

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/344114217>

On combinatorial optimization for dominating sets (literature survey, new models)

Preprint · September 2020

DOI: 10.13140/RG.2.2.34919.68006

CITATIONS

0

1 author:



Mark Sh. Levin

Russian Academy of Sciences

193 PUBLICATIONS 1,004 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Scheduling [View project](#)

On combinatorial optimization for dominating sets (literature survey, new models)

Mark Sh. Levin

The paper focuses on some versions of connected dominating set problems: basic problems and multi-criteria problems. A literature survey on basic problem formulations and solving approaches is presented. The basic connected dominating set problems are illustrated by simplified numerical examples. New integer programming formulations of dominating set problems (with multiset estimates) are suggested.

Keywords: combinatorial optimization, connected dominating sets, multicriteria decision making, solving strategy, heuristics, networking, multiset

1. Introduction

In recent decades, the significance of the connected dominating set problem has been increased (e.g., [43,60,62,66,124,149]). This problem consists in searching for the minimum sized connected dominating set for the initial graph (e.g., [43,60,62,66]). Main applications of the problems of this kind are pointed out in Table 1.

Table 1. Basic application domains of dominating set problems

No.	Application domain	Source(s)
1.	Communication networks (e.g., design of virtual backbone in mobile networks, connected dominating set based index scheme in WSNs)	[16,30,43,48,68,108] [116,121,141,149]
2.	Topology design for power (electricity) networked systems, electric power monitoring (e.g., minimum cardinality of a power dominating set)	[31,41,67,93]
3.	Converter placement problem in optical networks	[43]
4.	Placement of monitoring devices (e.g., surveillance cameras, fire alarms) as total domination in graphs	[70]
5.	Social networks (positive influence dominating sets)	[39,97,133]
6.	Multi-document summarization in information retrieval	[112]

This paper describes some basic dominating set problems and connected dominating set problems (a brief survey) and some new problem formulations (with multiset estimates). Some relations of dominating set problems and some other well-known combinatorial optimization problems are depicted in Fig. 1. Fig. 2 illustrates the dominating set problem (communication network application).

2. Preliminaries

Let graph $G = (A, R)$ be a connected undirected graph, where A is a vertex/node set and R is an edge set. A vertex subset $B \subseteq A$ is a *dominating set* if every vertex not in B has a neighbor in B . The vertices from B are called dominators. If the subgraph induced by B is connected, then B is called a *connected dominating set*. Two special indices are considered: (a) *domination number*: $\gamma(B)$ (minimum cardinality of B); and (b) *connected domination number*: $\gamma_c(B)$ (minimum cardinality of connected B). Clearly, $\gamma_c(B) \geq \gamma(B)$.

The basic dominating set (DS) problem is the following combinatorial optimization problem:

Find the minimum sized dominating set B (i.e., $\min \gamma(B)$) for the initial graph $G = (A, R)$.

The connected dominating set (CDS) problem is:

Find the minimum sized connected dominating set B (i.e., $\min \gamma_c(B)$) for the initial graph $G = (A, R)$.

The problems are proved to be NP-hard (e.g., [43,60,66]).

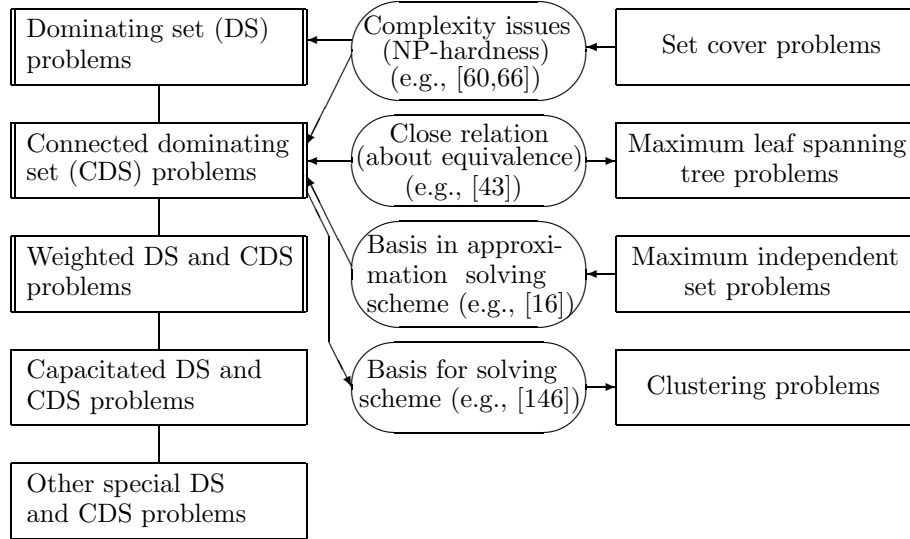


Fig. 1. Some relations of dominating sets and other combinatorial problems

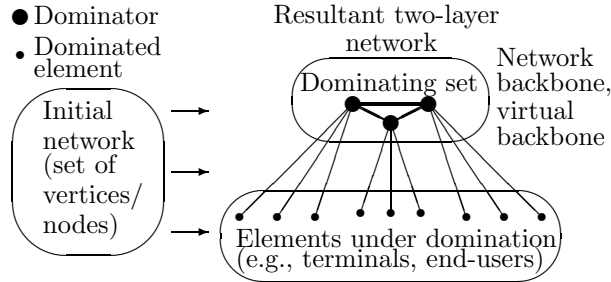


Fig. 2. Illustration for dominating set problem

Evidently, the following basic requirements are examined for the problem (e.g., [43,60,62,66,124,149]): (a) minimization of dominator set cardinality, (b) connectivity of the dominating vertices, (c) some special properties for the connectivity and domination (e.g., as k -connectivity, m -domination).

Some simplified illustrative examples of k -connected dominating networks are shown in figures:

- (a) 3-connected domination set example (3-connected dominating set is a clique, Fig. 3);
- (b) 1-connected dominating set example (1-connected dominating set is a tree, Fig. 4);
- (c) 2-connected dominating set example (2-connected dominating set is a ring, Fig. 5);
- (d) 2-connected 3-dominating example (i.e., (2, 3)-CDS problem, 2-connected dominating set is a ring, Fig. 6).

In addition, some special requirements are considered as well (Table 2). Note, the requirements can be used as constraints or as criteria in multicriteria (multiobjective) problem formulations.

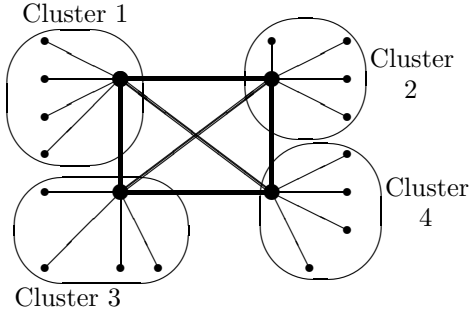


Fig. 3. 3-connected domination set example

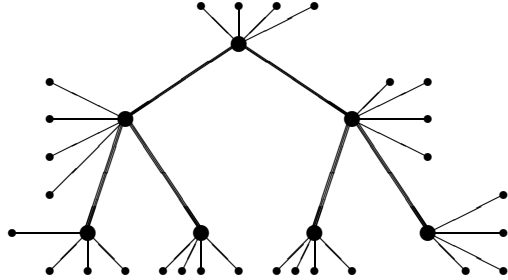


Fig. 4. 1-connected domination set example

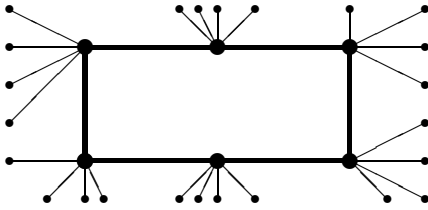


Fig. 5. 2-connected domination set example

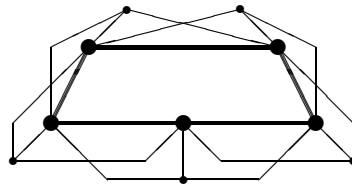


Fig. 6. 2-connected 3-domination set example

Table 2. Additional special requirements in dominating set problems

No.	Requirement	Source(s)
1.	Restricted diameter on connected dominating set	[22]
2.	Multipoint relays	[2,116,137]
3.	Routing cost constraint in networks	[44,45,47]
4.	Demand constraints	[82]
5.	Capacity constraints	[82]
6.	Fault-tolerance of connected dominating set	[154]

Two combinatorial problems are very close to the examined problems:

Close problem 1. The maximum independent set problem (e.g., [12,13,15,16,60,102,125,139]):

Find a maximum subset of vertices of an input graph (an independent or stable set) such that there is no edges between two vertices in the subset.

This problem is often used as a preliminary one in approximation two-phase approach for the designing the minimum connected dominating sets (e.g., [16,58,101,132]): (1) to construct a maximal independent set for the initial network; (2) to connect the nodes in it.

Close problem 2. The maximum leaves spanning tree problem: (e.g., [10,24,54,60,83]):

Find a spanning tree of an input graph so that the number of the tree leaves is maximal.

This problem is equivalent to the problem of computing $\gamma_c(G)$, because a vertex subset is a *connected dominating set* if and only its complement is (contained in) the set of leaves of a spanning tree (e.g., [24]).

Generally, the above-mentioned two problems belong the class of NP-hard problems as well (e.g., [60,66]). Thus, for the problems (i.e., minimum connecting dominating set, maximum independent set, maximum leaves spanning tree) the following approaches are used: (1) exact enumerative methods (e.g., Branch-and-Bound method) (e.g., [43,54,123]); (2) approximation heuristics (e.g., [23,62,118,155]); and (3) metaheuristics and hybrid methods (e.g., [9,16,112]). For some special cases of the problems polynomial approaches are suggested: (a) polynomial algorithms (e.g., [74,87,109,113,114]), and (b) polynomial time approximate schemes (PTAS) (e.g., [30,59,82,105,150]).

Some basic versions of connected set problems are listed in Table 3 (e.g., [43,60,62,66,124,149]).

Table 3. Basic dominating set and connected dominating set problems, part 1

No.	Problem type	Source(s)
1.	Basic surveys on problems and applications:	
1.1.	Connected dominating set: theory and applications	[43]
1.2.	Connected dominating sets in wireless ad hoc and sensor networks	[149]
1.3.	Connected dominating set in sensor networks and MANETs	[21]
1.4.	Connected dominating set in wireless networks	[46]
2.	Basic problems:	
2.1.	Dominating set problem, minimum dominating set problem	[43,60,77]
2.2.	Independent dominating sets in graphs, minimum independent dominating set	[61,79,98,115]
2.3.	Dominating sets in planar graphs	[50]
2.4.	Connected dominating set problems (e.g., minimum connected dominating set, i.e., minimum cardinality of the dominating set)	[16,21,22,43,46,60,62,95] [102,103,108,118,119,121] [123,125,136,139]
2.5.	Connected dominating set problems in unit disk graphs	[43,56,122,155]
2.6.	Planar connected dominating set problem	[99]
3.	Dominating set problems with special kinds of connectivity (e.g., weakly, strongly, k -connected):	
3.1.	Minimum size weakly-connected dominating sets	[3,13,28,29,43,148]
3.2.	Strongly connected dominating sets in networks with unidirectional links	[42,43]
3.3.	Total domination set problems	[7,27,33,70–72,119]
3.4.	Problems on total k -domination in graphs	[19,20]
3.5.	Problem on semitotal domination in graphs	[73]
3.6.	Problem on double domination in graphs	[63–65]
3.7.	Mixed domination problem in graphs	[86]
3.8.	k -connected m -dominating set problem ((k, m) -CDM problem)	[38,56,90,120,128]
3.9.	k -connected m -dominating set problem ((k, m) -CDM problem) with node weights	[56,107,122]
4.	Weighted dominating set problems:	
4.1.	Node weighted connected dominating set problem (e.g., vertex importance) (minimum weight connected dominating set problem)	[9,27,43,62,96,112] [130,135,155]
4.2.	Minimum weight k -connected m -fold dominating set (minimum weight (k, m) -CDS problem)	[56,107,122]
4.3.	Optimal degree constrained minimum-weight connected dominating set problem (network backbone formation)	[5]
4.4.	Weighted connected dominating set problem in unit disk graphs	[43]
4.5.	Minimum weight partial connected set cover problem	[92]
5.	Capacitated domination problems:	
5.1.	Capacitated domination set problems	[40,94]
5.2.	Minimum capacitated dominating set problem	[111]
5.3.	Capacitated (soft) domination problem (minimum cardinality of DS satisfying both the capacity and demand constraints)	[82]
5.4.	Capacitated b -edge dominating set problem	[17]
6.	Edge dominating set problems:	
6.1.	Edge dominating set problems	[32,52,60,78,144]
6.2.	2-edge connected dominating sets of a graph	[91]
6.3.	b -edge dominating set problem	[55]
6.4.	Capacitated b -edge dominating set problem	[17]
6.5.	Edge total domination problems	[152]
6.6.	Edge weighted dominating set problem	[26,53]
6.7.	Edge weighted connected dominating set problem	[62]
7.	Multi-hop dominating set problems:	
7.1.	Hop domination in graphs (k -step dominating sets)	[75,85]
7.2.	Connected k -hop dominating set problems	[34,59,104,106] [140,145,153]
7.3.	Connected dominating sets with multipoint relays	[2,116,137]
7.4.	Energy-efficient dominating tree (in multi-hop wireless networks)	[147]
7.5.	Connected dominating set for multi-hop wireless networks	[126]

Table 3. Basic dominating set and connected dominating set problems, part 2

No.	Problem type	Source(s)
8.	Steiner weighted dominating set problems:	
8.1.	Steiner connected dominating set problem	[6,62,101,138]
8.2.	Node weighted Steiner connected dominating set problem (minimum weight Steiner connected dominating set problem)	[4,62]
9.	Special dominating set problems:	
9.1.	k -fair domination in graphs	[25]
9.2.	Energy efficient stable connected dominating set	[116]
9.3.	Minimum connected dominating set for certain circulant networks	[110]
9.4.	Routing-cost constrained connected dominating set problems	[47]
9.5.	Highly connected multi-dominating sets problems	[56]
9.6.	Load balanced connected dominating set problem	[68]
9.7.	Dominating sets and connected dominating sets in dynamic graphs	[76]
9.8.	Reconfiguration of dominating sets	[127]

Table 4 contains a list of basic solving approaches.

Table 4. Basic solving approaches, part 1

No.	Approach	Source(s)
1.	Basic surveys on problems and applications:	
1.1.	Survey on problems and solving approaches	[43]
1.2.	Classification and comparison of connected DS construction algorithms	[149]
2.	Exact methods:	
2.1.	Branch-and-Bound algorithm for minimum connected dominating set problem	[123]
2.2.	Polynomial time algorithms for minimum paired-dominating sets (special graphs: tree, convex bipartite graph, strongly orderable graph, permutation graph)	[74,87,109]
2.3.	Enumeration algorithms for minimal dominating sets problems	[113,114]
2.4.	Exact algorithms for dominating set (exponential time algorithms, Branch and reduce algorithm)	[35]
2.5.	Exact algorithm for connected red-blue dominating set	[49,94,117]
3.	Heuristics, approximation algorithms:	
3.1.	Heuristic for the minimum connected dominating set problem on ad hoc wireless networks	[23]
3.2.	Extended localized algorithm for connected dominating set formation in Ad Hoc wireless networks	[37]
3.3.	Approximation algorithm for minimum size weakly-connected dominating sets	[28]
3.4.	Efficient algorithms for the minimum connected domination on trapezoid graphs	[129]
3.4.	Greedy approximation for minimum connected dominating sets	[118]
3.5.	Approximations for minimum-weighted dominating sets and minimum-weighted connected dominating sets	[155]
3.6.	Approximating minimum size weakly-connected dominating sets	[28]
3.7.	Approximation algorithms for 1- m -CDS and k - k -CDM problems	[128]
3.8.	Approximation algorithms for highly connected multi-dominating sets problems	[56]
3.9.	Constant-approximation for minimum-weight (connected) dominating sets	[14]
3.10.	Approximation for k -hop connected dominating set problem	[34]
3.11.	Two-phase approximation algorithms (for minimum CDS)	[132]
3.12.	Performance guaranteed approximation algorithm for minimum k -connected m -fold dominating set	[151]
3.13.	Approximation centralized algorithm for minimum CDS constructing virtual backbones in WSNs	[100]
3.14.	Approximation distributed algorithm for minimum CDS (in unit disk graphs)	[58]
3.15.	Simple heuristic for minimum connected dominating set in graphs (approximation algorithm)	[131]
3.16.	Approximation algorithm as distributed construction of connected dominating set (unit disk graphs)	[81]
3.17.	Efficient randomized distributed greedy algorithm for constructing small dominating sets (approximation)	[80]
3.18.	Approximation algorithms for k -connected m -dominating set problems	[107]

Table 4. Basic solving approaches, part 2

No.	Approach	Source(s)
4.	Metaheuristics:	
4.1.	Efficient mathheuristic for the minimum-weight dominating set problem.	[9]
4.2.	Hybrid metaheuristic algorithms for minimum weight dominating set	[112]
4.3.	Enhanced ant colony optimization metaheuristic (minimum dominating set problem)	[77]
4.4.	Two stage efficient centralized algorithm for connected dominating set on wireless networks	[51]
5.	Evolutionary methods:	
5.1.	Hybrid genetic algorithm for minimum dominating set problem	[69]
5.2.	Hybrid GA for minimum weight connected dominating set problem	[36]
5.3.	Effective hybrid memetic algorithm for the minimum weight dominating set problem	[96]
5.4.	Memetic algorithm for finding positive influence dominating sets in social networks	[97]
6.	Linear time algorithms:	
6.1.	Linear time algorithm for optimal k -hop dominating set problem (for tree, for bipartite permutation graphs)	[75,85]
6.2.	Linear time algorithms for finding a dominating set of fixed size in degenerated graphs	[11]
6.3.	Linear-time exact algorithm for tree (positive influence dominating set in social networks)	[39]
6.4.	Linear time algorithms for generalized edge dominating set problems	[18]
6.5.	Dynamic programming style linear time algorithm for k -power domination problem in weighted trees	[31]
7.	Polynomial time approximation schemes:	
7.1.	PTAS for minimum connected dominating set in ad hoc wireless networks	[30]
7.2.	PTAS for minimum connected dominating set in 3-dimensional WSNs	[150]
7.3.	PTAS for minimum d -hop connected dominating set in growth-bounded graphs	[59]
7.4.	PTAS for the minimum dominating set problem in unit disk graphs	[105]
7.5.	PTAS for minimum connected dominating set with routing cost constraint in WSNs	[44,45]
7.6.	PTAS for routing-cost constrained minimum connected dominating set (in growth bounded graphs)	[142]
7.7.	PTAS for capacitated domination problem on trees	[82]
7.8.	PTAS for the minimum weighted dominating set in growth bounded graphs	[134]
8.	Special approaches:	
8.1.	Search tree technique for dominating set on planar graphs	[8]
8.2.	Distributed algorithm for efficient construction and maintenance of connected k -hop dominating set	[145]
8.3.	Distributed learning automata approach for weakly connected dominating set	[3]
8.4.	Adaptive algorithms for connected dominating sets	[57]

3. Integer programming formulations of dominating set problems

The considered dominating set problem formulations are listed in Table 5.

3.1. Basic models

The minimum dominating set problem consists in finding the minimum dominating set $B \subseteq A$ in graph $G = (A, R)$, where $A = \{a_1, \dots, a_i, \dots, a_n\}$, $R = \{r_1, \dots, r_j, \dots, r_q\}$. The following binary variable x_i ($i = \overline{1, n}$) is used: $\forall a_i \in A$ $x_i = 1$ if a is selected for B (i.e., $a \in B$) and $x_i = 0$ otherwise. Thus, the solution of the dominating set problems is defined by binary vector $\bar{x} = (x_1, \dots, x_i, \dots, x_n)$. The following optimization models can be considered:

Model 1. The integer programming formulation of the basic minimum dominating set problem is:

$$\min \gamma(b) = \min \sum_{i=1}^n x_i$$

$$s.t. \quad \forall a_{\zeta_1} \in (A \setminus B) \quad \exists a_{\zeta_2} \in B \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

Model 2. The integer programming formulation of the basic minimum connected dominating set problem is (here $B_c \subseteq A$ is the connected dominating set):

$$\min \gamma_c(b) = \min \sum_{i=1}^n x_i$$

$$s.t. \quad \forall a_{\zeta'}, a_{\zeta''} \in B_c \quad \exists \text{ path } l(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle,$$

$$\forall a_{\zeta_1} \in (A \setminus B_c) \quad \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

In addition, vertex weights are considered: $w(a_i) > 0$ ($\forall a_i \in A$).

Table 5. Integer programming formulations of dominating set problems

No.	Model
I.	Basic models:
Model 1	Integer programming formulation of the basic minimum dominating set problem
Model 2	Integer programming formulation of the basic minimum connected dominating set problem
Model 3	Integer programming formulation of the minimum node weighted dominating set problem
Model 4	Integer programming formulation of the minimum vertex weighted connected dominating set problem
II.	Basic multicriteria models:
Model 5	Multicriteria (multiobjective) minimum node weighted dominating set problem
Model 6	Multicriteria (multiobjective) minimum node weighted connected dominating set problem
III.	Models of k -connected dominating set problems:
Model 7	Integer programming formulation of the basic minimum k connected dominating set problem
Model 8	Integer programming formulation of the minimum vertex weighted k -connected connected dominating set problem
Model 9	Multicriteria (multiobjective) minimum node weighted k -connected dominating set problem
IV.	Models of k -connected m -dominating set problems:
Model 10	Integer programming formulation of the basic minimum k -connected m -dominating set problem
Model 11	Integer programming formulation of the minimum vertex weighted k -connected connected m -dominating set problem
Model 12	Multicriteria (multiobjective) minimum node weighted k -connected m -dominating set problem
V.	Dominating problems with multiset estimates:
Model 13	Minimum node weighted dominating set problem with multiset estimates (the objective function is based on "generalized median")
Model 14	Minimum node weighted connected dominating set problem with multiset estimates (the objective function is based on "generalized median")
Model 15	Multicriteria (multiobjective) minimum node weighted k -connected dominating set problem with multiset estimates (the objective function is based on "generalized median")
Model 16	Multicriteria (multiobjective) minimum node weighted k -connected m -dominating set problem with multiset estimates (the objective function is based on "generalized median")

Model 3. The integer programming formulation of the minimum node weighted dominating set problem is:

$$\min \gamma^w(b) = \min \sum_{i=1}^n w(a_i) x_i$$

$$s.t. \quad \forall a_{\zeta_1} \in (A \setminus B) \quad \exists a_{\zeta_2} \in B \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

Model 4. The integer programming formulation of the minimum vertex weighted connected dominating set problem is (here $B_c \subseteq A$ is the connected dominating set):

$$\min \gamma_c^w(b) = \min \sum_{i=1}^n w(a_i) x_i$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists \text{ path } l(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle,$
 $\forall a_{\zeta_1} \in (A \setminus B_c) \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$

3.2. Basic multicriteria models

Further, vertex weight vectors can be considered: $\bar{w}(a_i) = (w(a_i^1), \dots, w^\kappa(a_i), \dots, w^\mu(a_i))$ ($w^\kappa(a_i) > 0 \forall \kappa = \bar{1}, \bar{\mu}, \forall a_i \in A$). The following multicriteria (multiobjective) models can be examined (here the Pareto-efficient solutions are searched for):

Model 5. Multicriteria (multiobjective) minimum node weighted dominating set problem:

$$\min \sum_{i=1}^n w^1(a_i) x_i, \dots, \min \sum_{i=1}^n w^\kappa(a_i) x_i, \dots, \min \sum_{i=1}^n w^\mu(a_i) x_i,$$

s.t. $\forall a_{\zeta_1} \in (A \setminus B) \exists a_{\zeta_2} \in B \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$

Model 6. Multicriteria (multiobjective) minimum node weighted connected dominating set problem:

$$\min \sum_{i=1}^n w^1(a_i) x_i, \dots, \min \sum_{i=1}^n w^\kappa(a_i) x_i, \dots, \min \sum_{i=1}^n w^\mu(a_i) x_i,$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists \text{ path } l(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle,$
 $\forall a_{\zeta_1} \in (A \setminus B_c) \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$

3.3. Models of k -connected dominating set problems

In the case of k -connected dominating set the models are extended by an additional constraint for k -connectivity of dominating set:

Model 7. The integer programming formulation of the basic minimum k connected dominating set problem is (here $B_c \subseteq A$ is the k -connected dominating set):

$$\min \gamma_c(b) = \min \sum_{i=1}^n x_i$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k \text{ vertex disjoint paths}$
 $l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$
 $\forall a_{\zeta_1} \in (A \setminus B_c) \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$

Model 8. The integer programming formulation of the minimum vertex weighted k -connected dominating set problem is (here $B_c \subseteq A$ is the k -connected dominating set):

$$\min \gamma_c^w(b) = \min \sum_{i=1}^n w(a_i) x_i$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k \text{ vertex disjoint paths}$
 $l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$
 $\forall a_{\zeta_1} \in (A \setminus B_c) \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$

Model 9. Multicriteria (multiobjective) minimum node weighted k -connected dominating set problem:

$$\min \sum_{i=1}^n w^1(a_i) x_i, \dots, \min \sum_{i=1}^n w^\kappa(a_i) x_i, \dots, \min \sum_{i=1}^n w^\mu(a_i) x_i,$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k$ vertex disjoint paths

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$\forall a_{\zeta_1} \in (A \setminus B_c) \exists a_{\zeta_2} \in B_c$ (i.e., $x_{a_{\zeta_2}} = 1$) such that $(a_{\zeta_1}, a_{\zeta_2}) \in R$.

3.4. Models of k -connected m -dominating set problems

In the case of k -connected m dominating set the models are extended by an additional constraints for k -connectivity and m -dominating:

Model 10. The integer programming formulation of the basic minimum k -connected m -dominating set problem is (here $B_c \subseteq A$ is the k -connected dominating set):

$$\min \gamma_c(b) = \min \sum_{i=1}^n x_i$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k$ vertex disjoint paths

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$\forall a_{\zeta_1} \in (A \setminus B_c) \exists m$ vertices $a_{\zeta_1}^1, \dots, a_{\zeta_1}^m \in B_c$ (i.e., $x_{a_{\zeta_1}^1} = 1, \dots, x_{a_{\zeta_1}^m} = 1$)

such that $(a_{\zeta_1}, a_{\zeta_1}^1) \in R, \dots, (a_{\zeta_1}, a_{\zeta_1}^m) \in R$.

Model 11. The integer programming formulation of the minimum vertex weighted k -connected m -dominating set problem is (here $B_c \subseteq A$ is the k -connected dominating set):

$$\min \gamma_c^w(b) = \min \sum_{i=1}^n w(a_i) x_i$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k$ vertex disjoint paths

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$\forall a_{\zeta_1} \in (A \setminus B_c) \exists m$ vertices $a_{\zeta_1}^1, \dots, a_{\zeta_1}^m \in B_c$ (i.e., $x_{a_{\zeta_1}^1} = 1, \dots, x_{a_{\zeta_1}^m} = 1$)

such that $(a_{\zeta_1}, a_{\zeta_1}^1) \in R, \dots, (a_{\zeta_1}, a_{\zeta_1}^m) \in R$.

Model 12. Multicriteria (multiobjective) minimum node weighted k -connected m -dominating set problem:

$$\min \sum_{i=1}^n w^1(a_i) x_i, \dots, \min \sum_{i=1}^n w^\kappa(a_i) x_i, \dots, \min \sum_{i=1}^n w^\mu(a_i) x_i,$$

s.t. $\forall a_{\zeta'}, a_{\zeta''} \in B_c \exists k$ vertex disjoint paths

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$\forall a_{\zeta_1} \in (A \setminus B_c) \exists m$ vertices $a_{\zeta_1}^1, \dots, a_{\zeta_1}^m \in B_c$ (i.e., $x_{a_{\zeta_1}^1} = 1, \dots, x_{a_{\zeta_1}^m} = 1$)

such that $(a_{\zeta_1}, a_{\zeta_1}^1) \in R, \dots, (a_{\zeta_1}, a_{\zeta_1}^m) \in R$.

3.5. Problems with multiset estimates

3.5.1. Interval multiset estimates

The fundamentals of the multiset theory are presented in [84,143]. Here a brief description of interval multiset estimates from [88] is described. The approach consists in assignment of elements $(1, 2, 3, \dots)$ into an ordinal scale $[1, 2, \dots, l]$. In the obtained multi-set based estimate, a basis set involves all levels of the ordinal scale: $\Omega = \{1, 2, \dots, l\}$ (the levels are linear ordered: $1 \succ 2 \succ 3 \succ \dots$) and the assessment problem (for each alternative/object under assessment) consists in selection of a multiset over set Ω while taking into account two conditions:

1. cardinality of the selected multiset equals a specified number of elements $\eta = 1, 2, 3, \dots$ (i.e., multisets of cardinality η are considered);
2. “configuration” of the multiset is the following: the selected elements of Ω cover an interval over scale $[1, l]$ (i.e., “interval multiset estimate”).

An estimate e for an alternative/object a (e.g., $a \in A$) is (scale $[1, l]$, position-based form or position form): $e(a) = (\eta_1, \dots, \eta_l, \dots, \eta_l)$, where η_l corresponds to the number of elements at the level l ($l = \overline{1, l}$),

or $e(a) = \{\underbrace{1, \dots, 1}_{\eta_1}, \underbrace{2, \dots, 2}_{\eta_2}, \underbrace{3, \dots, 3}_{\eta_3}, \dots, \underbrace{l, \dots, l}_{\eta_l}\}$. The number of multisets of cardinality η , with elements taken from a finite set of cardinality l , is called the “multiset coefficient” or “multiset number” ([84,143]): $\mu^{l, \eta} = \frac{l(l+1)(l+2)\dots(l+\eta-1)}{\eta!}$. This number corresponds to possible estimates (without taking into account interval condition 2). In the case of condition 2, the number of estimates is decreased. The certain assessment problem based on interval multiset estimates can be denoted as the following: $P^{l, \eta}$.

In addition, basic operations over multiset estimates are used: integration, vector-like proximity, aggregation, and alignment [88]. Integration of estimates (mainly, for composite systems) is based on summarization of the estimates by components (i.e., positions). Let us consider n estimates (position form): estimate $e^1 = (\eta_1^1, \dots, \eta_l^1, \dots, \eta_l^1)$, ..., estimate $e^\kappa = (\eta_1^\kappa, \dots, \eta_l^\kappa, \dots, \eta_l^\kappa)$, ..., estimate $e^n = (\eta_1^n, \dots, \eta_l^n, \dots, \eta_l^n)$. The integrated estimate is: estimate $e^I = (\eta_1^I, \dots, \eta_l^I, \dots, \eta_l^I)$, where $\eta_l^I = \sum_{\kappa=1}^n \eta_l^\kappa \quad \forall l = \overline{1, l}$. In fact, the operation \uplus is used for multiset estimates: $e^I = e^1 \uplus \dots \uplus e^\kappa \uplus \dots \uplus e^n$.

The vector-like proximity of two multiset based estimates $e(a_1), e(a_2)$ (e.g., $a_1, a_2 \in A$) is:

$$\delta(e(A_1), e(A_2)) = (\delta^-(A_1, A_2), \delta^+(A_1, A_2)),$$

where vector components are: (i) δ^- is the number of one-step changes: element of quality $l + 1$ into element of quality l ($l = \overline{1, l - 1}$) (this corresponds to “improvement”); (ii) δ^+ is the number of one-step changes: element of quality l into element of quality $l + 1$ ($l = \overline{1, l - 1}$) (this corresponds to “degradation”). It is assumed: $|\delta(e(A_1), e(A_2))| = |\delta^-(A_1, A_2)| + |\delta^+(A_1, A_2)|$.

For aggregation (as an analogue of summarization) of the specified set of multiset based estimates $E = \{e_1, \dots, e_\kappa, \dots, e_n\}$, two types of medians are considered. Here \overline{E} corresponds to the the set of all possible multiset based estimates ($E \subseteq \overline{E}$). The median estimates are:

- (a) “generalized median”: $M^g = \arg \min_{M \in \overline{E}} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|$;
- (b) “set median”: $M^s = \arg \min_{M \in E} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|$.

3.5.2. Models with multiset estimates

Here vertex weight vectors $\overline{w}(a_i) = (w(a_i^1), \dots, w^\kappa(a_i), \dots, w^\mu(a_i))$ ($w^\kappa(a_i) > 0 \quad \forall \kappa = \overline{1, \mu}, \forall a_i \in A$) is transformed (compressed) into multiset based estimate $e(a_i)$. The following models can be considered:

Model 13. Minimum node weighted dominating set problem with multiset estimates (the objective function is based on “generalized median”):

$$\min M^g = \arg \min_{M \in \overline{E}} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|;$$

$$s.t. \quad \forall a_{\zeta_1} \in (A \setminus B) \quad \exists a_{\zeta_2} \in B \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

Model 14. Minimum node weighted connected dominating set problem with multiset estimates (the

objective function is based on “generalized median”):

$$\min M^g = \arg \min_{M \in \bar{E}} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|;$$

$$s.t. \quad \forall a_{\zeta'}, a_{\zeta''} \in B_c \quad \exists \text{ path } l(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle,$$

$$\forall a_{\zeta_1} \in (A \setminus B_c) \quad \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

Model 15. Multicriteria (multiobjective) minimum node weighted k -connected dominating set problem with multiset estimates (the objective function is based on “generalized median”):

$$\min M^g = \arg \min_{M \in \bar{E}} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|;$$

$$s.t. \quad \forall a_{\zeta'}, a_{\zeta''} \in B_c \quad \exists k \text{ vertex disjoint paths}$$

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$$\forall a_{\zeta_1} \in (A \setminus B_c) \quad \exists a_{\zeta_2} \in B_c \text{ (i.e., } x_{a_{\zeta_2}} = 1) \text{ such that } (a_{\zeta_1}, a_{\zeta_2}) \in R.$$

Model 16. Multicriteria (multiobjective) minimum node weighted k -connected m -dominating set problem with multiset estimates (the objective function is based on “generalized median”):

$$\min M^g = \arg \min_{M \in \bar{E}} \sum_{\kappa=1}^n |\delta(M, e_\kappa)|;$$

$$s.t. \quad \forall a_{\zeta'}, a_{\zeta''} \in B_c \quad \exists k \text{ vertex disjoint paths}$$

$$l^1(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle, \dots, l^k(a_{\zeta'}, a_{\zeta''}) = \langle a_{\zeta'}, \dots, a_{\zeta''} \rangle;$$

$$\forall a_{\zeta_1} \in (A \setminus B_c) \quad \exists m \text{ vertices } a_{\zeta_1}^1, \dots, a_{\zeta_1}^m \in B_c \text{ (i.e., } x_{a_{\zeta_1}^1} = 1, \dots, x_{a_{\zeta_1}^m} = 1)$$

$$\text{such that } (a_{\zeta_1}, a_{\zeta_1}^1) \in R, \dots, (a_{\zeta_1}, a_{\zeta_1}^m) \in R.$$

4. Conclusion

In the paper, a brief survey on dominating set problems is presented. The integer programming formulations of dominating set problems are described. Some new multicriteria models and models based on multiset estimates are considered as well.

It may be reasonable to point out the following future research directions: (1) consideration of other application domains for the dominating set problems; (2) examination of multistage solving strategies; (3) further examination of various k -connected m -dominating set problems; (4) additional study of the described combinatorial models and their various versions (including studies of domination structures as trees, etc.); (5) special future studies of multi-hop dominating set problems on various graphs (including applications in networking); (6) design of a special software package (i.e., decision support system) for the dominating set problems (including descriptions of problems, descriptions of applied domains, solving methods/procedures, examples); and (7) using the described optimization models for dominating set problems in education (e.g., student projects).

The author states that there is no conflict of interest.

REFERENCES

1. F.N. Abu-Khzam, A.E. Mouawad, M. Liedloff, An exact algorithm for connected red-blue dominating set. *J. of Discrete Algorithms*, 9, 252–262, 2011.
2. C. Adjih, P. Jacquet, L. Viennot, Computing connected dominating sets with multipoint relays. *Ad Hoc & Sensor Wireless Networks*, 27–39, Mar. 2005.
3. J. Akbari Torkestani, M.R. Meybodi, Clustering the wireless Ad Hoc networks: distributed learning automata approach. *J. of Parallel and Distr. Computing*, 70(4), 394–405, 2010.
4. J. Akbari Torkestani, M.R. Meybodi, Weighted Steiner connected dominating set and its application to multicast routing in wireless MANETs. *Wireless Personal Communications*, 60(2), 145–169, 2011.
5. J. Akbari Torkestani, An adaptive backbone formation algorithm for wireless sensor networks. *Computer Communications*, 35(11), 1333–1344, 2012.
6. J. Akbari Torkestani, Algorithms for Steiner connected dominating set problem based on learning automata theory. *Int. J. of Foundations of Computer Science*, 26(06), 769–801, 2015.
7. R.B. Allan, R. Laskar, S.T. Hedetniemi, A note on total domination. *Discr. Math.*, 49(1), 7–13, 1984.
8. J. Alber, H. Fan, M.R. Fellows, R. Niederreier, F.A. Rosamond, U. Stege, A refined search tree technique for dominating set on planar graphs. *J. Comput. Syst. Sci.*, 71(4), 385–405, 2005.
9. M. Albuquerque, T. Vidal, An efficient mathheuristic for the minimum-weight dominating set problem. *Electr. prepr.*, 24 p., Aug. 28, 2018. <http://arxiv.org/abs/1808.09809> [cs.AI]
10. N. Alon, F. Fomin, G. Gutin, M. Krivelevich, S. Saurabh, Spanning directed trees with many leaves. *SIAM J. on Discr. Math.*, 23(1), 466–476, 2009.
11. N. Alon, S. Gutner, Linear time algorithms for finding a dominating set of fixed size in degenerated graphs. *Algorithmica*, 54(4), 544–556, 2009.
12. J.D. Alvarado, S. Dantas, E. Mohr, D. Rautenbach, On the maximum number of minimum dominating sets in forests. *Discr. Math.*, 342(4), 934–942, 2019.
13. K.M. Alzoubi, P.-J. Wan, O. Frieder, Maximal independent set, weakly connected dominating set, and induced spanners for mobile ad-hoc networks. *Int. J. of Foundations of Computer Science*, 14(2), 287–303, 2003.
14. C. Ambuhl, T. Erlebach, M. Mihalak, M. Nunkesser, Constant-factor approximation for minimum-weight (connected) dominating sets in unit disk graphs. In: *APPROX-RANDOM 2006*, LNCS 4110, Springer, pp. 3–14, 2006.
15. D.V. Andrade, M.G.C. Resende, R.F. Werneck, Fast local search for the maximum independent set problem. *J. of Heuristics*, 18, 525–547, 2012.
16. X. Bai, D. Zhao, S. Bai, Q. Wang, W. Li, D. Mu, Minimum connected dominating sets in heterogeneous 3D wireless Ad Hoc networks. *Ad Hoc Networks*, 97, art. 102023, 2020.
17. A. Berger, T. Fukunaga, H. Nagamochi, O. Parekh, Approximability of the capacitated b -edge dominating set problem. *Theor. Comp. Sci.*, 385(1–3), 202–213, 2007.
18. A. Berger, O. Parekh, Linear time algorithms for generalized edge dominating set problems. *Algorithmica*, 59(2), 244–254, 2008.
19. S. Bermudo, J.C. Hernandez-Gomez, J.M. Sigarreta, Total k -domination in strong product graphs. *Discr. Appl. Math.*, 263, 51–58, 2019.
20. S. Bermudo, A.C. Martinez, F.A. Hernandez Mira, J.M. Sigarreta, On the global total k -domination number of graphs. *Discr. Appl. Math.*, 263, 42–50, 2019.
21. J. Blum, M. Ding, A. Thaeler, X. Cheng, Connected dominating set in sensor networks and MANETs. In: D.-Z. Du, P.M. Pardalos (eds), *Handbook of Combinatorial Optimization*, Springer, pp. 329–369, 2005.
22. A. Buchanan, J.S. Sung, V. Boginski, S. Butenko, On connected dominating set of restricted diameter. *EJOR*, 236(2), 410–418, 2014.
23. S. Butenko, X. Cheng, C.A.S. Oliveira, P.M. Pardalos, A new heuristic for the minimum connected dominating set problem on ad hoc wireless networks. In: *Recent Developments in Cooperative Control and Optimization*, Springer, pp. 61–73, 2004.
24. Y. Caro, D.B. West, R. Yuster, Connected domination and spanning trees with many leaves. *SIAM J. on Discr. Math.*, 13(2), 202–211, 2000.
25. Y. Caro, A. Hansberg, M. Henning, Fair domination in graphs. *Discr. Math.*, 312, 2905–2914, 2012.
26. R. Carr, T. Fujito, G. Konjevod, O. Parekh, A $2\frac{1}{10}$ -approximation algorithm for a generalization of

- the weighted edge-dominating set problem. *J. of Combin. Optim.*, 5, 317–326, 2001.
27. M.-S. Chang, Weighted domination of cocomparability graphs. *Discr. Appl. Math.*, 80, 135–148, 1997.
 28. Y.P. Chen, A.L. Liestman, Approximating minimum size weakly-connected dominating sets for clustering mobile ad hoc networks. In: *MobiHoc'02*, pp. 165–172, 2002.
 29. Y.P. Chen, A.L. Liestman, Maintaining weakly connected dominating sets for clustering Ad-Hoc networks. *Ad Hoc Networks*, 3, 629–642, 2005.
 30. X. Cheng, X. Huang, D. Li, W. Wu, D.-Z. Du, A polynomial-time approximation scheme for minimum connected dominating set in ad hoc wireless networks. *Networks*, 42(4), 202–208, 2003.
 31. C.J. Cheng, C. Lu, Y. Zhou, The k -power domination problem in weighted trees. In: *AAIM 2018*, LNCS 11343, Springer, pp. 149–160, 2018.
 32. M. Chlebik, J. Chlebikova, Approximation hardness of edge dominating set problems. *J. of Comb. Optim.*, 11(3), 279–290, 2006.
 33. E.J. Cockayne, R. Dawes, S.T. Hedetniemi, Total domination in graphs. *Networks*, 10, 211–215, 1980.
 34. R.S. Coelho, P.F.S. Moura, Y. Wakabayashi, The k -hop connected dominating set problem: approximation and hardness. *J. of Combin. Optim.*, 34(4), 1060–1083, 2017.
 35. J.-F. Couturier, P. Heggenes, P. van 't Hof, D. Kratsch, Minimal dominating sets in graph classes: Combinatorial bounds and enumeration. *Theor. Comp. Sc.*, 487, 82–94, 2013.
 36. Z.A. Dagdeviren, D. Aydin, M. Cinsdikici, Two population-based optimization algorithms for minimum weight connected dominating set problem. *Appl. Soft Comput.*, 59, 644–658, 2017.
 37. F. Dai, J. Wu, An extended localized algorithm for connected dominating set formation in Ad Hoc wireless networks. *IEEE Trans. on Parallel and Distrib. Syst.*, 15(10), 908–920, 2004.
 38. F. Dai, J. Wu, On constructing k -connected k -dominating set in wireless ad hoc and sensor networks. *J. of Parallel and Distributed Computing*, 66(7), 947–958, 2006.
 39. T.N. Dinh, Y. Shen, D.T. Nguyen, M.T. Thai, On the approximability of positive influence dominating set in social networks. *J. of Com. Optim.*, 27(3), 487–503, 2014.
 40. M. Dom, D. Lokshtanov, S. Saurabh, Y. Villanger, Capacitated domination and covering: a parameterized perspective. In: *Proc. 3rd IWPEC*, LNCS 5018, Springer, pp. 78–90, 2008.
 41. M. Dorfling, M.A. Henning, A note on power domination in grid graphs. *Discr. Appl. Math.*, 154, 1023–1027, 2006.
 42. D.-Z. Du, M.T. Thai, Y. Li, D. Liu, S. Zhu, Strongly connected dominating sets in wireless sensor networks with unidirectional links. In: *APWeb 2006*, LNCS 3841, Springer, pp. 13–24, 2006.
 43. D.-Z. Du, P.-J. Wan, *Connected Dominating Set: Theory and Applications*. Springer, 2013.
 44. H. Du, Q. Ye, J. Zhong, Y. Wang, W. Lee, H. Park, PTAS for minimum connected dominating set with routing cost constraint in wireless sensor networks. In: *COCOA 2010, Part 1*, LNCS 6508, Springer, pp. 252–259, 2020.
 45. H. Du, Q. Ye, J. Zhong, Y. Wang, W. Lee, H. Park, Polynomial-time approximation scheme for minimum connected dominating set under routing cost constraint in wireless sensor networks. *Theor. Comp. Sci.*, 447, 38–43, 2012.
 46. H. Du, L. Ding, W. Wu, D. Kim, P.M. Pardalos, J. Willson, Connected dominating set in wireless networks. In: P.M. Pardalos, R.L. Graham, D.-Z. Du (eds), *Handbook of Combinatorial Optimization*. 2nd ed., Springer, pp. 783–834, 2013.
 47. H. Du, H. Luo, Routing-cost constrained connected dominating set. In: M.Y. Kao (ed), *Encyclopedia of Algorithms*, Springer, pp. 1879–1883, 2016.
 48. K. Erciyes, O. Dagdeviren, D. Cokeslu, D. Ozsoyeller, Graph theoretic clustering algorithms in mobile ad hoc networks and wireless sensor networks - survey. *Appl. Comput. Math.*, 6(2), 162–80, 2007.
 49. F.V. Fomin, D. Kratsch, G.J. Woeginger, Exact (exponential) algorithms for the dominating set problem. In: J. Hromkovic, M. Nagl, B. Westfechtel (eds), LNCS 3353, Springer, pp. 245–256, 2004.
 50. F.V. Fomin, D.M. Thilikos, Dominating sets in planar graphs: branch-width and exponential speed-up. *SIAM J. on Computing*, 36(2), 281–309, 2006.
 51. D. Fu, L. Han, L. Liu, Q. Gao, Z. Feng, An efficient centralized algorithm for connected dominating set on wireless networks. *Procedia CS*, 56, 162–167, 2015.
 52. T. Fujito, Approximability of the independent/connected edge dominating set problems. *Inform. Proc. Lett.*, 79, 261–266, 2001
 53. T. Fujito, H. Nagamochi, A 2-approximation algorithm for the minimum weight edge dominating set

- problem. *Discr. Appl. Math.*, 118(3), 199–207, 2002.
54. T. Fujie, An exact algorithm for the maximum leaf spanning tree problem. *Comp. and Oper. Res.*, 30, 1931–1944, 2003.
 55. T. Fukunaga, H. Nagamochi, Approximation algorithm for the b -edge dominating set problem and its related problems. In: *COCOON 2005*, LNCS 3595, Springer, pp. 747–756, 2005.
 56. T. Fukunaga, Approximation algorithms for highly connected multi-dominating sets in unit disk graphs. *Algorithmica*, 80(11), 3270–3292, 2018.
 57. T. Fukunaga, Adaptive algorithms for finding connected dominating sets in uncertain graphs. *Electr. prepr.*, 19 p., Dec 29, 2019. <http://arxiv.org/abs/1912.12665> [cs.DS]
 58. S. Funke, A. Kesselman, U. Meyer, M. Segal, A simple improved distributed algorithm for minimum CDS in unit disk graphs. *ACM Trans. on Sensor Networks*, 2(3), 444–453, 2006.
 59. X. Gao, W. Wag, Z. Zhang, S. Zhu, W. Wu, A PTAS for minimum d -hop connected dominating set in growth-bounded graphs. *Optim. Lett.*, 4(3), 321–333, 2010.
 60. M.R. Garey, D.S. Johnson, *Computers and intractability. The Guide to the theory of NP-completeness*. San Francisco: W.H. Freeman and Company, 1979.
 61. W. Goddard, J. Lyle, Independent dominating sets in triangle-free graphs. *J. of Comb. Optim.*, 23(1), 9–20, 2012.
 62. S. Guha, S. Khuller, Approximation algorithms for connected dominating sets. *Algorithmica*, 20(4), 374–387, 1998.
 63. M. Hajian, N.J. Rad, A new lower bound on the double domination number of a graph. *Discr. Appl. Math.*, 254, 280–282, 2019.
 64. J. Harant, M.A. Henning, On double dominating in graphs. *Discussiones Mathematicae*, 25, 29–34, 2005.
 65. F. Harary, T.W. Haynes, Double domination in graphs. *Ars Combin.*, 55, 201–213, 2000.
 66. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*. Marcel Dekker Inc., 1998.
 67. T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, M.A. Henning, Domination in graphs applied to electrical power networks. *SIAM J. on Discrete Math.*, 15(4), 519–529, 2002.
 68. J. He, S. Ji, P. Fan, Y. Pan, Y. Li, Constructing a load-balanced virtual backbone in wireless sensor networks. In: *2012 Int. Conf. on Computing, Networking and Communication (ICNC)*, pp. 959–963, 2012.
 69. A.-R. Hedar, R. Ismail, Hybrid genetic algorithm for minimum dominating set problem. In: *Computational Science and Its Applications - ICCSA 2010*, pp. 457–467, 2010.
 70. M.A. Henning, N.J. Rad, Locating-total domination in graphs. *Discr. Appl. Math.*, 160, 1986–1993, 2012.
 71. M.A. Henning, N.J. Rad, Bounds on neighborhood total domination in graphs. *Discr. Appl. Math.*, 161, 2460–2466, 2013.
 72. M.A. Henning, A. Yeo, *Total Domination in Graphs*. Springer, 2013.
 73. M.A. Henning, A.J. Marcon, On matching and semitotal domination in graphs. *Discr. Math.*, 324, 13–18, 2014.
 74. M.A. Henning, D. Pradhan, Algorithmic aspects of upper paired-domination in graphs. *Theor. Comp. Sci.*, 804, 98–114, 2020.
 75. M.A. Henning, S. Pal, D. Pradhan, Algorithm and hardness results on hop domination in graphs. *Inform. Proc. Lett.*, 153, 105872, 2020.
 76. N. Hjuler, G.F. Italiano, N. Parotsidis, D. Saulpic, Dominating sets and connected dominating sets in dynamic graphs. In: *STACS 2019*, pp. 35:1–35:17, 2019.
 77. C.K. Ho, Y.P. Singh, H.T. Ewe, An enhanced ant colony optimization metaheuristic for the minimum dominating set problem. *Applied Artificial Intelligence*, 20(10), 881–903, 2006.
 78. J. Horton, K. Kilakos, Minimum edge dominating sets. *SIAM J. Discr. Math.*, 6(3), 375–387, 1993.
 79. R.W. Irving, On approximating the minimum independent dominating set. *Inform. Process. Lett.*, 37(4), 197–200, 1991.
 80. L. Jia, R. Rajaraman, T. Suel, An efficient distributed algorithm for constructing small dominating sets. *Distrib. Comput.*, 15(4), 193–205, 2002.
 81. R.K. Jullu, P.R. Prasad, G.K. Das, Distributed construction of connected dominating set in unit disk

- graphs. *J. of Parallel and Distr. Comput.*, 104, 159–166, 2017.
82. M.J. Kao, C.S. Liao, D.T. Lee, Capacitated domination problem. *Algorithmica*, 60(2), 274–300, 2011.
 83. D.J. Kleitman, D.B. West, Spanning trees with many leaves. *SIAM J. on Discr. Math.*, 4(1), 99–106, 1991.
 84. D.E. Knuth, *The Art of Computer Programming. Vol. 2, Seminumerical Algorithms*. Addison Wesley, Reading, 1998.
 85. S. Kundu, S. Majumder, A linear time algorithm for optimal k -hop dominating set of a tree. *Inf. Process. Lett.*, 116(2), 197–202, 2016.
 86. J.K. Lan, G.J. Chang, On the mixed domination problem in graphs. *Theor. Comp. Sci.*, 476, 84–93, 2013.
 87. E. Lappas, S.D. Nikolopoulos, L. Palios, An $O(n)$ -time algorithm for paired-domination on permutation graphs. *Eur. J. Combin.*, 34(3), 593–608, 2013.
 88. M.Sh. Levin, *Modular System Design and Evaluation*. Springer, 2015.
 89. M.Sh. Levin, Note on decision support platform for modular systems. *Information processes*, 19(2), 132–141, 2019.
 90. Y. Li, Y. Wu, C. Ai, F. Beyah, On the construction of k -connected m -dominating sets in wireless networks. *J. of Comb. Optim.*, 23(1), 118–139, 2012.
 91. H. Li, Y. Yang, B. Wu, 2-edge connected dominating sets and 2-connected dominating sets of a graph. *J. of Combin, Optim.*, 31(2), 713–724, 2016.
 92. D. Liang, Z. Zhang, X. Liu, W. Wang, Y. Jiang, Approximation algorithms for minimum weight partial connected set cover problem. *J. of Combin, Optim.*, 31(2), 696–712, 2016.
 93. C.-S. Liao, T.-J. Hsieh, X.-C. Guo, C.-C. Chu, Hybrid search for the optimal pmu placement problem on a power grid. *EJOR*, 243(3), 985–994, 2015.
 94. M. Liedloff, I. Todinca, Y. Villanger, Solving capacitated dominating set by using covering by subsets and maximum matching. *Discr. Appl. Math.*, 168, 60–68, 2014.
 95. Z. Lin, H. Liu, X. Chu, Y.-W. Leung, I. Stojmenovic, Maximizing lifetime of connected-dominating set in cognitive radio. In: *NETWORKING 2012, Part II, LNCS 7290*, Springer, pp. 316–330, 2012.
 96. G. Lin, W. Zhu, M.M. Ali, An effective hybrid memetic algorithm for the minimum weight dominating set problem. *IEEE Trans. on Evolutionary Computation*, 20(6), 892–907, 2016.
 97. G. Lin, J. Guan, H. Feng, An ILP based memetic algorithm for finding positive influence dominating sets in social networks. *Physica A*, 500, 199–209, 2018.
 98. C.-H. Liu, S.-H. Poon, J.-Y. Lin, Independent dominating set problem revised. *Theor. Comp. Sci.*, 562, 1–22, 2015.
 99. D. Lokshtanov, M. Mnich, S. Saurabh, A linear kernel for planar connected dominating set. *Theor. Comp. Sci.*, 412, 2536–2543, 2011.
 100. C. Luo, W. Chen, J. Yu, Y. Wang, D. Li, A novel centralized algorithm for constructing virtual backbones in wireless sensor networks. *EURASIP J. on Wireless Communications and Networking*, art. 55, 2018.
 101. M. Min, H. Du, X. Jia, C.X. Huang, S.C.-H. Huang, W. Wu, Improving construction for connected dominating set with Steiner tree in Wireless Sensor Networks. *J. of Global Optim.*, 35, 111–119, 2006.
 102. J.P. Mohanty, C. Mandal, C. Reade, A. Das, Construction of minimum connected dominating set in wireless sensor networks. *Ad Hoc Netw.*, 42, 61–73, 2016.
 103. J.P. Mohanty, C. Mandal, C. Reade, Distributed construction of minimum Connected Dominating Set in wireless sensor network using two-hop information. *Computer Networks*, 123, 137–152, 2017.
 104. T.N. Nguen, D.T. Huynh, Connected d -hop dominating sets in mobile ad hoc networks. In: *Proc. 2005 4th Int. Symp. on Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks*, vols. 1 and 2, pp. 138–145, 2006.
 105. T. Nieberg, J. Hurink, A PTAS for the minimum dominating set problem in unit disk graphs. In: *WAOA 2005, LNCS 3879*, Springer, pp. 296–306, 2005.
 106. F.G. Nocetti, J.S. Gonzalez, I. Stojmenovic, Connectivity based k -hop clustering in wireless ad hoc networks. *Telecommunication Systems*, 22(1–4), 205–220, 2003.
 107. Z. Nutov, Improved approximation algorithms for k -connected m -dominating set problems. *Electr. prepr.*, 6 p., Mar. 13, 2017. <http://arxiv.org/abs/1703.04230> [cs.DC]
 108. C.A.S. Oliveira, P.M. Pardalos, *Ad Hoc networks: optimization problems and solution methods*.

- In: M.X. Cheng, Y. Li, D.-Z. Du (eds), *Combinatorial Optimization in Communication Networks*. Springer, pp. 147–170, 2006.
- 109.B.S. Panda, D. Pradhan, A linear time algorithm for computing a minimum paired-dominating set of a convex bipartite graph. *Discr. Appl. Math.*, 161, 1776–1783, 2013.
- 110.N. Parthiban, I. Rajasingh, R. Sundara Rajan, Minimum connected dominating set for certain circulant networks. *Procedia CS*, 57, 587–591, 2015.
- 111.P. Pinacho-Davidson, S. Bouamama, C. Blum, Application of CMSA to the minimum capacitated dominating set problem. In: *GECCO 2019*, pp. 321–328, 2019.
- 112.A. Potluri, A. Singh, Hybrid metaheuristic algorithms for minimum weight dominating set. *Appl. Soft Computing*, 13, 76–88, 2013.
- 113.D. Pradhan, B.S. Panda, Computing a minimum paired-dominating set in strongly orderable graphs. *Discr. Appl. Math.*, 253, 37–50, 2019.
- 114.H. Qiao, L. Kang, M. Gardei, D.-Z. Du, Paired-domination of trees. *J. Glob. Optim.*, 25(1), 43–54, 2003.
- 115.N.J. Rad, L. Volkmann, A note on the independent domination number in graphs. *Discrete Applied Math.*, 161, 3087–3089, 2013.
- 116.R. Ramalakshmi, S. Radhaktishnan, Energy efficient stable connected dominating set construction in mobile ad hoc networks. In: *CCSIT 2012, Part I, LNICST 84*, Springer, pp. 64–72, 2012.
- 117.J.M.M. van Rooij, H.L. Bodlaender, Exact algorithms for dominating set. *Discr. Appl. Math.*, 159, 2147–2164, 2011.
- 118.L. Ruan, H. Du, X. Jia, W. Wu, Y. Li, K.-I. Ko, A greedy approximation for minimum connected dominating sets. *Theor. Comp. Sci.*, 329(1–3), 325–330, 2004.
- 119.O. Schaudt, R. Schrader, The complexity of connected dominating sets and total dominating sets with specified induced subgraphs. *Inf. Proc. Lett.*, 112, 953–957, 2012.
- 120.W. Shang, F. Yao, P. Wan, X. Hu, On minimum m -connected k -dominating set problem in unit disc graph. *J. of Comb. Optim.*, 16(2), 99–106, 2008.
- 121.T. Shi, S. Cheng, Z. Cai, Y. Li, J. Li, Exploiting connected dominating sets in energy harvest networks. *IEEE/ACM Trans. on Networking*, 25(3), 1803–1817, 2017.
- 122.Y. Shi, Z. Zhang, D.-Z. Du, Approximation algorithm for minimum weight (k, m) -CDS problem in unit disk graph. *Electr. prepr.*, 18 p., Jan. 4, 2019. <http://arxiv.org/abs/1508.005515> [cs.DM]
- 123.L. Simonetti, A.S. da Cunha, A. Lucena, The minimum connected dominating set problem: formulation, valid inequalities and a Branch-and-Bound algorithm. In: *INOC 2011, LNCS 6701*, Springer, pp. 162–169, 2011.
- 124.I. Stojmenovic, M. Seddigh, J. Zunic, Dominating sets and neighbor elimination-based broadcasting algorithms in wireless networks. *IEEE Trans. on Paral. and Distr. Syst.*, 13(1), 14–25, 2002.
- 125.X. Sun, Y. Yang, M. Ma, Minimum connected dominating set algorithms for Ad Hoc networks. *Sensors*, Vol. 2019, 19(8), art. 1919, 2019.
- 126.S. Surendran, S. Vijayan, Distributed computation of connected dominating set for multi-hop wireless networks. *Procedia CS*, 63, 482–487, 2015.
- 127.A. Suzuki, A.E. Mouawad, N. Nishimura, Reconfiguration of dominating sets. *J. of Combin, Optim.*, 32(4), 1182–1195, 2016.
- 128.M. Thai, N. Zhang, R. Tiwari, X. Xu, On approximation algorithms of k -connected m -dominating sets in disk graphs. *Theor. Computer Science*, 385(1–3), 49–59, 2007.
- 129.Y.T. Tsai, Y.L. Lin, F.R. Hsu, Efficient algorithms for the minimum connected domination on trapezoid graphs. *Inform. Sci.* 177(12), 2405–2417, 2007.
- 130.F.J. Vazquez-Araujo, A. Dapena, M.J.S. Salorio, P.-M. Castro-Castro, Calculation of the connected dominating set considering vertex importance metrics. *Entropy*, 20(2), art. 87, 2018.
- 131.P.-J. Wan, K.M. Alzoubi, A simple heuristic for minimum connected dominating set in graphs. *Int. J. of Foundations of Computer Science*, 14(2), 323–333, 2003.
- 132.P.-J. Wan, L. Wang, F. Yao, Two-phase approximation algorithms for minimum CDS in wireless ad hoc networks. In: *IEEE ICDCS*, pp. 337–344, 2008.
- 133.F. Wang, E. Camacho, K. Xu, Positive influence dominating set in social networks. *Theor. Comp Sci.*, 412(3), 265–269, 2011.
- 134.Z. Wang, W. Wang, J.-M. Kim, B. Thuraisingham, W. Wu, PTAS for the minimum weighted domi-

- nating set in growth bounded graphs. *J. of Global Optim.*, 54(3), 641–648, 2012.
- 135.Y. Wang, W. Wang, X.-Y. Li, Weighted connected dominating set. In: M.-Y. Kao (ed), *Encyclopedia of Algorithms*, Springer, pp. 2359–2363, 2016.
- 136.J. Wu, H. Li, A dominating set based routing scheme in Ad Hoc wireless sensor networks. *Telecommunication Systems*, 18(1–3), 13–36, 2001,
- 137.J. Wu, W. Lou, Extended multipoint relays to determine connected dominating sets in MANETs. *IEEE Trans. on Computers*, 55, 334–347, 2006.
- 138.Y.-F. Wu, Y.-L. Xu, G.-L. Chen, Approximation algorithms for Steiner connected dominating set. *J. of Computer Science and Technology*, 20(5), 713–716, 2005.
- 139.W. Wu, H. Du, X. Jia, Y. Li, S.C.-H. Huang, Minimum connected dominating sets and maximal independent sets in unit disk graphs. *Theor. Comp. Sci.*, 352(1–3), 1–7, 2006.
- 140.Y. Wu, Y. Li, Connecting dominating sets. In: H. Liu, Y.W. Leung, X. Chu (eds), *Handbook of Ad Hoc and Sensor Wireless Networks: Architecture, Algorithms and Protocols*, pp. 19–39, 2009.
- 141.Y. Wu, X. Gao, Y. Li, A framework of distributed indexing and data dissemination in large scale wireless sensor networks. *Optim. Lett.*, 4(3), 335–345, 2010.
- 142.L. Wu, H. Du, W. Wu, Y. Hu, A. Wang, W. Lee, PTAS for routing-cost constrained minimum connected dominating set in growth bounded graphs. *J. of Comb. Optim.*, 30(1), 18–26, 2015.
- 143.R.R. Yager, On the theory of bags. *Int. J. of General Systems*, 23–37, 1986.
- 144.M. Yannakakis, F. Gavril, Edge dominating sets in graphs. *SIAM J. on Applied Math.*, 38(3), 364–372, 1980.
- 145.H.-Y. Yang, C.-H. Lin, M.-J. Tsai, Distributed algorithm for efficient construction and maintenance of connected k -hop dominating set in mobile ad hoc networks. *IEEE Trans. on Mobile Computing*, 7, 444–457, 2008.
- 146.J.Y. Yu, P.H.J. Chong, A survey of clustering schemes for mobile Ad Hoc networks. *IEEE Commun. Surveys & Tutorials*, 7(1), 32–47, First Quarter 2005.
- 147.R. Yu, X. Wang, S.K. Das, EEDTC: energy-efficient dominating tree construction in multi-hop wireless networks. *Pervasive and Mobile Computing*, 5(4), 318–333, 2009.
- 148.J. Yu, N. Wang, G. Wang, Constructing minimum extended weakly-connected dominating sets for clustering in ad hoc networks. *J. Parallel Distrib. Comput.*, 72(1), 35–47, 2012.
- 149.J. Yu, N. Wang, G. Wang, D. Yu, Connected dominating sets in wireless ad hoc and sensor networks – a comprehensive survey. *Computer Communications*, 36(2), 121–134, 2013.
- 150.Z. Zhang, X. Gao, W. Wu, D.-Z. Du, A PTAS for minimum connected dominating set in 3-dimensional wireless sensor networks. *J. Global Optimization*, 45, 451–458, 2009.
- 151.Z. Zhang, J. Zhou, X. Huang, D.-Z. Du, Performance guaranteed approximation algorithm for minimum k -connected m -fold dominating set. *Electr. prepr.*, 14 p., Aug. 27, 2016. <http://arxiv.org/abs/1608.07634> [cs.DM]
- 152.Y. Zhao, Z. Liao, L. Miao, On the algorithmic complexity of edge total domination. *Theor. Comp. Sci.*, 557, 28–33, 2014.
- 153.C. Zheng, L. Yin, S. Sun, Construction of d -hop connected dominating sets in wireless sensor networks. *Procedia Engineering*, 15, 3416–3420, 2011.
- 154.J. Zhou, Z. Zhang, W. Wu, K. Xing, A greedy algorithm for the fault-tolerant connected dominating set in a general graph. *J. of Combinatorial Optimization*, 28(1), 310–319, 2014.
- 155.F. Zou, Y. Wang, X.-H. Xu, X. Li, H. Du, P. Wan, W. Wu, New approximations for minimum-weighted dominating sets and minimum-weighted connected dominating sets on unit-disk graphs. *Theor. Comp. Sci.*, 412(3), 198–208, 2011.

Author work address: Mark Sh. Levin, Inst. for Information Transmission Problem, Russian Academy of Sciences, 19 Bolshoy Karetny lane, Moscow 127051, Russia

<http://www.mslevin.iitp.ru/> email: mslevin@acm.org

Author home address: Mark Sh. Levin, Sumskey Proezd 5-1-103, Moscow 117208, Russia