Shortest and Straightest Geodesics in Sub-Riemannian Geometry

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Dedicated to Jubilee of Joseph Krasil'shchik

Abstract

There are several different, but equivalent definitions of geodesics in a Riemannian manifold, based on two characteristic properties: geodesics as shortest curves and geodesics as straightest curves. They are generalized to sub-Riemannian manifolds, but become non-equivalent. We give an overview of different approaches to the definition, study and generalisation of sub-Riemannian geodesics and discuss interrelations between different definitions. For Chaplygin transversally homogeneous sub-Riemannian manifold Q, we prove that straightest geodesics (defined as geodesics of the Schouten partial connection) coincide with shortest geodesics (defined as the projection to Q of integral curves (with trivial initial covector) of the sub-Riemannian Hamiltonian system). This gives a Hamiltonization of Chaplygin systems in non-holonomic mechanics.
We consider a class of homogeneous sub-Riemannian manifolds, where straightest geodesics coincide with shortest geodesics, and give a description of all sub-Riemannian symmetric spaces in terms of affine symmetric spaces.

Contents

*The work was partially supported by the grant no. 18-00496S of the Czech Science Foundation

1 Introduction

The important role of Riemannian geometry in applications is based on the fact that many important equations, arising in mechanics, mathematical physics, biology, economy, information theory, image processing etc., can be reduced to the geodesic equation. Moreover, Riemannian geometry gives an effective tool to investigate the geodesic equation and other equations associated with the metric (Laplace, wave, heat and Schrödinger equations, Einstein equation, Yang-Mills equation etc). There are many equivalent definitions of geodesics in a Riemannian manifold. They are naturally generalised to sub-Riemannian manifolds, but become non-equivalent. H.R. Herz remarked that there are two main approaches to the definition of geodesics: geodesics as **shortest curves** based on Maupertruis principle of least action (variational approach) and geodesics as **straightest curves** based on d'Alembert's principle of virtual work (which leads to a geometric description, based on the notion of connection).

We consider three variational definitions of geodesics of a sub-Riemannian manifold (Q, D, g^D) (i.e. a manifold Q with a non-holonomic distribution D and a Riemannian metric g^D on D) as (locally) shortest curves (Euler-Lagrange (EL-geodesics), Pontryagin (P-geodesics) and Hamilton (H-geodesics)) and three geometric definitions of sub-Riemannian geodesics as straightest curves (d'Alembert (dA-geodesics), Schouten-Synge-Vranceanu (S-geodesics) and Morimoto (M-geodesics)) and discuss interrelations between them.
The definition of M-geodesics is based on E.Cartan frame bundle definition of Riemannian geodesics, which is naturally generalized to Cartan connections and G-structures of finite type. We give a short introduction to this theory in section 4. In section 5, we discuss the relation between Cartan connections and Tanaka structures (or non-holonomic G-structures). They are defined as a G-principal bundle $\pi : P \to Q = P/G$ of frames on a non-holonomic distribution $D \subset TQ$. In particular, a regular sub-Riemannian manifold (Q, D, g^D) (see Sect. 5.1) may be identified with a Tanaka structure $\pi : P \to Q$ of admissible orthonormal frames in D.
Using his theory of filtered manifold, T. Morimoto proved that this Tanaka structure admits a unique normal Cartan connection, i.e. a Cartan connection with coclosed curvature. The Morimoto geodesics are defined in terms of this Cartan connection. We give a simple description of all (not necessary normal) Cartan connections, associated to a regular sub-Riemannian manifold (Q, D, g^D) in term of admissible riggings V (some distribution, which is complement to D) and define Cartan-Morimoto (shortly, CM) geodesics in terms of such Cartan connections. CM-geodesics are horizontal geodesics of some Riemannian connection with torsion, which preserves the distribution. A necessary and sufficient condition that CM-geodesics coincides with S-geodesics (i.e. geodesics of the partial sub-Riemannian Schouten connection, associated with a given rigging V) is given.
A. Vershik and L. Faddeev [35] had formulated the problem how to characterize sub-Riemannian manifolds such that straightest S-geodesics "coincide"(more precisely, consistent) with shortest H-geodesics in the following sense.
An S-geodesic $\gamma(t)$ of a sub-Riemannian manifold (Q, D, g^D) is determined by the initial velocity $\dot\gamma(0) \in D_q \subset T_qQ$. The initial data for an H-geodesic is a pair $(\dot\gamma(0), \lambda) \in D_q \times D^0_q$ where $D^0 \subset T^*Q$ is the codistribution (the annihilator of the distribution D). The covector λ is called the initial codistribution covector.
Taking this into account, we say, following [35], that (straightest) S-geodesics coin-

cide with (shortest) H-geodesics if the class of S-geodesics coincides with the class of H-geodesics with zero initial codistribution covector.
Vershik and Faddeev showed that for generic sub-Riemannian manifolds almost all shortest geodesics are different from straightest geodesics. They gave the first example when shortest geodesics coincide with straightest geodesics with zero codistribution covector.
In the second part of the paper, we show that this is true for any Chaplygin system, that is G-invariant sub-Riemannian metric $(D = \ker(\varpi), g^D)$ on the total space of a G-principal bundle $\pi : Q \to M = Q/G$ over a Riemannian manifold (M, g^M) with a principal connection $\varpi : TQ \to \mathfrak{g}$, where g^D is the metric in D, induced by the Riemannian metric g^M.
Any left-invariant metric on the group G defines an extension of the sub-Riemannian metric g^D to a Riemannian metric g^Q on Q. We show that H-geodesics of sub-Riemannian Chaplygin metric are the horizontal lifts of the projection to M of geodesics of the Riemannian metric g^Q and S-geodesics are horizontal lift of geodesics of the Riemannian metric g^M. This is a generalization of results by R. Montgomery [23], who considered the case when the extended metric g^Q is defined by a bi-invariant metric on G. We give a simple proof of Wong results on the description of the evolution of charge particle in a classical Yang-Mills field in terms of geodesics of the bi-invariant extension g^Q of the Chaplygin sub-Riemannian metric.
In the last section, we describe some classes of invariant sub-Riemannian structures on homogeneous manifolds, where straightest geodesics coincide with shortest ones. We give also a simple description of all bracket generating symmetric sub-Riemannian manifolds, introduced by R.S. Strichartz [30], and show that any flag manifold of a compact semisimple Lie group G, associated to a gradation of depth $k > 1$ of the corresponding complex semisimple Lie algebra, has a structure of sub-Riemannian symmetric space.

Acknowledgment. I thank A.M. Vershik for his comments on non-holonomic geometry and explanation of his joint with L.D. Faddeev results and A. Spiro for useful discussions.

2 Sub-Riemannian geodesics as shortest curves

Here we briefly discuss three approaches to the definition of geodesics of sub-Riemannian manifolds : Euler-Lagrange variational approach, Pontryagin optimal control approach and Hamiltonian approach. We describe interrelation between corresponding notions of sub-Riemannian geodesics : EL-geodesics, P-geodesics and H-geodesics.

2.1 Euler-Lagrange sub-Riemannian geodesics

Recall that a rank-m distribution $D \subset TQ$ on a connected n-dimensional manifold Q is called **bracket generating** if the space ΓD of sections generates the Lie algebra $\mathfrak{X}(M)$ of vector fields.
According to Rashevsky-Chow theorem, any two points on such manifold can be joint by a horizontal (i.e. tangent to D) curve. Then any Lagrangian $L \in C^\infty(TQ)$ defines a nonholonomic variational problem:
Let $C^D(q_0, q_1)$ be the space of horizontal curves, connecting points q_0 and q_1. Find a curve $q(t)$, $t \in [0, T]$ in $C^D(q_0, q_1)$ which delivers a minimum or, more generally,

a critical point, for the action functional

$$A(q(t)) = \int_0^T L(q(t), \dot{q}(t))dt, \;\; q(t) \in C^D(q_0, q_1).$$

The Lagrangian $L(q, \dot{q})$ determines a horizontal 1-form

$$F_L = (\delta L)_i dq^i := (\frac{d}{dt} L_{\dot{q}_i} - L_{q_i})dq^i$$

on TQ, called the Lagrangian force [31], [35]. Locally the distribution D is the kernel of a system $(\omega^1, \cdots, \omega^k)$, $k = n - m$, of 1-forms. The 1-form

$$\omega_\lambda = \sum \lambda_a \omega^a = \sum \lambda_a(q, \dot{q}) \omega^a_i(q) dq^i$$

vanishes on D for any vector-function $\lambda(q, \dot{q}) = (\lambda_1, \cdots, \lambda_m)$ on TQ.

Then critical points $q(t)$ of the functional $A(q(t))$ are solution of the Euler-Lagrange equations [38]

$$F_L \equiv (\delta L)_i \dot{q}^i(t) = \mathcal{L}_{\dot{q}} \omega_\lambda = \dot{\lambda}_a \omega^a + \lambda_a \dot{q} \lrcorner d\omega^a \tag{1}$$

$$\omega_\lambda(\dot{q}) = 0$$

for unknown curve $q(t) \in C^D(q_0, q_1)$ and vector-function $\lambda(q(t), \dot{q}(t))$. Here $\mathcal{L}_{\dot{q}(t)}$ is the Lie derivative along the vector field $\dot{q}(t)$.

A **sub-Riemannian manifold** (Q, D, g^D) is a manifold with a distribution D and a Riemannian metric g^D on D .

Let (Q, D, g^D) be a sub-Riemannian manifold with bracket generating distribution D. An **Euler-Lagrange or EL non-parametrized geodesic** (resp., **EL-parametrized geodesic**) is a critical point of the length functional with $L = \sqrt{g(\dot{q}, \dot{q})}$) (resp., the energy functional with $L = \frac{1}{2} g(\dot{q}, \dot{q})$) in the space $C^D(q_0, q_1)$.

2.2 Pontryagin sub-Riemannian geodesics

Let (Q, D, g^D) be a bracket generating sub-Riemannian manifold as above. Denote by $(X_1, \cdots, X_m)$ a field of orthonormal frames in D. Then any horizontal curve $q(t) \in C^D(q_0, q_1)$ is a solution of the first order ODE

$$\dot{q}(t) = \sum_{i=1}^m u^i(t) X_i(q(t)), \;\; q(0) = q_0. \tag{2}$$

where the vector-function $u(t) = (u^1(t), \cdots, u^m(t))$ (called the **control**) consists of the coordinates of the velocity vector field $\dot{q}(t)$ with respect to the frame (X_i) .

The vector-function $u(t)$ is called an **admissible control** if the solution $q(t)$ of (2) belongs to $C^D(q_0, q_1)$.

The energy $E(u(t)) = \frac{1}{2} \int_0^T \sum u^i(t)^2 dt$ and the length $\ell(u) = \int_0^T \sqrt{\sum u^i(t)^2} dt$ of the solution $q(t)$ of (2) depend only on the control $u(t)$ and may be considered as the functionals (called the **cost functionals**) on the space of admissible controls.

A **parametrized** (respectively, **non-parametrized**) Pontryagin geodesic (shortly, **P-geodesic**) is defined as the integral curve $q^u(t) \in C^D(q_0, q_1)$ of the

equation (2) with an admissible control $u(t)$, which is a critical point of the cost functional $E(u(t))$ (respectively, $\ell(u(t))$).
A P-geodesic with an admissible control $u(t)$, which delivers a minimum of the cost functional is called a **minimizer** or a **minimal geodesic**. P-geodesics coincide with EL-geodesics and are locally minimizers, [8].

2.3 Hamiltonian sub-Riemannian geodesics

Let (Q, D, g^D) be a sub-Riemannian manifold. Denote by (X_i) an orthonormal frame in D and by (θ^i) the dual coframe. Then the restriction ξ_D of a covector $\xi \in T^*Q$ to D has coordinates $p_i(\xi) := \xi(X_i)$ and can be written as $\xi = p_i\theta^i$. The inverse $(g^D)^{-1}$ of the sub-Riemannian metric g^D is a non-degenerate metric in the dual to D vector bundle D^*. It defines a degenerate symmetric bilinear form $g^* \in \Gamma(S^2TQ)$ in T^*Q, called the **cometric**, which is given by

$$g^*(\xi,\xi) = (g^D)^{-1}(\xi_D, \xi_D) = \sum_i p_i(\xi)^2,\ \xi \in T^*Q.$$

The function $h_{g^D}(\xi,\xi) = \frac{1}{2}g^*(\xi,\xi) = \frac{1}{2}\sum p_i(\xi)^2$ on T^*Q is called the **sub-Riemannian Hamiltonian**. **H-geodesics** are projection to Q of orbits of Hamiltonian vector field $\vec{h} = \omega^{-1}dh \in \mathfrak{X}(T^*Q)$ with quadratic (degenerate) sub-Riemannian Hamiltonian $h_{g^D}(\xi,\xi) = \frac{1}{2}g^*(\xi,\xi)$. Here $\omega = dp_a \wedge dq^a$, $a = 1, \cdots, n$ is the standard symplectic form of T^*Q.

2.4 Pontryagin Maximum Principle

Recall that vector fields $X = X^a\partial_{q^a}$, where (x^a) are local coordinates in Q bijectively correspond to fiberwise linear functions

$$p_X : T^*Q \to \mathbb{R},\ \xi = p_a\partial_{q^a} \mapsto p(X) = X^a p_a$$

on the cotangent bundle T^*Q. The function p_X is the Hamiltonian of the Hamiltonian vector field

$$\vec{p}_X = X^a\partial_{q^a} - p_a\partial_{q^b}X^a\partial_{p_b},$$

which is the complete lift of X to T^*Q. The map

$$\mathfrak{X}(Q) \ni X \to \vec{p}_X \in \mathfrak{X}(T^*Q)$$

is an isomorphism of the Lie algebra $\mathfrak{X}(Q)$ of vector field onto the Lie algebra $\mathfrak{X}(T^*Q)^1$ of fiberwise linear vector fields on T^*Q.

Theorem 1 *(Pontryagin Maximum Principle) Let $q(t) \in C^D(q_0, q_1)$ be a minimal P-geodesic on a sub-Riemannian manifold $(Q, D = \mathrm{span}(X_i), g^D)$ with natural parametrization (s.t. $|\dot{q}(t)| = const$), which corresponds to a control $u(t) = (u^i(t))$:*

$$\dot{q}(t) = u^i(t)X_i(q(t)). \tag{3}$$

Denote by φ_t the (local) flow, generated by the non-autonomous vector field $X^u = u^i(t)X_i$. Then for some covector $\xi_0 \in T^*_{q_0}Q$ the curve

$$\xi(t) := \varphi^*_{-t}\xi_0 := \xi_0 \circ \varphi_{-t_*} \in T^*_{q(t)}Q$$

satisfies the equation

$$\dot{\xi}(t) = u^i(t)\vec{p}_i(\xi(t)) \tag{4}$$

where $p_i := p_{X_i}$ *and one of the following conditions holds*

$$\begin{array}{rcll} u^i(t) & \equiv & <\xi(t), X_i(q(t))> & (N) \\ 0 & \equiv & <\xi(t), X_i(q(t))>. & (A) \end{array}$$

Here the bracket $<\xi, X>$ denotes the pairing between covectors and vectors.

An extremal curve $\xi(t) \subset T^*Q$, which satisfies (N) (resp., (A)), is called a **normal** (resp., an **abnormal**) **extremal**, and its projection $q(t) \subset Q$ is called a **normal** (resp., an **abnormal**) **P-geodesic**. Note that abnormal extremals are curves in the codistribution D^0, considered as a submanifold of T^*Q.

2.4.1 Normal P-geodesics as H-geodesics

Pontryagin theorem shows that normal geodesics are H-geodesics. More precisely, we have

Corollary 2 *Let D be a rank-m bracket generating distribution with a sub-Riemannian metric g^D. A normal extremal $\xi(t) \subset T^*Q$ for (Q, D, g^D) is an integral curve of the Hamiltonian equation on T^*Q with the sub-Riemannian Hamiltonian*

$$h_{g^D}(\xi) = \frac{1}{2}g^*(\xi,\xi) = \frac{1}{2}\sum_{i=1}^{m} p_i(\xi)^2,$$

where $\xi = p_i(\xi)\theta^i,\ p_i(\xi) = \xi(X_i)$.

Proof: In the case of normal geodesic, the equation (4) take the form

$$\dot{\xi}(t) = \sum p_i(\xi(t))\vec{p}_i(\xi(t)) = \frac{1}{2}\omega^{-1}d\sum p_i^2(\xi(t)) = \vec{h}_{g^D}(\xi(t)).$$

□

Since the Hamiltonian vector field $\vec{h}_{g^D}$ preserves the Hamiltonian h_{g^D}, a normal extremal $\xi(t)$ belongs to a level set $L_c = \{h = c\} \subset T^*Q$ of the Hamiltonian $h = h_{g^D}$.

A curve $\xi(t) \subset L$ on a submanifold $L \subset T^*Q$ is called a **characteristic** if its velocity $\dot{\xi}(t)$ belongs to the kernel ker $(\omega|L)$ of the restriction of the symplectic form to L.

Corollary 3 *Assume that an extremal $\xi(t) \subset L_c$ belongs to a regular level set of the Hamiltonian $h = h_{g^Q}$, i.e. L_c is a smooth hypersurface. Then* ker $\omega|_{L_c}$ *is the 1-dimensional distribution generated by $\vec{h}$. In particular, the extremals $\xi(t)$ are the characteristic curves of L_c.*

Proof: The tangent space of the level set L_c is described as follows

$$T_\xi L_c = \{w \in T_\xi(T^*Q),\ 0 =< dh_\xi, w >=< \omega(\vec{h}_\xi), w >= \omega(\vec{h}_\xi, w)\}.$$

This shows that $T_\xi L_c$ consists of all ω-orthogonal to $\vec{h}_\xi$ vectors. Since the $\omega|T_\xi L_c$ has 1-dimensional kernel, it is generated by $\vec{h}_\xi$. □

2.4.2 Abnormal P-geodesics

Now we shortly discuss main properties of abnormal geodesics, following R. Montgometry. Denote by $C^D(q_0) = \{\gamma : [0,T] \to Q, \gamma(0) = q_0, \dot{\gamma} \in D\}$ the space of horizontal curves, starting from q_0 , where $D \subset TQ$ is a bracket generating distribution. A curve $\gamma(t) \in C^D(q_0)$ is called **singular** (resp.,**regular**) if the end-point map

$$\varepsilon : C^D(q_0) \to Q, \, \gamma(t) \mapsto \gamma(T)$$

is singular (resp., regular).
The following theorem by L.S. Pontryagin, L. Hsu and R. Montgomery shows that abnormal geodesics coincide with singular curves and they are projection on Q of characteristic curves of the codistribution D^0, considered as a submanifold of the symplectic manifold (T^*Q, ω).

Theorem 4 *(see [23], [24]) i) Abnormal geodesics of any sub-Riemannian metric g^D on D are exactly singular horizontal curves in Q.*
ii) A horizontal curve $\gamma \subset Q$ is singular if and only if it is a projection to Q of a characteristic curve of the submanifold $\tilde{D^0} := D^0 \setminus \{zero\ section\} \subset T^*Q$.

Now we give a description of characteristic curves in $\tilde{D^0}$, following [23]. To simplify notation, we will denote $\tilde{D}^0$ by D^0.

Denote by $\tau : T^*Q \to Q$, $\tau_* : T(T^*Q) \to T^*Q$ the natural projections.
We fix a complementary to D distribution V such that $TQ = D \oplus V$. Let (X_i), $i = 1, \cdots, m$ be a local frame in D, (Y_α), $\alpha = 1, \cdots, n-m$ a local frame in V and denote by $(\theta^i_Q, \eta^\alpha_Q)$ the dual coframe, such that

$$\theta^i_Q(X_j) = \delta^i_j, \eta^\alpha_Q(Y_\beta) = \delta^\alpha_\beta, \theta^i_Q(Y_\alpha) = \eta^\alpha_Q(X_i) = 0.$$

The Liouville tautological 1-form $\theta_\xi = \tau^*\xi = \xi \circ \tau_*$ in $T^*Q = D^* \oplus D^0$ at a point $\xi = h_i\theta^i_Q + k_\alpha\eta^\alpha_Q \in T^*Q$ can be written as

$$\theta_\xi = h_i\theta^i + k_\alpha\eta^\alpha \in (\tau_*)^*D^*_\xi \oplus (\tau_*)^*D^0_\xi \subset T^*(T^*Q),$$

where $\theta^i = \theta^i_Q \circ \tau_*$, $\eta^\alpha = \eta^\alpha_Q \circ \tau_*$ are the pull back of the 1-forms θ^i_Q, η^α_Q to T^*Q. We will consider h_i, k_α as fiberwise coordinates in the bundle T^*Q and in the bundle $\tau^*(T^*Q) \subset T^*(T^*Q)$ of horizontal 1-forms on T^*Q.

The restrictions θ^0, ω^0 of the Liuville form θ and the standard symplectic form $\omega = -d\theta$ on T^*Q to the submanifold $D^0 \subset T^*Q$ are given by

$$\theta^0_\xi = \xi|_D = k_\alpha\eta^\alpha, \;\; -\omega^0 = d\theta^0 = dk_\alpha \wedge \eta^\alpha + k_\alpha d\eta^\alpha.$$

Denote by

$$Ch(D^0) := \ker\, \omega^0 = T(D^0)^\perp \cap T(D^0) = \{v \in T_\eta D^0, \, \omega(v, T_\eta D^0) = 0\}$$

the characteristic submanifold of $T(D^0)$, where the vector bundle $T(D^0)^\perp \subset T(T^*Q)|_{D^0}$ is the ω-orthogonal complement to the tangent bundle $T(D^0)$ of D^0.
The fiber $Ch_\eta(D^0) = \ker\, \omega^0_\eta \subset T_\eta(Q^0)$ over a point $\eta \in Q^0$ is a vector space, but since the rank of ω^0_η may vary, the natural projection $Ch(D^0) \to D^0$ is not a vector bundle, in general.

By definition,**characteristic curves** are curves $\eta(t) \subset D^0$, tangent to the characteristic manifold $Ch(D^0) \subset T(D^0)$.

Lemma 5 *The vector bundle* $T(D^0)^\perp = \mathrm{span}\{\vec{h}_i,\, i = 1, \cdots, m\}$*, and the projection* $\tau_* : T(T^*Q) \to TQ)$ *induces for any* $\eta \in D^0$ *the isomorphism*

$$\tau_* : T_\eta(D^0)^\perp \to D_q,\; q = \tau(\eta),$$

$$u^i \vec{h}_i \mapsto u^i X_i|_q.$$

Proof: The submanifold $D^0 = \{\eta = k_\alpha \eta^\alpha\}$ is defined by the equations

$$h_i = 0,\; i = 1, \cdots, m.,$$

Hence,

$$T_\eta D^0 = \{v \in T_\eta(T^*Q), 0 =< dh_i, v >= \omega(\omega^{-1} dh_i, v) = \omega(\vec{h}_i, v)\}.$$

□

Since $D^0 = \{\eta = k_\alpha \eta^\alpha\}$, k_α are fiberwise coordinate of the bundle $D^0 \to Q$. We identify X_i, Y_α with "horizontal" vector fields in T^*Q, which annihilate the fiberwise coordinates h_i, k_α. Then $\partial_{k_\alpha}, X_i, Y_\alpha$ form a frame in the tangent bundle $T(D^0)$. The tangent vector to a curve $\eta(t) = k_\alpha(t)\eta^\alpha(t) \subset D^0$ with projection $\gamma(t) = \tau\eta(t)$ can be written as

$$\dot{\eta}(t) = \dot{k}_\alpha \partial_{k_\alpha} + \dot{\gamma}^i X_i(\gamma(t)) + \dot{\gamma}^\alpha Y_\alpha(\gamma(t)). \tag{5}$$

We need an explicit description of the restriction $-\omega^0 = dk_\alpha \wedge \eta^\alpha + d\eta^\alpha$ to D^0 of the symplectic 2-form $\omega = -d\theta$. We may write the 2-form $d\eta^\alpha$ as

$$d\eta^\alpha = -c^\alpha_{ij}\theta^i \wedge \theta^j - c^\alpha_{i\beta}\theta^i \wedge \eta^\beta - c^\alpha_{\beta\delta}\eta^\beta \wedge \eta^\delta$$

where $c^\alpha_{ij} = \eta^\alpha([X_i, X_j])$, $c^\alpha_{\beta i} = \eta^\alpha([Y_\beta, X_i])$, $c^\alpha_{\beta\delta} = \eta^\alpha(Y_\beta, Y_\delta)$.
Then

$$\omega^0 = -dk_\alpha \wedge \eta^\alpha + k_\alpha(c^\alpha_{ij}\theta^i \wedge \theta^j + c^\alpha_{i\beta}\theta^i \wedge \eta^\beta + c^\alpha_{\beta\delta}\eta^\beta \wedge \eta^\delta).$$

Now we are ready to write down the necessary and sufficient condition that a tangent vector $\dot{\eta}(t) \in T_{\eta(t)}D^0$ of a curve $\eta(t) \subset D^0$ belongs to $Ch_{\eta(t)}(D^0)$.
Calculating the contraction $\dot{\eta}\lrcorner\omega^0$, we get

$$\dot{\eta}\lrcorner\omega^0 = \dot{\gamma}^\alpha dk_\alpha + \kappa_\alpha(c^\alpha_{ij}\dot{\gamma}^j - c^\alpha_{i\beta}\dot{\gamma}^\beta)\theta^i + (k_\alpha c^\alpha_{i\beta}\dot{\gamma}^i + k_\alpha c^\alpha_{\beta\delta}\dot{\gamma}^\delta - \dot{k}_\beta)\eta^\beta.$$

In particular, a curve $\eta(t) = k_\alpha \eta^\alpha \subset D^0$ is a characteristic curve if and only if its velocity vector (5) satisfies the following equations

$$\begin{array}{lll} i) & \dot{\gamma}^\alpha & = 0, \\ ii) & \dot{k}_\beta - k_\alpha c^\alpha_{i\beta}\dot{\gamma}^i & = 0, \\ iii) & k_\alpha c^\alpha_{ij}\dot{\gamma}^i & = 0. \end{array}$$

For $q \in Q$, denote by $\Lambda^2 D^*_q$ the space of 2-forms in D_q and by D^0_q the fiber of the bundle $\tau : D^0 \to Q$. There is a natural linear map

$$\bar{d} : D_q \to \Lambda^2 D^*_q$$

$$\eta \mapsto d\tilde{\eta}|_{\Lambda^2 D_x},$$

where $\tilde{\eta}$ is an extension of η to a local 1-form. If $\tilde{X}, \tilde{X}'$ are extensions of vectors $X, X' \in D_q$ to local sections of D, then

$$\bar{d}\eta(X, X') = -\eta([\tilde{X}, \tilde{X}']).$$

This shows that the map $\bar{d}$ does not depend on extensions $\tilde{\xi}, \tilde{X}, \tilde{X}'$.
We set

$$K_\eta = \ker \bar{d}\eta \subset D_{\tau(\eta)}.$$

Proposition 6 *The projection $\tau_* : T_\eta(D^0) \to D_{\tau(\eta)}$ induces an isomorphism*

$$\tau_* : Ch_\eta(D^0) = \ker \omega^0_\eta \to K_\eta.$$

Proof: Lemma 5 shows that $\tau_* : T_\eta D^0 \to D_{\tau(\eta)}$ is an isomorphism. The conditions i), iii) may be rewritten as

$$\dot\gamma = \dot\gamma^i(t) X_i \in K_{\eta(t)} = K_{k_\alpha \eta^\alpha}.$$

Any characteristic vector $\dot\eta \in Ch_\eta(D^0) = \ker \omega^0_\eta$ can be written now as

$$\dot\eta = \dot k_\alpha \partial_{k_\alpha} + \dot\gamma = k_\beta c^\beta_{\alpha i} \dot\gamma^i \partial_{k_\alpha} + \dot\gamma^i X_i$$

and it is completely determined by the point $\eta = k_\alpha \eta^\alpha \in D^0_q$ and the tangent vector $\dot\gamma \in K_\eta \subset D_{\tau(\eta)}$. □

As a corollary, we get the following characterization of characteristic curves and abnormal geodesics.

Theorem 7 *i) A curve $\eta(t) = k_\alpha(t)\eta^\alpha(\gamma(t)) \subset D^0$ with the projection $\gamma(t) = \tau(\eta(t))$ is a characteristic and then $\gamma(t)$ is an abnormal geodesic if and only if the velocity vector field has the form*

$$\dot\eta(t) = k_\beta c^\beta_{\alpha i} \dot\gamma^i \partial_{k_\alpha} + \dot\gamma^i X_i$$

such that $\dot\gamma(t) = \dot\gamma^i X_i \in K_{\eta(t)}$.
ii) A horizontal curve $\gamma(t) \subset Q$ with velocity vector field $\dot\gamma(t) = \dot\gamma^i(t) X_i(\gamma(t))$ is an abnormal geodesic if and only if it can be lifted to a characteristic curve $\eta(t) \subset D^0$ such that $\dot\gamma(t) \in K_{\eta(t)}$.

3 Sub-Riemannian geodesics as straightest curves

3.1 d'Alembert's sub-Riemannian geodesics

Let (Q, D, g^D) be a sub-Riemannian manifold. To define d'Alembert's (shortly, dA) geodesics, we extend the sub-Riemannian metric g^D to a Riemannian metric g^Q.
The d'Alembert's principle of virtual displacements for a mechanical system may be formulated as follows, see [35].
1) The evolution of a mechanical system with a (smooth) configuration space Q is described by projection to Q of integral curves of a special vector field $X \in \mathfrak{X}(TQ)$ (the evolution field). A field X is called **special** if it corresponds to a second order equation, that is $\pi_* X_{(q,\dot q)} = \dot q$ where $\pi : TQ \to Q$ is the projection.
2) The vector field X is determined by the Lagrangian force, defined as the horizontal 1-form $F_L := (\delta L(q, \dot q))_i dq^i$ on TQ, associated with the Lagrangian $L(q, \dot q)$, and external forces.
3) d'Alembert's Principle states that the special vector field X, which describes the real dynamics of a mechanical system, is determined by the condition that the Lagrangian force is equal to the external force.
Assume that
i) the Lagrangian $L(q, \dot q)$ of the system with a configuration space Q is quadratic in

velocities $\dot{q}$ and positively defined (that is can be written as $L = \frac{1}{2}g(\dot{q}, \dot{q})$, where g is a Riemannian metric in Q) and that
ii) the only external force is the reaction of a non-holonomic constraint, defined by a rank-m distribution $D = \ker \eta^1 \cap \cdots \cap \ker \eta^k$, where $\eta^\alpha = \eta^\alpha_i dq^i$, $\alpha = 1, \cdots, k = n - m$ is a coframe of the codistribution D^0. The reaction of the constraint is the horizontal 1-form $\phi_\lambda = \lambda_\alpha(q, \dot{q})\eta^\alpha \in \Omega^1(TQ)$ defined by the condition that the equation $F_L - \phi_\lambda = 0$ corresponds to a vector field $X \in \mathfrak{X}(TQ)$ tangent to the distribution $D \subset TQ$.
In coordinates, this equation take the form [38]

$$F_L \equiv (\frac{d}{dt}L_{\dot{q}_i} - L_{q_i})dq^i = \lambda_\alpha(q(t), \dot{q}(t))\eta^\alpha_i dq^i$$

or

$$\frac{d}{dt}L_{\dot{q}_i} - L_{q_i} \equiv 0 \quad (\mathrm{mod} D^0).$$

The projection to Q of integral curves of this equation is called **dA-geodesic** of the sub-Riemannian metric (D, g^D), associated with an extension of g^D to a Riemannian metric g on Q. In general, the equation of dA-geodesics is neither Lagrangian nor Hamiltonian.

3.2 Schouten-Synge-Vranceanu sub-Riemannian geodesics

Recall that Levi-Civita associated to a Riemannian manifold (Q, g) the canonical torsion free connection ∇^g, which preserves the metric (called the Levi-Civita connection). According to Levi-Civita, a geodesic is defined as an autoparallel curve $q(t)$, such that the velocity vector field $\dot{q}(t)$ is parallel along $q(t)$, i.e. satisfies the geodesic equation

$$\nabla^g_{\dot{\gamma}}\dot{\gamma} \equiv \ddot{q}^i(t) + \Gamma^i_{jk}(q^j(t))\dot{q}^j(t)\dot{q}^k(t) = 0$$

where Γ^i_{jk} are the Christoffel symbols of the metric $g = g_{ij}(q)dq^i dq^j$. The extension of this definition to sub-Riemannian manifolds had been proposed independently by J.A. Schouten, J.L.Synge and G. Vranceanu, see [10].

3.2.1 Schouten partial connection of a sub-Riemannian manifold

Let $D \subset TQ$ be a distribution. A partial D-connection in D is an $\mathbb{R}$-bilinear map

$$\nabla^D : \Gamma D \times \Gamma D \to \Gamma D,\ (X, Y) \mapsto \nabla^D_X Y$$

which is $C^\infty(Q)$ linear in X and satisfies the Leibnitz rule in Y:

$$\nabla^D_X(fY) = f\nabla^D_X Y + (X \cdot f)Y,\ f \in C^\infty(Q).$$

Let e_i, $i = 1, \cdots, m$ be a frame of D defined in a neighborhood of a horizontal curve $q(t)$.
The Christoffel symbols of a partial connection ∇^D are the local functions $\Gamma^i_{jk}(q)$ on Q defined by

$$\nabla_{e_j} e_k = \Gamma^i_{jk}(q)e_i.$$

The value of the functions $\Gamma^i_{jk}(t) := \Gamma^i_{jk}(q(t))$ on a horizontal curve $q(t)$ depends only on the frame $e_i(t) := e_i(q(t))$ along the curve $q(t)$. Due to this, the partial connection defines a parallel transport of a vector $Y_0 \in D_{q_0}$ along a horizontal curve q_t as the solution $Y(t) = Y^c(t)e_c(t) \in D_{q_t}$ of the equation

$$0 = \nabla_{\dot{q}_t} Y(t) = \nabla_{\dot{q}_t}(Y^i(t)e_i(t)) = [\dot{Y}^i(t) + \Gamma^i_{jk}(t)q^j_t Y^k(t)]e_i(t).$$

I.A. Schouten showed that a complementary to D distribution V on a sub-Riemannian manifold (Q, D, g^D) (called a **rigging**) defines a partial connection ∇^S in D which preserves the metric g^D and has zero torsion T. The torsion tensor is defined by

$$T(X,Y) = \nabla^S_X Y - \nabla^S_Y X - [X,Y]_D,\ X, Y \in \Gamma D,$$

where X_D is the horizontal part of the vector

$$X = X_D + X_V \in T_qQ = D_q \oplus V_q.$$

In coordinate-free way, the Schouten partial connection of (Q, D, g^D) associated to a rigging V is defined by the Koszul formula

$$\begin{aligned} 2g(\nabla^S_X Y, Z) = \quad & X \cdot g(Y,Z) + Y \cdot g(X,Z) - Z \cdot g(X,Y) + \\ & g([X,Y]_D, Z) - g(Y,[X,Z]_D) - g(X,[Y,Z]_D), \\ & X, Y, Z \in \Gamma(D). \end{aligned}$$

Schouten defined the curvature tensor $R \in \mathfrak{so}(D) \otimes \Lambda^2 T^*M$ of the Schouten connection by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - [[X,Y]_V, Z]_D,\ X, Y, Z \in \Gamma D.$$

V.V. Wagner generalized this notion and defined Wagner curvature tensor, such that the vanishing of the Wagner tensor is equivalent to the flatness of the Schouten connection (that is the property that the associated parallel transport does not depend on the path, connecting two points), see [10], [17] for a modern exposition and generalization of this theory.

3.2.2 Sub-Riemannian S-geodesics and non-holonomic mechanics

Schouten-Synge-Vranceanu geodesics (S-geodesics) of a sub-Riemannian manifold (Q, D, g^D), associated to a rigging V, are defined as horizontal curves $\gamma(t)$ with parallel (w.r.t. Schouten connection) tangent vector field $\dot{\gamma}(t)$, i.e. solutions of the equation $\nabla^S_{\dot{\gamma}}\dot{\gamma} = 0$.
Assume that the sub-Riemannian metric g^D is extended to a Riemannian metric g on Q. Denote by $V = D^\perp$ the g-orthogonal complement to D. Then the Levi-Civita connection ∇^g induces a connection ∇^D in D given by

$$\nabla^D_X Y = \mathrm{pr}_D \nabla^g_X Y = (\nabla^g_X Y)_D,\ X \in TQ, Y \in \Gamma D.$$

where $\mathrm{pr}_D : TQ = D \oplus D^\perp \to D$ is the natural projection.

The connection ∇^D is an extension of the partial Schouten connection ∇^S associated to the rigging $D^\perp$.

Theorem 8 *(Vershik-Faddeev[35], [36]) Let* (Q, D, g^D) *be a* sub-Riemannian manifold, g an extension of g^D to a metric in Q and $V = D^\perp$ the orthogonal complement to D. Then S-geodesics coincide with dA-geodesics and they describe evolution of the free mechanical system with kinetic energy g in configuration space Q with nonholonomic linear constraint D.

4 Cartan frame bundle definition of geodesics

An important frame bundle definition of Riemannian geodesics had been proposed by E. Cartan. It is naturally generalized to a wide class of geometric structures. Below we consider Cartan approach to definition of geodesics for Cartan connections and G-structures of finite type. This will be used for definition of Morimoto geodesics on a regular sub-Riemannian manifold.

4.1 Cartan definition of Riemannian geodesics

A Riemannian metric g on a manifold Q can be considered as a $G = O_n$-structure, i.e. a principal G-subbundle $\pi : P \to Q = P/G$ of the bundle of orthonormal frames (i.e. isometries $f : \mathbb{R}^n = V \to T_xQ$) with the tautological soldering form

$$\theta : TP \to V, \quad \theta_f(X) := f^{-1}(\pi_* X).$$

The total space P of an O_n-structure admits a canonical O_n-equivariant absolute parallelism (Cartan connection)

$$\kappa = \theta \oplus \omega : TP \to V \oplus \mathfrak{so}(V),$$

which is an extension of the vertical parallelism $i_p : T^v_p P \simeq \mathfrak{so}_n$, $\forall p \in P$ (defined by the free action of O_n on P). Here $T^v P \subset TP$ is the vertical subbundle and $\omega : TP \to \mathfrak{so}_n$ is the connection form of the Levi-Civita connection.
Cartan geodesics (C-geodesics) are defined as the projection to Q of constant horizontal vector fields $X \in \kappa^{-1}(V) \subset \mathfrak{X}(P)$, see [21].

4.2 Normal Cartan connection and C-geodesics

We recall the definitions of a Cartan connection and associated C-geodesics.
Let $M_0 = L/G$ be an n-dimensional homogeneous manifold.
A **Cartan connection of type** $M_0 = L/G$ on an n-dimensional manifold Q is a principal G-bundle $\pi : P \to Q = P/G$ together with an $\mathfrak{l}$-valued G-equivariant (s.t. $r_g^* \kappa = \mathrm{Ad}_g^{-1} \circ \kappa$, $g \in G$) kernel-free 1-form $\kappa : TP \to \mathfrak{l}$ which extends the vertical parallelism $T^v_p P \simeq \mathfrak{g}$.
The form κ defines an absolute parallelism $\kappa_p : T_p P \simeq \mathfrak{l}$. Hence, tensor fields on P may be identified with tensor-valued functions.
In particular, the horizontal **curvature 2-form** $\Omega := d\kappa + \frac{1}{2}[\kappa, \kappa]$ on P can be identified with a function $K : P \to C^2(\mathfrak{n}, \mathfrak{l}) := \Lambda^2 \mathfrak{n}^* \otimes \mathfrak{l}$ where $\mathfrak{n} = \mathfrak{l}/\mathfrak{g}$.

One of the most powerful method for studying different (holonomic and non-holonomic) geometric structures and for constructing their invariants is based on construction of the associated canonically defined Cartan connection. In many cases, it is not difficult to associate to a given structure a family of Cartan connections. Then the problem comes down to finding suitable normalization conditions which uniquely specify a Cartan connection (called the **normal Cartan connection**) of this family. The standard way is to impose some normalization conditions on the curvature function K, for example, the condition that the curvature tensor K_p, $\forall p \in P$ is coclosed.
We explain this condition in the case when the homogeneous manifold $M_0 = L/G$ satisfies the following property :

(∗) The Lie algebra $\mathfrak{l}$ admits an ad $\mathfrak{g}$-invariant metric g and the g-orthogonal complement $\mathfrak{m}$ to $\mathfrak{g}$ in $\mathfrak{l}$ is a subalgebra.
This is sufficient to define Morimoto geodesics for regular sub-Riemannian structures. Let

$$\partial : C^1(\mathfrak{m},\mathfrak{l}) = \mathrm{Hom}(\mathfrak{m},\mathfrak{m}+\mathfrak{g}) \to C^2(\mathfrak{m},\mathfrak{l})$$

be the differential of the complex of exterior forms on the Lie algebra $\mathfrak{m}$ with values in $\mathfrak{m}$-module γ. Denote by

$$\partial^* : C^2(\mathfrak{m},\mathfrak{l}) \to C^1(\mathfrak{m},\mathfrak{l})$$

the dual codifferential, defined by means of the induced metrics on $C^j(\mathfrak{m},\mathfrak{l})$.
A **Cartan connection is called normal** if the curvature K is coclosed, i.e. $\partial^* K = 0$.

Theorem 9 *(see [25], [5], [12]) Let $(\pi : P \to Q,\ \kappa : TP \to \mathfrak{l} = \mathfrak{g}+\mathfrak{m})$ be a Cartan connection, which satisfies the condition (*). Then the bundle π admits a unique normal Cartan connection κ_0.*

More general sufficient conditions for the existence of a unique normal Cartan connection are given in [?], [14], [5], [12], [13].

Let $(\pi : P \to \mathfrak{l},\ \kappa : TP \to \mathfrak{l})$ be a Cartan connection and $\mathfrak{m}$ a fixed complementary to $\mathfrak{g}$ subspace of $\mathfrak{l}$. Then $\kappa^{-1}(\mathfrak{m}) \subset TP$ is a complementary to T^vP distribution, called the **horizontal distribution** and any vector $v \in \mathfrak{m}$ defines a horizontal vector field $X^v = \kappa^{-1}(v)$, called the **constant horizontal vector field associated to** v.

Like in Riemannian case, **C-geodesics of a Cartan connection** are defined as the projection to Q of integral curves of constant horizontal vector fields.

Assume that the homogeneous manifold $M_0 = L/G$ is **reductive**, i.e. there is a reductive decomposition $\mathfrak{l} = \mathfrak{g} \oplus V$, where V is an Ad_G-invariant complement to $\mathfrak{g}$. Let (π,κ) be Cartan connection of a reductive type $M_0 = L/G$. Denote by

$$\theta := \mathrm{pr}_V \circ \kappa : TP \to V \ \text{(resp., } \omega := \mathrm{pr}_{\mathfrak{g}} \circ \kappa : TP \to \mathfrak{g})$$

the horizontal part (resp., the vertical part) of the 1-form κ. Then θ is a soldering form, which turns π into a G-structure, and ω is a connection form, which defines a principal connection in π. The form ω defines a linear connection ∇ in the tangent bundle $TQ = P \times_G V$, see [21] and C-geodesics of the Cartan connection coincide with geodesics of ∇.

4.3 C-geodesics for G-structures

4.3.1 G-structures and their torsion function

We recall the definition of G-structure and its torsion function.

Let $G \subset GL(V)$, $V = \mathbb{R}^n$ be a linear Lie group. A **G-structure** on an n-dimensional manifold Q is a G-principal bundle $\pi : P \to Q = P/G$ with a **soldering 1-form** $\theta : TP \to V$ i.e. a strictly horizontal (ker $\theta = T^vP$) G-equivariant 1-form. Such 1-form allows to identify the G-principal bundle with a G-principal bundle of

frames on TQ.
Indeed, the soldering form at a point $p \in P$ defines a coframe, i.e. an isomorphism

$$\theta_p : T_{\pi(p)}Q \to V.$$

We denote by

$$\hat{p} = \theta_p^{-1} : V \to T_{\pi(p)}Q$$

the dual frame. This allows to identify the bundle π with a G-principal bundle of frames.
Denote by $j^1(\pi) : J^1 \to P$ the bundle of 1-jets of local sections $H = H_p = j^1_p(s)$, that is horizontal subspaces $H \subset T_pP$ such that $TpP = T^v_p P \oplus H$.
The differential $d\theta$ of the soldering 1-form defines a function $\tau : J^1 \to \mathrm{Tor}(V)$ with values in the space $\mathrm{Tor}(V) := V \otimes \Lambda^2(V^*)$ of V-valued 2-forms. It is called the **torsion function** and it associates with a horizontal space $H = H_p$ the 2-form τ_H defined by

$$\tau_H(u, v) = d\theta(u^H, v^H) \in V, \ u, v \in V$$

where u^H, v^H are the horizontal lifts to $H \subset T_pP$ of tangent vectors $\hat{p}u, \hat{p}v \in T_{\pi(p)}Q$.

4.3.2 C-geodesics of a G-structure of type $k = 0$

Assume that the linear Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{gl}(V)$ has type $k = 0$, i.e. has trivial first prolongation

$$\mathfrak{g}^{(1)} := \mathfrak{g} \otimes V^* \cap V \otimes S^2(V^*) = 0.$$

Then the Spencer differential

$$\partial : \mathfrak{g} \otimes V^* \to \mathrm{Tor}(V)$$

$$\partial(A \otimes \xi)(u, v) = \xi(u)Av - \xi(v)Au \in V$$

is an embedding. Assume that there is a G-invariant complementary subspace W to the image $\partial(\mathfrak{g} \otimes V^*)$ in $\mathrm{Tor}(V)$. Then the preimage $D := \tau^{-1}(W)$, where τ is the torsion function, see 4.3.1, is a G-invariant distribution of horizontal subspaces. More precisely, for any $p \in P$ there is a unique horizontal subspace $D = D_p$ such that $\tau_H \in W$ and the field $P \ni p \to H_p$ is G-invariant. Such distribution defines a linear connection in the frame bundle $\pi : P \to Q$ with the connection form $\omega : TP = T^vP \oplus D \to \mathfrak{g}$ which has kernel $\ker \omega = D$ and coincides with the vertical parallelism $T^vP \to \mathfrak{g}$ on T^vP. Then the sum

$$\kappa = \theta + \omega : TP \to V \oplus \mathfrak{g}$$

is a Cartan connection. Like in the Riemannian case, **C-geodesics** are defined as projection to Q of constant horizontal vector fields $X \in \kappa^{-1}(V)$, and they coincides with geodesics of the linear connection ω, see [21].

Example 10 *Let (Q, g) be a Riemannian manifold and $\pi : P \to Q$ the $O(V)$-bundle of orthonormal frames, i.e. $O(V)$ -structure.*
One may easily check that the first prolongation of the orthogonal Lie algebra $\mathfrak{so}(V)$ is trivial and that the map $\partial : \mathfrak{so}(V) \otimes V^* \to Tor(V)$ is an isomorphism. Taking $W = 0$, we get the distribution $D = \tau^{-1}(0)$, defined by the condition $\tau|D = 0$. The associated connection is the Levi-Civita connection.

More generally, let $G \subset O(V)$ be a closed subgroup of the orthogonal group $O(V)$ and $\pi : P \to Q$ a G-structure. Let $\mathfrak{g}^\perp$ be the orthogonal complement to the subalgebra $\mathfrak{g} = \mathrm{Lie}(G)$ in $\mathfrak{so}(V)$ with respect to the Killing form. Then $W = \partial(\mathfrak{g}^\perp \otimes V^*)$ is a G-invariant complement to $\partial(\mathfrak{g} \otimes V^*)$ in $\mathrm{Tor}(V)$. The corresponding distribution $D = \tau^{-1}(W)$ defines a linear connection ω^{can} in π with torsion in W. In the classical language, this means that at any point $q \in Q$, the torsion tensor T_q of the linear connection ω at a point $q \in Q$, calculated with respect to a frame $\hat{p}$, takes values in the subspace $W = \mathfrak{g}^\perp \oplus V^*$.

We call the connection ω^{can} the **canonical connection of the G-structure with $G \subset O(V)$.** We use this example for the definition of the sub-Riemannian geodesics in the sense of T. Morimoto.

4.3.3 C-geodesics for a G-structure of finite type $k > 0$

Assume that $G \subset GL(V)$ group G has the finite type k, that is its Lie algebra $\mathfrak{g}$ has non trivial k-th prolongation $\mathfrak{g}^{(k)}$ and $\mathfrak{g}^{(k+1)} = 0$. Then the full prolongation

$$\mathfrak{g}^{(\infty)} = \sum_{j=-1}^{\infty} \mathfrak{g}^{(j)} = V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)},\ V = \mathfrak{g}^{(-1)},\ \mathfrak{g} = \mathfrak{g}^{(0)}$$

is a finite dimensional $\mathbb{Z}$-graded Lie algebra, see [29]. The bundle π can be prolonged ([29]) to a bundle $\pi^{(k)} : P^{(k)} \to Q$ with absolute parallelism

$$\kappa : TP^{(k)} \to \mathfrak{g}^{(\infty)} = V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)}.$$

C-geodesics for the G-structure $\pi : P \to Q$ of finite type k are defined as the projection of orbits of constant vector fields $X \in \kappa^{-1}(V) \subset \mathfrak{X}(P^{(k)})$ to Q.

Remark 11 *In general, $\pi^{(k)} : P^{(k)} \to Q$ is not a principal bundle and κ is not a Cartan connection.*

Of particular interest is the case of G-structures, when $G \subset GL(V)$ is an irreducible linear Lie group of type $k = 1$. Then the full prolongation

$$\mathfrak{g}^\infty = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} = V \oplus \mathfrak{g} \oplus V^*$$

is a simple 3-graded Lie algebra. List of all such 3-graded Lie algebras is known and it is very short. In this case, the prolongation $\pi^{(1)} : P^{(1)} \to Q$ is a principal bundle with the structure group $G^{\geq 0} = G \cdot G^{(1)}$, associated to the Lie algebra $\mathfrak{g}^{\geq 0} = \mathfrak{g} + \mathfrak{g}^{(1)}$. Moreover, there exists a canonical choice of the absolute parallelism $\kappa : TP^{(1)} \to \mathfrak{g} + \mathfrak{g}^{(1)}$ (the **normal Cartan connection**) which is $G^{\geq 0}$-equivariant and has the coclosed curvature, see [20], [14], [5], [7].
The C-geodesics for such geometries form an interesting class of distinguished curves in Q, studied, for example, in [14], [15], [16], [**?**].
For the conformal structure, which can be considered as $\mathbb{R}^+ \cdot SO_n$-structure, generalized geodesics are conformal circles.

5 Regular sub-Riemannian manifolds and Morimoto geodesics

Now we consider less familiar approach for the definition and the study of sub-Riemannian geodesics, based on the theory of Cartan connections and Tanaka theory of non-holonomic G-structures (or Tanaka structures).

5.1 Regular distributions and regular sub-Riemannian structures

5.1.1 Regular distributions

Let $D \subset TM$ be a bracket generating distribution on M and $\mathcal{D}^{-1} := \Gamma\mathcal{D}$ the $C^\infty(M)$-module of sections. It generates a negative filtration

$$\mathcal{D}^{-1} \subset \mathcal{D}^{-2} \subset \cdots \subset \mathcal{D}^{-k} = \mathcal{X}(Q)$$

of the Lie algebra of vector fields, inductively defined by

$$\mathcal{D}^{-i-1} := \mathcal{D}^{-i} + [\mathcal{D}^{-i}, \mathcal{D}^{-i}],\ i = 1, 2, \cdots .$$

The restriction $D_q^{-i} = \mathcal{D}^{-i}|_q$ of vector fields to a point $q \in Q$ defines a flag of subspaces

$$D_q^{-1} \subset D_q^{-2} \subset \cdots \subset D_q^{-k} = T_qQ \tag{6}$$

of the tangent space. The associated graded space

$$T_q^{gr}Q = \mathfrak{m}_q = \mathfrak{m}_q^{-1} \oplus \mathfrak{m}_q^{-2} \oplus \cdots \oplus \mathfrak{m}_q^{-k} := D_q^{-1} \oplus D_q^{-2}/D_q^{-1} \oplus \cdots + \oplus D_q^{-k}/D_q^{-(k-1)}$$

has the structure of negatively graded metric Lie algebra, induced by the Lie bracket of vector fields. The graded Lie algebra $\mathfrak{m}_q$ is called the **symbol algebra of the distribution D at a point q** or **graded tangent space at q.**
The distribution D is called a **regular distribution of type $\mathfrak{m}$ and depth k**, if all symbol algebras $\mathfrak{m}_q$, $q \in Q$ are isomorphic to a fixed negatively graded Lie algebra $\mathfrak{m} = \mathfrak{m}^{-1} \oplus \cdots \oplus \mathfrak{m}^{-k}$. Then (6) defines the **derived flag** of vector bundles

$$D^{-1} = D \subset D^{-2} \subset \cdots \subset D^{-k} = TQ.$$

Note that $\mathfrak{m}$ is a **fundamentally graded Lie algebra**, i.e. it is generated by $\mathfrak{m}^{-1}$.

5.1.2 Regular sub-Riemannian manifolds

Let (Q, D, g) be a sub-Riemannian manifold, where D is a regular distribution of type $\mathfrak{m}$.

Then the graded tangent space $T_q^{gr}Q = \mathfrak{m}_q$ has the structure of a **negatively graded metric Lie algebra**, i.e. a graded Lie algebra $\mathfrak{m}_q = \sum_{i=-1}^{-k} \mathfrak{m}_q^i$ with an Euclidean metric $g_q^{\mathfrak{m}}$ such that the graded spaces $\mathfrak{m}_q^i$ are mutually orthogonal.

The metric $g_q^{\mathfrak{m}}$ is a natural extension of the sub-Riemannian metric g_q^D in D_q, which is described in the following elementary lemma.

Lemma 12 *Let $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$ be a negatively graded fundamental Lie algebra. Then an Euclidean metric g on $\mathfrak{m}^{-1}$ has a natural extension to an Euclidean metric $g^{\mathfrak{m}}$ in $\mathfrak{m}$.*

A sub-Riemannian manifold (Q, D, g^D) with a regular distribution D of type $\mathfrak{m}$ is called a **regular** sub-Riemannian manifold of the metric type $(\mathfrak{m}, g^{\mathfrak{m}})$ if all metric Lie algebras $(\mathfrak{m}_q, g_q^{\mathfrak{m}})$ are isomorphic to the metric graded Lie algebra $(\mathfrak{m}, g^{\mathfrak{m}})$.

5.1.3 Regular sub-Riemannian structure as Tanaka structure

Let $D \subset TQ$ be a regular rank-m distribution of type $\mathfrak{m}$, and $\mathrm{Aut}(\mathfrak{m})$ the group of graded preserving automorphisms of $\mathfrak{m}$.

An **admissible frame** of D is an isomorphism

$$f : \mathfrak{m} \to T_q^{gr}Q = \mathfrak{m}_q$$

of graded Lie algebras. The automorphism group $\mathrm{Aut}(\mathfrak{m})$ acts freely and properly on the manifold $\mathrm{Fr}(D)$ of admissible frames on D with the orbit space $\mathrm{Fr}(D)/\mathrm{Aut}(\mathfrak{m}) = Q$. Hence, $\mathrm{Fr}(D) \to Q$ is a principal bundle (called **the bundle of admissible frames on** D).
Let $G^0 \subset \mathrm{Aut}(\mathfrak{m})$ be a Lie subgroup. A **Tanaka** G^0**-structure** (or **a relative** G^0**-structure**) is a G^0-principal subbundle $\pi : P \to Q = P/G^0$ of the bundle of admissible frames on D.

The classical identification of Riemannian manifolds with O_n-structures is extended to the sub-Riemannian case:

Proposition 13 *A regular sub-Riemannian manifold* (Q, D, g^D) *of type* $(\mathfrak{m}, g^{\mathfrak{m}})$ *is identified with a Tanaka* G^0*-structure with the structure group* $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}}) \subset O(\mathfrak{m})$*. Conversely, any* $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$*-Tanaka structure defines a regular sub-Riemannian manifold* (Q, D, g^D) *of type* $(\mathfrak{m}, g^{\mathfrak{m}})$*.*

Proof: The Tanaka G^0-structure, associated to (Q, D, g^D), consists of all admissible frames $f : \mathfrak{m} \to \mathfrak{m}_q$, which are isomorphisms of the metric Lie algebras.
Conversely, let $(\pi : P \to Q,\ D)$ be a Tanaka G^0-structure with $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$. Then the associated sub-Riemannian metric on D is defined by the condition that for any admissible frame $f \in P$, its restriction

$$f_{\mathfrak{m}^{-1}} : \mathfrak{m}^{-1} \to D_q = \mathfrak{m}_q^{-1}$$

is an isometry. □

5.2 Morimoto geodesics of sub-Riemannian manifolds

5.2.1 Tanaka prolongation of non-holonomic G-structures

N. Tanaka generalised the theory of G-structures to Tanaka structures. In particular, he defined the full prolongation of a non-positively graded Lie algebra $\mathfrak{g} = \sum_{i=-k}^{0} \mathfrak{g}^i = \mathfrak{g}^{-k} + +\mathfrak{g}^{-1} + \mathfrak{g}^0$ as a maximal $\mathbb{Z}$-graded Lie algebra of the form

$$\mathfrak{g}^{(\infty)} = \sum_{i=-k}^{\infty} \mathfrak{g}^i = \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \cdots$$

such that for any $X \in \mathfrak{g}^{(i)}$, $i > 0$ the condition $[X, \mathfrak{g}^{-1}] = 0$ implies $X = 0$.
A non-positively graded Lie algebra $\mathfrak{g}$ is called a **Lie algebra of finite type** ℓ , if $\ell < \infty$ is the maximal number such that $\mathfrak{g}^{(\ell)} \neq 0$.

Theorem 14 *(Tanaka)(see [14],[33], [42],[6]). Let $\pi : P \to Q$ be a Tanaka G^0-structure on (Q, D) where D is a regular distribution of type $\mathfrak{m} = \mathfrak{m}^{-1} + \cdots + \mathfrak{m}^{-k}$ and $G^0 \subset \mathrm{Aut}(\mathfrak{m})$ a connected closed subgroup of the automorphism group with Lie algebra $\mathfrak{g}^0 \subset \mathfrak{der}(\mathfrak{m})$. Assume that the non-positively graded Lie algebra $\tilde{\mathfrak{m}} := \mathfrak{m} + \mathfrak{g}^0$ has finite type ℓ. Then there is a canonical bundle $P^\infty \to Q$, constructed by successive prolongations, with an absolute parallelism $\kappa : TP^\infty \to \tilde{\mathfrak{m}}^\infty$. If the first prolongation $\mathfrak{g}^{(1)} = 0$, then the absolute parallelism $\kappa : TP \to \tilde{\mathfrak{m}} = \mathfrak{m} + \mathfrak{g}^0$ is a Cartan connection.*

5.2.2 Morimoto definition of sub-Riemannian geodesics

The Morimoto definition of sub-Riemannian geodesics is based on the following important theorem , which he proved in the framework of his remarkable theory of filtered manifolds [**?**].

Theorem 15 *(T. Morimoto [25]) i) Let $(\mathfrak{m} = \sum_{-k}^{-1} \mathfrak{m}^i, g^{\mathfrak{m}})$ be a fundamental negatively graded metric Lie algebra and $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$ the linear Lie group of orthogonal automorphisms with the Lie algebra $\mathfrak{g}^0 = \mathfrak{der}(\mathfrak{m}, g^{\mathfrak{m}})$. Then the full prolongation of the non-positively graded Lie algebra $\tilde{\mathfrak{m}} = \mathfrak{m} + \mathfrak{g}^0$ coincides with $\tilde{\mathfrak{m}}$.*
ii) Let (Q, D, g^D) be a regular sub-Riemannian manifold of the metric type $(\mathfrak{m}, g^{\mathfrak{m}})$. Then the associated Tanaka structure $\pi : P \to Q = P/G^0$ admits a canonically defined 1-form

$$\kappa : TP \to \tilde{\mathfrak{m}} = \mathfrak{m} + \mathfrak{g}^0 \tag{7}$$

such that (π, κ) is a normal Cartan connection of type L/G^0, where L is the simply connected Lie group associated with the Lie algebra $\tilde{\mathfrak{m}}$. Moreover, the horizontal part $\theta = \mathrm{pr}_{\mathfrak{m}} \circ \kappa$ of κ is a soldering form and the vertical part $\omega = \mathrm{pr}_{\mathfrak{g}} \circ \kappa$ is a principal connection.

The **Morimoto sub-Riemannian geodesics** (shortly, **M-geodesics**) of a regular sub-Riemannian manifold $((Q, D, g^D)$ is defined as projection to Q of the integral curves of a constant vector fields $X \in \kappa^{-1}(\mathfrak{m}^{-1})$, where κ is the associated Cartan connection (7).

5.3 Admissible rigging, associated Cartan connections and Cartan-Morimoto geodesics

Here we develop an elementary approach for constructing Cartan connections (in particular, normal Cartan connection) for regular sub-Riemannian manifold, considered as Tanaka structurs. It is working also for other Tanaka structures with trivial first prolongation. It is based on the notion of admissible rigging, see [7], and results from [5].
Let (Q, D, g^D) be a regular sub-Riemannian manifold of metric type $(\mathfrak{m}, g^{\mathfrak{m}})$, and $D^{-1} = D \subset D^{-2} \subset \cdots \subset D^{-k} = TQ$ the derived flag of distributions.
A complementary to D distribution V with a direct sum decomposition $V = V^{-2} \oplus \cdots \oplus V^{-k}$ as called an **admissible rigging** if V^{-j} as a complementary to D^{-j+1} subdistribution in D^{-j}. In other words,

$$TQ = D \oplus V^{-2} \oplus \cdots \oplus V^{-k},\ D^{-j} = D^{-j+1} \oplus V^{-j},\ j = 2, \cdots, k.$$

Since $\mathfrak{m}^{-j} := D^{-j}/D^{-j+1} = (D^{-j+1} \oplus V^{-j})/D^{-j} \simeq V^{-j}$, the admissible rigging V defines an isomorphism $\psi_V : T^{gr}Q \to TQ$ of vector bundles. It induces an isomorphism $\hat{\psi}_V : P \to P^V$ of the Tanaka structure $\pi : P \to Q = P/G^0$, associated to the sub-Riemannian manifold, onto a G^0-structure, which we denote by $\pi^V : P^V \to P^V/G^0$. Identifying these principal bundles, we will consider the soldering form θ^V of the G^0-structure π^V as a soldering form on P. It turns the Tanaka principal bundle $\pi : P \to Q$ into a G-structure, with the structure group $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}}) \subset O(\mathfrak{m})$. We denote by g the associated Riemannian metric in Q and by ω^V the canonical connection form.The principal connection ω^V defines a Riemannian connection ∇^{ω^V} with torsion. Denote by $\tau_{\theta^V} : P \to W := \partial(\mathfrak{g}^{\perp} \otimes \mathfrak{m}^*) \subset Tor(\mathfrak{m})$ the associated torsion function. Then the 1-form

$$\kappa^V := \theta^V + \omega^V : TQ \to \mathfrak{m} + \mathfrak{g}$$

defines a structure of Cartan connection in the principal bundle $\pi : P \to Q$.
This Cartan connection induces the Tanaka structure $\pi : P \to Q$ via the isomorphism $\psi : TQ \to T^{gr}Q$. Moreover, any Cartan connection, which induces the Tanaka structure π, is associated with some admissible rigging, see [5], proposition 2.
Like in the case of normal connections, we define sub-Riemannian geodesics as the projection to Q of the integral curves of constant vector fields from $\kappa^{-1}(\mathfrak{m}^{-1})$. We call such geodesics **CM-geodesics associated to an admissible rigging**. Since the connection ∇^{ω^V} preserves the sub-Riemannian metric (D, g^D), CM-geodesics are D-horizontal geodesics of the connection ∇^{ω^V} or, in other words, the geodesics of the partial connection on D, which is the restriction of ∇^{ω^V} to D. Note that this partial connection coincides with the Schouten partial connection ∇^S associated to the rigging V if and only if the torsion function τ of the soldering form θ^V satisfies the condition $\tau_p(D, D) \subset V$, $\forall p \in P$. We get

Proposition 16 *Let V be an admissible rigging of a regular sub-Riemannian manifold (Q, D, g^D). Then CM-geodesics of the Cartan connection κ^V coincide with the S-geodesics of the Schouten partial connection ∇^S, defined by the rigging V, if and only if the torsion function τ of the soldering form θ^V satisfies the condition $\tau_p(D, D) \subset V$, $\forall p \in P$.*

5.3.1 Admissible riggings and the normal Cartan connection

Here we apply the results from [5], to prove the existence and the uniqueness of an admissible rigging V which defines the normal Cartan connection κ^V associated to a regular sub-Riemannian manifold. It reduces the problem of constructing normal Cartan connections to an appropriate deformation of an admissible rigging.

The following theorem is an elaboration of the Morimoto theorem.

Theorem 17 *Let $\pi : P \to Q$ be the Tanaka G^0-structure, associated to a regular sub-Riemannian manifold (Q, D, g^D) with a metric symbol $(\mathfrak{m}, g^{\mathfrak{m}})$. Then there is a uniquely defined admissible rigging V_0, which defines a normal Cartan connection $(\pi : P = P^V \to Q, \kappa^{V_0} = \theta^{V_0} + \omega^{V_0})$.*

Proof: The proof follows from [5], theorems 1 and 2. For the uniqueness of the Cartan connection, we have to check that the first cohomology group for the cocycles of positive degree vanishes: $H^1(\mathfrak{m}, \mathfrak{m} + \mathfrak{g})_1$‘ $= 0$. The degree of a cocycle from

$(\mathfrak{m}^{-i})^* \otimes \mathfrak{m}^{-j}$ is defined as $i - j$. The vanishing of this cohomology is a simple exercise. □

Corollary 18 *Let (Q, D, g^D) be a regular sub-Riemannian manifold with metric symbol algebra $(\mathfrak{m}, g^{\mathfrak{m}})$ and $\pi : P \to Q$ the associated Tanaka structure with the structure group $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}}) \subset O(\mathfrak{m})$. Then*
i) there exists the unique soldering form $\theta^0 : TP \to \mathfrak{m}$ which together with the associated canonical connection $\omega^{can} : TP \to \mathfrak{g}$ defines the normal Cartan connection $\kappa^{can} = \theta^0 + \omega^{can}$.
ii) The sub-Riemannian metric g^D admits a canonical extension to the Riemannain metric g on Q, defined by the condition that the isomorphism (coframes) $\theta^0_p : T_{\pi(p)}Q \to \mathfrak{m}$ is an isometry onto the Euclidean space $(\mathfrak{m}, g^{\mathfrak{m}})$ for $p \in P$.
iii) The isometry group $A = \mathrm{Iso}(Q, D, g^D)$ is a Lie group of dimension $\dim A \leq n + \dim G \leq n + \frac{m(m-1)}{2}$ where $m = \dim \mathfrak{m}$, which preserves the Riemannian metric g and acts freely with closed orbits on P. The stability subgroup A_q of a point $q \in Q$ has the exact isotropy representation $j : A_q \to GL(T_qQ)$ and the isotropy group $j(A_q)$ is identified with a subgroup of the group $G^0 = \mathrm{Aut}(\mathfrak{m}, g^{\mathfrak{m}})$ and has dimension $\dim A_q \leq \frac{m(m-1)}{2}$.

Proof: i) and ii) directly follow from the Theorem. iii) The isometry group A preserves the absolute parallelism κ^{can} on P and the Riemannian metric g. By the Kobayashi theorem about the automorphism group of an absolute parallelism [29], it acts freely with closed orbits on P. Hence $\dim A \leq \dim P = n + \dim G \leq \frac{m(m-1)}{2}$, since G is identified with a subgroup of the orthogonal group $O(\mathfrak{m})$. The stability subgroup A_q has exact isotropy representation $j : A_q \to GL(T_qQ)$ and since $j(A_q)$ preserves the metric of T_qQ and the derived flag $D_q \subset D_q^{-2} \subset \cdots \subset D_q^{-k}$, it acts freely in $T_qQ \simeq T^{gr}Q = \mathfrak{m}_q$ and preserves the Lie algebra structure in $\mathfrak{m}_q$. This implies that $j(A_q)$ is identified with a subgroup of G. □

6 Geodesics of Chaplygin transversally homogeneous sub-Riemannian manifolds

6.1 Principal connection and its curvature

Let $\pi : Q \to M = Q/G$ be a G-principal bundle with a right action $R_g q = qg$ of a Lie group G.
For $a \in \mathfrak{g} = \mathrm{Lie(G)}$, we denote by $a^* : q \mapsto qa := \frac{d}{dt} q \exp(ta)|_{t=0} \in T_qQ$ the fundamental vector field (the velocity vector field of $R_{\exp ta}$).
Recall that the principal connection is a G-equivariant $\mathfrak{g}$-valued 1-form $\varpi : TQ \to \mathfrak{g}$, which is an extension of the vertical parallelism, defined by $T_qQ \ni a^*_q \mapsto a \in \mathfrak{g}$. The equivariancy means that $R_g^* \varpi = \mathrm{Ad}_g^{-1} \circ \varpi$, $g \in G$.

The connection ϖ is completely determined by the horizontal G-invariant distribution $D = \ker \varpi$.
Since $T_qQ = D_q \oplus T_q^v Q$, any vector $X \in T_qQ$ is decomposed as $X = X^h \oplus X^v$ into the horizontal and the vertical parts. Moreover, any vector filed $X \in \mathfrak{X}(M)$ has the canonical (G-invariant) horizontal lift $X^D \in \Gamma D$ due to the isomorphism $\pi_* : D_q \to T_{\pi(q)}M$. Recall the following formulas, [21],

$$[a^*, b^*] = [a, b]^*, [a^*, X^D] = 0, \, [X^D, Y^D] = [X, Y]^D + [X^D, Y^D]^v. \tag{8}$$

where $a, b \in \mathfrak{g}$, $X, Y \in \mathfrak{X}(M)$.
Note that $[X^D, Y^D]^v$ is a G-invariant vertical vector field and its restriction to a fiber $\pi^{-1}(x)$ depends only on vectors X_x, Y_x. So we have a linear skew-symmetric map

$$\mathcal{A} : T_xM \times T_xM \to \mathfrak{X}(\pi^{-1}(x))^G, \ (X, Y) \mapsto \mathcal{A}(X, Y) := \frac{1}{2}[X^D, Y^D]^v$$

into the space of R_G-invariant vector fields along the fiber $\pi^{-1}(x)$.
For $X \in \mathfrak{X}(M)$, we defines a $C^\infty(Q)$ linear map

$$\mathcal{A}_X : \Gamma(D) \to \Gamma T^vQ, \, Y^D \mapsto [X^D, Y^D]^v = \mathcal{A}(X, \pi_* Y^D)$$

which maps G-invariant horizontal vector fields Y^D into G-invariant vertical vector fields from $\Gamma T^v(Q)$. The dual map

$$\mathcal{A}^*_X : \Gamma(T^v(Q)) \to \Gamma D$$

sends G -invariant vertical vector fields into G-invariant horizontal vector fields.
The curvature of ϖ is the $\mathfrak{g}$-valued horizontal 2-form $F = d\varpi + \frac{1}{2}\varpi \wedge \varpi$. Denote by e^*_α the basis of fundamental vector fields, associated to a basis e_α of $\mathfrak{g}$. Then the curvature form is related with the tensor $\mathcal{A}(X, Y)$ as follows:

$$\begin{aligned} F_q(X^D, Y^D) &= F_q(X^D, Y^D)^\alpha e_\alpha &&= -\varpi_q([X^D, Y^D]) + (\varpi_q \wedge \varpi_q)(X^D, Y^D) \\ &= -2\varpi_q(\mathcal{A}_q(X, Y)) &&= -2\varpi_q(\mathcal{A}^\alpha_q(X, Y)e^*_\alpha) \\ &= (\mathcal{A}_q(X, Y)^\alpha e_\alpha. \end{aligned} \tag{9}$$

6.1.1 Formulas in coordinates

To write formulas in coordinates, we fix a section s of π. To simplify notations, we assume that the section $s : M \to Q$ is global. It defines a trivialization

$$M \times G = Q, \ (x, g) = s(x)g$$

of the principal bundle, where the group G acts on $M \times G$ as

$$R_g(x, g_1) = (x, g_1 g).$$

A fundamental vector field a^*, $a \in \mathfrak{g}$ is identified with the left invariant vector fields

$$a^L : (x, g) \mapsto (x, ga) := (L_g)_* a.$$

We denote by $a^R : (x, g) \mapsto (x, ag)$ the right invariant vector field, generated by $a \in \mathfrak{g}$.
We identify the tangent bundle TG with the trivial bundle $\mathfrak{g} \times G$ (which we denote simply as $\mathfrak{g}$), using left translations

$$\mathfrak{g} \times G \ni (\dot{g}, g) = g^{-1}\dot{g} := (L_g)^{-1}_* \dot{g} \in T_gG$$

and the tangent bundle $T(M \times G) = TM \times TG$ with the bundle $TM \otimes \mathfrak{g}$ over $M \times G$. The tangent vector $\dot{x} + ga := \dot{x} + (L_g)^* a \in T_{(x,g)}(M \times G)$ will be denoted by $(\dot{x}, a)$, where $\dot{x} \in T_xM$, $a \in \mathfrak{g}$.
The pull back $\omega^s := s^*\varpi$ of the connection form ϖ to M is a $\mathfrak{g}$-valued 1-form

$\omega^s = A_i^\alpha(x)dx^i \otimes e_\alpha \in \Omega^1(M, \mathfrak{g})$ on M. The horizontal distribution along $M \times \{e\}$ is given by

$$D_{(x,e)} = \{\dot{x} - \omega^s(\dot{x})\} = \{\partial_i - A_i^\alpha(x)e_\alpha\}.$$

Since the distribution D is R_G-invariant, we get

$$D_{(x,g)} = (R_g)_* D_{(x,e)} = \{\dot{x} - A_i^\alpha(x)e_\alpha^R\} = \{\dot{x} - A_i^\alpha(x)(\mathrm{Ad}\,_g)_\alpha^\beta e_\beta^L\}$$

Note that $\varpi(e_\alpha^*) \equiv \varpi(e_\alpha^L) = e_\alpha$. This shows that $\varpi|_{TG}$ coincides with the left invariant Maurer-Cartan form $\mu(\dot{g}) = g^{-1}\dot{g}$. Hence, the connection form may be written as $\varpi = \mu + A$, where $A = A_i^\alpha(x,g)dx^i \otimes e_\alpha$ is a 1-form, which vanishes on TG. Solving the equations

$$0 = \varpi(\partial_i^D) = \varpi(\partial_i - A^\alpha(x)e_\alpha) = -\mu(A_i^\alpha(x)e_\alpha) + A_i^\alpha(x,g)e_\alpha =$$

$$-A_i^\alpha(x)(Ad_g^{-1})_\alpha^\beta e_\beta) + A_i^\alpha(x,g)e_\alpha,$$

we find that the connection form ϖ on $Q = M \times G$ is given by

$$\varpi = \mu + A,\ A = A_i^\alpha(x,g)dx^i \otimes e_\alpha = A_i^\alpha(x)dx^i \otimes (\mathrm{Ad}\,_g^{-1})_\alpha^\beta e_\beta. \tag{10}$$

6.2 Chaplygin metric and its standard extension

Let $\pi : Q \to M = Q/G$ be a principal bundle with a connection ϖ and $D = \ker \varpi$ the horizontal distribution.

A Riemannian metric g^M on the base manifold M defines a canonical invariant sub-Riemannian metric g^D on D such that the projection $\pi_* : D_q \to T_{\pi(q)}M$ is an isometry.

The sub-Riemannian metric (D, g^D) is called a Chaplygin metric and the sub-Riemannian manifold (Q, D, g^D) is called a **Chaplygin system** or a **transversally homogeneous** sub-Riemannian manifold.

Let (Q, D, g^D) be a Chaplygin system associated to a principal bundle $(\pi : Q \to M, \varpi)$ over a Riemannian manifolds (M, g^M) as above and $g^{\mathfrak{g}}$ an Euclidean metric on the Lie algebra $\mathfrak{g}$. It defines a degenerate metric

$$g^F(X,Y) = g^{\mathfrak{g}}(\varpi(X), \varpi(Y))$$

on Q with kernel D, whose restriction to a fiber $\mathcal{F}(x) = \pi^{-1}(x)$ is a Riemannian metric. We will consider also g^D as a degenerate metric on Q with $\ker g^D = T^vQ$.

Then

$$g^Q = g^F \oplus g^D \tag{11}$$

is a Riemannian metric in Q. It is called the **standard extension of the Chaplygin metric g^D.**

Note that the metric g^Q is G-invariant if and only if the degenerate metric g^F is invariant or, equivalently, the metric $g^{\mathfrak{g}}$ is Ad_G-invariant. In this case the metric g^Q is called the **bi-invariant extension** of the sub-Riemannian metric g^D.

Denote by ∇^M (resp., ∇^Q) the Levi-Civita connection of g^M (resp., g^Q), and by ∇^F the Levi-Civita connection of the induced metric g^F on a fiber, which is a totally geodesic submanifold of (Q, g^Q).

The Koszul formula implies the following O'Neill formulas for the covariant derivative of fundamental vector field b^* and the horizontal lift X^D of a vector field $X \in \mathfrak{X}(M)$, see [11].

$$
\begin{array}{lll}
i) & \nabla^Q_{a^*} b^* = & \nabla^F_{a^*} b^*, \ \ a, b \in \mathfrak{g} \\
ii) & \nabla^Q_{a^*} X^D = & \nabla^Q_{X^D} a^* = (\nabla^Q_{a^*} X^D)^h = \mathcal{A}^*_X a^*, \\
iii) & \nabla^Q_{X^D} Y^D = & (\nabla^M_X Y)^D + \mathcal{A}(X,Y), \ \ X, Y \in \mathfrak{X}(M).
\end{array}
$$

The connection ∇^F is described in terms of Lie brackets as follows

$$
2g(\nabla^F_{a^*} b^*, c^*) = g^{\mathfrak{g}}([a,b],c) - g^{\mathfrak{g}}(b,[a,c]) - g^{\mathfrak{g}}(a,[b,c]), \ a, b, c \in \mathfrak{g}.
$$

6.2.1 S-geodesics of a Chaplygin metric

The O'Neill formulas imply the following relations between S-geodesics of the Chaplygin sub-Riemannian metric and geodesics of the Riemannian metrics g^Q, g^M, see also [11].

Theorem 19 *i) The principal bundle $\pi : Q \to M$ with a standard metric g^Q associated to a Riemannian metric g^M is a Riemannian submersion with totally geodesic fibers.*
ii)A Riemannian geodesic $\gamma(s)$ of (Q, g^Q) which is horizontal at one point is horizontal and it projects onto the geodesic $\pi\gamma(t)$ of the base manifold (M, g^M).
iii) S-geodesics of the Chaplygin metric are precisely horizontal geodesics of (Q, g^Q) and they are horizontal lifts of geodesics of the base manifold.

Proof: Recall that S-geodesics are geodesics of the Schouten connection $\nabla^S_X Y = \mathrm{pr}_D \nabla^Q_X Y$, $X, Y \in \Gamma D$, that is horizontal curves $\gamma(s)$ which satisfy the equation

$$
0 = \nabla^S_{\dot\gamma(s)} \dot\gamma(s) = \mathrm{pr}_D \nabla^Q_{\dot\gamma(s)} \dot\gamma(s) = (\nabla^M_{\dot x} \dot x)^D.
$$

The result follows from the O'Neill formula iii) and skew-symmetry of the tensor $\mathcal{A}(X,Y)$. □

6.2.2 H-geodesics of a Chaplygin metric

Now we consider H-geodesics of the Chaplygin metric g^D on the principal bundle $(\pi : Q \to M, \varpi)$ over a Riemannian manifold (M, g^M) and study their relation with geodesics of the standard extension g^Q and the metric g^M of the base manifold.
As above, we fix a trivialization $Q = M \times G$ of the principal bundle π, defined by a section s. Then any fiber $F_x = x \times G$ with the metric g^F is identified with the Lie group G with the left invariant metric (still denoted by g^F) defined by the metric $g^{\mathfrak{g}}$. Note that the horizontal distribution $D = \ker \varpi$ on $Q = M \times G$ and the sub-Riemannian metric g^D are R_G-invariant, but not L_G-invariant, the fiberwise metric g^F is L_G-invariant, but not R_G-invariant and the metric g^Q is neither R_G-invariant, nor L_G-invariant.

The orthogonal decomposition $g^Q = g^D \oplus g^F$ of the metric g^Q defines the orthogonal decomposition $g_Q^{-1} = g_D^{-1} \oplus g_F^{-1}$ of the associated contravariant metric, which we denote by g_Q^{-1}. Here g_D^{-1} (resp., g_F^{-1})is the contravariant metric on D (resp., on T^vQ). They may be locally written as

$$
g_D^{-1} = \sum_{i=1}^m X_i \otimes X_i, \quad g_F^{-1} = \sum e^*_\alpha \otimes e^*_\alpha
$$

where (X_i) is a local orthonormal frame in D and (e_α) is an orthonormal basis of $\mathfrak{g}$. We will consider g_D^{-1} and g_F^{-1} as functions on T^*Q (cometrics).

The Hamiltonian $h_Q = \frac{1}{2} g_Q^{-1} \in C^\infty(T^*Q)$ of the geodesic flow of the Riemannian metric g^Q is the sum $h_Q = h_D + h_F$ of the Hamiltonian $h_D(\xi) := \frac{1}{2} g_D^{-1}(\xi, \xi)$ of the sub-Riemannian metric and the Hamiltonian $h_F(\xi) = \frac{1}{2} g_F^{-1}(\xi, \xi)$ of the fiberwise metric g^F.

Now we describe the relations between sub-Riemannian H-geodesics and Riemannian geodesics of the metric g^M on the base M and left invariant metric g^F on the group G. In the case, when the extension g^Q in bi-invariant, i.e. the metric $g^{\mathfrak{g}}$ is Ad_G invariant, they had been proved by R. Montgomery [23], Theorem 11.2.5 (the main theorem).

Lemma 20 *Let $g^Q = g^D \oplus g^F$ be a standard extension of a Chaplygin metric g^D. Then the Hamiltonians h_Q, h_F, h_D Poisson commute*

$$\{h_F, h_D\} = \{h_F, h_Q\} = 0,$$

and the associated Hamiltonian vector fields $\vec{h}_F, \vec{h}_D, \vec{h}_F$ commute.

Proof: A fundamental field a^* commutes with the horizontal lift X^D of a basic vector field, see (8), and preserves the decompositions $TQ = T^vQ + D$, $T^*Q = (T^vQ)^* + D^*$. We calculate the Lie derivative of the sub-Riemannian metric g^D as follows:

$$\begin{aligned}&(\mathcal{L}_{a^*} g^D)(X^D, Y^D) \\ &= a^* \cdot (g^D(X^D, Y^D)) - g^D([a^*, X^D], Y^D) + g^D(X^D, [a^*, Y^D]) \\ &= a^* \cdot g^M(X, Y) = 0.\end{aligned}$$

This shows that the fundamental field a^* preserves the sub-Riemannian metric g^D and the dual cometric g_D^{-1}. This means that

$$0 = \mathcal{L}_{a^*} g_D^{-1} = \{a^*, g_D^{-1}\} \equiv \{p_{a^*}, g_D^{-1}\}$$

where p_{a^*} is the Hamiltonian of the fundamental field a^*. Then the Leibnitz rule for Poisson bracket shows that $\{a^* \otimes a^*, g_D^{-1}\} \equiv \{p_{a^*}^2, g_D^{-1}\} = 0$. Hence the Hamiltonian $h_F = \frac{1}{2} \sum a_\alpha^* \otimes a_\alpha^* = \frac{1}{2} \sum (p_{a_\alpha})^2$ commute with $h_D = \frac{1}{2} g_D^{-1}$. □

Using the same arguments as in [23], we get

Theorem 21 *i)The sub-Riemannian geodesic flow of the sub-Riemannian metric g^D is a composition $\exp t\vec{h}_D = \exp t\vec{h}_Q \circ \exp(-t\vec{h}_F)$ of the Riemannian geodesic flows of the metric g^Q and the fiberwise metric g^F.*
ii) Denote by $g_a(t) \subset G$ the geodesic of the group G with the left invariant metric g^F as above with initial conditions $g_a(0) = e, \dot{g}_a(0) = a \in \mathfrak{g}$ and by $\gamma_w(t)$ the geodesic of the standard metric g^Q with initial conditions $\gamma(0) = q \in Q$, $\dot{\gamma}(0) = w = w^v + w^h \in T_qQ$. Then the curve $q(t) = \gamma_w(t) g_a(t)$ is a sub-Riemannian H-geodesic if and only if it has horizontal velocity $\dot{q}(0) = w + a_q^* = w^h \in D_q$ that is $\varpi(w) = -a$.
iii) Horizontal geodesics of g^Q are sub-Riemannian geodesics and they project to geodesics of (M, g^M).
iv) Sub-Riemannian geodesics are horizontal lifts of the projection of geodesics $\gamma_w(t)$ of g^Q to M.

Proof: i) is obvious. ii) The restriction $g^F|_{F_x}$ of g^F to any fiber $F_x = (x, G)$ is identified with the left invariant Riemannian metric (denoted again by g^F) on G, defined by the metric $g^{\mathfrak{g}}$.
Note that the projection of integral curves of the g^Q-geodesic flow to Q are geodesics of g^Q. The projection of integral curves of $\exp t\vec{h}_F$ to Q are geodesics of the metric g^F, hence also of g^Q, since the fibers are totally geodesics, see O'Neill formulas. The projection $q(t) = \tau \circ \xi(t)$ to Q of the composed curves

$$\xi(t) = \exp t\vec{h}_F \circ \exp t\vec{h}_Q \circ (\xi),\ \xi \in T_q^*Q$$

are curves of the form $q(t) = R_{g(t)}\gamma(t) = \gamma(t)g(t)$ where $\gamma(t) = \tau \circ \exp t\vec{h}_Q(\xi)$ is a geodesic of g^Q and $g(t) \subset G$ is a geodesic of the metric g^F on G. We may assume that $g(0) = e$ and $\dot{g}(0) = a \in \mathfrak{g}$. If $\dot{\gamma}(0) = w$, then the curve $q(t)$ is a sub-Riemannian geodesic if and only if its velocity vector $\dot{q}(0) = w + a_q^*$ is horizontal, that is $\varpi(w + a_q^*) = \varpi(w) + a = 0$. This proves ii), which implies iii). Now iv) follows from the remark that the transformation $R_{g(t)}$ deforms the geodesic $\gamma(t)$ in vertical directions. Hence $q(t)$ and $\gamma(t)$ have the same projection to M (which are geodesics if and only if $\gamma(t)$ is a horizontal geodesic.) □
Let $(\pi : Q \to M, g^Q)$ be a Riemannian submersion and $D \subset TQ$ a transversal to fibers distribution. Necessary and sufficient conditions when the projection to M of g^Q-geodesics coincides with projection of geodesics of the sub-Riemannian manifold $(Q, D, g|_D)$ are given in [22].

A sub-Riemannian geodesic $q(t) = \tau(\xi(t))$ through a point $q = q(0)$ is determined by the initial covector $\xi(0) \in T_q^*Q$ which may be decomposed as $\xi(0) = \xi(0)_D + \lambda$, where $\lambda \in D_q^0 = \mathrm{Ann}(D)_q$ is the **codistribution covector** and $\xi(0)_D \in D_q^*$ is determined by the velocity vector $\dot{q}(0) \in D_q$. The sub-Riemannian geodesics, which are horizontal geodesics of g^Q are characterized as geodesics with trivial codistribution covector. Comparing theorem 21 and theorem 19, we get

Theorem 22 *Let g^D be a Chaplygin sub-Riemannian metric in a principal bundle $(\pi : Q \to M, \varpi)$ and g^Q the standard extension of the sub-Riemannian metric g^D. Then sub-Riemannian S-geodesics coincide with H-geodesics with trivial codistribution covector.*

6.2.3 Bi-invariant extension of Chaplygin metric and Yang-Mills dynamics

Assume now that g^Q is a bi-invariant extension of the Chaplygin sub-Riemannian metric g^D, defined by an Ad_G-invariant metric $g^{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$. Such metric exists only when $\mathfrak{g}$ is the Lie algebra of a compact Lie group. Then the associated left invariant metric on the Lie group G is also right-invariant and the metric $g^F(a^*, b^*) = g^{\mathfrak{g}}(a, b)$ on a fiber $\pi^{-1}(x)$ and the extended Riemannian metric $g^Q = g^F + g^D$ are also R_G-invariant.

In this case, the geodesic Hamiltonian system with Hamiltonian h_Q has a nice physical interpretation as dynamical system, which describes the evolution of a charged particle in the base manifold M in the presence of the Yang-Mills field, defined by the principal connection $\varpi : TQ \to \mathfrak{g}$, see [41],[23].
Recall that with respect to a trivialisation $Q = M \times G$, the connection form may be written as

$$\varpi = \mu + A = (e_L^\alpha + A_i^\alpha dx^i) \otimes e_\alpha$$

where (e_α) is an orthonormal basis of $\mathfrak{g}$, (e^L_α) (resp. , (e^α_L)) is the corresponding left invariant field of frames (resp., coframes) on G and $A = A^\alpha_i(x,g)dx^i \otimes e_\alpha$ the Yang-Mills potential, given by (10). The horizontal (R_G-invariant) lifts of the coordinate vector fields $\partial_i := \partial_{x^i}$ has the form $\partial^D_i := \partial_i - A^\alpha_i e^R_\alpha$. Together with the fundamental fields $e^*_\alpha = e^L_\alpha$, they form a frame in $Q = M \times G$. The sub-Riemannian metric is characterized by the conditions $g^D(\partial^D_i, \partial^D_j) = g^M(\partial_i, \partial_j) = g_{ij}$. The vertical metric g^F is defined by $g^F(e^L_\alpha, e^L_\beta) = g^{\mathfrak{g}}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. The metric $g^Q = g^F + g^D$ is R_G-invariant and the fundamental fields $e^*_\alpha = e^L_\alpha$ are Killing vector fields. The associated contravariant metric $g^{-1}_Q = g^{-1}_F + g^{-1}_D$ is defined by

$$g^{-1}_F = \sum e^L_\alpha \otimes e^L_\alpha, \quad g^{-1}_D = g^{ij}(x)\partial^D_i \otimes \partial^D_j = g^{ij}(\partial_i - A^\alpha_i e^L_\alpha)(\partial_j - A^\alpha_j e^L_\alpha).$$

Denote by (x^i, p_i) the local coordinates in T^*M with $T^*M \ni p = p_i dx^i$ and by $(g^\alpha, \lambda_\alpha)$ the local coordinates in T^*G, where g^α are local coordinates in G and $T^*_g G \ni \lambda = \lambda_\alpha e^\alpha_L$. Note that the linear forms $\lambda_\alpha \in T^*_g G$ are identified with $e^L_\alpha|_g$. The left invariant vector fields $\partial_i, \partial_{p_i}, \partial_{\lambda_\alpha}, e^\alpha_L$ form a frame on $T^*Q = T^*M \times T^*G$. The quadratic in momenta Hamiltonians $h_M, h_F, h_D, h_Q = h_F + h_D$ can be written as follows

$$\begin{aligned} h_M &= \tfrac{1}{2} g^{ij}(x) p_i p_j \\ h_F &= \tfrac{1}{2}\sum e^L_\alpha e^L_\alpha \\ h_D &= \tfrac{1}{2} g^{ij}(x)(p_i - A^\alpha_i(x)\lambda_\alpha)(p_j - A^\beta_j(x)\lambda_\beta). \end{aligned}$$

Using formula for the Poisson structure on T^*G, one can easily calculate the Hamiltonian vector fields and the geodesic equation. We consider another approach, based on the O'Neill formulas.

Lemma 23 *The angle between a geodesic $\gamma(t)$ of g^Q and a fundamental field a^*, $a \in \mathfrak{g}$ is constant. In particular, the orthogonal projection $\mathrm{pr}_{T^vQ}\dot\gamma(t)$ of the velocity vector field $\dot\gamma$ to vertical subbundle is the restriction to $\gamma(t)$ of some fundamental vector field a^* and the velocity vector field can be written as*

$$\dot\gamma(t) = a^*(\gamma(t)) + \dot x^D(\gamma(t))$$

where $\dot x^D(\gamma(t))$ is the horizontal lift of the velocity vector filed $\dot x(t)$ of the projection $x(t)$ of $\gamma(t)$ to M.

Remark 24 *Physically, the angles φ_α between a geodesic $\dot\gamma$ and the basic fundamental fields e^*_α characterise the charges of a particle with respect to components of the Yang-Mills field and the conditions $\varphi_\alpha = const$ are called the conservation of charges. In particular, the evolution of neutral particles is described by horizontal geodesics.*

Proof: Let $\gamma(t)$ be a geodesic and $x(t) = \mathrm{pr}_M \gamma(t)$ its projection to M. Then $\dot x(t) = \mathrm{pr}_{TM}(\dot\gamma(t))$ and the horizontal part of the velocity vector field is $\dot\gamma(t)^h = \dot x(t)^D$. Hence, we can write

$$\dot\gamma(t) = \dot x(t)^D + u^a(t) e^*_a(\gamma(t)).$$

Then

$$\begin{aligned} \tfrac{d}{dt} g^Q(e^*_\beta, \dot\gamma(t)) &= \dot u^\beta(t) \\ &= \nabla_{\dot\gamma} g^Q(e^*_\beta, \dot\gamma(t)) \\ &= g^Q(\nabla_{\dot\gamma} e^*_\beta, \dot\gamma(t)) + g^Q(e^*_\beta, \nabla_{\dot\gamma}\dot\gamma(t)) \\ &= 0, \end{aligned}$$

since the covariant derivative $\nabla . e^*_\beta$ of a Killing vector field $e^*_\beta = e^L_\beta$ is a skew-symmetric operator. $\square$

The following theorem describes the relation between geodesics of the Riemannian metric g^M and geodesics of its bi-invariant extension g^Q.

Theorem 25 *A curve $\gamma(t) \subset Q$ with projection $x(t) = \mathrm{pr}_M \gamma(t)$ and velocity vector field $\dot\gamma(t) = a^*(\gamma(t)) + \dot x^D(\gamma(t))$ is a geodesic of g^Q if and only if it satisfies the equation*

$$\nabla^Q_{\dot\gamma(t)} \dot\gamma(t) = (\nabla^M_{\dot x} \dot x)^D + 2\mathcal{A}^*_{\dot x^D} a^* = 0. \tag{12}$$

Proof: Using O'Neill formulas, we calculate the covariant derivative $\nabla^Q_{\dot\gamma}\dot\gamma$ of the velocity field $\dot\gamma = a^*(\gamma(t)) + \dot x^D(\dot\gamma(t))$ as follows

$$\nabla^Q_{\dot\gamma}\dot\gamma = (\nabla^Q_{a^*} a^*)(\gamma(t)) + \nabla^Q_{\dot x^D} \dot x^D + 2\nabla^Q_{(\dot x)^D} a^* = (\nabla^M_{\dot x} \dot x)^D(\gamma(t)) + 2\mathcal{A}^*_{\dot x^D} a^*(\gamma(t)).$$

We use the fact that geodesics of the bi-invariant metric on a Lie group G are orbits of 1-parameter subgroups, which implies $\nabla^Q_{a^*} a^* = \nabla^F_{a^*} a^* = 0$. $\square$

Recall that $2\mathcal{A}^*_{X^D} a^* = -F^*_X \lambda$, $\lambda = g^Q \circ a^* \in D^0$ where $F^*_X : \mathfrak{g}^* \to \Gamma D$ is the linear map, dual to the map $F_{X_q} : D_q \to \mathfrak{g}$, associated with the curvature 2-form F.

The equation (12) is equivalent to the equation

$$\nabla^M_{\dot x} \dot x = g_M^{-1} \lambda_b F^b_i(\dot x) \tag{13}$$

where the right hand side is the vector field metrically dual to the 1-form $\lambda_b F^b_i(\dot x)$ and $\lambda \in \mathfrak{g}^*$ is a constant covector (a charge). $\square$

The equation (13) describes the motion of a charged particle in the Yang-Mills field ϖ with the strength tensor F. In the case when $\pi : Q \to M$ is a circle bundle, the connection ϖ defines the Maxwell field (if g^M has Lorentz signature) and the equation reduces to the Lorentz equation for a charge particle in the electromagnetic field, defined by the curvature 2-form F.

7 Homogeneous sub-Riemannian manifolds

We consider some class of homogeneous sub-Riemannian manifolds, for which S-geodesics coincide with H-geodesics and describe sub-Riemannian symmetric spaces.

7.1 Chaplygin system on homogeneous spaces

7.1.1 Chaplygin system of a Lie group

Let $\pi : G \to M = G/H$ be the principal bundle associated to a homogeneous Riemannian manifold $(M = G/H, g)$. A reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, $\mathfrak{m} = T_oM$, $o = eH$, defines a principal connection with connection form $\varpi = \mathrm{pr}_{\mathfrak{h}} \circ \mu^L$, which is the projection to $\mathfrak{h}$ of the left invariant Maurer-Cartan form μ^L. Denote by $(D = \ker \varpi, g^D)$ the associated Chaplygin sub-Riemannian metric. Since the stability subalgebra $\mathfrak{h}$ is compact, it admits a bi-invariant Euclidean metric $g^{\mathfrak{h}}$. We denote by g^G the associated bi-invariant extension of the sub-Riemannian metric. It is a left G-invariant and right H-invariant metric on G.

The distribution $D := \ker \varpi$ with $D_0 = \mathfrak{m}$ is bracket generating if and only if $\mathfrak{m}$ generates the Lie algebra $\mathfrak{g}$. The Jacobi identity shows that $\mathfrak{g}' := [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is a subalgebra of $\mathfrak{g}$. It generates a subgroup $G' \subset G$ which acts transitively in M. Hence, changing G to G', we may always assume that D is bracket generating.

Proposition 26 *Let $(M = G/H, g^M)$ be a homogeneous Riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that $\mathfrak{m}$ generates $\mathfrak{g}$. Then the principal connection $\varpi := \mathrm{pr}_{\mathfrak{h}} \circ \mu^L$ defines a Chaplygin left invariant sub-Riemannian structure (G, D, g^D) on the Lie group G with the connection ϖ. A bi-invariant metric $g^{\mathfrak{h}}$ on $\mathfrak{h}$ defines a bi-invariant extension of the sub-Riemannuian metric g^D to a left invariant Riemannian metric on G.*

7.1.2 Chaplygin systems on homogeneous manifolds

Now we consider a generalisation of the above construction.

Assume that the stabilizer H of a Riemannian homogeneous manifold $M = G/H$ is an almost direct product $H = K \cdot L$ of two compact normal subgroups and $\mathfrak{k}, \mathfrak{l}$ are associated Lie subalgebras. Then $\pi : Q = G/K \to M = G/K \cdot L$ is an L-principal bundle with the right action of L and $M = G/H$ has the reductive (i.e. Ad_H-invariant) decomposition of the form

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{k} + \mathfrak{l}) + \mathfrak{m}.$$

The projection $\varpi^G = \mathrm{pr}_{\mathfrak{l}} \mu^L : TG \to \mathfrak{l}$ of the left invariant Maurer-Cartan form μ to $\mathfrak{l}$ is a left invariant $\mathfrak{l}$-valued 1-form. The form ϖ^G is right K-invariant and right L-equivariant, that is $R_\ell^* \varpi = \mathrm{Ad}_\ell^{-1} \circ \varpi$, $\ell \in L$. Hence it projects to a G-invariant principal connection form $\varpi : TQ \to \mathfrak{l}$. The principal bundle $\pi : Q = G/K \to M = G/K \cdot L$ with the connection form $\varpi : TQ \to \mathfrak{l}$ defines a Chaplygin sub-Riemannian metric $(D = \ker \varpi, g^D)$. As above, it admits a bi-invariant extension. We get

Proposition 27 *A homogeneous Riemannian manifold $(M = G/H, g^M)$ with non simple stabilizer $H = K \cdot L$ defines an invariant sub-Riemannian Chaplygin metric (D, g^D) on the total space of the principal L-bundle $\pi : Q = G/K \to M = G/K \cdot L$ with the connection form $\varpi : TQ \to \mathfrak{l}$, which is the projection to Q of the form $\varpi^G = \mathrm{pr}_{\mathfrak{l}} \circ \mu^L$ on G. The sub-Riemannian metric admits a bi-invariant extension to an invariant metric g^Q on Q.*

7.1.3 Homogeneous contact sub-Riemannian manifolds

The above construction may be applied to homogeneous Sasaki manifolds. We consider the case of regular compact homogeneous Sasaki manifolds, described as follows. Let $M = G/H$ be a flag manifold (i.e. an adjoint orbit of a compact semisimple Lie group G) and $g = \omega(\cdot, J\cdot)$ an invariant Hodge-Kähler metric on M, where J is an invariant complex structure and ω an integer invariant symplectic form (the Kähler form). Then there exists a homogeneous principal circle bundle $\pi : Q = G/K \to M = G/H = G/K \cdot S^1$ with a principal connection $\varpi : TQ \to \mathbb{R} = \mathrm{LieS}^1$, whose curvature form is ω. The Kähler metric is naturally extended to an invariant Sasaki metric g^Q, such that the fundamental field Z of the S^1-bundle π is a Killing field.
This Sasaki metric g^Q is the bi-invariant extension of the Chaplygin sub-Riemannian metric (D, g^D) associated to the principal bundle $\pi : Q \to M$ with the connection ϖ.

From physical point of view, the principal S^1-bundle $\pi : Q \to M$ with Sasaki metric corresponds to Kaluza-Klein description of electromagnetic field. The projection to M of geodesics of Sasaki metric are solutions of the Lorentz equation which describes the evolution of electric charges in the electromagnetic field ω.

7.2 Symmetric sub-Riemannian manifolds

Strichartz [30] defined the notion of sub-Riemannian symmetric space as a homogeneous sub-Riemannian manifold $(Q = G/H, D, g^D)$ such that the stabilizer H contains an involutive element σ (called the sub-Riemannian symmetry) which acts on the subspace D_o at the point $o = eH \in Q$ as $-\mathrm{id}$.
He classified 3-dimensional sub-Riemannian symmetric spaces and stated the problem of extension of this classification to higher dimensions.
P. Bieliavsky, E. Falbel and C. Gorodski [9] classified symmetric sub-Riemannian manifolds of contact type. W. Respondek and A.J. Maciejewski [27] describe all integrable sub-Riemannian metrics on 3-dimensional Lie groups with integrable H-geodesic flow. They are exhausted by sub-Riemannian symmetric spaces.
Below we recall basic properties of affine symmetric spaces and give a construction of sub-Riemannian symmetric spaces in terms of affine symmetric spaces: Any bracket generating sub-Riemannian symmetric space is the total space $M = G/K$ of a homogeneous bundle $\pi : M = G/K \to S = G/H$ over an affine symmetric space $S = G/H$, determined by a compact subgroup K of the stability group H.

7.2.1 Affine symmetric spaces

Let (M, ∇) be a (connected) manifold with a linear connection ∇. A non-trivial involutive automorphism $\sigma = \sigma_x$ of $(M, \nabla))$ is called a **cental symmetry with center** $x \in M$ if σ preserves x and acts as $-\mathrm{id}$ in the tangent space T_xM. The manifold (M, ∇) is called an **(affine) symmetric space** if any point is the center of some central symmetry σ_x. A product $\sigma_x\sigma_y$ of two central symmetries with sufficiently closed to each other centers x, y is a shift along the geodesics, connecting these points. This implies that the group G, generated by all central symmetries is a transitive Lie group, called the **transvection group**. The manifold M is identified with the quotient space $M = G/H$, where H is the stabilizer of a point $o \in M$. Then the central symmetry $\sigma = \sigma_o$ defines an involutive automorphism $s = \mathrm{Ad}_{\sigma_0} : g \mapsto s(g) := \sigma_0 \circ g \circ \sigma_0$ of the Lie group G, which acts trivially on the connected component H^0 of H. We denote by s also the induced involutive automorphism of the Lie algebra $\mathfrak{g} = Lie(G)$. Its eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-, \ \ s|_{\mathfrak{g}_\pm} = \pm\mathrm{id} \ ,$$

where $\mathfrak{g}_+ = \mathfrak{h} = \mathrm{Lie}(H)$, is called the **symmetric decomposition**. It is characterized by the conditions

$$[\mathfrak{g}_+, \mathfrak{g}_-] \subset \mathfrak{g}_-, \ \ [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+.$$

Moreover, if G is the transvection group, then

$$[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+. \tag{14}$$

The geodesics through the point $o = eH$ are orbits $e^{tX}o$ of 1-parametric subgroups $e^{tX} \in G$ generated by elements $X \in \mathfrak{g}_-$.

The following well known result establishes a bijection between symmetric decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ of a Lie algebra $\mathfrak{g}$ with (14) and simply connected affine symmetric spaces $S = G/H$, where G is the simply connected transvection group with $\mathrm{Lie}(G) = \mathfrak{g}$.

Theorem 28 *Let $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ be a symmetric decomposition with (14) associated to an involutive automorphism s. Denote by G the simply connected Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$ and by H^0 the connected subgroup of G, generated by $\mathfrak{h} = \mathfrak{g}_+$. Then $S = G/H^0$ is a simply connected affine symmetric space. The invariant torsion free linear connection ∇ in S is defined by the condition*

$$\nabla_X Y^*|_o = -\frac{1}{2}[X, Y]_o, \ X, Y \in \mathfrak{g}_- = T_o S$$

where Y^* denote the velocity vector field of a 1-parameter subgroup e^{tY}, $Y \in \mathfrak{g}_-$. The central symmetry with the center $o = eH^0$ is defined by

$$\sigma : x = gH^0 \mapsto \sigma x := s(g)H^0$$

where s is the involutive automorphism of the Lie group G generated by the automorphism s of $\mathfrak{g}$. Moreover, **any** *affine symmetric space, associated with the above symmetric decomposition, has the form G/H where H is a closed subgroup such that $H^0 \subset H \subset G^\sigma$. Here G^σ is the fixed point set of σ.*

7.2.2 Sub-Riemannian symmetric spaces associated with an affine symmetric space

Let $(S = G/H, \nabla, \sigma)$ be a simply connected affine symmetric space with the transvection group G.Without loss of generality, we may assume that the central symmetry $\sigma = \sigma_o$ belongs to the center $Z(H)$ of the stability subgroup.Then the associated involutive automorphism $s = \mathrm{Ad}_\sigma$ of G acts trivially on H and defines a symmetric decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ where $\mathfrak{g}_+ = \mathfrak{h} = Lie(H)$. Let $K \subset H$ be a compact subgroup of H which contains σ.
The homogeneous manifold $Q = G/K$ has a reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{k} + (\mathfrak{p} + \mathfrak{g}_-)$$

where $\mathfrak{k} = Lie(K)$ and $\mathfrak{g}_+ = \mathfrak{k} + \mathfrak{p}$ is a reductive (i.e. Ad_K-invariant) decomposition of $\mathfrak{g}_+$. We identify $\mathfrak{m}$ with the tangent space T_oQ at the point $o = eK$. Then the isotropy representation of K in T_oQ is identified with $\mathrm{Ad}_K|_\mathfrak{m}$. The Ad_K-invariant subspace $\mathfrak{g}_-$ of the tangent space $\mathfrak{m} = T_oQ$ is naturally extended to an invariant distribution $D \subset TQ$. More precisely, for $x = aK \in G/K$, the subspace $D_x = (L_a)_*\mathfrak{g}_-$, where $L_a : bK \to abK$ is the action of G in $Q = G/K$.
The distribution D is invariant with respect to the action of involution $\sigma \in K$ and the isotropy action $\mathrm{Ad}_\sigma|_\mathfrak{m}$ of σ acts on $\mathfrak{g}_- \subset \mathfrak{m}$ as $-\mathrm{id}$. Since G is the transvection group, $[\mathfrak{g}_-, \mathfrak{g}_-] = \mathfrak{g}_+$ and D is bracket generating distribution. Since the group K is compact, there exist an Ad_K-invariant Euclidean metric g in $\mathfrak{g}_-$. It is naturally extended to an invariant sub-Riemannian metric g^D in D, defined by

$$g^D_x(X, Y) := g((L_a^{-1})_* X, (L_a^{-1})_* Y), \ \ a \in G, \, x = aK, \, X, Y \in T_xQ = (L_a)_*\mathfrak{g}_-.$$

Hence, the invariant sub-Riemannian manifold $(Q = G/K, D, \sigma)$ is a sub-Riemannian symmetric space. This proves the first claim of the following theorem.

Theorem 29 *i) Let $(S = G/H, \nabla, \sigma)$ be a simply connected affine symmetric space with the transvection group G and $K \subset H$ a compact Lie subgroup, which contains σ as a central element. Let g be an Ad_K-invariant Euclidean metric in $\mathfrak{g}_-$. Then the Euclidean space $(\mathfrak{g}_-, g)$ is extended to an invariant sub-Riemannian structure (D, g^D) in $Q = G/K$ such that $(Q = G/K, D, g^D, \sigma)$ is a bracket generating sub-Riemannian symmetric space.*

ii) Conversely, up to a covering any bracket generating sub-Riemannian symmetric space can be obtained by this construction.

Proof: ii) Let $(Q = G/K, D, g^D, \sigma)$ be a bracket generating sub-Riemannian symmetric space, where $\sigma \in K$ is the sub-Riemannian symmetry with center $o = eK$ and $s = \mathrm{Ad}_\sigma$ the associated involutive automorphism of G and $\mathfrak{g}$. We may chose a reductive decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = (\mathfrak{k}_+ + \mathfrak{k}_-) + (\mathfrak{m}_+ + \mathfrak{m}_-)$$

of G/K which is consistent with the symmetric decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$, defined by s, such that $\mathfrak{g}_+ = \mathfrak{k}_+ + \mathfrak{m}_+$, $\mathfrak{g}_- = \mathfrak{k}_- + \mathfrak{m}_-$. By definition,the subspace $D_o = D|_o \subset T_oQ = \mathfrak{m}$ belongs to $\mathfrak{m}_-$. Since the distribution D is bracket generating, we may assume that the subalgebra $\bar{\mathfrak{g}}$ generated by Δ_o coincides with $\mathfrak{g}$. But $\bar{\mathfrak{g}} = [D_o, D_o] + D_o \subset \mathfrak{g}_+ + D_0$. This implies that $\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{m}_+ + \mathfrak{g}_-$, and $\mathfrak{k} = \mathfrak{k}_+, D_o = \mathfrak{g}_-, \mathfrak{m} = \mathfrak{m}_+ + \mathfrak{g}_- = \mathfrak{g}_-$. Denote by G_+ the connected subgroup of G, generated by $\mathfrak{g}_+$, Since it commutes with σ, it is the connected component of the group $H = G_+ \cup \sigma G_+$. The manifold $S = G/H$ is an affine symmetric space with the symmetry $\sigma \in H$, belonging to the center. Consider the subgroup $K' = K \cap H$ with the Lie algebra $\mathfrak{k}$. It also contains σ as a central element. The claim i) shows that the space $Q' = G/K'$ has a structure of sub-Riemannian symmetric space which is locally isomorphic to the initial sub-Riemannian symmetric space Q. $\square$

7.2.3 Compact sub-Riemannian symmetric space associated to a graded complex semisimple Lie algebra

We show that any flag manifold of depth > 1 admits the structure of bracket generating symmetric sub-Riemannian manifold.

Let $\mathfrak{g} = \sum_{i=-d}^{d} \mathfrak{g}_i$ be a **fundamentally graded** complex semisimple Lie algebra of depth $d \geq 2$ (s.t. $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$) and $\mathfrak{p} := \sum_{i \geq 0} \mathfrak{g}_i$ the associated parabolic subalgebra. The associated (complex compact simply connected) homogeneous manifold $F = G/P$, where $G \supset P$ are the Lie groups associated to Lie algebras $\mathfrak{g} \supset \mathfrak{p}$, is called a **flag manifold**.
Denote by τ the anti-linear involution of $\mathfrak{g}$, which defines the compact real form $\mathfrak{g}^\tau$ s.t. $\mathfrak{g}^\tau = \mathfrak{g}_0^\tau + \sum_{i>0} \mathfrak{m}_i,\ \mathfrak{m}_i := (\mathfrak{g}_{-i} + \mathfrak{g}_i)^\tau$.
The Lie algebra $\mathfrak{g}^\tau$ has the symmetric decomposition

$$\mathfrak{g}^\tau = \mathfrak{g}_{ev}^\tau + \mathfrak{g}_{odd}^\tau = (\mathfrak{g}_0^\tau + \sum_{i \equiv 0 (\mathrm{mod} 2)} \mathfrak{m}_i) + \sum_{i \equiv 1 (\mathrm{mod} 2)} \mathfrak{m}_i.$$

We denote by s the associated involution of the Lie algebra $\mathfrak{g}^\tau$ and the corresponding simply connected compact Lie group G^τ. Denote by $H \subset G$ the connected compact subgroup generated by $\mathfrak{h} = \mathfrak{g}_0^\tau$. The group G^τ acts transitively on the flag manifold F with stability subgroup H and has the reductive decomposition

$$\mathfrak{g}^\tau = \mathfrak{h} + \mathfrak{m} = \mathfrak{g}_0^\tau + \sum_{i>0} \mathfrak{m}_i.$$

The involutive automorphism s acts by $\sigma|\mathfrak{h} = \mathrm{id}$, $\sigma|\mathfrak{m}_i = (-1)^i\mathrm{id}$.
Denote by $D \subset TF$ the (bracket generating) invariant distribution generated by $\mathfrak{m}_1$ and by g^D the invariant sub-Riemannian metric in D defined by an Ad_H-invariant metric in $\mathfrak{m}_1$. Then (D, g^D) is an invariant sub-Riemannian metric of $F = G/H$. Moreover, $(F = G^\tau/H, D, g^D)$ is a sub-Riemannian symmetric space, where the symmetry, defined by the involutive automorphism s.

This implies

Theorem 30 *Let $\mathfrak{g} = \sum_{i=-k}^{k} \mathfrak{g}_i$ be a fundamental depth $k > 1$ gradation of a complex semisimple Lie algebra and let $F = G/P$ be the associated flag manifold. Denote by*

$$\mathfrak{g}^\tau = \mathfrak{h} + \mathfrak{m} = \mathfrak{h} + \sum_{i=1}^{k} \mathfrak{m}_i,\ \mathfrak{h} = \mathfrak{g}_0^{\tau,} \mathfrak{m}_i = (\mathfrak{g}_{-i} + \mathfrak{g}_i)^\tau$$

the associated decomposition of the compact real form $\mathfrak{g}^\tau$ and by $g^{\mathfrak{m}_1}$ an $\mathrm{ad}_{\mathfrak{h}}$-invariant Euclidean metric in $\mathfrak{m}_1$.
Then the pair $(\mathfrak{m}_1, g^{\mathfrak{m}_1})$ defines an invariant bracket generating sub-Riemannian metric (D, g^D) on the flag manifold $F = G^\tau/H$ considered as a homogeneous manifold of the compact real form G^τ of G. Moreover, the sub-Riemannian manifold $(F = G^\tau/H, D, g^D)$ is a sub-Riemannian symmetric space with the symmetry defined by the involutive automorphism s of $\mathfrak{g}$, associated with the symmetric decomposition $\mathfrak{g}^\tau = \mathfrak{g}^\tau_{ev} + \mathfrak{g}^\tau_{odd}$.

Example Let

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_9 + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,\ \dim \mathfrak{g}_{\pm 2} = 1$$

be the contact gradation of a complex simple Lie algebra $\mathfrak{g}$, i.e. the eigenspace decomposition of ad_{H_μ} where H_μ is the coroot associated to the maximal root μ of $\mathfrak{g}$. Then the symmetric space G^τ/G^τ_{ev} is the quaternionic Kähler symmetric space (the Wolf space) and the flag manifold $F = G^\tau/H$, where $Lie(H) = \mathfrak{h}_0^\tau$, is the associated twistor space. The distribution D is the holomorphic contact distribution and g^D is the unique (up to scaling) invariant sub-Riemannian metric on D (for $\mathfrak{g} \neq \mathfrak{sl}_n(\mathbb{C})$). It is the restriction of the invariant Kähler-Einstein metric on F.

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