

$$R^* = \bigcup_{k=0}^{\infty} R^k,$$

$$R(w) = \{u \mid wRu\}.$$

Notice that $R^*$ is the reflexive transitive closure of $R$.

A Kripke frame with $n$ relations is a tuple $F = (W, R_1, \ldots, R_n)$, where $W$ is a non-empty set and $R_i \subseteq W \times W$ is a relation on $W$ for each $i \in \{1, \ldots, n\}$.

**Remark 2.5**

We will sometimes write $w \in F$ as a shortcut for $w \in W$ and $F = (W, R_1, \ldots, R_n)$.

A frame $F$ with a valuation $V : PROP \rightarrow 2^W$ is called a model $M = (F, V)$.

For a Kripke frame $F = (W, R_1, \ldots, R_n)$ we define the subframe generated by $w \in W$ as the frame $F^w = (W', R_1|_{w'}, \ldots, R_n|_{w'})$, where $W' = (R_1 \cup \ldots \cup R_n)^\circ (w)$ and $R_i|_{w'} = R_i \cap W' \times W'$. A frame $F$ is called rooted if $F = F^w$ for some $w$.

The truth of a formula in a model $M$ at a point $x \in W$ is defined, as usual, by induction on the length of the formula:

- $M, x \not \models \bot$,
- $M, x \models p$ $\iff x \in V(p)$;
- $M, x \models A \rightarrow B$ $\iff M, x \not \models A$ or $M, x \models B$;
- $M, x \models \Box_r A$ $\iff \forall y \,(xR_{ry} \Rightarrow M, y \models A)$.

A formula is true in a (Kripke) model $M$ if it is true at all points of $M$ (notation $M \models A$). A formula is valid on a (Kripke) frame $F$ if it is true in all models based on $F$ (notation $F \models A$). We write $F \models L$ if, for any $A \in L$, $F \models A$. The logic of a class of Kripke frames $C$ is $\text{Log}(C) = \{A \mid F \models A$ for all $F \in C\}$. For a logic $L$ we also define $\mathcal{V}(L) = \{F \mid F$ is a Kripke frame and $F \models L\}$. Note that if there is no $F$ such that $F \models L$, then $\mathcal{V}(L) = \emptyset$.

**Definition 2.6**

Let $F = (W, R_1, \ldots, R_n)$ and $G = (U, S_1, \ldots, S_n)$ be Kripke frames. A function $f : W \rightarrow U$ is called a $p$-morphism (Notation: $f : F \rightarrow G$) if

1. $f$ is surjective;
2. [monotonisity] for any $w, v \in W$ $wR_iv$ implies $f(w)S_if(v)$;
3. [lifting] for any $w \in W$ and $v' \in U$ such that $f(w)S_iv'$ there exists $v \in W$ such that $wR_iv$ and $f(v) = v'$.
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The following p-morphism lemma is well known (see [2, Proposition 2.14]):

**Lemma 2.7**
Let $f : F \rightarrow G$ and $V$ be a valuation on $G$. We define a valuation on $F$ by $[f^{-1}(V)](p) = f^{-1}(V(p))$.
Then for any $w \in F$ and any formula $A$

$$F, f^{-1}(V), w \models A \iff G, V, f(w) \models A.$$  

The following is a straightforward corollary.

**Corollary 2.8**
If $f : F \rightarrow G$, then $\text{Log}(F) \subseteq \text{Log}(G)$.

For a consistent modal logic $L$ with $n$ modalities we define the canonical model (c.f. [2]) $M_L = (F_L, V_L)$, where $F_L = (W, R_1, \ldots, R_n)$ such that

$$W = \{x \mid x \text{ is an } L\text{-complete set of formulas}\},$$

$$xR_i y \iff \forall A (\Box_i A \in x \Rightarrow A \in y),$$

$$x \in V(p) \iff p \in x.$$  

The classical result on canonical models is

**Lemma 2.9**
For any formula $A$ and any consistent logic $L$

$$M_L, x \models A \iff A \in x.$$  

We also define 0-canonical frame $\mathcal{F}_0^0$ being the counterparts of canonical frame in the modal language without variables. More precisely,

$$\mathcal{F}_0^0 = (W_0, R_1', \ldots, R_n'),$$

$$W_0 = \{\bar{x} \mid \bar{x} \text{ is an } L\text{-complete set of closed formulas}\},$$

$$\bar{x}R_i\bar{y} \iff \forall A (\Box_i A \in \bar{x} \Rightarrow A \in \bar{y}).$$  

Note that there are no 0-canonical models since there are no variables in closed formulas. So the lemma for canonical model transforms into

**Lemma 2.10**
For any closed formula $A$ and any logic $L$

$$\mathcal{F}_0^0, \bar{x} \models A \iff A \in \bar{x}.$$  

Here we describe a construction of continuum unravelling. It is similar to the construction in [4, Lemma 4.9].
DEFINITION 2.11
Let $F = (W, R) = F^{w_0}$ be a rooted Kripke frame, $S$ be a non-empty set and $x_0 \in S$ be a point in it (the starting point). Then

$$F \cdot S = (W \times S, R \cdot S);$$

$$(w, x) R \cdot S (v, y) \iff w R v;$$

$$F_S = (F \cdot S)^{(w_0, x_0)} = (W_S, R_S)$$

—a rooted subframe. $F_S$ is called the thickening of $F$ by $S$.

The proof of the following lemma is straightforward.

LEMMA 2.12
The first projection $p_1(w, x) = w$ is a p-morphism $p_1 : F_S \rightarrow F$.

The following construction is well known (c.f. [2]).

DEFINITION 2.13
Let $F = (W, R_1, \ldots, R_n) = F^{w_0}$ be a rooted Kripke frame.

We define the unravelling of it and a map $\pi$ as follows

$$F^\sharp = (W^\sharp, R_1^\sharp, \ldots, R_n^\sharp);$$

$$W^\sharp = \{w_0 R_{j_1} w_1 \ldots R_{j_m} w_m \ | \ \forall i \in \{1, \ldots, m\} (w_{i-1} R_{j_i} w_i) \};$$

$$\pi(w_0 w_1 \ldots w_m) = w_m, \quad (\pi : W^\sharp \rightarrow W);$$

$$\alpha R_j^\sharp \beta \iff \beta = \alpha w_{m+1} \text{ and } \pi(\alpha) R_j \pi(\beta).$$

LEMMA 2.14
The map $\pi$ is a p-morphism: $\pi : F^\sharp \rightarrow F$.

The proof is straightforward.

DEFINITION 2.15
Let $F = F^{w_0}$ be a rooted frame. Then we define the continuum unravelling of it as $F^\sharp_{\mathbb{R}} = (F_{\mathbb{R}})^\sharp$ (the unravelling of the thickening by $\mathbb{R}$ with 0 as the starting point).

Now we turn to neighbourhood frames (c.f. [13] and [3] or a recent book [10]).

DEFINITION 2.16
Let $X$ be a non-empty set, then $\mathcal{F} \subseteq 2^X$ is a filter on $X$ if

1. $X \in \mathcal{F}$;
2. if $U_1, U_2 \in \mathcal{F}$, then $U_1 \cap U_2 \in \mathcal{F}$;
3. if $U_1 \in \mathcal{F}$ and $U_1 \subseteq U_2$, then $U_2 \in \mathcal{F}$. 
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It is usually required that $\emptyset \notin \mathcal{F}$ ($\mathcal{F}$ is a proper filter), but we will not require this in our paper.

**Definition 2.17**

A (normal) neighbourhood frame (or an n-frame) is a pair $\mathfrak{X} = (X, \tau)$, where $X$ is a non-empty set and $\tau : X \rightarrow 2^2X$ such that $\tau(x)$ is a filter on $X$ for any $x$. The function $\tau$ is called the neighbourhood function of $\mathfrak{X}$, and sets from $\tau(x)$ are called neighbourhoods of $x$. A neighbourhood model (n-model) is a pair $(\mathfrak{X}, V)$, where $\mathfrak{X} = (X, \tau)$ is an n-frame and $V : PROP \rightarrow 2^X$ is a valuation. In a similar way, we define neighbourhood 2-frame (n-2-frame) as $(X, \tau_1, \tau_2)$ such that $\tau_i(x)$ is a filter on $X$ for any $x$, and an n-2-model.

**Remark 2.18**

Many authors consider neighbourhood semantics for non-normal and even non-monotone logics. In this case a set of neighbourhoods can be an arbitrary set of sets. Other authors consider monotone neighbourhood frames and require only item 3 from Definition 2.16 (a set of neighbourhoods is closed under supersets).

**Definition 2.19**

The valuation function in an n-model can be extended to all formulas by induction. For Boolean connectives the definition is as usual, so we omit it. For modalities the definition is as follows:

$$M, x \models \Box_i A \iff \exists U \forall y(y \in U \in \tau_i(x) \Rightarrow M, y \models A).$$

A formula is true in an n-model $M$ if it is valid at all points of $M$ (notation $M \models A$). A formula is valid on an n-frame $\mathfrak{X}$ if it is true in all models based on $\mathfrak{X}$ (notation $\mathfrak{X} \models A$). We write $\mathfrak{X} \models L$ if for any $A \in L$, $\mathfrak{X} \models A$. We define the logic of a class of n-frames $\mathcal{C}$ as $\text{Log}(\mathcal{C}) = \{A \mid \mathfrak{X} \models A \text{ for all } \mathfrak{X} \in \mathcal{C}\}$. For a logic $L$ we also define $\mathcal{Vn}(L) = \{\mathfrak{X} \mid \mathfrak{X} \text{ is an n-frame and } \mathfrak{X} \models L\}$. Note that if there is no $\mathfrak{X}$ such that $\mathfrak{X} \models L$, then $\mathcal{Vn}(L) = \emptyset$.

**Definition 2.20**

Let $F = (W, R)$ be a Kripke frame. We define the n-frame $\mathcal{N}(F) = (W, \tau)$ in the following way

$$\tau(w) = \{U \mid R(w) \subseteq U \subseteq W\}.$$

**Lemma 2.21**

Let $F = (W, R)$ be a Kripke frame. Then

$$\text{Log}(\mathcal{N}(F)) = \text{Log}(F).$$

The proof is straightforward (see [3] or [10]).

**Definition 2.22**

Let $\mathfrak{X} = (X, \tau_1, \ldots)$ and $\mathfrak{Y} = (Y, \sigma_1, \ldots)$ be n-frames. Then a function $f : X \rightarrow Y$ is a p-morphism (notation $f : \mathfrak{X} \rightarrow \mathfrak{Y}$) if

1. $f$ is surjective;
2. for any $x \in X$ and $U \in \tau_i(x)$, we have $f(U) \in \sigma_i(f(x))$;
3. for any $x \in X$ and $V \in \sigma_i(f(x))$, we have $f^{-1}(V) \in \tau_i(x)$.
REMARK 2.23
According to Lemma 2.21, a Kripke frame is a special case of a neighbourhood frame. It is easy to check that for any two Kripke frames \( F \) and \( G \) a function \( f \) is a p-morphism (Definition 2.6) from \( F \) to \( G \) iff \( f \) is a p-morphism (Definition 2.22) from \( N(F) \) to \( N(G) \). So, a p-morphism for n-frames is a natural generalization of the notion of p-morphism for Kripke frames. This is why we use the same name for these two formally different notions.

LEMMA 2.24
Let \( X = (X, \tau_1, \ldots) \), \( Y = (Y, \sigma_1, \ldots) \) be n-frames and \( f : X \rightarrow Y \). Let \( V \) be a valuation on \( Y \). We define \( [ f^{-1}(V) ] (p) = f^{-1}(V(p)) \). Then
\[
X, f^{-1}(V), x \models A \iff Y, V, f(x) \models A.
\]
The proof is by induction on the length of formula \( A \) (c.f. [10]). The following is a straightforward corollary.

COROLLARY 2.25
If \( f : X \rightarrow Y \), then \( \text{Log}(X) \subseteq \text{Log}(Y) \).

3 Products: from Kripke to neighbourhood frames

DEFINITION 3.1
Let \( F_i = (W_i, R_i) \) \((i = 1, 2)\) be two Kripke frames. We define their product (see [4]) as a frame with two relations: \( F_1 \times F_2 = (W_1 \times W_2, R_1^h, R_2^v) \), where
\[
(x, y)R_1^h(z, t) \iff xR_1z \land y = t;
\]
\[
(x, y)R_2^v(z, t) \iff x = z \land yR_2t.
\]

DEFINITION 3.2
Let \( X_1 = (X_1, \tau_1) \) and \( X_2 = (X_2, \tau_2) \) be two n-frames. Then the product of these n-frames is the n-2-frame defined as follows:
\[
X_1 \times X_2 = (X_1 \times X_2, \tau_1^h, \tau_2^v);
\]
\[
\tau_1^h(x_1, x_2) = \{ U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_1(x_1) \land V \times \{x_2\} \subseteq U) \};
\]
\[
\tau_2^v(x_1, x_2) = \{ U \subseteq X_1 \times X_2 \mid \exists V (V \in \tau_2(x_2) \land \{x_1\} \times V \subseteq U) \}.
\]

REMARK 3.3
Note that \( \tau_1^h(x_1, x_2) \) and \( \tau_2^v(x_1, x_2) \) are closed under supersets. So it is possible to generalize the definition of the product to monotone neighbourhood frames. This was done in [11]. However, we consider only normal n-frames in this paper, because we use Kripke semantics in the proof of completeness.
DEFINITION 3.4
For two unimodal logics \( L_1 \) and \( L_2 \) such that \( \forall n(L_1) \neq \emptyset \) and \( \forall n(L_2) \neq \emptyset \), we define the \( n \)-product of them as follows:

\[
L_1 \times_n L_2 = \text{Log} \left( \{ \mathcal{X}_1 \times \mathcal{X}_2 \mid \mathcal{X}_1 \in \forall n(L_1) \land \mathcal{X}_2 \in \forall n(L_2) \} \right).
\]

If we forget about one of its neighbourhood functions, say \( \tau^2 \), then \( \mathcal{X}_1 \times \mathcal{X}_2 \) will be the disjoint union of \( L_1 \) \( n \)-frames.  

Hence,

PROPOSITION 3.5 ([11])
For two unimodal normal logics \( L_1 \) and \( L_2 \)

\[
L_1 \ast L_2 \subseteq L_1 \times_n L_2.
\]

In Section 8 we will show that for many logics with the seriality axiom (including \( S4, D4, T, D \)) their \( n \)-product coincides with the fusion. But this is not the case for logic \( K \):

PROPOSITION 3.6
\( K \times_n K \neq K \ast K \).

PROOF. Let \( \mathcal{X}_1 = (X_1, \tau_1) \) and \( \mathcal{X}_2 = (X_2, \tau_2) \) be two \( n \)-frames and \( \mathcal{X}_1 \times \mathcal{X}_2 = (X_1 \times X_2, \tau^h_1, \tau^v_2) \).
Consider formula \( \square_1 \bot \rightarrow \square_2 \square_1 \bot \). It is non-derivable from \( K_2 \) but valid in any product of two \( n \)-frames. Indeed, consider a Kripke frame \( F = (W, R_1, R_2) \), where

\[
W = \{x_1, x_2, y\},
\]

\[
x_1 R_2 x_2,
\]

\[
R_1(x_1) = \emptyset,
\]

\[
x_2 R_1 y.
\]

Then \( x_2 \models \neg \square_1 \bot, x_1 \models \square_1 \bot \) and \( x_1 \models \neg \square_2 \square_1 \bot \).

Since this formula has no variables, the truth of this formula does not depend on the valuation. So

\[
\mathcal{X}_1 \times \mathcal{X}_2, (x, y) \models \square_1 \bot \iff \emptyset \in \tau^h_1(x, y) \iff \emptyset \in \tau_1(x) \iff \forall y' \in X_2 \left( \emptyset \in \tau^h_1(x, y') \right) \iff \forall y' \in X_2 \left( \mathcal{X}_1 \times \mathcal{X}_2, (x, y') \models \square_1 \bot \right) \implies \mathcal{X}_1 \times \mathcal{X}_2, (x, y) \models \square_2 \square_1 \bot.
\]

Hence, \( \mathcal{X}_1 \times \mathcal{X}_2 \models \square_1 \bot \rightarrow \square_2 \square_1 \bot \).  

Moreover,

\[
\text{Indeed an } n\text{-frame } (\mathcal{X}_1 \times \mathcal{X}_2, \tau^h_1) \text{ is isomorphic to the disjoint union of copies of } (\mathcal{X}_1, \tau_1). \text{ Neighbourhoods in the union are all supersets of all neighbourhoods of its components.} 
\]
Lemma 3.7
For any two n-frames $\mathcal{X}_1$ and $\mathcal{X}_2$ (i) if $B$ is a closed formula without $\Box_2$, then for any two n-frames $\mathcal{X}_1$ and $\mathcal{X}_2$

$$\mathcal{X}_1 \times \mathcal{X}_2 \models B \rightarrow \Box_2 B,$$

(ii) if $B$ is a closed formula without $\Box_1$, then

$$\mathcal{X}_1 \times \mathcal{X}_2 \models B \rightarrow \Box_1 B.$$

Proof. We prove only (i) because (ii) can be proved analogously. Since $B$ contains neither $\Box_2$, nor variables, its value does not depend on the second coordinate. Let $F = \mathcal{X}_1 \times \mathcal{X}_2$. So if $F, (x, y) \models B$, then $\forall y' (F, (x, y') \models B)$, hence, $F, (x, y) \models \Box_2 B$. □

Definition 3.8
We put

$$\Delta_1 = \{B_1 \rightarrow \Box_2 B_1 \mid B_1 \text{ is closed and } \Box_2\text{-free}\};$$

$$\Delta_2 = \{B_2 \rightarrow \Box_1 B_2 \mid B_2 \text{ is closed and } \Box_1\text{-free}\};$$

$$\Delta = \Delta_1 \cup \Delta_2.$$

For two unimodal logics $L_1$ and $L_2$, we define the weak commutator of them as

$$\langle L_1, L_2 \rangle = L_1 \ast L_2 + \Delta.$$

From Lemma 3.7 and Proposition 3.5 follows

Lemma 3.9
For any two normal modal logics $L_1$ and $L_2$, $L_1 \parallel L_2 \subseteq L_1 \times_n L_2$.

Corollary 3.10
$\langle K, K \rangle \subseteq K \times_n K$.

The converse inclusion also holds but the proof requires some work.

4 Dense neighbourhood frames
To prove the completeness of a logic w.r.t. neighbourhood frames we rely on Kripke completeness. It is possible that using purely neighbourhood semantics constructions one can prove more general results for non-normal logics. But we leave this for the future.

Starting from a Kripke frame we will construct a neighbourhood frame so that the neighbourhood frame will be dense. An n-frame is called dense if no point in it has a minimal neighbourhood. This is important, because otherwise n-frames will be equivalent to Kripke frames, and any product of Kripke frames satisfies the commutativity axioms and the Church–Rosser axiom.
DEFINITION 4.1
Let \( \Sigma \) be a non-empty finite set (alphabet). A finite sequence of elements from \( \Sigma \) is called a word; by \( \epsilon \) we denote the empty word. Let \( \Sigma^* \) be the set of all words. We will write words without brackets or commas, e.g. \( a_1a_2\ldots a_n \in \Sigma^* \). The length of a word is the number of elements in it:

\[
\text{len}(a_1a_2\ldots a_n) = n, \quad \text{len}(\epsilon) = 0.
\]

We also define the concatenation of words:

\[
a_1a_2\ldots a_n \cdot b_1b_2\ldots b_m = a_1a_2\ldots a_nb_1b_2\ldots b_m.
\]

DEFINITION 4.2
For a frame \( F = (W, R) \) with a fixed root \( a_0 \) we define a (rooted) path with stops as a word in alphabet \( W \cup \{0\} : a_1\ldots a_n \), so that \( a_i \in W \) or \( a_i = 0 \), and after dropping zeros, each point is related to the next one by relation \( R \), and the first one is a successor of the root. The empty word \( \epsilon \) is allowed and it corresponds to the root \( a_0 \). In other words, a path with stops is a word of the following type:

\[
0^{i_1}b_10^{i_2}\ldots 0^{i_m}b_m, \text{ where } b_j \in W, \quad i_j \geq 0, \quad 0^j = 00\ldots 0; \text{ } \text{ }\text{ }\text{ } i \text{ times}
\]

and \( f_0 (0^{i_1}b_10^{i_2}\ldots 0^{i_m}b_m) = a_0Rb_1R\ldots Rb_m \in W^\omega \).

Let us consider some examples. \( f_0(\epsilon) = a_0 \). If the root is reflexive, then \( a_0 \) is a path with stops, and \( f_0(a_0) = a_0Ra_0 \). In the frame \( (W, R), W = \{a_0, b\}, R = \{(a_0, b)\} \) the following are paths with stops: \( \epsilon, 000, 00b, 0b00 \).

We also consider infinite paths with stops that end with infinitely many zeros. We call these sequences pseudo-infinite paths (with stops). A pseudo-infinite path can be presented uniquely in the following way:

\[
\alpha = 0^{i_1}b_10^{i_2}\ldots 0^{i_m}b_m0^\omega, \text{ where } b_j \in W, \quad i_j \geq 0.
\]

Let \( W_\omega \) be the set of all pseudo-infinite paths in \( W \).

The function of dropping zeros can be extended to \( W_\omega \) as \( f_0 : W_\omega \to W^\omega \) in the following way: for a pseudo-infinite path \( \alpha = a_1\ldots a_n\ldots \) we define

\[
\text{st}(\alpha) = \min \left\{ N \mid \forall k > N(a_k = 0) \right\};
\]

\[
\alpha|_k = a_1\ldots a_k; \quad \alpha|_0 = \epsilon;
\]

\[
f_0(\alpha) = f_0(\alpha|_{\text{st}(\alpha)}), \text{ i.e. } f_0(0^{i_1}b_10^{i_2}\ldots 0^{i_m}b_m0^\omega) = a_0Rb_1R\ldots Rb_m.
\]

In order to introduce a neighbourhood function on \( W_\omega \), we define

\[
U_k(\alpha) = \left\{ \beta \in W_\omega \mid \alpha|_m = \beta|_m \& f_0(\alpha)R^\omega f_0(\beta), \quad m = \max(k, \text{st}(\alpha)) \right\}.
\]

LEMMA 4.3
\( U_k(\alpha) \subseteq U_m(\alpha) \) whenever \( k \geq m \).

PROOF. Let \( \beta \in U_k(\alpha) \). Since \( \alpha|_k = \beta|_k \) and \( k \geq m \), \( \alpha|_m = \beta|_m \). Hence, \( \beta \in U_m(\alpha) \). \( \square \)
DEFINITION 4.4
Due to Lemma 4.3, sets $U_n(\alpha)$ form a filter base. So we can define

\[ \tau(\alpha) — \text{the filter with the base } \{U_n(\alpha) \mid n \in \mathbb{N}\}; \]

\[ \mathcal{N}_\omega(F) = (W_\omega, \tau) — \text{is a dense n-frame based on } F. \]

The frame $\mathcal{N}_\omega(F)$ is dense unlike $\mathcal{N}(F)$. Indeed,

\[ \bigcap_n U_n(\alpha) = \emptyset \not\in \tau(\alpha). \]

LEMMA 4.5
For any $\alpha$ and any $k$ we have that $f_0(U_k(\alpha)) = R^2(f_0(\alpha))$.

PROOF.
By definition for $k < st(\alpha)$ $U_k(\alpha) = U_{st(\alpha)}(\alpha)$, hence we can assume that $k \geq st(\alpha)$.

Let $x \in f_0(U_k(\alpha))$. Then

\[ \exists \beta (\alpha |_k = \beta |_k & f_0(\alpha)R^2f_0(\beta) = x), \]

so $x \in R^2(f_0(\alpha))$.

Let $x \in R^2(f_0(\alpha))$, then for some $b' x = f_0(\alpha) Rb'$. We put

\[ \beta = \alpha^k - st(\alpha) b'. \]

It is easy to show that $\beta \in U_k(\alpha)$ and $f_0(\beta) = x$. \(\square\)

LEMMA 4.6
Let $F = (W, R)$ be a Kripke frame with root $a_0$, then

\[ f_0 : \mathcal{N}_\omega(F) \to \mathcal{N}(F^\omega). \]

PROOF. From now on in this proof we will omit the subindex in $f_0$. Since for any $b \in W$ there is a path $a_0a_1...a_{n-1}b$ and hence for a pseudo-infinite path $\alpha = a_1...b0^\omega \in X$, $f(\alpha) = b$ and $f$ is surjective.

Assume that $\alpha \in W_\omega$ and $U \in \tau(\alpha)$. We need to prove that $R^2(f(\alpha)) \subseteq f(U)$. There exists an $m$ such that $U_m(\alpha) \subseteq U$, and since $f(U_m(\alpha)) = R^2(f(\alpha))$, we have

\[ R^2(f(\alpha)) = f(U_m(\alpha)) \subseteq f(U). \]

Assume that $\alpha \in W_\omega$ and $V$ is a neighbourhood of $f(\alpha)$, i.e. $R^2(f(\alpha)) \subseteq V$. We need to prove that there exists $U \in \tau(\alpha)$ such that $f(U) \subseteq V$. For $U$, we take $U_m(\alpha)$ for some $m \geq st(\alpha)$, then

\[ f(U_m(\alpha)) = R^2(f(\alpha)) \subseteq V. \]

\(\square\)

COROLLARY 4.7
For any frame $F$, $\text{Log} \left( \mathcal{N}_\omega(F) \right) \subseteq \text{Log} (F)$. 

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On neighbourhood product of some Horn axiomatizable logics

PROOF. It follows from Lemmas 2.21, 4.6, 2.14 and Corollary 2.25 that

\[ \text{Log} \left( N_\omega(F) \right) \subseteq \text{Log} \left( N^2(F) \right) = \text{Log} \left( F^2 \right) \subseteq \text{Log} \left( F \right). \]

Let us remark that it is possible that \( \text{Log} \left( N_\omega(F) \right) \neq \text{Log} \left( F \right) \). For example, let us consider the natural numbers with the ‘next’ relation. It is convenient here to regard a number as a word in a one-letter alphabet:

\[ G = ( \{ 1 \}^*, S ), \quad 1^n 1^m \iff m = n + 1. \]

Obviously \( G \models \lozenge p \rightarrow \square p \).

Since in \( G \) every point, except for the root point, has only one predecessor, we can identify a point and a path from the root to this point, i.e. \( G^2 = G \). Therefore, points in \( N_\omega(G) \) can be presented as infinite sequences of 0 and 1 with only zeros at the end.

PROPOSITION 4.8

\( N_\omega(G) \not\models \lozenge p \land \lozenge \neg p \)

PROOF. Consider valuation \( V(p) = \{ 0^{2n} 1^{10^m} \mid n \in \mathbb{N} \} \). In every neighbourhood of point 0\(^\omega\) there are points, where \( p \) is true and there are points where \( p \) is false. For example, if \( k \) is even, then in \( U_k(0^\omega) \) there is a point \( 0^k 1^{10^m} \) where \( p \) is true, and a point \( 0^{k+1} 1^0 \) where \( p \) is false. Hence,

\[ N_\omega(G) \models \lozenge p \land \lozenge \neg p. \]

This formula does not preserved under the \( N_\omega \) operation and probably any formula that restricts branching does not preserved under the \( N_\omega \) operation. In Section 7 we define some formulas that are preserved.

5 Weak product of Kripke frames

In order to prove the completeness w.r.t. n-frames, we first establish completeness w.r.t. a special kind of Kripke frames. The weak products of Kripke frames were introduced in [8] for this purpose. Here we modify the definition. Nevertheless frames from the new construction are isomorphic to frames from the old one, but their description is better in some respect.

DEFINITION 5.1

Let \( F_1 = (W_1, R_1) \) and \( F_2 = (W_2, R_2) \) be two Kripke frames with the roots \( x_0 \) and \( y_0 \), respectively, and \( W_1 \cap W_2 = \emptyset \). Let \( \Sigma = W_1 \cup W_2 \). Then we define the functions \( p_1, p_2 : \Sigma^* \to \Sigma^* \) and \( \pi : \Sigma^* \setminus \{\epsilon\} \to \Sigma \) by induction

\[ p_1(\epsilon) = \epsilon, \]
\[ p_2(\epsilon) = \epsilon, \]
\[ p_1(au) = p_1(a) \cdot u \text{ for } a \in \Sigma^*, u \in W_1; \]
\[ p_2(au) = p_1(a) \text{ for } a \in \Sigma^*, u \in W_2; \]
\[ p_1(au) = p_2(a) \text{ for } a \in \Sigma^*, u \in W_1; \]
\[ p_2(au) = p_2(a) \cdot u \text{ for } a \in \Sigma^*, u \in W_2; \]
\[ \pi(au) = u \text{ for } a \in \Sigma^*, u \in \Sigma. \]
Intuitively $p_1$ is dropping all symbols not belonging to $W_1$ (the same for $p_2$) and $\pi$ maps a non-empty word to its final symbol.

Since $F_1$ and $F_2$ are frames with roots and have only one relation, we will assume that paths in them do not contain relations and start from the roots:

$$W_1^x = \{x_1 \ldots x_n \mid x_0 R_1 x_1 R_1 \ldots R_1 x_n \text{ is a path in the usual sense }\};$$
$$W_2^y = \{y_1 \ldots y_n \mid y_0 R_2 y_1 R_2 \ldots R_2 y_n \text{ is a path in the usual sense }\}.$$

We define the entanglement of $F_1$ and $F_2$ as follows:

$$F_1 \bowtie F_2 = \{a \in \Sigma^* \mid p_1(a) \in W_1^x \text{ and } p_2(a) \in W_2^y\}.$$  

We define the weak product of frames $F_1$ and $F_2$ as follows:

$$\langle F_1, F_2 \rangle = (F_1 \bowtie F_2, R_1^\prec, R_2^\prec),$$

$$a R_1^\prec b \iff \exists u \in W_1 (b = au);$$

$$a R_2^\prec b \iff \exists v \in W_2 (b = av).$$

**Proposition 5.2**

For any two rooted frames $F_1$ and $F_2$ $\langle F_1, F_2 \rangle \models \Delta$.

**Proof.** Let $B$ be a closed $\Box_2$-free formula and $\langle F_1, F_2 \rangle, a \models B$, then for any $v \in W_2$ we need to show that $\langle F_1, F_2 \rangle, av \models B$. Indeed, the frames

$$\left( (R_1^\prec)^*(a), R_1^\prec \right)$$

and

$$\left( (R_1^\prec)^*(av), R_1^\prec \right)$$

are isomorphic to $\left( F_1^x \right)^{p_1(a)}$. Then, since $B$ is closed and does not contain $\Box_2$, $\langle F_1, F_2 \rangle, av \models B$.

For $\Delta_2$ the proof is similar. □

The aim of this section is to prove the following theorem:

**Theorem 5.3**

The logic $\langle K, K \rangle$ is complete with respect to the class of all weak products of Kripke frames.

Let $F = F^{x_0}$ be a rooted subframe of the canonical frame of a logic $L$ with two modalities. By $\Upsilon$ we define all closed (variable-free) modal formulas of the modal language. For a point $x \in F$ we define $\bar{x} = x \cap \Upsilon$. Then let $F_0^{x_0}$ be a rooted subframe of the 0-canonical frame.

We define $\Upsilon_i$ as the set of all closed formulas in the language with only $\Box_i$ modality.
LEMMA 5.4
Let $\mathcal{F}_0 = (\bar{W}, \bar{R}_1, \bar{R}_2)$ be the 0-canonical frame for logic $L$, such that $\Delta_1 \subseteq L$. Then

$\bar{x}\bar{R}_1\bar{y} \Rightarrow \bar{x} \cap \gamma_2 = \bar{y} \cap \gamma_2,$

$\bar{x}\bar{R}_2\bar{y} \Rightarrow \bar{x} \cap \gamma_1 = \bar{y} \cap \gamma_1.$

PROOF.
We prove only one half, since the other half is similar.
For any $A \in \gamma_2$

$A \rightarrow \Box_1 A \in \Delta$ and $\neg A \rightarrow \Box_1 \neg A \in \Delta.$

So

$A \in \gamma_2 \cap \bar{x} \Rightarrow A \rightarrow \Box_1 A \in \bar{x} \Rightarrow A \in \bar{y},$

$A \in \gamma_2$ and $A \notin \bar{x} \Rightarrow \neg A \rightarrow \Box_1 \neg A \in \bar{x} \Rightarrow \neg A \in \bar{y} \Rightarrow A \notin \bar{y}.$

□

By a straightforward induction we get

COROLLARY 5.5
Let $\mathcal{F}_0 = (\bar{W}, \bar{R}_1, \bar{R}_2)$ be the 0-canonical frame for logic $L$, such that $\Delta \subseteq L$. Then

$\bar{x}(\bar{R}_1 \cup \bar{R}_1^{-1})^* \bar{y} \Rightarrow \bar{x} \cap \gamma_2 = \bar{y} \cap \gamma_2,$

$\bar{x}(\bar{R}_2 \cup \bar{R}_2^{-1})^* \bar{y} \Rightarrow \bar{x} \cap \gamma_1 = \bar{y} \cap \gamma_1.$

Since any closed formula is canonical, the following holds:

LEMMA 5.6
Let $L_1$ and $L_2$ be two canonical logics. Then $\{L_1, L_2\}$ is also canonical.

LEMMA 5.7
Let $L$ be a 2-modal logic, $L_i = \{A \mid A \in L \cap M_i\}$ $(i = 1, 2)$ be the 1-modal fragments of it, $\mathcal{F}_L = (W, R_1, R_2)$ be the canonical frame of $L$ and $a \in \mathcal{F}_L$. Let $F_i = \left(\left(\mathcal{F}_L^0\right)^{a \cap \gamma_i}\right)^* \bar{R}_i(a, R_1|\gamma_i(a))$ be the continuum unravelling of the rooted subframe of the 0-canonical model of logic $L_i$ with the root $a \cap \gamma_i$ ($i = 1, 2$). Then for any $a_0 \in F_1$ such that $\pi(a_0) = (a \cap \gamma_1, r)$ for some $r \in R$ there exist a p-morphism of 1-Kripke frames $f : F_1^{a_0} \rightarrow (R_i^*(a), R_1|\gamma_i(a))$ with the following property.

$\forall b \in F_1^{a_0} \forall b \in R_i^*(a) \left(f(b) = b \Rightarrow \exists l \in R(\pi(b) = (b \cap \gamma_i, l))\right).$

The same holds for $F_2$.

PROOF. We will describe the construction only for $F_1$ because for $F_2$ it is similar.
To simplify formulas we assume that $G = (R_1^*(a), R_1|\gamma_i(a)) = (W, R)$ and $F_1 = (W', R')$. Since $F_1$ and $G$ are rooted we can define a map $f : F_1 \rightarrow G$ recursively.

Base: $f(e) = x_0$. 
Step: We assume that \( f(b) = x, \pi(b) = (x \cap \Upsilon_1, r) \) and \( c \in R'(b) \). We need to choose the image for \( c \) from \( R(x) \).
For \( y, z \in R(x) \) we define a relation
\[
y \sim z \iff y \cap \Upsilon_1 = z \cap \Upsilon_1.
\]
It is obviously an equivalence relation. Let \( U = R(x)/\sim \) be the quotient set of \( R(x) \) by \( \sim \).

Since the cardinality of each equivalence class \([y]\) \( \in U \) is no greater than the cardinality of the canonical frame which is no greater than continuum, then there exists a partition of \( \mathbb{R} \) indexed by elements of \([y]\) into sets of continuum cardinality:
\[
\mathbb{R} = \bigsqcup_{z \in [y]} V_z^y \text{ and for each } z \in [y] \quad |V_z^y| = |\mathbb{R}|.
\]
This is due to the standard result of Set Theory: \( |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}| \).

For a fixed \( c \in R'(b) \) there exists \( y \in R(x) \) and \( r' \) such that
\[
\pi(c) = (y \cap \Upsilon_1, r'), \quad r' \in V_y^y.
\]
We define
\[
f(c) = y.
\]
Each point in \((\mathbb{F}^0)_{\mathbb{R}}^y\) is reachable from \((\tilde{x}_0, 0)\) in finitely many steps. A point reachable in \( m \)-th iteration. So the function \( f \) is defined correctly.
Let us check that \( f \) is a p-morphism.

Monotonicity. It is obvious from the construction.

Lifting. Let \( f(a) = x \) and \( xRy \); then for any \( r' \in V_y^y \) and \( b = aR(\tilde{y}, r') \) we have \( aR'b \) and \( f(b) = y \).

Surjectivity. Since \( F_1 \) and \( G \) are rooted, and the root maps to the root, surjectivity follows from the lifting property.

\[\square\]

Lemma 5.8
Let \( L_1 \) and \( L_2 \) be two unimodal logics and \( F = \mathbb{F}_{\mathbb{R}}^{0} \) be a rooted subframe of the canonical frame for logic \((L_1, L_2)\); then there exist two rooted frames \( F_1, F_2 \) and a p-morphism \( f : (F_1, F_2) \rightarrow \mathbb{F} \).

Proof. We take \( (\mathbb{F}_{L_1})_{\mathbb{R}}^{\tilde{x}_0} \) and \( (\mathbb{F}_{L_2})_{\mathbb{R}}^{\tilde{y}_0} \) as \( F_1 \) and \( F_2 \), respectively, where \( \tilde{x}_0 = a_0 \cap \Upsilon_1 \) and \( \tilde{y}_0 = a_0 \cap \Upsilon_2 \). Let \( F_1 = (W_1, R_1) \), \( F_2 = (W_2, R_2) \) and \( \mathbb{F} = (W, R_1, R_2) \).

Using Lemma 5.7 for each \( a \in \{F_1, F_2\} \) we fix two p-morphisms:
\[
g_1^a : F_1^{p_1(a)} \rightarrow (R_1^*, R_1 |_{R_1(a)}), \quad g_2^a : F_2^{p_2(a)} \rightarrow (R_2^*, R_2 |_{R_2(a)}).
\]
We also make sure that they are coordinated in the following way
\[
aR_1^c bR_2^c c \implies g_1^a(c) = g_2^b(c),
\]
LEMMA 6.1

The function $g_F$ in a rooted subframe of the canonical frame $\Box$

Let $F_1, F_2$ be two rooted frames. Assume that $W_1 \cap W_2 = \emptyset$. Consider the product of n-frames $X_1 = (X_1, \tau_1) = N_\omega(F_1)$ and $X_2 = (X_2, \tau_2) = N_\omega(F_2)$

$$X = (X_1 \times X_2, \tau_1, \tau_2) = N_\omega(F_1) \times N_\omega(F_2).$$

We define a function $g : X_1 \times X_2 \rightarrow [F_1, F_2]$ by induction, as follows.

Let $(\alpha, \beta) \in X_1 \times X_2$, so that $\alpha = x_1x_2\ldots$ and $\beta = y_1y_2\ldots$, $x_i \in W_1 \cup \{0\}, y_j \in W_2 \cup \{0\}$. We define $g(\alpha, \beta)$ to be the finite sequence that we obtain after dropping all zeros from the infinite sequence $x_1y_1x_2y_2\ldots$.

**Lemma 6.1**

The function $g$ defined above is a p-morphism:

$$g : X_1 \times X_2 \rightarrow N([F_1, F_2]).$$

**Proof.** First, we need to check that for any $\alpha \in N_\omega(F_1)$ and any $\beta \in N_\omega(F_2)$ we have that $g(\alpha, \beta) \in F_1 \circ F_2$. This follows from the equalities:

$$p_1(g(\alpha, \beta)) = p_1(f_0(\alpha)), p_2(g(\alpha, \beta)) = p_2(f_0(\beta)).$$

To prove surjectivity, we take $z = z_1\ldots z_n \in F_1 \circ F_2$. For $i \leq n$ we define

$$x_i = \begin{cases} z_i, & \text{if } z_i \in W_1; \\ 0, & \text{if } z_i \in W_2; \end{cases} \quad y_i = \begin{cases} 0, & \text{if } z_i \in W_1; \\ z_i, & \text{if } z_i \in W_2. \end{cases}$$

Let $\alpha = x_1x_2\ldots x_n0^\omega$ and $\beta = y_1y_2\ldots y_n0^\omega$, then $g(\alpha, \beta) = z$. Hence, $g$ is surjective.

We check the next two conditions only for $\tau_1$, since for $\tau_2$ it is similar. We assume that $(\alpha, \beta) \in X_1 \times X_2$ and $U \in \tau_1(\alpha, \beta)$. We need to prove that $R_1^<(g(\alpha, \beta)) \subseteq g(U)$. There exists
\( m > \max \{ \text{st}(\alpha), \text{st}(\beta) \} \) such that \( U_m(\alpha) \times \{ \beta \} \subseteq U \) and, since \( g(U_m(\alpha) \times \{ \beta \}) = \mathcal{R}_1^\prec(g(\alpha, \beta)) \), then

\[
\mathcal{R}_1^\prec(g(\alpha, \beta)) = g(U_m(\alpha) \times \{ \beta \}) \subseteq g(U),
\]

where \( U_m(\alpha) \) is the corresponding neighbourhood from \( \mathfrak{X}_1 \).

We assume that \((\alpha, \beta) \in \mathfrak{X}_1 \times \mathfrak{X}_2 \) and \( \mathcal{R}_1^\prec(g(\alpha, \beta)) \subseteq V \). We need to prove that there exists \( U \in \tau_1(\alpha, \beta) \) such that \( g(U) \subseteq V \). For \( U \), we take \( U_m(\alpha) \times \{ \beta \} \) for some \( m > \max \{ \text{st}(\alpha), \text{st}(\beta) \} \), then

\[
g(U_m(\alpha) \times \{ \beta \}) = \mathcal{R}_1^\prec(g(\alpha, \beta)) \subseteq V.
\]

\[\square\]

**Corollary 6.2**

Let \( F_1 = (W_1, R_1) \) and \( F_2 = (W_2, R_2) \), then \( \log(\mathcal{N}_\omega(F_1) \times \mathcal{N}_\omega(F_2)) \subseteq \log([F_1, F_2]) \).

This immediately follows from Lemma 6.1 and Corollary 2.25.

**Theorem 6.3**

The logic \( \langle K, K \rangle \) is complete with respect to products of normal neighbourhood frames, i.e.

\[
\langle K, K \rangle = K \times_n K.
\]

**Proof.** The inclusion from left to right of (1) was proved in Corollary 3.10.

The converse inclusion follows from Theorem 5.3 and Corollary 6.2. Indeed,

\[
K \times_n K = \bigcap_{\mathfrak{X}_1, \mathfrak{X}_2 \in \jmath n(\mathfrak{K})} \log(\mathfrak{X}_1 \times \mathfrak{X}_2)
\]

\[
\subseteq \bigcap_{F_1, F_2 - \text{Kripke frames}} \log([\mathcal{N}_\omega(F_1), \mathcal{N}_\omega(F_2)])
\]

\[
\subseteq \bigcap_{F_1, F_2 - \text{Kripke frames}} \log([F_1, F_2]) \subseteq \langle K, K \rangle.
\]

\[\square\]

### 7 Horn axioms

**Definition 7.1**

Following [4], we define a **universal strict Horn sentence** as a first-order closed formula of the form

\[
\forall x \forall y \forall z_1 \ldots \forall z_n (\phi(x, y, z_1, \ldots, z_n) \rightarrow \psi(x, y)),
\]

where \( \phi(x, y, z_1, \ldots, z_n) \) is quantifier-free positive (i.e. it is built from atomic formulas by using \( \land \) and \( \lor \)) and \( \psi(x, y) \) is an atomic formula in the signature \( \Omega = \{ R_1(2), \ldots, R_m(2) \} \), where \( R_i(2) \) is the propositional letter that corresponds to the relation \( R_i \).

**Definition 7.2**

A logic \( \mathbf{L} \) is called an **HTC-logic** (from Horn preTransitive Closed logic) if it can be axiomatized by closed formulas and formulas of the type \( \Box p \rightarrow \Box^n p \), \( n \geq 0 \). These formulas correspond to universal strict Horn sentences (see [4]).
Let $\Gamma$ be a set of universal strict Horn formulas and $F$ be a Kripke frame. By $F^\Gamma$ we define the $\Gamma$-closure of $F$, that is the minimal (in terms of inclusion of relations) frame such that all formulas from $\Gamma$ are valid in it. Such a frame exists due to [4]:

**Lemma 7.3 ([4, Prop 7.9])**
For any Kripke frame $F = (W, R_1, \ldots, R_n)$ and a set of universal strict Horn formulas $\Gamma$, there exists $F^\Gamma = (W, R_1^\Gamma, \ldots, R_n^\Gamma)$ such that
- $R_i \subseteq R_i^\Gamma$ for all $i \in \{1, \ldots, n\}$;
- $F^\Gamma \models \Gamma$;
- if $G \models \Gamma$ and $f : F \rightarrow G$ then $f : F^\Gamma \rightarrow G$.

**Definition 7.4**
Let $\Gamma$ be a set of universal strict Horn formulas, $F = (W, R)$ be a rooted frame, $\alpha \in W^\omega$ and $f_0 : W^\omega \rightarrow W^\sharp$ be the ‘zero-dropping’ function. Then we define

$$U_k^\Gamma(\alpha) = \{ \beta \in W^\omega \mid \alpha|_m = \beta|_m \& f_0(\alpha)(R^\sharp)^\Gamma f_0(\beta), \ m = \max(k, st(\alpha)) \},$$

$$\tau^\Gamma(\alpha) = \{ V \mid \exists k (U_k^\Gamma(\alpha) \subseteq V) \},$$

$$\mathcal{N}_\omega^\Gamma(F) = (W^\omega, \tau^\Gamma).$$

We also need the following obvious lemmas:

**Lemma 7.5**
For any closed modal formula $A$ and a p-morphism of Kripke frames $f : F \rightarrow G$

$$F, x \models A \iff G, f(x) \models A.$$ And its neighbourhood analogue:

**Lemma 7.6**
For any closed modal formula $A$ and a p-morphism of n-frames $f : \mathcal{X} \rightarrow \mathcal{Y}$

$$\mathcal{X}, x \models A \iff \mathcal{Y}, f(x) \models A.$$ In [4], the product matching was proved for a large class of Horn axiomatizable logics, including $S5$. But in our case, $S5 \times_n S5 \neq \langle S5, S5 \rangle$. In fact, since neighbourhood frames correspond to topological spaces in case of transitive and reflexive logics, and due to [6],

$$S5 \times_n S5 = S5 \times S5 = [S5, S5] = S5 \ast S5 + \Box_1\Box_2 p \leftrightarrow \Box_2\Box_1 p + \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p.$$
On neighbourhood product of some Horn axiomatizable logics

PROOF. Let \( M = (\mathcal{N}_\omega^\Gamma (F), V) \) be a neighbourhood model. We assume that \( M, \alpha \not\models \Box^n p \), and then we prove that \( M, \alpha \not\models \Box p \), i.e.

\[
\forall m \exists \beta \in U_m^\Gamma (\alpha) (\beta \not\models p).
\]

Let us fix \( m \). Then

\[
\exists \alpha_1 \in U_m^\Gamma (\alpha) (\alpha_1 \not\models \Box^{n-1} p) \Rightarrow \\
\exists \alpha_2 \in U_m^\Gamma (\alpha_1) (\alpha_2 \not\models \Box^{n-2} p) \Rightarrow \\
\vdots \\
\exists \alpha_n \in U_m^\Gamma (\alpha_{n-1}) (\alpha_n \not\models p).
\]

By the definition of \( U_m^\Gamma (\alpha) \)

\[
f_0 (\alpha) (R^\Sigma)^\Gamma f_0 (\alpha_1) (R^\Sigma)^\Gamma \cdots (R^\Sigma)^\Gamma f_0 (\alpha_n)
\]

and

\[
\alpha |_m = \alpha_1 |_m = \ldots = \alpha_n |_m.
\]

Since \( (W^\Sigma, (R^\Sigma)^\Gamma) \models \Box p \rightarrow \Box^n p \),

\[
f_0 (\alpha) (R^\Sigma)^\Gamma f_0 (\alpha_n).
\]

It follows that \( \alpha_n \in U_m^\Gamma (\alpha) \).

\[ \square \]

**Lemma 7.8**

Let \( L \) be an HTC-logic, \( \Gamma \) be the corresponding set of Horn formulas and \( F \models L \). Then

\[
f_0 : \mathcal{N}_\omega^\Gamma (F) \rightarrow \mathcal{N} (F^{\sharp \Gamma}).
\]

PROOF. From now on in this proof we will omit the subindex in \( f_0 \). The surjectivity was established in Lemma 4.6.

Assume that \( \alpha \in W_\omega \) and \( U \in \tau^\Gamma (\alpha) \). We need to prove that \( R^{\sharp \Gamma} (f(\alpha)) \subseteq f(U) \). There exists \( m \) such that \( U_m^\Gamma (\alpha) \subseteq U \), and since \( f (U_m^\Gamma (\alpha)) = R^{\sharp \Gamma} (f(\alpha)) \), then

\[
R^{\sharp \Gamma} (f(\alpha)) = f (U_m^\Gamma (\alpha)) \subseteq f(U).
\]

Assume that \( \alpha \in W_\omega \) and \( V \) is a neighbourhood of \( f(\alpha) \), i.e. \( R^{\sharp \Gamma} (f(\alpha)) \subseteq V \). We need to prove that there exists \( U \in \tau^\Gamma (\alpha) \) such that \( f(U) \subseteq V \). For \( U \), we take \( U_m^\Gamma (\alpha) \) for some \( m \geq s(\alpha) \), then

\[
f (U_m^\Gamma (\alpha)) = R^{\sharp \Gamma} (f(\alpha)) \subseteq V.
\]

\[ \square \]

**Corollary 7.9**

Let \( L \) be an HTC-logic and \( F \models L \); then \( \mathcal{N}_\omega^\Gamma (F) \models L \).
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**Lemma 7.10**
Let $F_1$ and $F_2$ be two frames, $\Gamma_1$ and $\Gamma_2$ be two sets of Horn sentences that correspond to HTC-logics, then

$$\mathcal{N}^{T_1}_{\omega}(F_1) \times \mathcal{N}^{T_2}_{\omega}(F_2) \rightarrow \mathcal{N}(\{F_1,F_2\}^{\Gamma_1 \cup \Gamma_2}).$$

The proof is similar to Lemma 6.1. The underlining sets are the same and we can take the same function $g$. So, surjectivity follows. Monotonicity and lifting are proved similarly.

**Theorem 7.11**
Let $L_1$ and $L_2$ be two HTC-logics then

$$L_1 \times_n L_2 = \langle L_1, L_2 \rangle.$$

**Proof.** By Lemma 3.9 $\langle L_1, L_2 \rangle \subseteq L_1 \times_n L_2$.

Let $\Gamma_1$ and $\Gamma_2$ be the sets of Horn sentences corresponding to $L_1$ and $L_2$. Let $A \notin \langle L_1, L_2 \rangle$; then there is a rooted subframe $F$ of the canonical frame of logic $\langle L_1, L_2 \rangle$ such that $F \not\models A$. Then by Lemma 5.8 there are frames $F_1$ and $F_2$ such that

$$\{F_1,F_2\} \rightarrow F.$$  

Since $L_1$, $L_2$ and $\langle L_1, L_2 \rangle$ are canonical then

$$\{F_1,F_2\}^{\Gamma_1 \cup \Gamma_2} \rightarrow F.$$  

By Lemma 7.10

$$\mathcal{N}^{T_1}_{\omega}(F_1) \times \mathcal{N}^{T_2}_{\omega}(F_2) \rightarrow \mathcal{N}(\{F_1,F_2\}^{\Gamma_1 \cup \Gamma_2}).$$

By Corollary 7.9

$$\mathcal{N}^{T_1}_{\omega}(F_1) \models L_1$$  

and

$$\mathcal{N}^{T_2}_{\omega}(F_2) \models L_2.$$  

At the same time

$$\mathcal{N}^{T_1}_{\omega}(F_1) \times \mathcal{N}^{T_2}_{\omega}(F_2) \not\models A.$$  

So $L_1 \times_n L_2 \subseteq \langle L_1, L_2 \rangle$.  

\[\square\]

**8 Seriality axiom**

Consider the seriality axiom $\neg \square \bot$. By induction on the length of a formula, one can easily prove

**Lemma 8.1**
If $\neg \square \bot \in L$ then any closed formula is $L$-equivalent to $\bot$ or $\top$.

The base is obvious and the inductive step follows from

$$\vdash \square \top \leftrightarrow \top, \quad \neg \square \bot \vdash \square \bot \leftrightarrow \bot.$$
LEMMA 8.2
For a bimodal logic \( L \) if \( L \vdash \neg \Box_1 \bot \) then \( L \vdash B \rightarrow \Box_2 B \) for any closed formula \( B \in \mathcal{ML}_{\Box_1} \).

This is a simple exercise.

COROLLARY 8.3
If \( L_1 \) and \( L_2 \) are HTC-logics and \( \neg \Box_1 \bot \in L_1, \neg \Box_1 \bot \in L_2 \) then
\[
\langle L_1, L_2 \rangle = L_1 \ast L_2.
\]

From Corollary 8.3 and Theorem 7.11 it follows.

THEOREM 8.4
Let \( L_1 \) and \( L_2 \) be HTC-logics with seriality then
\[
L_1 \times_n L_2 = L_1 \ast L_2.
\]

Note that this theorem covers the results from [7], since the logics \( D, T, D4 \) and \( S4 \) are all HTC-logics with seriality.

PROPOSITION 8.5
If \( L_1 \) and \( L_2 \) are finitely axiomatizable, and have only finitely many non-equivalent closed formulas then \( \langle L_1, L_2 \rangle \) is finitely axiomatizable.

To prove this proposition it is enough to show that the set of formulas \( \Delta \) has only finitely many non-equivalent formulas.

9 Derivational semantics
The derivational semantics studied by many authors (see, e.g. [1, 14]) can be equivalently defined as follows.

Let \( \mathfrak{X} = (X, T) \) be a topological space. We define
\[
\tau_d^\mathfrak{X}(x) = \{ U \mid U' \setminus \{ x \} \subseteq U, x \in U' \in T \}.
\]

Then for any valuation \( V \) on \( X \) the following holds
\[
\mathfrak{X}, V, x \models_d A \iff (X, \tau_d^\mathfrak{X}), V, x \models_n A,
\]
where \( \models_d \) corresponds to the derivational semantics, and \( \models_n \) corresponds to the neighbourhood semantics. We define \( \mathcal{N}_d(\mathfrak{X}) = (X, \tau_d^\mathfrak{X}) \).

For a class of topological spaces \( C \) and logics \( L_1 \) and \( L_2 \) we put
\[
Log_d(C) = \{ A \mid \forall \mathfrak{X} \in C(\mathfrak{X} \models_d A) \},
\]
\[
L_1 \times_d L_2 = Log_d(\{ \mathfrak{X}_1 \times \tau_1 \mathfrak{X}_2 \mid \mathfrak{X}_1, \mathfrak{X}_2 \text{— topological spaces, } \mathfrak{X}_1 \models_d L_1, \mathfrak{X}_2 \models_d L_2 \}).
\]

Here \( \times \) is the bitopological product defined in [17].
We say that $\text{Log}_d(C)$ is the $d$-logic of $C$.

**Theorem 9.1**

1. $K4 \times_d K4 = \{K4, K4\}$,
2. $K4 \times_d D4 = \{K4, D4\}$,
3. $D4 \times_d K4 = \{D4, K4\}$,
4. $D4 \times_d D4 = D4 \ast D4$.

**Proof.** This follows from Theorems 7.11 and 8.4. But, it is not a straightforward corollary because for a logic $L$ the set of $L$-n-frames and the set of all n-frames that correspond to $L$-topological spaces do not coincide. Indeed, in a topological space $X = (X, T)$ the family of $\tau_d^X$-neighbourhoods of a point $x$ always contains set $X \setminus \{x\}$ and it is not the case for n-frames.

So to prove this theorem it is sufficient to say that all the logics mentioned in this theorem are not reflexive and the unravellings are irreflexive. So let $F = (W, R)$, then $F^\Gamma$ is irreflexive, and $N_{\omega}^T(F^\Gamma)$ can be obtained as $N_d(x)$, where $x = (W_\omega, T)$, $\Gamma$ is the Horn sentence expressing transitivity, and sets $U_\Gamma$ form the base for topology $T$. $\square$

**Theorem 9.2**
The d-logic of the class of all products of $T_1$ spaces is $\{K4, K4\}$.

It is enough to check that the topological space corresponding to $N_{\omega}^T(F)$ is a $T_1$ space, whenever $F$ is the unravelling of a rooted $S4$-frame and $\Gamma$ corresponds to transitivity. This can be easily checked.

10 Conclusions

We are still in the beginning of the road of studying products of neighbourhood frames.

This topic can be interesting from different points of view. It is interesting in itself because it is a natural way to combine modal logics, and the result is weaker then the product of logics based on Kripke semantics. It is also interesting because using the products we can express new properties, e.g. $\mathcal{Q}$ and $\mathcal{R}$ are indistinguishable in the unimodal language with topological semantics, whereas the logics of $Q \times_t Q$ and $R \times_t R$ are different (see [5]). It is also possible that this construction will be useful for epistemic modal logic as semantics for multi-agents systems.

There are a lot of open questions in this area, to name a few:

- find other sufficient conditions for product matching;
- investigate products of type $L \times_n S5$; we give a partial answer to this question in a forthcoming paper: for any HTC-logic $L$ (this result is announced at Advances in Modal Logic ’16 conference);
- find the logics of $\mathcal{R} \times_t \mathcal{R}$ and $\mathcal{C} \times_t \mathcal{C}$, where $\mathcal{C}$ is the Cantor space;
- find the n-products of well-known logics like $S4.1$, $S4.2$, $S4.3$, $GL$, Grz, $DL$ and others.

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References


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