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Proof of a Counterexample to the Finiteness Conjecture in the Spirit of the Theory of Dynamical Systems

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Abstract

In 1995 J. C. Lagarias and Y. Wang conjectured that the generalized spectral radius of a finite set of square matrices can be attained on a finite product of matrices. The first counterexample to this Finiteness Conjecture was given in 2002 by T. Bousch and J. Mairesse and their proof was based on measure-theoretical ideas. In 2003 V. D. Blondel, J. Theys and A. A. Vladimirov proposed another proof of a counterexample to the Finiteness Conjecture which extensively exploited combinatorial properties of permutations of products of positive matrices.

In the paper, it is proposed one more proof of a counterexample of the Finiteness Conjecture fulfilled in a rather traditional manner of the theory of dynamical systems. It is presented description of the structure of trajectories with the maximal growing rate in terms of extremal norms and associated with them so called extremal trajectories. The construction of the counterexample is based on a detailed analysis of properties of extremal norms of two-dimensional positive matrices in which the technique of the Gram symbols is essentially used. At last, notions and properties of the rotation number for discontinuous orientation preserving circle maps play significant role in the proof.

Contents

1 Introduction

Let $\boldsymbol{A} = \{A_1, \dots, A_r\}$ be a finite set of real $m \times m$ matrices, and $\|\cdot\|$ be a norm in $\mathbb{R}^m$. Associate with any finite sequence $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \dots, \sigma_n\} \in \{1, \dots, r\}^n$ the matrix

$$A_{\boldsymbol{\sigma}} = A_{\sigma_n} \cdots A_{\sigma_2} A_{\sigma_1},$$

and define for any $n \geq 1$ two quantities:

$$\rho_n(\boldsymbol{A}) = \max_{\boldsymbol{\sigma} \in \{1,\dots,r\}^n} \|A_{\boldsymbol{\sigma}}\|^{1/n}, \qquad \bar{\rho}_n(\boldsymbol{A}) = \max_{\boldsymbol{\sigma} \in \{1,\dots,r\}^n} \rho(A_{\boldsymbol{\sigma}})^{1/n}.$$

Then there exists the limit

$$\rho(\boldsymbol{A}) = \limsup_{n \to \infty} \rho_n(\boldsymbol{A}),$$

which does not depend on the choice of the norm $\|\cdot\|$. This limit is called *the joint spectral radius* of the matrix set $\boldsymbol{A}$. Analogously, there exists the limit

$$\bar{\rho}(\boldsymbol{A}) = \limsup_{n \to \infty} \bar{\rho}_n(\boldsymbol{A}),$$

which is called *the generalized spectral radius* of the matrix set $\boldsymbol{A}$. As is shown in [2], for finite matrix sets $\boldsymbol{A}$ the quantities $\rho(\boldsymbol{A})$ and $\bar{\rho}(\boldsymbol{A})$ coincide with each other, and for any n the following inequalities hold

$$\bar{\rho}_n(\boldsymbol{A}) \leq \bar{\rho}(\boldsymbol{A}) = \rho(\boldsymbol{A}) \leq \rho_n(\boldsymbol{A}).$$

In [14] J. C. Lagarias and Y. Wang conjectured that the value $\bar{\rho}(\boldsymbol{A})$ in fact coincides with $\rho(A_{\boldsymbol{\sigma}})^{1/n}$ for some n and $\boldsymbol{\sigma} \in \{1, \dots, r\}^n$. The first counterexample to this conjecture (which got the name Finiteness Conjecture) was proposed in [5], and the corresponding proof was essentially based on measure-theoretical ideas. Later, another proof [3, 4] of counterexample to the Finiteness Conjecture appeared which extensively exploited combinatorial properties of permutations of products of positive matrices.

In this paper, it is given one more proof of the counterexample to the Finiteness Conjecture fulfilled in a rather traditional manner of the theory of dynamical systems. The proof is based on the technique of the so called extremal norms (closely related with the usage of functionals Mañé in [5]) and associated with them extremal trajectories for analysis of "the fastest growing trajectories" generated by matrix sets. To our knowledge, it was N. Barabanov [1] who first realized the role of extremal norms in the analysis of properties of matrix products. At a later time, the technique of extremal norm was used in different problems related to the investigation of properties of matrix products (see, e.g., [15]). In this paper, we give a more detailed analysis of the properties of extremal norms of two-dimensional positive matrices in which the technique of the Gram symbols, borrowed from [5], is essentially used. At last, in the proposed proof, the notion and properties of the rotation number for discontinuous preserving orientation circle maps [6, 8, 9] play significant role.

2 Trajectories of Matrix Sets

One of the important problem in the study of properties of matrix sets $\boldsymbol{A} = \{A_1, \dots, A_r\}$ is how the joint (generalized) spectral radius $\rho(\boldsymbol{A})$ is related with the rate of growth of solutions of the difference inclusion

$$x_{n+1} \in \{A_1, \dots, A_r\}x_n, \tag{1}$$

in which the value of x_{n+1} is chosen from the set of vectors $\{A_1x_n, \dots, A_rx_n\}$. Notice that each solution of inclusion (1) is defined for all $n \geq 0$ and, with some choice of the index sequence $\{\sigma_n\}$, satisfies the equation

$$x_{n+1} = A_{\sigma_n}x_n, \qquad \sigma_n \in \{1, \dots, r\}. \tag{2}$$

Clearly, the converse is also true, which means that each solution of the difference equation of the type (2) corresponding to some index sequence $\{\sigma_n\}$ is a solution of the difference equation (1). To formulate further properties of the solutions of inclusion (1) we recall some definitions and commonly known facts.

In what follows solutions of inclusion (1) will be referred to as *trajectories* defined by the matrix set $\boldsymbol{A}$ or simply trajectories of the matrix set $\boldsymbol{A}$. The set of all trajectories of the matrix set $\boldsymbol{A}$ will be denoted as $\mathcal{T}(\boldsymbol{A})$, the set of all trajectories $\boldsymbol{x} = \{x_n\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ satisfying the initial condition $x_0 = x$ will be denoted as $\mathcal{T}(\boldsymbol{A}, x)$. In general, for $r > 1$ the map

$$x \mapsto \mathcal{T}(\boldsymbol{A}, x)$$

is set-valued, as well as plenty of other maps arising in the field of difference inclusions. In connection with this recall some definitions and basic facts of the theory of set-valued maps (see, e.g., [13, §18]).

Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces and f be a map associating with each element $x \in \mathbb{X}$ a set $f(x) \subseteq \mathbb{Y}$. Then the map f is called set-valued or multi-valued. The map f is called *upper semi-continuous* at a point $x \in \mathbb{X}$ if for any open set $\mathcal{U} \ni f(x)$ there is an open set $\mathcal{V} \ni x$ such that $f(\mathcal{V}) \subseteq \mathcal{U}$.[1] *The graph of the map* f is the set

$$\mathrm{Gr}(f) = \{(x, y) :\ x \in \mathbb{X}, y \in f(x)\} \subseteq \mathbb{X} \times \mathbb{Y}.$$

The map f is called *closed* (*compact*) if for any closed (compact) set $\mathcal{G} \subseteq \mathbb{X}$ the set $f(\mathcal{G}) \subseteq \mathbb{Y}$ is also closed (compact). Clearly, each compact map is closed.

For the convenience of the reader recall without proofs some commonly known properties of set-valued maps.

Lemma 1 *Let $x \in \mathbb{X} \mapsto f(x) \subseteq \mathbb{Y}$ be a set-valued map and let the space $\mathbb{Y}$ be regular.*[2] *Then the following statements are valid:*

(i) if the map f is closed and upper semi-continuous then its graph is closed in $\mathbb{X} \times \mathbb{Y}$;

(ii) if the map f is compact and its graph is closed then it is upper semi-continuous;

(iii) the map f is compact and upper semi-continuous if and only if, given a converging sequence $\{x_n \in \mathbb{X}\}$, any sequence $\{y_n \in \mathbb{Y}\}$ satisfying $y_n \in f(x_n)$ is compact and the limiting elements x_* and y_* of the sequences $\{x_n\}$ and $\{y_n\}$, respectively, are bounded by the inclusion $y_* \in f(x_*)$.

Denote the set of all ordered r-tuples $\boldsymbol{A} = \{A_1, \ldots, A_r\}$ of real $m \times m$ matrices by $\mathcal{M}_{m,r}$. Then the set $\mathcal{M}_{m,r}$ may be identified in a natural way

[1] Here, the notation $f(\mathcal{V})$ is used to denote the set $\cup_{y \in \mathcal{V}} f(y)$.

[2] A topological space $\mathbb{X}$ is called regular if for any its closed set $\mathcal{G}$ and point $x \notin \mathcal{G}$ there are open sets $\mathcal{U}$ and $\mathcal{V}$ such that $x \in \mathcal{U}$, $\mathcal{G} \subset \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. For example, any metric space is regular. In particular, spaces $\mathbb{R}^m$ and $\mathcal{M}_{m,r}$ are regular.

with $\mathbb{R}^{rm^2}$ if to treat entries of the matrices from $\boldsymbol{A}$ enumerated in some predefined order as coordinates in $\mathbb{R}^{rm^2}$. This allows to treat $\mathcal{M}_{m,r}$ as a topological or, when needed, a metric space.

Denote the space of sequences $\{x_n\}_{n=0}^{\infty}$ endowed with the topology of point-wise convergence by $\Omega(\mathbb{R}^m)$. At last, the subset

$$\Omega_n = \{x : \ \exists\ \boldsymbol{x} = \{x_n\}_{n=0}^{\infty} \in \Omega : \ x_n = x\}.$$

of $\mathbb{R}^m$ consisting of n-th elements of the sequences from the set $\Omega \subseteq \Omega(\mathbb{R}^m)$ will be called *the n-section* of the set Ω. Point out that the set Ω is compact in the space $\Omega(\mathbb{R}^m)$ provided that each its section Ω_n is bounded.

Now, we are able to formulate properties of the trajectories of matrix sets needed in what follows.

Lemma 2 *For any matrix set $\boldsymbol{A}$ the set of trajectories $\mathcal{T}(\boldsymbol{A})$ is closed in the space $\Omega(\mathbb{R}^m)$, and the map $(\boldsymbol{A}, x) \mapsto \mathcal{T}(\boldsymbol{A}, x)$ is compact and upper semi-continuous.*

This Lemma is a simple corollary of the compactness criterium in the sequence space $\Omega(\mathbb{R}^m)$, so its proof is omitted.

In what follows, our prime point of interest will be so-called irreducible matrix sets. In connection with this, recall that the matrix set $\boldsymbol{A}$ is called *irreducible* if the matrices from $\boldsymbol{A}$ have no common invariant spaces except $\{0\}$ and $\mathbb{R}^m$. In [10–12] such a matrix set was called *quasi-controllable.* Basic properties of irreducible matrix sets in the context of the Finiteness Conjecture will be studied later on, but now let us formulate an auxiliary statement.

Let $x \in \mathbb{R}^m$. Denote the n-section of the set $\mathcal{T}(\boldsymbol{A}, x)$ by $\mathcal{T}_n(\boldsymbol{A}, x)$. Also, define for any $n = 0, 1, 2 \ldots$ the sets

$$\mathcal{T}_n^*(\boldsymbol{A}, x) = \bigcup_{k=0}^{n} \mathcal{T}_k(\boldsymbol{A}, x).$$

Recall that in the Introduction every finite sequence $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \in \{1, \ldots, r\}^n$ was associated with the matrix $A_{\boldsymbol{\sigma}} = A_{\sigma_n} \cdots A_{\sigma_2} A_{\sigma_1}$, and it was supposed implicitly that $n \geq 1$. In the sequel it will be convenient to extend the notation $A_{\boldsymbol{\sigma}}$ on the degenerate case in which the sequence $\boldsymbol{\sigma}$ is empty, i.e. consists of zero amount of elements. So, we set $\{1, \ldots, r\}^0 = \emptyset$. In this case it is naturally to identify $\boldsymbol{\sigma} \in \{1, \ldots, r\}^0$ with the empty set and to denote $A_\emptyset = I$.

Lemma 3 *The set $\mathcal{T}_n^*(\boldsymbol{A},x)$ coincides with the set of all possible vectors of the form $A_{\boldsymbol{\sigma}}x$, where $\boldsymbol{\sigma} \in \{1,\dots,r\}^k$ for some, possibly zero, integer $k \le n$.*

If $\boldsymbol{A}$ is an irreducible set of $m \times m$ matrices and $x \ne 0$ then the set $\mathcal{T}_n^*(\boldsymbol{A},x)$ contains at least $\min\{n+1,m\}$ linearly independent elements one of which may be assumed to be coinciding with x.

PROOF. Only the second claim of this Lemma has to be proved. Denote the linear hull of the set $\mathcal{T}_n^*(\boldsymbol{A},x)$ by $\mathcal{L}_n(\boldsymbol{A},x)$. Then the dimension of the subspace $\mathcal{L}_n(\boldsymbol{A},x)$ will be equal to the amount of linearly independent vectors in the set $\mathcal{T}_n^*(\boldsymbol{A},x)$. Since, in addition, for any $n \ge 0$ the inclusion $\mathcal{T}_n^*(\boldsymbol{A},x) \subseteq \mathcal{T}_{n+1}^*(\boldsymbol{A},x)$ holds then $\mathcal{L}_n(\boldsymbol{A},x) \subseteq \mathcal{L}_{n+1}(\boldsymbol{A},x)$. So,

$$1 = \dim \mathcal{L}_0(\boldsymbol{A},x) \le \dim \mathcal{L}_1(\boldsymbol{A},x) \le \ldots \le \dim \mathcal{L}_n(\boldsymbol{A},x) \le \ldots .$$

Consequently, Lemma 3 will be proved if we show that

$$\dim \mathcal{L}_n(\boldsymbol{A},x) \ge n+1, \qquad n = 0,1,\dots,m-1. \tag{3}$$

Prove inequalities (3) by induction. For $n = 0$ inequalities (3) hold since the subspace $\mathcal{L}_0(\boldsymbol{A},x)$ coincides with the linear hull of the vector x, and so $\dim \mathcal{L}_0(\boldsymbol{A},x) = 1$. Suppose that the assertion of Lemma 3 is valid for some $n = k < N-1$, i.e. $\dim \mathcal{L}_k(\boldsymbol{A},x) \ge k+1$. Then, due to the supposition on irreducibility of the matrix set $\boldsymbol{A}$, the subspace $\mathcal{L}_n(\boldsymbol{A},x)$ can not be invariant for all the matrices $A_1,\dots,A_r$. Therefore, there is a matrix A_i such that $A_i\mathcal{L}_n(\boldsymbol{A},x) \not\subseteq \mathcal{L}_n(\boldsymbol{A},x)$. Hence $\mathcal{L}_{k+1}(\boldsymbol{A},x) \ne \mathcal{L}_k(\boldsymbol{A},x)$. From this it follows that $\dim \mathcal{L}_{k+1}(\boldsymbol{A},x) \ge \dim \mathcal{L}_{k+1}(\boldsymbol{A},x) + 1 \ge k+2$. So, the induction step is justified, and the proof of Lemma 3 is completed. $\square$

3 Extremal Norms and Trajectories: General Case

In the analysis of the properties of the joint spectral radius ideas introduced by N. E. Barabanov in [1] play an important role. These ideas were further developed in a number of publications amongst which we distinguish [15].

Theorem 1 (N. E. Barabanov) *Let the matrix set $\boldsymbol{A} = \{A_1,\dots,A_r\}$ be irreducible. Then the quantity ρ is the joint (generalized) spectral radius of $\boldsymbol{A}$ if and only if there exists a norm $\|\cdot\|$ in $\mathbb{R}^m$ such that*

$$\rho\|x\| = \max\left\{\|A_0x\|, \|A_1x\|, \dots, \|A_rx\|\right\}. \tag{4}$$

A norm satisfying (4) will be called *an extremal norm* for the matrix set $\boldsymbol{A}$. Remark that in Theorem 1 it is sufficient to suppose that $\|\cdot\|$ in the condition (4) is not a norm but only a semi-norm. The validity of this statement follows from the next Lemma.

Lemma 4 *Let the matrix set $\boldsymbol{A}$ be irreducible. Then any semi-norm $\|\cdot\|$ satisfying (4) is a norm provided that it does not equal identically to zero.*

PROOF. As is easy to see the kernel of the semi-norm $\|\cdot\|$ is a subspace $\mathcal{L}$ which, due to the supposition that the semi-norm is not identically zero, does not coincide with the whole space $\mathbb{R}^m$, i.e. $\mathcal{L} \neq \mathbb{R}^m$. If additionally $\mathcal{L} \neq \{0\}$ then from the irreducibility of the matrix set $\boldsymbol{A}$ it follows the existence of such a vector $x_* \in \mathcal{L}$ for which $A_i x_* \notin \mathcal{L}$ for some i. Then by the definition of the subspace $\mathcal{L}$ the following two relations should be valid simultaneously: $\|x_*\| = 0$ and $\|A_i x_*\| \neq 0$, which contradicts to (4). From the obtained contradiction it follows that $\mathcal{L} = \{0\}$, and so $\|\cdot\|$ is a norm. Lemma 4 is proved. □

Note that the set of extremal norms possesses a variety of strong properties which will be shown below.

3.1 Boundedness of the Set of Extremal Norms

Let $\|\cdot\|_0$ be a norm in $\mathbb{R}^m$ which will play the role of a calibrating norm, i.e. such a norm with which all other norms in $\mathbb{R}^m$ are compared.

As is known, all norms in $\mathbb{R}^m$ are equivalent, so for any norm $\|\cdot\|$ there are constants $\Delta, \delta > 0$ such that

$$\delta\|x\|_0 \leq \|x\| \leq \Delta\|x\|_0.$$

Clearly, in general, there are no universal constants $\Delta, \delta > 0$ since for any given constants $\Delta, \delta > 0$ the multiplication of the norm $\|\cdot\|$ by a number easily breaks the above inequalities. Hence, it is meaningful to compare with $\|\cdot\|_0$ only such norms which are calibrated beforehand, i.e. which, for example, take the same values at a some predefined point $x_0 \neq 0$.

In this case the following question may be posed: are there constants $\Delta, \delta > 0$ for which the inequalities

$$\delta\|x\|_0 \leq \frac{\|x\|}{\|x_0\|} \leq \Delta\|x\|_0 \tag{5}$$

hold? Still, even in this case the question posed above has the negative answer for arbitrary norms $\|\cdot\|$. At the same time, as it will be shown below if we consider only extremal norms $\|\cdot\|$ then *there are universal constants* $\Delta, \delta > 0$ *for which the inequalities (5) hold.* In this Section, the proof of the existence of the required constant Δ will be given.

Choose an arbitrary nonzero vector $x_0 \in \mathbb{R}^m$ and an irreducible set of $m \times m$ matrices $\boldsymbol{A}$. Then, by Lemma 3, the set $\mathcal{T}^*_{m-1}(\boldsymbol{A}, x_0)$ contains N linearly independent vectors $x_0, x_1, \ldots, x_{m-1}$. Then the balanced convex set[3]

$$S_\sharp = \mathrm{co}\{\pm x_0, \pm x_1, \ldots, \pm x_{m-1}\} \tag{6}$$

contains the origin in its interior and so it may be treated as the unit ball in the norm $\|\cdot\|_\sharp$ in $\mathbb{R}^m$ determined by the inequality

$$\|x\|_\sharp = \inf\{t:\ t > 0,\ x \in tS_\sharp\}, \tag{7}$$

i.e. $S_\sharp = \{x:\ \|x\|_\sharp \le 1\}$.

Lemma 5 *Let $\|\cdot\|_\sharp$ be the norm introduced in (6)–(7) and determined by an irreducible set of $m \times m$ matrices $\boldsymbol{A}$ and by a vector $x_0 \neq 0$. Then for any norm $\|\cdot\|$ extremal with respect to the matrix set $\boldsymbol{A}$ the following estimate holds:*

$$\frac{\|x\|}{\|x_0\|} \le (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x\|_\sharp \qquad \forall x \in \mathbb{R}^m. \tag{8}$$

Proof. By Lemma 3 each of the vectors $x_0, x_1, \ldots, x_{m-1}$ in (6) may be represented in the form

$$x_i = A_{\boldsymbol{\sigma}(i)} x_0, \qquad i = 0, 1, \ldots, m-1,$$

where $\boldsymbol{\sigma}(i) \in \{1, \ldots, r\}^{k_i}$ for some, possibly zero, integer $k_i \le m-1$. Therefore, for arbitrary norm $\|\cdot\|$ extremal with respect to the matrix set $\boldsymbol{A}$ the inequalities

$$\|x_i\| \le (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x_0\|, \qquad i = 0, 1, \ldots, m-1.$$

are valid. The obtained inequalities show that

$$S_\sharp \subseteq \{x:\ \|x\| \le (\max\{1, \rho(\boldsymbol{A})\})^{m-1} \|x_0\|\}$$

from which the estimate (8) follows. Lemma 5 is proved. $\square$

[3] Recall that a set in a linear space is called balanced if with each its element x it contains also the element $-x$.

Lemma 6 *Let $\|\cdot\|_0$ be a norm and $x_0 \neq 0$ be a vector in $\mathbb{R}^m$. Let also $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there is a number $\Delta < \infty$ and a neighborhood $\mathcal{A}$ of the matrix set $\boldsymbol{A}$ such that for any norm $\|\cdot\|'$ extremal with respect to the matrix set $\boldsymbol{A}' \in \mathcal{A}$ the following estimate is valid*

$$\frac{\|x\|'}{\|x_0\|'} \leq \Delta\|x\|_0 \qquad \forall x \in \mathbb{R}^m. \tag{9}$$

Proof. Let $\boldsymbol{A} = \{A_1, \ldots, A_r\}$. Then by Lemma 3 the set $\mathcal{T}^*_{m-1}(\boldsymbol{A}, x_0)$ contains the linearly independent vectors $x_0, x_1, \ldots, x_{m-1}$ of the form

$$x_i = x_i(\boldsymbol{A}) = A_{\boldsymbol{\sigma}(i)}x_0, \qquad i = 0, 1, \ldots, m-1,$$

where $\boldsymbol{\sigma}(i) \in \{1, \ldots, r\}^{k_i}$ for some, possibly zero, $k_i \leq m-1$. In this case, for any matrix set $\boldsymbol{A}' = \{A'_1, \ldots, A'_r\}$ from a sufficiently small neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ the vectors

$$x'_i = x_i(\boldsymbol{A}') = A'_{\boldsymbol{\sigma}(i)}x_0, \qquad i = 0, 1, \ldots, m-1,$$

are also linearly independent.

For each $\boldsymbol{A}' \in \mathcal{A}$ we denote by $S_\sharp(\boldsymbol{A}')$ the balanced convex set

$$S_\sharp(\boldsymbol{A}') = \mathrm{co}\{\pm x_0(\boldsymbol{A}'), \pm x_1(\boldsymbol{A}'), \ldots, \pm x_{m-1}(\boldsymbol{A}')\},$$

which contains the origin in its interior. As it was noted above, such a set may be treated as the unit ball in the norm $\|\cdot\|'_\sharp$ in $\mathbb{R}^m$ determined by the equation

$$\|x\|'_\sharp = \inf\{t : \ t > 0, \ x \in tS_\sharp(\boldsymbol{A}')\}.$$

Then from Lemma 5 it follows that

$$\frac{\|x\|}{\|x_0\|} \leq \left(\max\{1, \rho(\boldsymbol{A}(\lambda))\}\right)^{m-1} \|x\|'_\sharp \qquad \forall x \in \mathbb{R}^m, \quad \forall \boldsymbol{A}' \in \mathcal{A}. \tag{10}$$

To complete the proof, it remains only to note that the vectors $x_0(\boldsymbol{A}')$, $x_1(\boldsymbol{A}')$, $\ldots$, $x_{m-1}(\boldsymbol{A}')$ depend continuously on $\boldsymbol{A}'$ and are linearly independent at the point $\boldsymbol{A}' = \boldsymbol{A}$. Hence the intersection of the sets $S_\sharp(\boldsymbol{A}')$ with $\boldsymbol{A}' \in \mathcal{A}$ has a nonempty interior to which the origin belongs. Therefore, there exists a constant μ such that

$$\{x : \ \|x\|_0 \leq 1\} \subseteq \mu \bigcap_{\boldsymbol{A}' \in \mathcal{A}} S_\sharp(\boldsymbol{A}'),$$

and then

$$\|x\|'_\sharp \leq \mu\|x\|_0 \qquad \forall x \in \mathbb{R}^m, \quad \forall \boldsymbol{A}' \in \mathcal{A}. \tag{11}$$

From (10) and (11) we readily obtain the statement of Lemma 6 with the constant Δ defined as

$$\Delta = \mu \sup_{\boldsymbol{A}' \in \mathcal{A}} \left(\max\{1, \rho(\boldsymbol{A}')\}\right)^{m-1}.$$

Here, one can assume that the constant Δ is finite since the supremum in the right hand part of the latter formula is bounded in any bounded neighborhood of the matrix set $\boldsymbol{A}$, while the neighborhood $\mathcal{A}$ is supposed to be rather small and hence bounded. Lemma 6 is proved. □

3.2 Compactness of the Set of Extremal Norms

Let $\boldsymbol{A}$ be an irreducible matrix set. Denote by $N_{\mathrm{ext}}(\boldsymbol{A}, x_0)$, where $x_0 \neq 0 \in \mathbb{R}^m$, the set of all norms $\|\cdot\|$ which are extremal with respect to the matrix set $\boldsymbol{A}$ and satisfy the calibrating condition $\|x_0\| = 1$. The notation $C_{\mathrm{loc}}(\mathbb{R}^m)$ will be used for the linear topological space of continuous functions defined on $\mathbb{R}^m$ with the topology of uniform convergence on bounded subsets from $\mathbb{R}^m$.

Theorem 2 *Let $x_0 \neq 0 \in \mathbb{R}^m$ and let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there exists a compact neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that the map*

$$\boldsymbol{A}' \mapsto N_{\mathrm{ext}}(\boldsymbol{A}', x_0), \qquad \boldsymbol{A}' \in \mathcal{A} \tag{12}$$

is compact and upper semi-continuous.

Proof. Given a norm $\|\cdot\|_0$ in $\mathbb{R}^m$, define $\mathcal{A}$ as such a compact neighborhood of the matrix set $\boldsymbol{A}$ whose existence has been established Lemma 6. Introduce the set of norms

$$\mathcal{N} := \bigcup_{\boldsymbol{A}' \in \mathcal{A}} N_{\mathrm{ext}}(\boldsymbol{A}', x_0),$$

and show that this set is compact in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$.

Indeed, by Lemma 6 for some $\Delta < \infty$ the following estimates hold

$$\|x\| \leq \Delta \|x\|_0 \qquad \forall x \in \mathbb{R}^m, \quad \forall \|\cdot\| \in \mathcal{N},$$

and so the values of the norms from $\mathcal{N}$ are uniformly bounded on each bounded set from $\mathbb{R}^m$. Besides, again by Lemma 6 we have

$$\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq \Delta \|x - y\|_0 \qquad \forall x, y \in \mathbb{R}^m, \quad \forall \|\cdot\| \in \mathcal{N},$$

and hence the norms from $\mathcal{N}$ are functions satisfying a uniform Lipschitz condition on $\mathbb{R}^m$. Thus, the norms from $\mathcal{N}$ form a set of uniformly bounded and equicontinuous functions on each closed bounded set from $\mathbb{R}^m$, from which by the Arzela-Ascoli theorem the compactness of the set $\mathcal{N}$ in the space $C_{\rm loc}(\mathbb{R}^m)$ follows.

Now we prove that the graph of the map (12) is closed in the space $\mathcal{A} \times C_{\rm loc}(\mathbb{R}^m)$. Let $\{(\boldsymbol{A}^{(n)}, \|\cdot\|^{(n)})\}$, where $\boldsymbol{A}^{(n)} \in \mathcal{A}$, be a sequence of elements belonging to the graph of the map (12) and converging to some element $(\boldsymbol{A}^*, \nu(\cdot)) \in \mathcal{M}_{m,r} \times C_{\rm loc}(\mathbb{R}^m)$. Then the compactness of $\mathcal{A}$ implies the inclusion $\boldsymbol{A}^* \in \mathcal{A}$. At the same time, we may state that the function $\nu(\cdot)$, being a limit in $C_{\rm loc}(\mathbb{R}^m)$ of a sequence of norms $\|\cdot\|^{(n)}$, is only a semi-norm.

From the definition of the sequence $\{(\boldsymbol{A}^{(n)}, \|\cdot\|^{(n)})\}$ it follows that $\|\cdot\|^{(n)} \in N_{\rm ext}(\boldsymbol{A}^{(n)}, x_0)$ for each value of n and therefore

$$\rho(\boldsymbol{A}^{(n)})\|x\|^{(n)} = \max\left\{\|A_1^{(n)}x\|^{(n)}, \ldots, \|A_r^{(n)}x\|^{(n)}\right\} \qquad \forall x \in \mathbb{R}^m\ , \forall n. \quad (13)$$

Here, due to the assumption about the irreducibility of the matrix set $\boldsymbol{A}$, without loss of generality, one can assume that each of the matrix sets $\boldsymbol{A}^{(n)}$ is also irreducible. In this case it holds (see [7]) $\rho(\boldsymbol{A}^{(n)}) \to \rho(\boldsymbol{A}^*)$ and, by passing to limit in (13), we obtain

$$\rho(\boldsymbol{A}^*)\nu(x) = \max\{\nu(A_0^*x), \nu(A_1^*x), \ldots, \nu(A_r^*x)\} \qquad \forall x \in \mathbb{R}^m,$$

with $\nu(x_0) = \lim_{n\to\infty} \|x_0\|^{(n)} = 1$. Hence, the semi-norm ν is extremal with respect to the irreducible matrix set $\boldsymbol{A}^*$ and does not equal identically to zero. Then by Lemma 4 this semi-norm is in fact an extremal norm, i.e. $\nu(\cdot) = \|\cdot\|^* \in N_{\rm ext}(\boldsymbol{A}^*, x_0)$, which means that the graph of the map (12) is closed.

So, it is proved that the graph of the map (12) is closed and that the set $\mathcal{N}$ is compact. From this we get by Lemma 1 the compactness and upper semi-continuity of the map (12). Theorem 2 is proved. □

3.3 Uniform Equivalence of Extremal Norms

Now, we are able to prove the left-hand side of the inequalities (5).

Lemma 7 *Given a norm $\|\cdot\|_0$ and a vector $x_0 \neq 0$ in $\mathbb{R}^m$, let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there exist a number $\delta > 0$ and a neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that for any norm $\|\cdot\|'$ extremal with respect to the matrix set $\boldsymbol{A}' \in \mathcal{A}$ the following estimate hold:*

$$\delta\|x\|_0 \leq \frac{\|x\|'}{\|x_0\|'} \qquad \forall x \in \mathbb{R}^m.$$

PROOF. Define $\mathcal{A}$ as the neighborhood of the matrix set $\boldsymbol{A}$ determined by Theorem 2. Then, supposing that Lemma 7 in not true, one may choose matrix sets $\boldsymbol{A}^{(n)} \in \mathcal{A}$ and corresponding to them extremal norms $\|\cdot\|^{(n)} \in N_{\mathrm{ext}}(\boldsymbol{A}^{(n)}, x_0)$ as well as vectors $x^{(n)}$ such that $\|x^{(n)}\|_0 = 1$ and

$$\frac{\|x^{(n)}\|^{(n)}}{\|x_0\|^{(n)}} \to 0 \quad \text{as} \quad n \to \infty. \tag{14}$$

Now, by Theorem 2 one can suppose that the sequences $\{\boldsymbol{A}^{(n)}\}$ and $\{\|\cdot\|^{(n)}\}$ are convergent, i.e. $\boldsymbol{A}^{(n)} \to \boldsymbol{A}^* \in \mathcal{A}$ and $\|\cdot\|^{(n)} \to \|\cdot\|^* \in N_{\mathrm{ext}}(\boldsymbol{A}^*, x_0)$. The sequence $\{x^{(n)}\}$ also can considered as convergent: $x^{(n)} \to x^* \neq 0$. Then, by passing in (14) to limit, we obtain the equality

$$\frac{\|x_*\|^*}{\|x_0\|^*} = 0, \qquad x_*, x_0 \neq 0,$$

which is impossible since $\|\cdot\|^*$ is a norm. The contradiction completes the proof of Lemma 7. □

Properties of extremal norms proven in Lemmas 6 and 7 can be easily reformulated in a universal form which does not depend on the choice of an auxiliary vector x_0.

Theorem 3 *For any irreducible set of $m \times m$ matrices $\boldsymbol{A}$ there are neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ and constants $0 < \delta \leq \Delta < \infty$ such that for any pair of norms $\|\cdot\|'$ and $\|\cdot\|''$ extremal for the matrix sets $\boldsymbol{A}', \boldsymbol{A}'' \in \mathcal{A}$, respectively, the following estimates*

$$\frac{\delta^2}{\Delta^2}\frac{\|x\|''}{\|y\|''} \leq \frac{\|x\|'}{\|y\|'} \leq \frac{\Delta^2}{\delta^2}\frac{\|x\|''}{\|y\|''} \qquad \forall x, y \neq 0 \in \mathbb{R}^m,$$

are valid.

This theorem is a direct corollary of Lemmas 6 and 7, so its proof is omitted.

Note in conclusion that in addition to topological properties formulated above, extremal norms possess also some algebraic structure.

Lemma 8 *Let $\|\cdot\|'$ and $\|\cdot\|''$ be extremal norms corresponding to a matrix set. Then $\|x\| = \max\{\|x\|', \|x\|''\}$ is also an extremal norm corresponding to the same matrix set.*

Proof of this Lemma is evident.

3.4 Extremal Trajectories and Their Generators

Introduce some notions. A trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ will be called *characteristic* if there are constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \leq \rho^{-n}(\boldsymbol{A})\|x_n\| \leq c_2 \qquad \forall n.$$

Remark that the definition of a characteristic trajectory does not depend on the choice of the norm $\|\cdot\|$ in $\mathbb{R}^m$. An important particular case of characteristic trajectories are so-called extremal trajectories. A trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ will be called *extremal* if in some extremal norm $\|\cdot\|$ the following identities hold:

$$\rho^{-n}(\boldsymbol{A})\|x_n\| \equiv \text{const.} \tag{15}$$

In contrast to the definition of characteristic trajectories the definition of extremal trajectories depends on the choice of the extremal norm. So, a trajectory extremal in one norm may be not extremal in another. Nevertheless, as will be shown below in Lemma 10 for an irreducible matrix set one can always find extremal trajectories which are *universal* in the that such trajectories are extremal in each extremal norm.

Now we prove that the set of extremal trajectories, and consequently the corresponding set of characteristic trajectories is not empty in the case when the matrix set $\boldsymbol{A}$ is irreducible.

Lemma 9 *For any vector $x \neq 0 \in \mathbb{R}^m$ and any extremal norm $\|\cdot\|$ there is an extremal trajectory $\{x_n\}$ satisfying $x_0 = x$.*

Proof. Construct recursively the trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ satisfying $x_0 = x$. Suppose that the element x_n is already found. Then, by the definition of the extremal norm, the following equality is valid:

$$\rho(\boldsymbol{A})\|x_n\| = \max\left\{\|A_0 x_n\|, \|A_1 x_n\|, \ldots, \|A_r x_n\|\right\}.$$

Hence, there exists an index σ_n for which $\rho(\boldsymbol{A})\|x_n\| = \|A_{\sigma_n} x_n\|$. So, in order to satisfy conditions (1), (15) it is sufficient to define the element x_{n+1} by the equality $x_{n+1} = A_{\sigma_n} x_n$. Lemma 9 is proved. □

Corollary 1 *If the matrix set $\boldsymbol{A}$ is irreducible then the set of its extremal trajectories as well as the set of its characteristic trajectories is nonempty.*

The proof of this Corollary immediately follows from Theorem 1 asserting that for an irreducible matrix set the set of extremal norms is not empty, and from Lemma 9 according to which in this case the set of corresponding extremal trajectories is also nonempty. □

Lemma 10 *For any irreducible matrix set $\boldsymbol{A}$ there are trajectories which are extremal with respect to any norm extremal for the matrix set $\boldsymbol{A}$.*

PROOF. Let $\{x_n\}$ be a trajectory of the matrix set $\boldsymbol{A}$ which is extremal in some extremal norm $\|\cdot\|_0$. Consider the sequence of "shifted" trajectories $\boldsymbol{x}_k = \{x_n^{(k)}\}$ defined as follows

$$x_n^{(k)} = \rho^{-k} x_{n+k}, \qquad n = 0, 1, 2, \dots \ .$$

Then for each fixed $n = 0, 1, \dots$ the set of elements $\{x_n^{(k)}\}$ is uniformly bounded

$$\|x_n^{(k)}\|_0 = \|\rho^{-k} x_{n+k}\|_0 = \rho^n \|x_0\|, \qquad k = 0, 1, \dots \ ,$$

where $\rho = \rho(\boldsymbol{A})$. Hence, by Lemma 2 the sequence of trajectories $\boldsymbol{x}_k$ is compact in the space $\Omega(\mathbb{R}^m)$. Therefore, without loss of generality one may suppose that for each $n = 0, 1, \dots$ there exists the limit

$$x_n^* = \lim_{k\to\infty} x_n^{(k)} = \lim_{k\to\infty} \rho^{-k} x_{n+k}. \tag{16}$$

Note that by Lemma 2 the set of all trajectories of the matrix set $\boldsymbol{A}$ is closed in the space $\Omega(\mathbb{R}^m)$, and so the limiting sequence $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is also a trajectory of the matrix set $\boldsymbol{A}$.

At last, show that the trajectory $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is extremal in any extremal norm of the matrix set $\boldsymbol{A}$. Fix an arbitrary norm $\|\cdot\|_*$ extremal for the matrix set $\boldsymbol{A}$. Then, by the definition of an extremal norm, for the trajectory $\{x_n\}$ the following inequalities hold

$$\|x_0\|_* \geq \rho^{-1}\|x_1\|_* \geq \rho^{-2}\|x_2\|_* \geq \ldots \geq \rho^{-n}\|x_n\|_* \geq \ldots \geq c_1 > 0.$$

Hence, the sequence $\{\rho^{-n}\|x_n\|_*\}$ monotonously decreases and consequently there exists limit

$$\lim_{n\to\infty} \rho^{-n}\|x_n\|_* = \omega \geq c_1 > 0.$$

Together with (16) the latter relation implies

$$\rho^{-n}\|x_n^*\|_* = \lim_{k\to\infty} \rho^{-(n+k)}\|x_{n+k}\|_* = \omega, \qquad n = 0, 1, \dots \ .$$

So, the trajectory $\boldsymbol{x}^* = \{x_n^*\}_{n=0}^{\infty}$ is extremal in the norm $\|\cdot\|_*$. Lemma 10 is proved. $\square$

Denote by $\mathcal{E}(\boldsymbol{A}, x)$ the set of all extremal trajectories $\{x_n\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ satisfying the initial condition $x_0 = x \neq 0$.

Theorem 4 *Let $\mathcal{X} \subset \mathbb{R}^m$ be a compact set which does not contain the origin and let $\boldsymbol{A}$ be an irreducible set of $m \times m$ matrices. Then there is a compact neighborhood $\mathcal{A}$ of $\boldsymbol{A}$ such that the map*

$$\big(\boldsymbol{A}', x\big) \mapsto \mathcal{E}(\boldsymbol{A}', x), \qquad \boldsymbol{A}' \in \mathcal{A},\ x \in \mathcal{X}, \tag{17}$$

is compact and upper semi-continuous.

Proof. Let $\mathcal{A}$ be such a closed neighborhood of the matrix set $\boldsymbol{A}$ whose existence is asserted by Lemma 6. Consider the sets

$$\mathcal{E} = \bigcup_{\boldsymbol{A}' \in \mathcal{A}} \bigcup_{x \in \mathcal{X}} \mathcal{E}(\boldsymbol{A}', x), \qquad \mathcal{T} = \bigcup_{\boldsymbol{A}' \in \mathcal{A}} \bigcup_{x \in \mathcal{X}} \mathcal{T}(\boldsymbol{A}', x),$$

and observe that the set $\mathcal{E}$, being a subset of a compact by Lemmas 1 and 2 the set $\mathcal{T} \subseteq \Omega(\mathbb{R}^m)$, is also compact in the space $\Omega(\mathbb{R}^m)$.

Now we show that the graph of the map (17) is closed in $\mathcal{A} \times \mathcal{X} \times \Omega(\mathbb{R}^m)$. Choose a sequence of elements $\big(\boldsymbol{A}^{(k)}, x^{(k)}, \boldsymbol{x}^{(k)}\big)$ with $\boldsymbol{A}^{(k)} \in \mathcal{A}$ and $x^{(k)} \in \mathcal{X}$ belonging to the graph of the map (17) and converging to some element $\big(\boldsymbol{A}^*, x^*, \boldsymbol{x}^*\big) \in \mathcal{A} \times \mathcal{X} \times \Omega(\mathbb{R}^m)$. Then the sequence $\big(\boldsymbol{A}^{(k)}, x^{(k)}, \boldsymbol{x}^{(k)}\big)$ belongs also to the graph of the map $\mathcal{T}(\boldsymbol{A}, x)$. In this case, due to the compactness and upper semi-continuity of the map $\mathcal{T}(\boldsymbol{A}, x)$ (see Lemma 2), the limiting element $\big(\boldsymbol{A}^*, x^*, \boldsymbol{x}^*\big)$ also belongs to the graph of the map $\mathcal{T}(\boldsymbol{A}, x)$:

$$\boldsymbol{x}^* \in \mathcal{T}(\boldsymbol{A}^*, x^*).$$

Hence, $\boldsymbol{x}^*$ is a trajectory of the matrix set $\boldsymbol{A}^* \in \mathcal{A}$ satisfying the initial condition $x^* \in \mathcal{X}$. It remains only to prove that the trajectory $\boldsymbol{x}^*$ is extremal.

By construction, $\boldsymbol{x}^{(k)} = \{x_n^{(k)}\}$ is a trajectory of the matrix set $\boldsymbol{A}^{(k)}$ which is extremal in some matrix norm $\|\cdot\|^{(k)}$. Then

$$\|x_0^{(k)}\|^{(k)} = \rho^{-1}(\boldsymbol{A}^{(k)})\|x_1^{(k)}\|_k = \ldots = \rho^{-n}(\boldsymbol{A}^{(k)})\|x_n^{(k)}\|^{(k)} = \ldots\ , \tag{18}$$

where by Theorem 2 one may assume that the sequence of extremal norms $\|\cdot\|^{(k)}$ converges to some extremal norm $\|\cdot\|^*$ of the matrix set $\boldsymbol{A}^*$. Therefore, taking the limit in (18) we obtain[4]:

$$\|x_0^*\|^* = \rho^{-1}(\boldsymbol{A}^*)\|x_1^*\|^* = \ldots = \rho^{-n}(\boldsymbol{A}^*)\|x_n^*\|^* = \ldots\ .$$

The obtained relations justify that the trajectory $\boldsymbol{x}^* = \{x_n^*\}$ of the matrix set $\boldsymbol{A}^*$ is extremal in the extremal norm $\|\cdot\|^*$.

[4]Remark that $\rho(\boldsymbol{A}^{(k)}) \to \rho(\boldsymbol{A}^*)$ as $k \to \infty$ since the generalized spectral radius depends continuously [7] on the irreducible matrix set.

So, it is proved that the graph of the map (17) is closed and that the set $\mathcal{E}$ is compact. By Lemma 1 this implies the compactness and upper semi-continuity of the map (17). Theorem 4 is proved. $\square$

In order to describe completely an extremal trajectory $\boldsymbol{x} = \{x_n\}$ one should know not only the information about the sequence $\{x_n\}$ but also the information about the related index sequence $\{\sigma_n\}$. Below, it will be proposed a construction which determines extremal trajectories as all possible trajectories of some set-valued nonlinear dynamical system. Such a construction will allow us to avoid the necessity to describe explicitly the index sequence $\{\sigma_n\}$.

Let $\rho = \rho(\boldsymbol{A})$ and let $\|\cdot\|_*$ be an extremal norm for the matrix set $\boldsymbol{A} = \{A_1, \ldots, A_r\}$. Denote for each $x \in \mathbb{R}^m$ the map $g(x)$ by setting

$$g(x) := \{w : \ \exists i \in \{1, \ldots, r\} \text{ for which } w = A_i x, \text{ with } \|A_i x\|_* = \rho\|x\|_*\}.$$

By the definition of an extremal norm the set $g(x)$ for each $x \in \mathbb{R}^m$ is not empty and consists of no more than m elements. Note also that each map $g(x)$ has a closed graph and for it the following identity is valid

$$\|g(x)\|_* \equiv \rho\|x\|_*. \tag{19}$$

Lemma 11 *The sequence $\boldsymbol{x} = \{x_n\}$ is extremal for the matrix set $\boldsymbol{A}$ in the extremal norm $\|\cdot\|_*$ if and only if it satisfies the inclusions*

$$x_{n+1} \in g(x_n) \qquad \forall n.$$

The proof of this Lemma immediately follows from the definitions of the extremal norm and the map g. $\square$

According to Lemma 11 each trajectory of the set-valued map $g(\cdot)$ is extremal in the norm $\|\cdot\|_*$. This motivates us to call the map $g(\cdot)$ as *the generator of extremal trajectories*. In general, the map $g(\cdot)$ can not be described explicitly. Nevertheless, in Section 5 for the sets of 2×2 matrices we will be able to obtain a rather detailed description of the properties of the generators of extremal trajectories.

4 Extremal Norms: the Case of a Pair of Two-dimensional Matrices

In this Section, for the case when the set $\boldsymbol{A}$ consists of two 2×2 matrices some additional properties of extremal norms and extremal trajectories are established.

4.1 Definition and Properties of the Matrix Set

Consider the pair of matrices

$$A_0 = \alpha \begin{Vmatrix} a & b \\ 0 & 1 \end{Vmatrix}, \quad A_1 = \beta \begin{Vmatrix} 1 & 0 \\ c & d \end{Vmatrix}, \tag{20}$$

where $\alpha, \beta > 0$ and

$$bc \geq 1 > a, d > 0. \tag{21}$$

In this case, iterations of an arbitrary point under the action of the matrix A_0 tend to the subspace $L_0 = \{(x_0, x_1) : bx_1 = (1-a)x_0\}$, and iterations of an arbitrary point under the action of the matrix A_1 tend to the subspace $L_1 = \{(x_0, x_1) : (1-d)x_1 = cx_0\}$ (see. Fig. 1). In the case when $bc > (1-a)(1-d)$, and all the more when conditions (21) are fulfilled, the limiting subspaces L_0 and L_1 differ from each other and their mutual disposition is such as in Fig. 1.

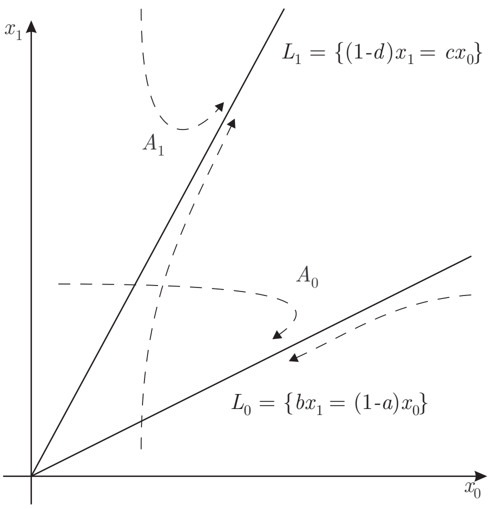

Figure 1: Action of matrices A_0, A_1

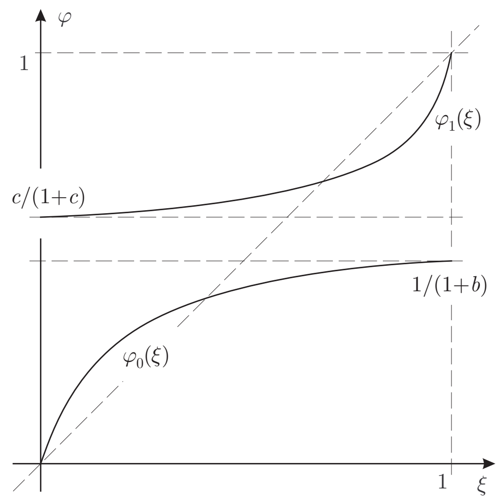

Figure 2: Plots of functions $\varphi_0(\xi)$, $\varphi_1(\xi)$

Associate the ray $t(x_0, x_1)$, $t > 0$, passing the point $(x_0, x_1) \neq 0$, $x_0, x_1 \geq 0$, with the number $\xi = x_1/(x_0 + x_1) \in [0, 1]$. Under such association the semi-axis of abscisses, i.e. the ray $t(1, 0)$, is represented by the number $\xi = 0$, while the semi-axis of ordinates, i.e. the ray $t(0, 1)$, is represented by the number $\xi = 1$. Then the matrix A_0 maps the ray with the coordinate ξ at the ray with the coordinate $\varphi_0(\xi)$, where

$$\varphi_0(\xi) = \frac{\xi}{a(1-\xi) + b\xi + \xi}, \tag{22}$$

while the matrix A_1 maps the ray with the coordinate ξ at the ray with the coordinate $\varphi_1(\xi)$:

$$\varphi_1(\xi) = \frac{c(1-\xi)+d\xi}{c(1-\xi)+d\xi+1-\xi}. \tag{23}$$

Under the condition $bc \geq 1$, for any $0 \leq \xi, \zeta \leq 1$ the inequalities $\varphi_1(\xi) \geq \varphi_0(\zeta)$ hold. Moreover, the function $\varphi_0(\xi)$ strictly increases when $a > 0$ while the function $\varphi_1(\xi)$ strictly increases when $d > 0$. Hence, under conditions (21) the graphs of the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ look like those plotted in Fig. 2.

Consider also the pair of matrices conjugate to the matrices A_0 and A_1:

$$A_0' = \alpha \begin{Vmatrix} a & 0 \\ b & 1 \end{Vmatrix}, \quad A_1' = \beta \begin{Vmatrix} 1 & c \\ 0 & d \end{Vmatrix}.$$

In this case the iterations of an arbitrary point under the action of the matrix A_0' tend to the axis $\{x_0 = 0\}$, and the iterations of an arbitrary point under the action of the matrix A_1' tend to the axis $\{x_1 = 0\}$ (see. Fig. 3).

The subspace $\{x_0 = 0\}$ is mapped by the matrix A_1' in the subspace $L_1' = \{(x_0, x_1) : cx_1 = dx_0\}$, while the subspace $\{x_1 = 0\}$ is mapped by the matrix A_0' in the subspace $L_0' = \{(x_0, x_1) : ax_1 = bx_0\}$. Such disposition of the spaces which plotted in Fig. 3 is achieved when $bc > ad$ and all the more when conditions (21) are fulfilled.

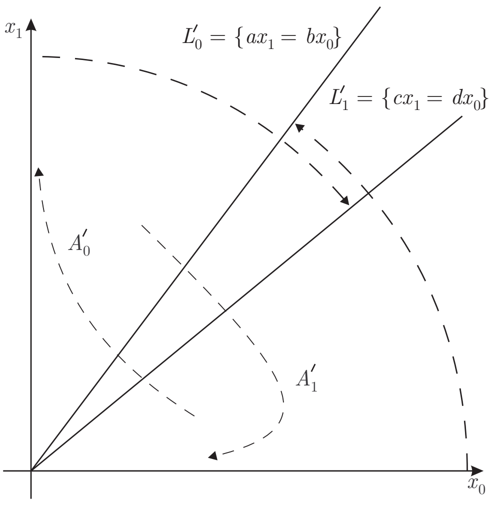

Figure 3: Action of matrices A_0', A_1'

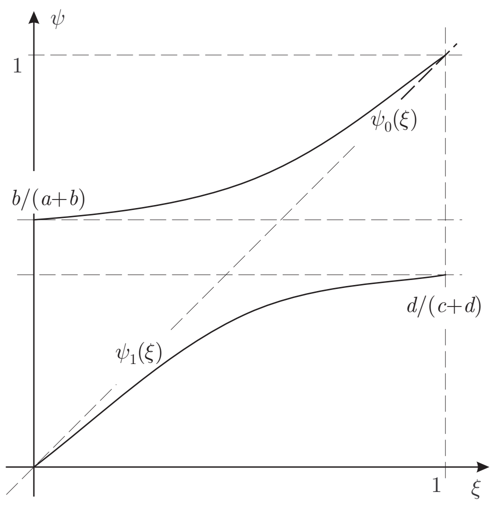

Figure 4: Plots of functions $\psi_0(\xi)$, $\psi_1(\xi)$

The matrix A_0' maps the ray with the coordinate ξ at the ray with the coordinate $\psi_0(\xi)$, where

$$\psi_0(\xi) = \frac{b(1-\xi)+\xi}{a(1-\xi)+b(1-\xi)+\xi},$$

while the matrix A_1' maps the ray with the coordinate ξ at the ray with the coordinate $\psi_1(\xi)$:

$$\psi_1(\xi) = \frac{d\xi}{1-\xi+c\xi+d\xi}.$$

Graphs of the functions $\psi_0(\xi)$ and $\psi_1(\xi)$ are plotted in Fig. 4.

4.2 Boundary of the Unit Ball of the Extremal Norm

Denote by $\mathcal{M}^\sharp \subset \mathcal{M}_{2,2}$ the set of all matrix sets $\boldsymbol{A}$ consisting of the matrices A_0 and A_1 of the form (20) satisfying conditions (21). From the description of the invariant spaces for the matrices A_0 and A_1 it follows that each matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$ is irreducible. Given some extremal norm $\|\cdot\|$ in $\mathbb{R}^2$ corresponding to $\boldsymbol{A}$, denote by $\mathbb{S}$ the unit ball in the norm $\|\cdot\|$. Recall that the linear functional $l(x)$, $x \in \mathbb{R}^2$ is called *the support functional* for the unit ball $\mathbb{S}$ if

$$\sup_{x\in\mathbb{S}} |l(x)| \le 1, \qquad \text{and} \quad \exists\, u_* \in \mathbb{S}: \quad l(u_*) = 1.$$

By the Khan-Banach theorem for any point $u_* \in \mathbb{S}$, $\|u_*\| = 1$, there is a support functional l_* for which $l_*(u_*) = 1$. Remark that each functional $l(x)$ can be represented by a linear form:

$$l(x) \equiv \langle l, x\rangle := l_0x_0 + l_1x_1, \quad \text{where} \quad l = (l_0, l_1),\ x = (x_0, x_1) \in \mathbb{R}^2,$$

and so, the values l_0, l_1 may be treated as the coordinates of the functional $l(x)$.

Lemma 12 *Let $\|\cdot\|$ be an extremal norm for the matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$. Then for any vector $u \in \mathbb{S}$, $\|u\| = 1$, with non-negative coordinates the support functional $l(x) = \langle l, x\rangle$ satisfying $l(u) = 1$ is also has non-negative coordinates. In other words, the unit ball in the norm $\|\cdot\|$ in the first quadrant has the form like that presented in Fig. 5.*

Proof. Let $\|\cdot\|$ be an extremal norm for the matrix set $\boldsymbol{A} \in \mathcal{M}^\sharp$ and let $u_* = (0, u_{*,1})$ be a point lying on the boundary of the ball $\mathbb{S} = \{x : \|x\| = 1\}$, i.e. $\|u_*\| = 1$. Show that in this case for any point $x = (x_1, x_2) \in \mathbb{S}$ lying in the first quadrant (i.e. such that $x_0, x_1 \ge 0$) the following relation holds

$$x_1 \le u_{*,1}.$$

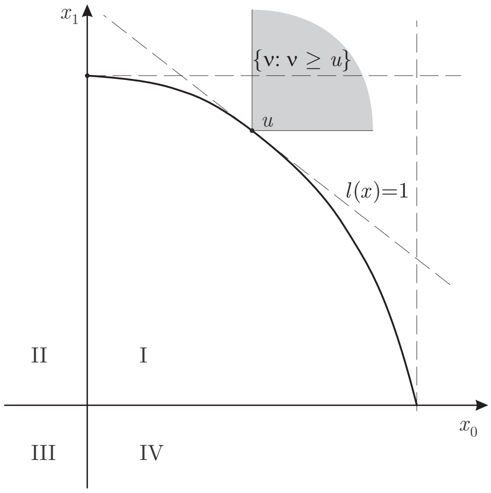

Figure 5: The unit ball of an extremal norm

Suppose the contrary, i.e. that there are points $u_n = (u_{n,0}, u_{n,1})$, $\|u_n\| = 1$, for which $u_{n,1} > u_{*,1}$. Denote by $l_0(x)$ the support functional to the ball $\mathbb{S}$ satisfying $l_0(u_*) = 1$. Then by the definition of the points u_n the following relation will be valid

$$l_0(u_n) \leq l_0(u_*) = 1, \tag{24}$$

and since the ball $\mathbb{S}$ contains a neighborhood of the origin then

$$l_0(u) \leq l_0(u_*) = 1 \qquad \forall u : \ \|u\| \text{ is sufficiently small.} \tag{25}$$

From (24), (25) it follows that the support line l_0 is parallel to a line lying in the first quadrant. Therefore, *all the vectors $u_n - u_*$ belong to the interior of the first quadrant.*

Now, note that $\rho = \rho(\boldsymbol{A}) \geq \rho(A_1) = \beta$, and since $A_1 u_* = \beta d u_*$ then $\|A_1 u_*\| < \rho \|u_*\|$. Then from the definition of an extremal norm we get the equality

$$\rho \|u_*\| = \max\{\|A_0 u_*\|, \|A_1 u_*\|\} = \|A_0 u_*\|.$$

Hence the vector $v = \rho^{-1} A_0 u_*$ belongs to the boundary of the ball $\mathbb{S}$ (see Fig. 6) and $v_1 = \rho^{-1} \beta u_{*,1} \leq u_{*,1}$.

Denote by $l_1(x)$ the support functional to the ball $\mathbb{S}$ satisfying $l_1(v) = 1$. Then by the definition of the point u_*

$$l_1(u_*) \leq l_1(v) = 1, \tag{26}$$

and since the ball $\mathbb{S}$ contains a neighborhood of the origin then

$$l_1(u) \leq l_0(v) = 1 \qquad \forall u : \ \|u\| \text{ is sufficiently small.} \tag{27}$$

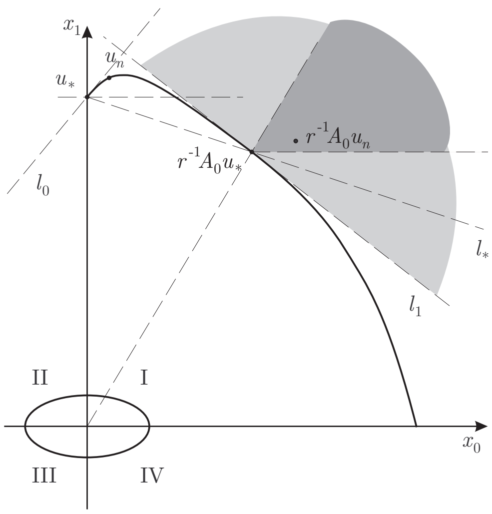

Figure 6: Boundary of the unit ball: impossible case

From (26), (27) it follows that the set $\{x : l_1(x) > 1\}$ contains the sector formed by the ray originated from the point $v = \rho^{-1}A_0u_*$ and directed parallel to the positive half-axis of abscisses and by the ray tv, $t \geq 1$ (see the dark-gray sector in Fig. 6).

Now, consider the sequence of the points $u_n \to u_*$, $\|u_n\| = 1$ for which $u_{n,1} > u_{*,1}$. According to the definition of an extremal norm

$$\rho\|u_n\| = \max\{\|A_0u_n\|, \|A_1u_n\|\}. \tag{28}$$

Since here $u_n \to u_*$ then $\|A_1u_n\| \to \|A_1u_*\| < \rho$, and (28) implies $\rho = \rho\|u_n\| = \|A_0u_n\|$ or, what is the same,

$$\|\rho^{-1}A_0u_n\| = 1 \quad \text{for all sufficiently large } n. \tag{29}$$

On the other hand, as was shown above the vector $u_n - u_*$ belongs to the interior of the first quadrant. Then simple calculations show that the vector $A_0(u_n - u_*)$ belongs to the interior of the sector formed by the positive semi-axis of abscisses and by the ray $tv = tA_0u_*$. In this case the vectors

$$\rho^{-1}A_0u_n = \rho^{-1}A_0u_* + \rho^{-1}A_0(u_n - u_*)$$

should belong to the sector marked in Fig. 6 by dark-gray color which contradicts to (29). The obtained contradiction proves impossibility of the situation plotted in Fig. 6.

At last, consider the vector $v_* = (v_{*,0}, 0)$, $\|v_*\| = 1$. Then, analogously to what has been done above, one can prove that the intersection of the ball $\mathbb{S}$

with the first quadrant lays entirely to the left from the vertical line passing the point $v_* = (v_{*,0}, 0)$ (see Fig. 5). From this the assertion of Lemma 12 immediately follows. $\square$

From Lemma 12 an important corollary follows. Call the norm $\|\cdot\|$ *monotone* (with respect to the cone of the vectors with non-negative coordinates) if for any pair of vectors u and v the relations $v \geq u \geq 0$, where the inequalities are understood coordinate-wise, imply the inequality $\|v\| \geq \|u\|$.

Lemma 13 *Any extremal norm for a matrix set* $\boldsymbol{A} \in \mathcal{M}^\sharp$ *is monotone.*

THE PROOF of this Lemma immediately follows from the description of the structure of the boundary of the unit ball of the extremal norm given in Lemma 12, and from Fig. 5 on which mutual disposition of the point sets $\{v : v \geq u\}$ and $\{x : \|x\| \leq \|u\|\}$ is plotted. $\square$

Define the sets

$$X_0 = \{x : \ \|A_0 x\| = \rho\|x\|\}, \qquad X_1 = \{x : \ \|A_1 x\| = \rho\|x\|\}. \tag{30}$$

Each of these sets is closed, conic (i.e. contains any vector tx along with the vector $x \neq 0$), and by the definition of an extremal norm $X_0 \cup X_1 = \mathbb{R}^2$.

The set $\Theta = X_0 \cap X_1$ will be called *the switching set* of the extremal norm $\|\cdot\|$. To analyze the structure of the sets X_0, X_1 and Θ we will need additional data.

4.3 The Gram Symbol

Given a pair of vectors $x, y \in \mathbb{R}^2$ and a pair of linear functionals

$$u(w) = \langle u, w\rangle, \quad v(w) = \langle v, w\rangle, \qquad u, v, w \in \mathbb{R}^2.$$

Then *the Gram symbol* of the ordered four-tuple $\{u, v, x, y\}$ is the expression

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} := u(x)v(y) - u(y)v(x) \equiv \langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle. \tag{31}$$

Lemma 14

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} = 0 \qquad \Longleftrightarrow \qquad u = tv \quad \text{or} \quad x = ty,$$

and

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} \geq 0 \qquad \text{if} \quad x = u, \ y = v, \tag{32}$$

$$\left\{\begin{matrix} u & x \\ v & y \end{matrix}\right\} \leq 0 \qquad \text{if} \quad x = v, \ y = u. \tag{33}$$

PROOF. By the definition (31), the Gram symbol of the four-tuple $\{u, v, x, y\}$ vanishes if and only if

$$\langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle = \langle u, x\langle v, y\rangle - y\langle v, x\rangle\rangle = 0.$$

If here $x\langle v, y\rangle - y\langle v, x\rangle = 0$ then $x = ty$, and Lemma 14 is proved. Therefore, we will suppose that $w = x\langle v, y\rangle - y\langle v, x\rangle \neq 0$. Then

$$\langle u, w\rangle = 0, \qquad w \neq 0, \tag{34}$$

and besides,

$$\langle v, w\rangle = \langle v, x\langle v, y\rangle - y\langle v, x\rangle\rangle \equiv 0. \tag{35}$$

Clearly, in a two-dimensional space the equalities (34), (35) with nonzero w may be valid only in the case when $u = tv$.

The inequalities (32) and (33) immediately follow from the following relations

$$\left\{\begin{matrix} u & u \\ v & v \end{matrix}\right\} = -\left\{\begin{matrix} u & v \\ v & u \end{matrix}\right\} = \langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2 \geq 0.$$

Lemma 14 is proved. $\square$

Lemma 14 implies that, under non-degenerate deformations of ordered pairs of the vectors $\{u, v\}$ and $\{x, y\}$ satisfying $u \neq tv$ and $x \neq ty$, the sign of the Gram symbol does not change. Moreover, each ordered pair of the vectors $\{u, v\}$ and $\{x, y\}$ may be deformed either at the ordered pair of the vectors $\{e_1, e_2\}$, or at the pair $\{e_2, e_1\}$, where

$$e_1 = (1, 0), \qquad e_2 = (0, 1).$$

So, the geometrical sense of the Gram symbol is that *the ordered pair of the vectors $\{x, y\}$ has the same orientation as the ordered pair of the vectors $\{u, v\}$ if and only if the Gram symbol of the corresponding ordered four-tuple of the vectors $\{u, v, x, y\}$ is positive.*

4.4 Structure of the Switching Set

Let $\mathbb{S}$ be the unit ball of an extremal norm $\|\cdot\|$ and let

$$\mathbb{S}' = \{u \in \mathbb{R}^2 : \sup_{x\in\mathbb{S}} |\langle u, x\rangle| \leq 1\}.$$

Denote by K_+ the cone of vectors in $\mathbb{R}^2$ with the non-negative coordinates.

Lemma 15 *Let* $\|\cdot\|'$ *be the norm the unit ball of which coincides with* $\mathbb{S}'$. *Then*

$$|\langle u, x\rangle| \leq \|x\|\|u\|'. \tag{36}$$

Moreover, for each vector $x \neq 0$ *there is a vector* $u \neq 0$ *such that* $\langle u, x\rangle = \|x\| \cdot \|u\|'$, *and furthermore, if* $x \in K_+$ *then* $u \in K_+$.

Proof. The inequality (36) is a well known fact in the theory of topological vector spaces and it directly follows from the definition of the dual norm $\|u\|'$. The fact that the equality $\langle u, x\rangle = \|x\| \cdot \|u\|'$ with $x \in K_+$ is valid for some $u \in K_+$ follows from Lemma 12. $\square$

Now, let $x, y \neq 0$ be a pair of the vectors satisfying $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$. Then due to the non-negativity of entries of the matrices A_0 and A_1,

$$A_0x \in K_+, \quad \|A_0x\| = \rho\|x\|, \qquad A_1y \in K_+, \quad \|A_1y\| = \rho\|y\|,$$

and by Lemma 15 such vectors $u, v \in K_+$ can be found for which

$$\begin{aligned} \langle u, A_0x\rangle &= \|u\|'\|A_0x\| = \rho\|u\|'\|x\|, & (37)\\ \langle v, A_1y\rangle &= \|v\|'\|A_1y\| = \rho\|v\|'\|y\|. & (38)\end{aligned}$$

On the other hand, (36) and the definition of an extremal norm imply

$$\begin{aligned} \langle u, A_0y\rangle &\leq \|u\|'\|A_0y\| = \rho\|u\|'\|y\|, & (39)\\ \langle v, A_1x\rangle &\leq \|v\|'\|A_1x\| = \rho\|v\|'\|x\|. & (40)\end{aligned}$$

From (37), (38), (39) and (40) we get

$$\langle u, A_0x\rangle\langle v, A_1y\rangle = \rho^2\|u\|'\|v\|'\|x\|'\|y\| \geq \langle u, A_0y\rangle\langle v, A_1x\rangle.$$

Then

$$\left\{ \begin{array}{cc} A_0'u & x \\ A_1'v & y \end{array} \right\} = \langle A_0'u, x\rangle\langle A_1'v, y\rangle - \langle A_0'u, y\rangle\langle A_1'v, x\rangle \geq 0. \tag{41}$$

So, we have proved the following

Lemma 16 *Let* $x, y \neq 0$ *be a pair of the vectors satisfying* $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$. *Then there are such nonzero vectors* $u, v \in K_+$ *for which relation (41) is valid.*

This Lemma is a key point in the analysis of the structure of the sets $X_0 \cap K_+$ and $X_1 \cap K_+$.

Theorem 5 *Let $\boldsymbol{A} = \{A_0, A_1\}$ be the matrix set defined by equalities (20) and satisfying conditions (21), and let $\|\cdot\|$ be an extremal norm of the matrix set $\boldsymbol{A}$. Then each of the sets $X_0 \cap K_+$ and $X_1 \cap K_+$ is a sector located as is shown in Fig. 7, and the intersection of these sectors is the ray*

$$\varTheta = X_0 \cap X_1 \cap K_+ = \{t\vartheta : \ t \in \mathbb{R}_+\} \tag{42}$$

passing a nonzero vector $\vartheta \in K_+$, $\|\vartheta\| = 1$. The vector ϑ belongs to the sector formed in K_+ by the straight lines

$$L_0 = \{(x_0, x_1) : bx_1 = (1-a)x_0\} \qquad L_1 = \{(x_0, x_1) : (1-d)x_1 = cx_0\}.$$

Moreover, the vector ϑ is a single solution to the system of equations

$$\|A_0 x\| = \|A_1 x\|, \quad \|x\| = 1, \qquad x \in K_+, \tag{43}$$

and it continuously depends on the matrices A_0, A_1 and the norm $\|\cdot\|$.

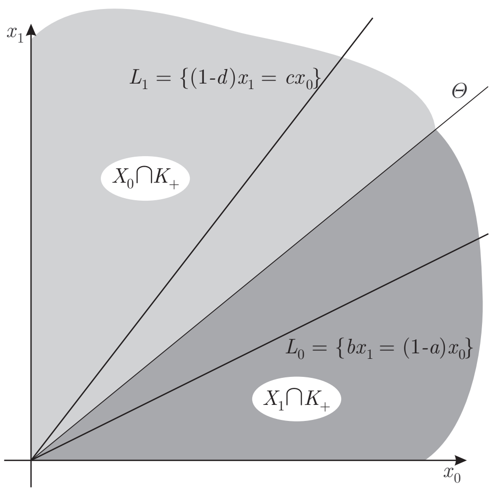

Figure 7: Location of the sectors $X_0 \cap K_+$ and $X_1 \cap K_+$

Proof. By Lemma 16 for a pair of nonzero vectors $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$ non-proportional to each other there are such nonzero vectors $u, v \in K_+$ for which the Gram symbol of the four-tuple $\{A_0'u, A_1'v, x, y\}$ is non-negative. This means that the ordered pair of vectors $\{x, y\}$ has the same orientation as the pair $\{A_0'u, A_1'v\}$. On the other hand, under conditions (21) for the pair of matrices A_0, A_1 the ordered pair of vectors $\{A_0'u, A_1'v\}$, as is seen from Fig 3, *is always oriented negatively*, i.e. the vector $A_1'v$ can be obtained by rotating the vector $A_0'u$ counter clockwise on the angle not exceeding π

and by appropriate stretching or contracting. Therefore, the ordered pair of vectors $\{x, y\}$ should also be oriented negatively.

So, any ordered pair of nonzero vectors $x \in X_0 \cap K_+$, $y \in X_1 \cap K_+$ not proportional to each other is negatively oriented. Since in addition, the sets $X_0 \cap K_+$ and $X_1 \cap K_+$ are closed and conic, i.e. contain with each its nonzero element the whole ray passing this element, then they should be such as it is asserted in Theorem 5.

Now, chose a nonzero vector x_* from the set $K_+ \cap L_0$. Then x_* is an eigenvector of the matrix A_0 corresponding to its eigenvalue αa, i.e. $A_0 x_* = \alpha a x_*$. Therefore,

$$\|A_0 x_*\| = \alpha a \|x_*\| < \rho \|x_*\|,$$

since due to (21) $a < 1$ and $\rho = \rho(\boldsymbol{A}) \geq \rho(A_0) = \alpha$. From this it follows, due to the definition of the extremal norm $\|\cdot\|$, that $x_* \notin X_0$. Therefore, $x_* \in X_1$, and so

$$K_+ \cap L_0 \subset X_1. \tag{44}$$

Analogously, one can obtain that each nonzero vector x^* from the set $K_+ \cap L_1$ should belong to the set X_0, i.e.

$$K_+ \cap L_1 \subset X_0. \tag{45}$$

The assertion of Theorem 5 that the vector ϑ belongs to the sector formed in the cone K_+ by the straight lines L_0 and L_1 follows now from inclusions (44) and (45).

The fact that the vector ϑ is the only solution of the system of equations (43) directly follows from the definitions (30), (42) of the sets $X_0, X_1, \varTheta$ and from the fact that the set $\varTheta$ is a ray. Therefore, to complete the proof of Theorem 5 we need only show that the vector ϑ depends continuously on the matrices A_0, A_1 and the extremal norm $\|\cdot\|$.

Let $\{A_0^{(n)}\}$, $\{A_1^{(n)}\}$ be sequences of the matrices (20) satisfying (21), and let $\{\|\cdot\|^{(n)}\}$ be a sequence of extremal norms corresponding to these matrices. Suppose that

$$A_0^{(n)} \to A_0^{(0)}, \qquad A_1^{(n)} \to A_1^{(0)}, \qquad \|\cdot\|^{(n)} \to \|\cdot\|^{(0)},$$

where convergence of the norms is understood as convergence in the space $C_{\mathrm{loc}}(\mathbb{R}^m)$. Denote by $\{\vartheta^{(n)}\}$ the sequence of vectors satisfying the system of equations

$$\|A_0^{(n)} \vartheta^{(n)}\|^{(n)} = \|A_1^{(n)} \vartheta^{(n)}\|^{(n)}, \quad \|\vartheta^{(n)}\|^{(n)} = 1, \qquad \vartheta^{(n)} \in K_+, \tag{46}$$

To prove that $\vartheta^{(n)} \to \vartheta^{(0)}$ it is sufficient to show that any limiting point ϑ^* of the sequence $\{\vartheta^{(n)}\}$ coincides with the element $\vartheta^{(0)}$. But it is really so, since by passing to limit in (46) one can be convinced readily that ϑ^* satisfies the equations

$$\|A_0^{(0)}x\|^{(0)} = \|A_1^{(0)}x\|^{(0)}, \quad \|x\|^{(0)} = 1, \qquad x \in K_+.$$

Since the only solution of the latter system is, by the definition, the vector $\vartheta^{(0)}$ then $\vartheta^* = \vartheta^{(0)}$.

So, the continuous dependance of the vector ϑ on the matrices A_0, A_1 and the extremal norm $\|\cdot\|$ is established, and the proof of Theorem 5 is completed. □

5 Frequency Properties of Extremal Trajectories: the Case of Two-dimensional Matrices

In this Section, the analysis of the properties of the extremal trajectories of the matrix sets $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ will be continued. Our prime goal will be to prove the following statement.

Theorem 6 (on the switching frequency) *For any extremal trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ determined by the equation*

$$x_{n+1} = A_{\sigma_n} x_n, \qquad n = 0, 1, \dots\ ,$$

it is defined the frequency (the switching frequency of the trajectory)

$$\sigma = \lim_{n\to\infty} \frac{\sum_{i=1}^n \sigma_i}{n}$$

of applying the matrix A_1 in the process of computation of the trajectory $\{x_n\}$.

The frequency σ does not depend on the choice of the extremal trajectory $\{x_n\}$ or on the index sequence $\{\sigma_n\}$, and so, it may be denoted as $\sigma(\boldsymbol{A})$. In addition, $\sigma(\boldsymbol{A})$ depends continuously on matrices of the matrix set $\boldsymbol{A}$ and takes rational values if and only if the matrix set $\boldsymbol{A}$ has an extremal trajectory corresponding to a periodic index sequence $\{\sigma_n\}$.

To prove Theorem 6, we will need auxiliary statements and constructions.

5.1 Generator of Extremal Trajectories

Fix in $\mathbb{R}^2$ a norm $\|\cdot\|$ extremal for the matrix set $\boldsymbol{A}$, and denote by X_0 and X_1 the sets (30) determined by this norm. In this case, the generator of extremal trajectories $g(\cdot)$ (see the definition in Section 3.4) will take the form

$$g(x) = \begin{cases} \rho^{-1}A_0x, & \text{if} \quad x \in X_0 \setminus X_1, \\ \rho^{-1}A_1x, & \text{if} \quad x \in X_1 \setminus X_0, \\ \{\rho^{-1}A_0x, \rho^{-1}A_1x\}, & \text{if} \quad x \in X_0 \cap X_1. \end{cases} \tag{47}$$

where $\rho = \rho(\boldsymbol{A})$.

Let us study the structure of the map $g(\cdot)$ in the first quadrant, i.e. in the cone $K_+ := \{x = (x_1, x_2) : \; x_1, x_2 \geq 0\}$, in more details. Introduce in K_+ the coordinate system (λ, ξ) by setting

$$\lambda(x) = \|x\|, \quad \xi(x) = \frac{x_2}{x_1 + x_2}, \qquad x \neq 0 \in K_+. \tag{48}$$

As was noted above (see (19)), for the map $g(\cdot)$ the identity $\|g(x)\|_* \equiv \|x\|_*$ is valid. Besides, by Theorem 5 the sets $X_0 \cap K_+$, $X_1 \cap K_+$ and $X_0 \cap X_1 \cap K_+$ are transferred by the map $\xi(\cdot)$ in the intervals $[\theta, 1]$, $[0, \theta]$ and a point θ, respectively, i.e.

$$\xi(X_1 \cap K_+) = [0, \theta], \qquad \xi(X_0 \cap K_+) = [\theta, 1], \qquad \xi(X_0 \cap X_1 \cap K_+) = \theta.$$

Then in the coordinate system (λ, ξ) the map f takes the form of a map with separable variables

$$f : (\lambda, \xi) \mapsto (\lambda, \varPhi), \tag{49}$$

where

$$\varPhi = \varPhi_\theta(\xi) = \begin{cases} \varphi_1(\xi), & \text{if} \quad \xi \in [0, \theta), \\ \{\varphi_0(\theta), \varphi_1(\theta)\} & \text{if} \quad \xi = \theta, \\ \varphi_0(\xi), & \text{if} \quad \xi \in (\theta, 1]. \end{cases} \tag{50}$$

Here the functions $\varphi_0(\xi)$ and $\varphi_1(\xi)$ are defined by (22) and (23), and have the appearance plotted in Fig 2. The graph of the set-valued function $\varPhi_\theta(\xi)$ is presented in Fig. 8.

Remark that the coordinate $\lambda(x)$ characterizes "remoteness" of the vector x from the origin of coordinates, while the coordinate $\xi(x)$ characterizes the "direction" of the vector x. In accordance with this, $\varPhi_\theta(\xi)$ can be treated as *the direction function* of the generator of extremal trajectories.

From Lemma 11, Theorem 5 and the representation (49), (50) of the map $g(\cdot)$ one can get the following description of the extremal trajectories.

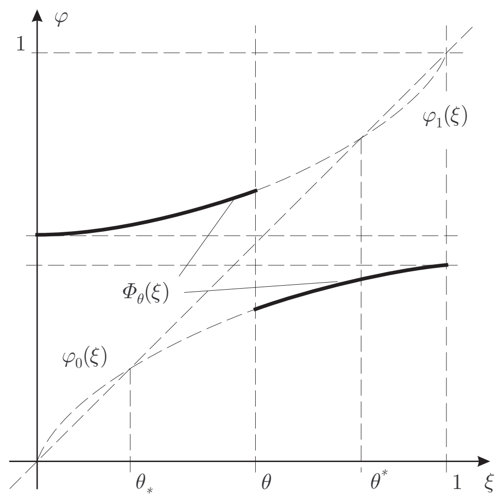

Figure 8: Graph of the direction function $\Phi_\theta(\xi)$

Lemma 17 *A nonzero trajectory $\{x_n\} \subseteq K_+$ is extremal for the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ in the extremal norm $\|\cdot\|$ if and only if its elements in the coordinate system (λ, ξ) can be represented in the form $x_n = (\lambda_n, \xi_n)$, where $\lambda_n \equiv \lambda_0$, and $\{\xi_n\}$ is a trajectory of the set-valued map $\Phi_\theta(\cdot)$, i.e.*

$$\xi_{n+1} \in \Phi_\theta(\xi_n), \qquad n = 0, 1, \dots\ ,$$

whose parameter θ satisfies the estimates $\theta_* \leq \theta \leq \theta^*$ with constants $\theta_*, \theta^* \in (0,1)$ defined by the equalities $\theta_* = \varphi_0(\theta_*)$ and $\theta^* = \varphi_1(\theta^*)$.

In addition, the trajectory $\{x_n\}$ satisfies the equation

$$x_{n+1} = A_{\sigma_n} x_n, \qquad n = 0, 1, \dots\ ,$$

with some index sequence $\{\sigma_n\}$ if and only if the trajectory $\{\xi_n\}$ satisfies the equation

$$\xi_{n+1} = \varphi_{\sigma_n}(\xi_n), \qquad n = 0, 1, \dots\ .$$

Remark that in spite of the fact that the extremal norm $\|\cdot\|_*$ is, in general, not known explicitly, the direction function $\Phi_\theta(\xi)$ of the generator of extremal trajectories is "defined rather unambiguously" which means that according to (50) it is uniquely defined by the triplet $(\varphi_0, \varphi_1, \theta)$ with the only unknown parameter θ.

When it will be needed to emphasize the dependance of the function $\Phi_\theta(\xi)$ on the triplet $(\varphi_0, \varphi_1, \theta)$, we will use the notation

$$\Phi_\theta(\xi) = \Phi[\varphi_0, \varphi_1, \theta](\xi). \tag{51}$$

In its turn, the triplet $(\varphi_0, \varphi_1, \theta)$ depends on the choice of the matrix set $\boldsymbol{A}$ and the corresponding extremal norm $\|\cdot\|$. Therefore, consider in more details the question on how the direction function $\Phi_\theta(\xi)$ depends on the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ and the related extremal norm $\|\cdot\|$.

According to (22) and (23), the function φ_0 is uniquely determined by entries of the matrix A_0, while the function φ_1 is completely determined by entries of the matrix A_1. To point out this dependance we will use the notation

$$\varphi_0(\xi) = \varphi_0[A_0](\xi), \qquad \varphi_1(\xi) = \varphi_1[A_1](\xi).$$

At the same time, by Theorem 5 and relations (47), (50) the parameter θ is a single-valued function of the matrix set $\boldsymbol{A}$ and the related extremal norm $\|\cdot\|$, i.e.

$$\theta = \theta[\boldsymbol{A}, \|\cdot\|]. \tag{52}$$

From (51), (52) one can see that the direction function $\Phi_\theta(\xi)$ is determined, in the long run, by the matrix set $\boldsymbol{A} = \{A_0, A_1\}$ and by the extremal norm $\|\cdot\|$ corresponding to this set; in the cases when we need stress this dependance it will be used the notation

$$\Phi_\theta(\xi) = \Phi[\boldsymbol{A}, \|\cdot\|](\xi).$$

As will be shown in Lemma 18 below, the direction function $\Phi[\boldsymbol{A}, \|\cdot\|]$ depends continuously on the matrix set $\boldsymbol{A}$ and the extremal norm $\|\cdot\|$. To make said above meaningful, define first the notion of closeness between set-valued functions on the interval $[0,1]$.

Denote by $\mathcal{F} = \mathcal{F}([0,1])$ the set of all set-valued functions $f : [0,1] \mapsto 2^{\mathbb{R}}$ with the closed graphs. In this case the graph $\mathrm{Gr}(f)$ of the function f is a closed bounded subset of the set $[0,1] \times \mathbb{R} \subset \mathbb{R}^2$, and hence, for any pair of functions $f, g \in \mathcal{F}$ it is defined and finite the value

$$\chi(f,g) = \max\Big\{ \sup_{x \in \mathrm{Gr}(f)} \inf_{y \in \mathrm{Gr}(g)} |x - y|, \sup_{y \in \mathrm{Gr}(g)} \inf_{x \in \mathrm{Gr}(f)} |x - y| \Big\},$$

where $|\cdot|$ is some norm in $\mathbb{R}^2$. The value χ is called *the Hausdorff distance* between the graphs of the maps f and g, it is a metric in the space $\mathcal{F}$. In its turn, the space $\mathcal{F}$, being equipped with the metric χ, is complete.

Lemma 18 *Let $x_0 \in \mathbb{R}^2$ be a nonzero vector. Then for any pair $\big(\boldsymbol{A}, \|\cdot\|\big)$, where $\boldsymbol{A} \in \mathcal{M}^\sharp$ and $\|\cdot\| \in N_{\mathrm{ext}}(\boldsymbol{A}, x_0)$, the map*

$$\big(\boldsymbol{A}, \|\cdot\|\big) \mapsto \Phi[\boldsymbol{A}, \|\cdot\|], \tag{53}$$

is uniquely defined and continuous by the metric of the space $\mathcal{F}$.

Proof. The fact that the map (53) is defined uniquely by the pair $\big(\boldsymbol{A}, \|\cdot\|\big)$, directly follows from (51), (52).

Remark now that by the definition (50) of the direction function $\varPhi$, continuity of the map (53) will be established if we show that both of the functions $\varphi_0 = \varphi_0[A_0]$ and $\varphi_1 = \varphi_1[A_1]$ depend continuously on the matrices A_0 and A_1 in the metric of the space $C[0,1]$, while the parameter $\theta = \theta[\boldsymbol{A}, \|\cdot\|]$ depends continuously on $\boldsymbol{A}$ and $\|\cdot\|$.

Continuous dependance of the functions $\varphi_0 = \varphi_0[A_0]$ and $\varphi_1 = \varphi_1[A_1]$ on defining them matrices immediately follows from the definitions (22), (23). Besides, continuous dependency of the parameter $\theta = \theta[\boldsymbol{A}, \|\cdot\|]$ on $\boldsymbol{A}$ and $\|\cdot\|$ follows from the fact that θ is the ξ-coordinate (see (48)) of the vector ϑ defined in Theorem 5, which depends continuously on $\boldsymbol{A}$ and $\|\cdot\|$ by Theorem 5.

So, the map (53) is continuous, and the proof of Lemma 18 is completed. $\square$

Properties of maps, graphs of which are like those presented in Fig. 8, are studied below in more details.

5.2 Orientation Preserving Discontinuous Circle Maps

Maps of the interval $[0,1)$ in itself is convenient to treat as maps of the circle $\mathbb{S} \equiv \mathbb{R}/\mathbb{Z}$. Below, we will study, primarily, discontinuous maps of the interval $[0,1)$. Such maps were studied by different authors (see, e.g., [6, 8, 9] and the bibliography therein), but unfortunately no one of results, known to the author, can be immediately applied to the analysis of the properties of the map $\varPhi_\theta(\xi)$. For example, in [6] main results are established for the set-valued maps with connected images while in [8, 9] properties of the single-valued discontinuous maps are investigated, whereas in our case $\varPhi_\theta(\xi)$ is a set-valued map with disconnected images. In connection with this, in what follows we will recall basic facts of the theory of orientation preserving discontinuous circle maps, following primarily to the work [6], and then deduce from these results properties of the map $\varPhi_\theta(\xi)$ needed below.

Let $\eta : [0,1) \to [0,1)$ be some, in general, discontinuous, set-valued function. The function $h : \mathbb{R} \to \mathbb{R}$ is called *the lift* of η if it satisfies conditions

$$h(\xi + 1) \equiv h(\xi) + 1, \tag{54}$$

and

$$\eta(\xi) = h(\xi) \pmod 1 \qquad \xi \in [0,1). \tag{55}$$

As is easy to see, each circle map has a lift, and conversely, each map h of the straight line in itself satisfying (54) is a lift of the circle map $\eta(\cdot)$ defined by the equality (55). Note that there is a plenty of properties of the circle maps which are more convenient to formulate in terms of corresponding lifts than in terms of the original circle maps.

The map $\eta : [0,1) \to [0,1)$, treated as a map of the circle $\mathbb{S} \equiv \mathbb{R}/\mathbb{Z}$ in itself, will be called *orientation preserving* if it has a strictly increasing lift[5]. A strictly increasing lift h of the map η will be called *standard* if it satisfies $h(0) = \eta(0)$. The orientation preserving map $\eta : [0,1) \to [0,1)$ will be called *closed* or *connectedly closed* if it has a strictly increasing lift with the closed graph, or the graph of some of its strictly increasing lift is a connected and closed set, respectively.

To illustrate notions introduced above, associate with the strictly increasing lift h of the map η the auxiliary maps

$$h_+(\xi) = \lim_{\bar{\xi} \downarrow \xi} h(\bar{\xi}), \qquad h_-(\xi) = \lim_{\bar{\xi} \uparrow \xi} h(\bar{\xi}),$$

where notations $\bar{\xi} \downarrow \xi$ and $\bar{\xi} \uparrow \xi$ are used to denote convergence of the variable $\bar{\xi}$ to ξ strictly from above or from below, correspondingly. Define also the maps

$$h_*(\xi) = \{h_-(\xi), h_+(\xi)\}, \qquad h_c(\xi) = [h_-(\xi), h_+(\xi)].$$

Directly from the definitions of the maps $h_+(\xi)$, $h_-(\xi)$, $h_*(\xi)$ and $h_c(\xi)$ it follows that all these maps are strictly increasing. The maps $h_+(\xi)$ and $h_-(\xi)$ are single-valued, and the map $h_+(\xi)$ is continuous from the right at each point, while the map $h_-(\xi)$ is continuous from the left at each point. The maps $h_*(\xi)$ and $h_c(\xi)$ are, in general, set-valued and their values coincide with the values of the map $h(\xi)$ at the points, in which the map $h(\xi)$ is single-valued and continuous. In all other points the values of $h_*(\xi)$ consist of exactly two points while the values of $h_c(\xi)$ consist of closed intervals. Besides, the graphs of the both maps $h_*(\xi)$ and $h_c(\xi)$ are closed. It should be noted also that

$$h_+(\xi), h_-(\xi) \in h_*(\xi) \subseteq h_c(\xi) \qquad \forall \xi.$$

In addition, if the graph of the map $h(\xi)$ is closed then $h_*(\xi) \subseteq h(\xi) \subseteq h_c(\xi)$. Therefore, it is natural to call the map $h_*(\xi)$ the *minimal closure* of the map $h(\xi)$ while the map $h_c(\xi)$ can be called the *connected* or *maximal closure* of

[5]Remark that the lift of a circle map is determined non-uniquely. Nevertheless, just as is in the case of continuous lifts of the circle homeomorphisms, any two strictly increasing lifts of the same circle map (provided that they exist) can differ from each other only on an integer constant [9, Lemma 2]. A detailed description of the structure of single-valued discontinuous orientation preserving circle maps and their lifts can be found in [8,9]. The role of the demand of strict increasing of a lift is discussed in Remark 1.

the map $h(\xi)$. Respectively, the map $h(\xi)$ will be called *minimally closed* if $h(\cdot) = h_*(\cdot)$, and it will be called *connectedly* or *maximally closed* if $h(\cdot) = h_c(\cdot)$.

Theorem 7 (see [6]) *Let $\eta : [0,1) \to [0,1)$ be an orientation preserving circle map with a connectedly closed lift h. Let $\{\xi_n\}$ be a trajectory of the map h, i.e.*

$$\xi_{n+1} \in h(\xi_n), \qquad n = 0, 1, \dots \ . \tag{56}$$

Then the following assertions are valid:

(i) there is a number τ, not depending on the initial value ξ_0, for which the estimates hold

$$\left| \frac{\xi_n}{n} - \tau \right| \le \frac{1}{n}, \qquad n = 1, 2, \dots \ ,$$

and hence

$$\tau = \lim_{n\to\infty} \frac{\xi_n}{n};$$

(ii) if the number τ is rational of the form $\tau = p/q$ with coprime p and q then the map $\eta(\cdot)$ has a periodic point of period q, and any trajectory (56) converges to a periodic trajectory of period q as $n \to \infty$;

(iii) if the number τ is irrational then all the trajectories (56) have the same limiting set which is either coincide with the whole circle or is the Cantor set;

(iv) the number τ depends continuously on the graph of the map h in the Hausdorff metric[6]*.*

According to this Theorem the number τ is uniquely determined by the map h and does not depend neither on the choice of the initial point ξ_0 of the trajectory $\{\xi_n\}$ nor on arbitrariness in the construction of the trajectory $\{\xi_n\}$ by formula (56). So, it is reasonable to denote the number τ by $\tau(h)$; this number is called *the rotation number* of the lift h. The value $\tau(h)$ is often called also the rotation number of the circle map η. One should only bear in mind that the rotation number for a circle map is defined modulo

[6]The statement means that for any orientation preserving circle map $\hat{\eta}$ with a connectedly closed lift $\hat{h}$ the values of $\hat{\tau}$ tend to τ when the graph of the map $\hat{h}$ tends to the graph of the map h by the Hausdorff metric. Point out that due to condition (54) the Hausdorff distance between the maps h and $\hat{h}$ is defined correctly in spite of the fact that the graphs of these maps are not bounded.

integer additives since lifts of the circle map are also defined modulo integer additives. Therefore, sometimes the rotation number of a circle map is defined as $\tau(h) \pmod 1$.

Remark 1 *An orientation preserving circle map was defined above as such a circle map which has a strictly increasing lift. Theorem 7 will be no longer valid if to omit the requirement that the corresponding lift increases strictly.*

Proof. Validity of Remark follows from the fact that a circle map with a non-decreasing lift may have simultaneously periodic points of different coprime periods as is plotted in Fig. 9 and 10. □

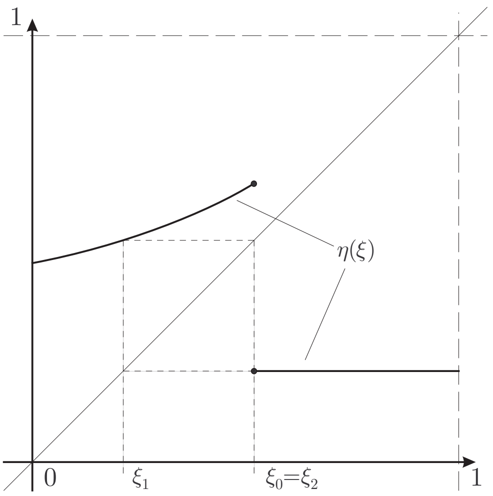

Figure 9: Periodic point of period 2

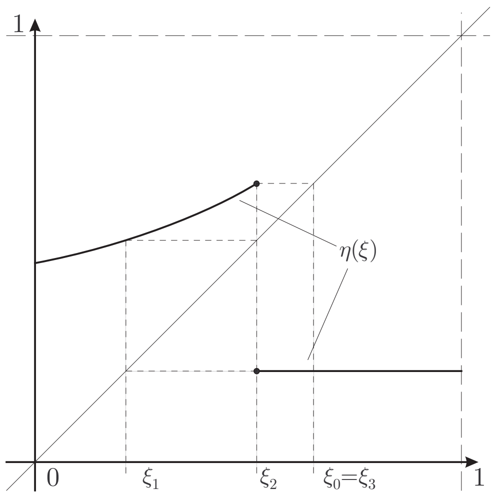

Figure 10: Periodic point of period 3

The next Remark shows that in Theorem 7 the requirement of the connectedness of the graph of the lift h is not essential. What is important is the closeness of the graph.

Remark 2 *All the statements of Theorem 7 continue to be valid for any circle map possessing a strictly increasing closed lift.*

Proof. Let the circle map $\eta(\xi)$ has a strictly increasing closed lift $h(\xi)$. Consider the connected closure $h_c(\xi)$ of the map $h(\xi)$. Then from the inclusions $h(\xi) \subseteq h_c(\xi)$ valid for any $\xi \in \mathbb{R}$ it follows that each trajectory $\{\xi_n\}$ of the map $h(\xi)$ is also a trajectory of the map $h_c(\xi)$. Hence, the rotation number $\tau(h)$ of the map h is correctly defined and coincides with $\tau(h_c)$, and besides, the limiting set of the trajectory $\{\xi_n\}$ does not depend on the choice of the trajectory in the case when $\tau(h)$ is irrational.

If the number $\tau(h)$ is rational then the trajectory $\{\xi_n\}$ of the map h, being at the same time a trajectory of the map h_c, by assertion (iii) of Theorem 7 converges to some periodic trajectory of the map h_c. But in view of closeness of the graph of the map h the corresponding limiting trajectory will be a trajectory of the map h, from which assertion (iii) of Theorem 7 for the map h follows.

At last, assertion (iv) of Theorem 7 for the map h follows from the already established identity $\tau(h) \equiv \tau(h_c)$ and from the remark that for any two strictly increasing maps h and $\hat{h}$ with the closed graphs the Hausdorff distance between their graphs coincide with the Hausdorff distance between the graphs of the maps h_c and $\hat{h}_c$. □

One of weak points in the definition of the rotation number $\tau(\eta)$ for the circle map $\eta(\cdot)$ is that one need perform intermediate steps (such as to construct the lift $h(\cdot)$ and to build the trajectory $\{\xi_n\}$ of the map $h(\cdot)$) to calculate the limit $\tau(\eta) = \lim_{n\to\infty} \xi_n/n$. It is desirable to find a method to calculate the rotation number $\tau(\eta)$ directly in terms of the map η and its trajectories. To do it, we first investigate in more details properties of the orientation preserving circle maps (cf. [9, Lemma 1]).

Lemma 19 *Let η be a closed orientation preserving circle map and let h be its standard lift. Then for any $\xi \in [0,1)$ and any pair of elements $\eta_\xi \in \eta(\xi)$, $h_\xi \in h(\xi)$ satisfying $\eta_\xi = h_\xi \pmod 1$ the following relation is valid:*

$$h_\xi = \eta_\xi + \nu(\eta_\xi), \tag{57}$$

where

$$\nu(\xi) = \begin{cases} 1 & \text{if} \quad 0 \le \xi < \omega, \\ 0 & \text{if} \quad \omega \le \xi < 1, \end{cases} \tag{58}$$

with $\omega = \min\{y : \; y \in \eta(0)\}$ (see Fig. 11)[7].

Conversely, if for a pair of elements $\eta_\xi \in \eta(\xi)$ and h_ξ relation (57) holds then $h_\xi \in h(\xi)$.

Proof. Fix a point $\xi \in [0,1)$ and choose a pair of elements $\eta_\xi \in \eta(\xi)$ and $h_\xi \in h(\xi)$ satisfying the relation $\eta_\xi = h_\xi \pmod 1$. Since, by Lemma's conditions, $h(\cdot)$ ia a standard lift of the map $\eta(\cdot)$ then $h(0) = \eta(0) \in [0,1)$.

[7]Remark that the function $\nu(\xi)$ is identically equal to zero if $\omega = 0$. In this case $h(\xi) \equiv \eta(\xi)$ on the interval $[0,1)$, and so, the function $\eta(\xi)$ strictly increases on $[0,1)$.

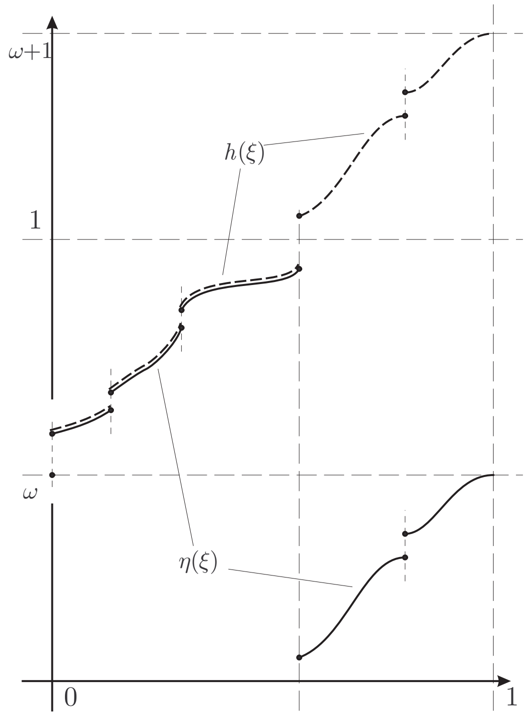

Figure 11: Orientation preserving closed circle map $\eta(\xi)$ and its standard lift $h(\xi)$.

Then from the fact that the map $h(\cdot)$ is strictly increasing we obtain the estimates

$$0 \leq \eta(0) = h(0) \leq h_\xi < h(1) = h(0) + 1 = \eta(0) + 1 < 2, \qquad \xi \in [0, 1),$$

i.e. $h_\xi \in [0, 2)$.

If $h_\xi \in [0, 1)$ then the equality $\eta_\xi = h_\xi \pmod 1$ implies $\eta_\xi = h_\xi$, and by monotony of the function $h(\cdot)$

$$\omega = \min\{y : \ y \in \eta(0)\} = \min\{y : \ y \in h(0)\} \leq h_\xi = \eta_\xi < 1.$$

Hence, in this case $\nu(\eta_\xi) = 0$ from which we obtain that $h_\xi = \eta_\xi + \nu(\eta_\xi)$.

But if $h_\xi \in [1, 2)$ then the equality $\eta_\xi = h_\xi \pmod 1$ implies $\eta_\xi = h_\xi - 1$. In this case by monotony of the function $h(\cdot)$ the following relations take place

$$\begin{aligned} 0 \leq \eta_\xi &= h_\xi - 1 < \min\{y : \ y \in h(1)\} - 1 = \min\{y : \ y \in h(0) + 1\} - 1 = \\ &= \min\{y : \ y \in h(0)\} = \min\{y : \ y \in \eta(0)\} = \omega. \end{aligned}$$

Hence $\nu(\eta_\xi) = 1$ which again implies $h_\xi = \eta_\xi + \nu(\eta_\xi)$. So, in one direction Lemma 19 is proved.

Now, let $\eta_\xi \in \eta(\xi)$ and h_ξ be elements for which relation (57) is fulfilled. By the definition of the lift of a circle map, the sets $\eta(\xi)$ and $h(\xi)$ satisfy the relation $\eta(\xi) = h(\xi) \pmod 1$. Consequently, the set $h(\xi)$ contains such an element h_* that $\eta_\xi = h_* \pmod 1$. But then, due to the already proven first part of Lemma, the relation $h_* = \eta_\xi + \nu(\eta_\xi)$ should be valid. But by supposition, for the elements η_ξ and h_ξ the analogous relation (57) is also true, i.e. $h_\xi = \eta_\xi + \nu(\eta_\xi)$, from which we immediately obtain $h_\xi = h_* \in h(\xi)$. Lemma 19 is completely proved. □

At last, we are able to present the definition of the rotation number of the circle map $\eta(\cdot)$ directly in terms of the map $\eta(\cdot)$ (to be precise, the definition of the rotation number of the standard lift $h(\cdot)$ of the map $\eta(\cdot)$).

Theorem 8 *Let $\eta : [0,1) \to [0,1)$ be an orientation preserving circle map with the closed standard lift h. Let $\{\zeta_n\}$ be a trajectory of the map η, i.e.*

$$\zeta_{n+1} \in \eta(\zeta_n), \qquad n = 0, 1, \dots \ .$$

Then the uniform estimates hold

$$\left| \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n} - \tau(h) \right| \le \frac{2}{n}, \qquad n = 1, 2, \dots \ , \tag{59}$$

and so,

$$\tau(h) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n}.$$

Proof. Define the sequence $\{\xi_n\}_{n=0}^{\infty}$ by setting $\xi_0 = \zeta_0$ and

$$\xi_n = \zeta_n + \sum_{i=1}^{n} \nu(\zeta_i), \qquad n = 1, 2, \dots \ .$$

Prove by induction that $\{\xi_n\}$ satisfies the inclusions

$$\xi_{n+1} \in h(\xi_n), \qquad n = 0, 1, \dots \ , \tag{60}$$

and so, it is a trajectory of the map h.

Indeed, by the definition, $\xi_1 = \zeta_1 + \nu(\zeta_1)$, where $\zeta_1 \in \eta(\zeta_0)$. Therefore, by Lemma 19 $\xi_1 \in h(\zeta_0) = h(\xi_0)$, and the statement of Theorem 8 is true for $n = 0$.

Perform the step of induction. Suppose that the statement of Theorem 8 is valid for $n = k \geq 0$ and show that this imply its validity for $n = k+1$. By the definition of the element ξ_{k+1},

$$\xi_{k+1} = \zeta_{k+1} + \sum_{i=1}^{k+1} \nu(\zeta_i)$$

or, what is the same,

$$\xi_{k+1} - \sum_{i=1}^{k} \nu(\zeta_i) = \zeta_{k+1} + \nu(\zeta_{k+1}).$$

Since here, by the definition of the trajectory $\{\zeta_n\}$, the inclusion $\zeta_{k+1} \in \eta(\zeta_k)$ with $\zeta_k \in [0,1)$ holds, then by Lemma 19 $\zeta_{k+1} + \nu(\zeta_{k+1}) \in h(\zeta_k)$. Hence,

$$\xi_{k+1} - \sum_{i=1}^{k} \nu(\zeta_i) \in h(\zeta_k)$$

or, what is the same,

$$\xi_{k+1} \in h(\zeta_k) + \sum_{i=1}^{k} \nu(\zeta_i) = h(\zeta_k + \sum_{i=1}^{k} \nu(\zeta_i)).$$

Here, by the supposition of induction, the argument of the function h in the right-hand part coincides with ξ_k which implies $\xi_{k+1} \in h(\xi_k)$.

So, the step of induction is justified and inclusions (60) are proved. To complete the proof of Theorem 8 it remains to note only that by Theorem 7 and Remark 2 for the trajectory $\{\xi_n\}$ the estimates hold

$$\left| \frac{\xi_n}{n} - \tau(h) \right| \leq \frac{1}{n}, \qquad n = 1, 2, \ldots\ ,$$

while by the definition of trajectory $\{\xi_n\}$ it is valid the equality

$$\frac{\xi_n}{n} = \frac{\zeta_n}{n} + \frac{\sum_{i=1}^{n} \nu(\zeta_i)}{n},$$

where $\zeta_n \in [0,1)$. Estimates (59) now directly follow from the latter relations. Theorem 8 is proved. $\square$

5.3 Frequency Properties of the Direction Function

In this Section, we make use of the properties of the circle maps obtained in Section 5.2 to analyze the properties of the direction function Φ_θ of the generator of extremal trajectories introduced in Section 5 (see (50)).

Note that the function $\Phi_\theta(\xi)$ differs from a function representing an orientation preserving circle map only in that it is defined on the closed interval $[0,1]$ but not on the semiopen one $[0,1)$ as is the case for a circle map. Let us show that the indicated difference is not essential, and for the function $\Phi_\theta(\xi)$ the notion of the rotation number can be defined with all the "good" properties intrinsical to the rotation number of the circle maps.

Theorem 9 *Let $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ be the set of 2×2 matrices (20) satisfying conditions (21), let Φ_θ be the direction function (50) of some generator of extremal trajectories for the matrix set $\boldsymbol{A}$ and let $\nu(\cdot)$ be the function defined by the equality (58). Then for any trajectory $\{\xi_n\}_{n=0}^\infty$ of the map Φ_θ there are valid the non-equalities $\xi_n \neq 0, 1$, where $n \geq 1$, and there is defined the frequency*

$$\tau = \lim_{n\to\infty} \frac{\sum_{i=1}^n \nu(\xi_i)}{n} \tag{61}$$

with which the elements of the trajectory $\{\xi_n\}$ hit the interval $[0,\omega)$, where $\omega = \varphi_0(1)$.

The frequency τ does not depend neither on the choice of the trajectory $\{\xi_n\}$ nor on the choice of the function Φ_θ. So the frequency τ may be denoted as $\tau(\boldsymbol{A})$. In addition, for $\tau(\boldsymbol{A})$ assertions (i)–(iii) of Theorem 7 are valid, and besides, $\tau(\boldsymbol{A})$ depends continuously on the matrices of the set $\boldsymbol{A}$.

Proof. Construct the map $\eta_\theta(\cdot)$ of the semiopen interval $[0,1)$ in itself with the help of following equalities

$$\eta_\theta(\xi) = \begin{cases} \varphi_1(0) \cup \varphi_0(1), & \text{if} \quad \xi = 0, \\ \varphi_1(\xi), & \text{if} \quad \xi \in (0,\theta], \\ \varphi_0(\xi), & \text{if} \quad \xi \in [\theta, 1). \end{cases}$$

This map can be treated as an orientation preserving circle map with the closed graph since it has the strictly increasing lift with a closed graph $h_\theta(\cdot)$ defined for $\xi \in [0,1)$ by the relation[8]

$$h_\theta(\xi) = \begin{cases} \varphi_1(0) \cup \varphi_0(1), & \text{if} \quad \xi = 0, \\ \varphi_1(\xi), & \text{if} \quad \xi \in (0,\theta], \\ \varphi_0(\xi) + 1, & \text{if} \quad \xi \in [\theta, 1). \end{cases}$$

[8]The lift $h_\theta(\cdot)$ can be extended on other values of $\xi \in \mathbb{R}$ with the preservation of the identity $h_\theta(\xi+1) \equiv h_\theta(\xi) + 1$.

Point out that the map $\eta_\theta(\cdot)$ takes two values at each of the points $\xi = 0, \theta$.

Now, let $\{\xi_n\}$ be a trajectory of the map Φ_θ, i.e.

$$\xi_{n+1} \in \Phi_\theta(\xi_n), \qquad n = 0, 1, \dots \ . \tag{62}$$

By Lemma 17 for the parameter θ of the map $\Phi_\theta(\xi)$ the estimates $\theta_* \le \theta \le \theta^*$ hold in which θ_* and θ^* are some numbers from the interval $(0,1)$. Then, as can be seen, e.g., from Fig. 8, the values of the function $\Phi_\theta(\xi)$ are separated from 0 and 1, i.e. one can find such $\mu > 0$ for which for all the elements of the trajectory $\{\xi_n\}$ will be valid the estimates $\mu \le \xi_n \le 1 - \mu$, except maybe for the element ξ_0. From here and from (62), and taking into account that the values of the functions $\Phi_\theta(\xi)$ and $\eta_\theta(\xi)$ coincide with each other for $0 < \xi < 1$, we deduce that the trajectory $\{\xi_n\}$ satisfies the inclusions $\xi_{n+1} \in \eta_\theta(\xi_n)$ for $n = 1, 2, \dots$. Defining now the sequence $\{\zeta_n\}$ by setting

$$\zeta_n = \begin{cases} \xi_0 \pmod 1, & \text{if} \quad n = 0, \\ \xi_n, & \text{if} \quad n \ge 1, \end{cases}$$

one can easily verify that this sequence satisfies the inclusions $\zeta_{n+1} \in \eta_\theta(\zeta_n)$ for $n = 1, 2, \dots$. From here by Theorem 8 it follows the existence of such a number τ for which the estimates hold

$$\left| \frac{\sum_{i=1}^n \nu(\xi_i)}{n} - \tau \right| = \left| \frac{\sum_{i=1}^n \nu(\zeta_i)}{n} - \tau \right| \le \frac{2}{n}, \qquad n = 1, 2, \dots \ , \tag{63}$$

where the function $\nu(\cdot)$ by Lemma 19 has the form

$$\nu(\xi) = \begin{cases} 1 & \text{if} \quad 0 \le \xi < \omega, \\ 0 & \text{if} \quad \omega \le \xi < 1. \end{cases}$$

Here $\omega = \min\{h_\theta(0)\} = \Phi_\theta(1)$ which means that in a formal sense the number ω depends on θ. But since by Lemma 17 the number θ satisfies the inclusion $\theta \in (0,1)$ then $\Phi_\theta(1) \equiv \varphi_0(1)$. Therefore, in fact the number ω, as well as the function $\nu(\cdot)$, does not depend on θ.

Estimates (63) imply the existence of the limit (61). Note that for a given direction function Φ_θ the number τ by Theorem 8 does not depend on the choice of the trajectory $\{\xi_n\}$, and so, τ is a function of the only argument θ, i.e. $\tau = \tau(\theta)$. Show that in fact *the number τ does not depend on θ*, too, but it is uniquely determined by the matrix set $\boldsymbol{A}$, i.e. $\tau = \tau(\boldsymbol{A})$.

Let $\Phi_{\theta_1}(\xi)$ and $\Phi_{\theta_2}(\xi)$ be the direction functions of some generators of extremal trajectories $g_1(x)$ and $g_2(x)$ corresponding to different extremal norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then by Lemma 10 there is a trajectory $\{x_n\}$ of the matrix set $\boldsymbol{A}$ which is extremal as in the norm $\|\cdot\|_1$ as in the norm $\|\cdot\|_2$. By the definition of a generator of extremal trajectories, this trajectory should satisfy as the inclusions

$$x_{n+1} \in g_1(x_n), \qquad n = 0, 1, 2, \dots ,$$

as the inclusions

$$x_{n+1} \in g_2(x_n), \qquad n = 0, 1, 2, \dots .$$

Then, by the definition of the direction function, the sequence $\{\xi_n\}$ defined by

$$\xi_n = \frac{x_{1,n}}{x_{1,n} + x_{2,n}}, \qquad n = 0, 1, 2, \dots ,$$

should satisfy as the inclusions

$$\xi_{n+1} \in \varPhi_{\theta_1}(\xi_n), \qquad n = 0, 1, 2, \dots , \tag{64}$$

as the inclusions

$$\xi_{n+1} \in \varPhi_{\theta_2}(\xi_n), \qquad n = 0, 1, 2, \dots . \tag{65}$$

We can use now formula (61) to calculate the number $\tau(\theta_1)$ for the sequence $\{\xi_n\}$ treating the latter as the sequence satisfying (64). Analogously, we can use the same formula (61) to calculate the number $\tau(\theta_2)$ for the same sequence $\{\xi_n\}$ but this time treating it as the sequence satisfying (65). Since in the both cases the calculations (61) are performed with the same sequence $\{\xi_n\}$ then we conclude that $\tau(\theta_1) = \tau(\theta_2)$, from which independence of the number τ from θ follows.

Validity of assertions (i)–(iii) for $\tau(\boldsymbol{A})$ follows from the definition of the number $\tau(\boldsymbol{A})$ and from Theorem 7. Therefore, to complete the proof of Theorem 9 it remains only to establish the continuous dependance of the function $\tau(\boldsymbol{A})$ on the matrix set $\boldsymbol{A}$. Let $\{\boldsymbol{A}^{(n)} \in \mathcal{M}^\sharp\}$ be a sequence of matrix sets converging to the matrix set $\boldsymbol{A}^* \in \mathcal{M}^\sharp$. Fix a vector $x_0 \neq 0 \in \mathbb{R}^2$ and choose for each n an extremal norm $\|\cdot\|^{(n)} \in N_{\text{ext}}(\boldsymbol{A}^{(n)}, x_0)$, and then build the direction function $\varPhi_{\theta^{(n)}}$ of the generator of extremal trajectories corresponding to the matrix set $\boldsymbol{A}^{(n)}$ and the norm $\|\cdot\|^{(n)}$.

By Theorem 2 one can suppose that the sequence $\{\|\cdot\|^{(n)}\}$ converges in the space $C_{\text{loc}}(\mathbb{R}^m)$ to some extremal norm $\|\cdot\|^*$ of the matrix set $\boldsymbol{A}^*$. Then by Lemma 18 the sequence $\{\varPhi_{\theta^{(n)}}\}$ converges by the metric of the space $\mathcal{F}$ to the direction function $\varPhi_{\theta^*}$ of the generator of extremal trajectories corresponding to the matrix set $\boldsymbol{A}^*$ and the norm $\|\cdot\|^*$. Hence

$$\tau(\boldsymbol{A}^{(n)}) = \tau(\varPhi_{\theta^{(n)}}) \to \tau(\varPhi_{\theta^*}) = \tau(\boldsymbol{A}^*). \tag{66}$$

Here, convergence of the numerical sequence $\{\tau(\Phi_{\theta(n)})\}$ to $\tau(\Phi_{\theta^*})$ follows from convergence of the sequence of functions $\{\Phi_{\theta(n)}\}$ to the function Φ_{θ^*} by the metric of the space $\mathcal{F}$ (i.e. in the sense of convergence of the graphs of these functions in the Hausdorff metric) and from Theorems 7 and 8. Equalities in (66) follows from the already proven fact that the number $\tau(\boldsymbol{A})$ does not depend on the choice of the direction function of the generator of extremal trajectories of the matrix set $\boldsymbol{A}$.

Thus, continuous dependance of the number $\tau(\boldsymbol{A})$ on the matrix set $\boldsymbol{A}$ is proved, and so the proof of Theorem 9 is completed. □

5.4 Switching Frequency for Extremal Trajectories and Construction of the Counterexample

Now all is ready to prove Theorem 6. Let $\{x_n\}$ be an extremal trajectory of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ and let $\{\sigma_n\}$ be the corresponding index sequence, i.e. the sequences $\{x_n\}$ and $\{\sigma_n\}$ satisfy the equalities $x_{n+1} = A_{\sigma_n} x_n$ for $n = 0, 1, \ldots$. Then by Lemma 17 the numerical sequence $\xi_n = \xi(x_n)$, where the function $\xi(\cdot)$ is defined by the equality (48), satisfies the relations

$$\xi_{n+1} = \varphi_{\sigma_n}(\xi_n) \in \Phi_\theta(\xi_n), \qquad n = 0, 1, \ldots ,$$

with some direction function Φ_θ. At the same time, by Theorem 9 there is defined the frequency

$$\tau = \lim_{n\to\infty} \frac{\sum_{i=1}^n \nu(\xi_i)}{n},$$

and besides, $\xi_n \neq 0, 1$ for $n \geq 1$. Therefore, for $n \geq 1$ the value $\xi_{n+1} \in (0,1)$ is obtained from $\xi_n \in (0,1)$ by the formula $\xi_{n+1} = \varphi_0(\xi_n)$ if and only if $0 < \xi_{n+1} < \varphi_0(1)$ or, what is the same, if and only if $\nu(\xi_{n+1}) = 1$. Consequently, $\sigma_n = 1 - \nu(\xi_{n+1})$ for $n \geq 1$ and by Theorem 9 there is the limit

$$\sigma(\boldsymbol{A}) = \lim_{n\to\infty} \frac{\sum_{i=1}^n \sigma_i}{n} = 1 - \lim_{n\to\infty} \frac{\sum_{i=1}^n \nu(\xi_{i+1})}{n} = 1 - \tau(\boldsymbol{A}).$$

Now, all the assertions of Theorem 6 follow from analogous assertions of Theorem 9. □

At last, start constructing the counterexample to the Finiteness Conjecture.

Lemma 20 (on unattainability of the generalized spectral radius) *Let the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ be such that the number $\sigma(\boldsymbol{A})$ is irrational. Then for any finite sequence of indices $\sigma_k \in \{0,1\}$, $k = 1, 2, \ldots, n$, the strict inequality $\rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1}) < \rho^n(\boldsymbol{A})$ is valid.*

Proof. Since the matrices A_0 and A_1 are non-negative then by the Perron-Frobenius theorem there is a vector x_0 with non-negative coordinates such that

$$\rho^n x_0 = A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1} x_0, \tag{67}$$

where $\rho = \rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1})$.

Extend the finite index sequence $\{\sigma_k\}_{k=1}^{n}$ to the infinite periodic one with period n and then consider the corresponding sequence $\{x_k\}_{k=0}^{\infty}$:

$$x_1 = A_{\sigma_1} x_0, \quad \ldots, \quad x_{n-1} = A_{\sigma_{n-1}} x_{n-2}, \quad x_n = A_{\sigma_n} x_{n-1}, \quad \ldots\ .$$

Then from (67) we get $x_n = \rho^n x_0$, and in any extremal norm $\|\cdot\|$ the following inequalities will be valid

$$\|x_1\| \le \rho(\boldsymbol{A})\|x_0\|, \quad \ldots, \quad \|x_n\| = \rho^n \|x_0\| \le \rho(\boldsymbol{A})\|x_{n-1}\|, \quad \ldots\ , \tag{68}$$

from which $\rho \le \rho(\boldsymbol{A})$. Here, the equality $\rho = \rho(\boldsymbol{A})$ may take place only in the case when each of inequalities (68) is in fact equality, i.e. when the sequence $\{x_n\}$ is extremal in the norm $\|\cdot\|$. However, by Theorem 9 periodicity of the index sequence of at least one of the extremal trajectories of the matrix set $\boldsymbol{A}$ implies the rationality of the number $\sigma(\boldsymbol{A})$ which contradicts the condition of Theorem. The obtained contradiction is caused by the supposition that $\rho = \rho(A_{\sigma_n} A_{\sigma_{n-1}} \cdots A_{\sigma_1}) = \rho(\boldsymbol{A})$. Lemma 20 is proved. □

From Lemma 20 it follows that in order to construct the counterexample to the Finiteness Conjecture it is sufficient to prove the existence of at least one of the matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ for which $\sigma(\boldsymbol{A})$ is irrational.

Lemma 21 *For any set of parameters a, b, c, d satisfying conditions (21) there are positive numbers $\gamma_* = \gamma_*(a, b, c, d)$ and $\gamma^* = \gamma^*(a, b, c, d)$ such that for the corresponding matrix set $\boldsymbol{A} = \{A_0, A_1\} \in \mathcal{M}^\sharp$ the relations are valid*

$$\sigma(\boldsymbol{A}) = \begin{cases} 0 & \text{if} \quad \alpha/\beta > \gamma^*, \\ 1 & \text{if} \quad \alpha/\beta < \gamma_*. \end{cases}$$

Proof. Denote by $\mathcal{K}$ the set (cone) of all vectors from the first quadrant lying between the straight lines $L_0 = \{(x_0, x_1) : bx_1 = (1-a)x_0\}$ and $L_1 = \{(x_0, x_1) : (1-d)x_1 = cx_0\}$, i.e.

$$\mathcal{K} := \left\{ (x_0, x_1) :\ x_0, x_1 \ge 0,\ \frac{1-a}{b} x_0 \le x_1 \le \frac{c}{1-d} x_0 \right\}.$$

Then the direct verification shows that

$$A_0 \mathcal{K} \subseteq \mathcal{K}, \qquad A_1 \mathcal{K} \subseteq \mathcal{K}. \tag{69}$$

Fix now some set of parameters a, b, c, d satisfying conditions (21) and a number $\gamma > 1$, and show that for sufficiently large values of the quotient α/β the following relation is valid

$$A_0 x > \gamma A_1 x, \qquad x \neq 0 \in \mathcal{K}, \tag{70}$$

where the inequality is understood coordinate-wise.

Indeed, the vector inequality (70) is equivalent to two scalar inequalities

$$\alpha(ax_0 + bx_1) > \gamma\beta x_0, \quad \alpha x_1 \geq \gamma\beta(cx_0 + dx_1), \qquad x \neq 0 \in \mathcal{K},$$

or, what is the same,

$$\frac{\alpha}{\beta} > \gamma \sup_{x \neq 0 \in \mathcal{K}} \left\{ \frac{x_0}{ax_0 + bx_1}, \; \frac{cx_0 + dx_1}{x_1} \right\}. \tag{71}$$

But as is easy to see, the supremum in the right-hand part of the inequality (71) is finite, from which follows the validity of the vector inequality (70) for sufficiently large values of the quotient α/β.

Finalize now the proof of Lemma. Let $\|\cdot\|$ be an arbitrary extremal norm for the matrix set $\boldsymbol{A}$, let x^* be a nonzero vector from the cone $\mathcal{K}$, and let the parameters α and β be such that the inequality (70) holds. Then by Lemma 9 there is an extremal trajectory $\{x^{(n)}\}_{n=0}^{\infty}$ of the matrix set $\boldsymbol{A}$ which starts from the point x^*, i.e.

$$x^{(0)} = x^* \neq 0 \in \mathcal{K}, \qquad x^{(n+1)} = A_{\sigma_n} x^{(n)}, \quad n = 0, 1, \ldots \;,$$

and from (69) it follows that $x^{(n)} \in \mathcal{K}$ for $n = 0, 1, \ldots$. Show that in this case for the index sequence $\{\sigma_n\}$ the identity $\sigma_n \equiv 0$ takes place.

Indeed, in the opposite case $\sigma_{n_0} = 1$ for some n_0. Then by the definition of the extremal trajectory

$$\|x^{(n_0+1)}\| = \|A_1 x^{(n_0)}\| = \rho \|x^{n_0}\|, \tag{72}$$

where $\rho = \rho(\boldsymbol{A})$, and at the same time the following inequality

$$\|A_0 x^{(n_0)}\| \leq \rho \|x^{n_0}\|, \tag{73}$$

should be valid. But by Lemma 13 the extremal norm $\|\cdot\|$ is monotone and then by (70)

$$\|A_0 x^{(n_0)}\| \geq \|\gamma A_1 x^{(n_0)}\|,$$

where $\gamma > 1$, which contradicts to the relations (72) and (73).

So, it is shown that $\sigma_n \equiv 0$, from which by Theorem 6 $\sigma(\boldsymbol{A}) = 0$. For small values of the quotient α/β the proof of this Lemma can be accomplished analogously. Lemma 21 is proved. $\square$

Complete the construction of the counterexample to the Finiteness Conjecture. Fix some set of numbers a, b, c, d satisfying conditions (21), and consider the family of the matrix sets $\boldsymbol{A}$ depending on α and β as on parameters. Then by Lemma 21 $\sigma(\boldsymbol{A}) = 0$ for large values of α/β and $\sigma(\boldsymbol{A}) = 1$ for small values of α/β. But by Theorem 6 the value $\sigma(\boldsymbol{A})$ depends continuously on the matrix set $\boldsymbol{A}$, and then on α and β. Hence, $\sigma(\boldsymbol{A})$ takes all the intermediate values between 0 and 1 when α and β vary. In particular, for some α and β the number $\sigma(\boldsymbol{A})$ takes an irrational value. Then by Lemma 20 for such α and β the generalized spectral radius $\rho(\boldsymbol{A})$ can not be attained on finite products of matrices from the set $\boldsymbol{A}$.

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