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On the weight distribution of the cosets of MDS codes

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Abstract. The weight distribution of the cosets of maximum distance separable (MDS) codes is considered. In 1990, P.G. Bonneau proposed a relation to obtain the full weight distribution of a coset of an MDS code with minimum distance d using the known numbers of vectors of weights $\leq d-2$ in this coset. In this paper, the Bonneau formula is transformed to a more structured and convenient form. The new version of the formula allows to consider effectively cosets of distinct weights W. (The weight W of a coset is the smallest Hamming weight of any vector in the coset.) For each of the considered W or regions of W, special relations more simple than the general one are obtained. We proved that all the MDS code cosets of weight $W = 1$ (as well as $W = d-1$) have the same weight distribution. The cosets of weight $W = 2$ or $W = d-2$ may have non-identical weight distributions; in this case, we proved that the distributions are symmetrical in some sense. The weight distributions of cosets of MDS codes corresponding to arcs in the projective plane $\mathrm{PG}(2, q)$ are also considered. For MDS codes of covering radius $R = d-1$ we obtained the number of the weight $W = d-1$ cosets and their weight distribution that gives rise to a certain classification of the so-called deep holes. We show that any MDS code of covering radius $R = d-1$ is an almost perfect multiple covering of the farthest-off points (deep holes); moreover, it corresponds to a multiple saturating set in the projective space $\mathrm{PG}(N, q)$.

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1 Introduction. The main results

Let $\mathbb{F}_q$ be the Galois field with q elements, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let $\mathbb{F}_q^n$ be the space of n-dimensional vectors over $\mathbb{F}_q$. We denote by $[n,k,d]_qR$ an $\mathbb{F}_q$-linear code of length n, dimension k, minimum distance d, and covering radius R. In this notation, one may omit R and call "minimum distance" simply "distance". If $d = n-k+1$, it is a maximum distance separable (MDS) code. The generalized Reed-Solomon (GRS) codes, including generalized doubly-extended Reed-Solomon (GDRS) codes and generalized triple-extended Reed-Solomon (GTRS) codes, form an important class of MDS codes. For an introduction to coding theory, see [9,27,34,36]. For preliminaries, see Section 2.2.

Let PG(N,q) be the N-dimensional projective space over $\mathbb{F}_q$. An *n-arc* in PG(N,q) is a set of n points such that no $N+1$ points belong to the same hyperplane of PG(N,q). An n-arc is complete if it is not contained in an $(n+1)$-arc. Arcs and MDS codes are equivalent objects, see e.g. [3,19,32,34]. For an introduction to projective spaces over finite fields and connections between projective geometry, coding theory, and combinatorics, see [2,3,19,23–25,32].

A *coset* of a code is a translation of the code. A coset $\mathcal{V}$ of an $[n,k,d]_q$ code $\mathcal{C}$ can be represented as

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{F}_q^n \,|\, \mathbf{x} = \mathbf{c} + \mathbf{v}, \mathbf{c} \in \mathcal{C}\} \subset \mathbb{F}_q^n \tag{1.1}$$

where $\mathbf{v} \in \mathcal{V}$ is a vector fixed for the given representation and $\mathbf{c}$ is a codeword; see e.g. [9,27,34,36]. The *weight W of a coset* is the smallest Hamming weight of any vector in the coset. For preliminaries, see Section 2.1.

The weight distribution of the code cosets, their classification, the number of the cosets with distinct distributions, are interesting by themselves; they are important combinatorial properties of a code. In particular, the weight distribution serves to estimate decoding results. Knowledge of the weight distributions of the code cosets gives information on the distance distribution of the code itself. There are many papers connected with distinct aspects of the weight distribution of code cosets, see e.g. [1, 9, 10, 12–16], [17, Section 6.3], [18,22], [27, Section 7], [28,30,31,33], [34, Sections 5.5, 6.6, and 6.9], [37,39,40], and the references therein.

In [13], for $[n,k,d]_q$ MDS codes, the integrated weight distribution for the union of all cosets of weight $W \le \lfloor (d-1)/2 \rfloor$ is obtained. In [14], using results of [13], the weight distributions for the unions of all cosets of weight 1 and all cosets of weight 2 of MDS codes are given. In [15], the weight distributions of all cosets (without any union) of the GDRS codes of codimension 4 are obtained.

In a few works, see e.g. [10, 16, 27, 34], it is shown that for an MDS code of distance d, the weight distribution of any coset is uniquely determined if, in the coset, the numbers of vectors of weights $1, 2, \ldots, d-2$ are known. Methods to obtain the weight distribution of the coset using these $d-2$ known numbers are considered in [10], [27, Section 7]. The approach of [27], applied in [15], can be used for distinct codes, not necessary MDS.

For MDS codes, the approach of [10] is more simple that the one of [27]. In Bonneau's paper [10], it is proposed a remarkable direct relation between the $d-2$ known numbers of vectors in a coset and the full weight distribution of this cosets, see (2.2)–(2.4) in Section 2.2. However, in the literature, as far as it is known to the authors, the Bonneau formula is not developed and applied. Note also that the Bonneau formula seems to be slightly "non-transparent"; its application to specific cases not always gives impressive results.

In this paper, we transform the Bonneau formula (2.2)–(2.4) of [10] to a more structured and convenient form (1.2), (1.3). Our result is given in Theorem 1.1 based on transformations in Section 3.

Theorem 1.1. (Bonneau formula transformed) *Let $\mathcal{C}$ be an $[n, k, d]_q$ MDS code. Let $\mathcal{V}$ be one of its cosets. Let $A_w(\mathcal{C})$ be the number of weight w codewords of $\mathcal{C}$. Let $B_w(\mathcal{V})$ be the number of weight w vectors in the coset $\mathcal{V}$. Assume that all the values of $B_v(\mathcal{V})$ with $0 \leq v \leq d-2$ are known. Then, for $w \geq d-1$, the weight distribution of $\mathcal{V}$ is as follows:*

$$B_w(\mathcal{V}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \sum_{v=0}^{d-2} \Omega_w^{(v)}(\mathcal{C}) B_v(\mathcal{V}), \;\; w = d-1, d, \ldots, n, \tag{1.2}$$

where

$$\Omega_w^{(v)}(\mathcal{C}) = (-1)^{w-d} \binom{n-v}{w-v} \binom{w-1-v}{d-2-v}. \tag{1.3}$$

Comparing the original Bonneau's formula (2.2)–(2.4) of [10], see Section 2.2, with the new version (1.2), (1.3) of Theorem 1.1, we see that the double sum of (2.4) is transformed to a single one in (1.2) with regular combinatorial coefficients $\Omega_w^{(v)}(\mathcal{C})$ for $B_v(\mathcal{V})$. Also, now the number $A_w(\mathcal{C})$ of codewords of the given weight is directly included in the relation for the corresponding value $B_w(\mathcal{V})$ for a coset.

The new version (1.2), (1.3) of the formula allows us to consider effectively specific cases of the weight distribution of the cosets of $[n, k, d]_q$ MDS codes, see Section 4. We consider separately cosets of weights W with $W = 1$, $2 \leq W \leq \lfloor (d-1)/2 \rfloor$, $\lfloor (d+1)/2 \rfloor \leq W \leq d-3$, $W = d-2$, $W = d-1$. For each of these W or regions of W we obtain special relations more simple than (1.2), (1.3). For the weight 1 and weight $d-1$ cosets we obtain formulas of the weight distributions depending only on code parameters n, d, q. This means that *all the MDS code cosets of weight* 1 (*as well as* $d-1$) *have the same* *weight distribution*. The weight $d-2$ cosets may have non-identical weight distributions; in this case, the distributions, as we prove, are symmetrical in some sense.

Also, we consider the weight distribution of cosets of MDS codes of distance $d=3,4$, see Section 5. Here we use the connections of these codes with the conics and hyperovals in the projective plane $\mathrm{PG}(2,q)$.

The weight distribution of the weight 2 cosets of MDS codes of distance $d\geq 5$ is considered in detail, see Section 6. We show that the distribution is uniquely determined by the number of weight $d-2$ vectors in this coset. Formulas for identical and non-identical weight distributions are obtained. A necessary condition for identity of weight distributions is proved. Again, as well as for weight $d-2$ cosets, we prove that non-identical weight distributions of weight 2 cosets are symmetrical.

The *symmetry of non-identical weight distributions* of weight 2 and weight $d-2$ cosets of MDS codes, proved in this paper, is interesting and slightly unexpected.

In coding theory, *farthest-off points* or *deep holes*, i.e. vectors of $\mathbb{F}_q^n$ lying at distance R from an $[n,k,d]_qR$ code, play an important role. There are useful relations between deep holes and the bounds on the size of the lists in the list decoding of GRS codes, see e.g. [26,29,30], [36, Chapter 9], [39,40] and the references therein. In particular, in [30,40], the classification of deep holes of GRS codes with redundancy (codimension) $n-k=3$ and 4 is considered.

In this context, linear multiple covering codes, called *multiple coverings of the farthest-off points* (*MCF codes for short*) or *multiple coverings of deep holes*, are of great interest. For an introduction to this topic, including a one-to-one correspondence between MCF codes and multiple saturating sets in the spaces $\mathrm{PG}(N,q)$, see [4,5,7,11] and the references therein. For preliminaries with the definitions of perfect and almost perfect MCF codes, see Section 7.1.

We show, see Section 7.2, that any $[n,k,d]_qR$ MDS code of maximal possible covering radius $R=d-1$ is an almost perfect MCF code such that for each farthest-off vector $\mathbf{x}\in\mathbb{F}_q^n$ there are exactly $\binom{n}{d-1}$ codewords at Hamming distance $d-1$ from $\mathbf{x}$. It is important that MDS codes of covering radius $R=d-1$ are a wide class of codes, see Theorem 7.6.

The weight distribution of the farthest-off cosets of weight $d-1$ given by Theorem 4.6, and the number of the such cosets given by Theorem 7.6, can be considered as the *classification of the farthest-off vectors* (*deep holes*) of MDS codes.

Note that MDS codes of covering radius $R=d-1$ provide almost perfect MCF codes with any covering radius R and the corresponding multiple (ρ,μ)-saturating sets with any parameter ρ, see Section 7. In the literature, as far as it is known to the authors, almost perfect MCF codes with $R>3$ are not described.

So, we consider the weight distributions of the cosets of MDS codes and some related problems. The results obtained extend the knowledge on cosets of MDS codes.

The paper is organized as follows. Section 2 contains preliminaries. In Section 3 we transform the original Bonneau's formula (2.2)–(2.4) to the new version (1.2), (1.3). In Section 4, we consider specific cases of the weight distribution of the cosets of MDS codes. In Section 5, the weight distribution of cosets of MDS codes of distance $d = 3, 4$ is investigated. In Section 6, the weight distribution of the weight 2 cosets of MDS codes of distance $d \geq 5$ is considered. In Section 7, we study MDS codes as multiple coverings of deep holes and the corresponding multiple saturating sets.

2 Preliminaries

We introduce notations and remind known definitions and properties connected with linear codes, their cosets, and the normal rational curves in the spaces $\mathrm{PG}(N, q)$, see [2, 3, 6, 8–10, 16, 20, 21, 27, 34, 36] and the references therein.

2.1 Cosets of a linear code

Notation 1. For an $[n, k, d]_q$ code $\mathcal{C}$ and its cosets $\mathcal{V}$ of the form (1.1), we use the following notations and definitions:

$t(\mathcal{C}) = \lfloor (d-1)/2 \rfloor$	the number of errors correctable by the code $\mathcal{C}$;
weight of a vector	Hamming weight of the vector;
$wt(\mathbf{x})$	the Hamming weight of a vector $\mathbf{x} \in \mathbb{F}_q^n$;
$\#M$	the cardinality of a set M;
$A_w(\mathcal{C})$	the number of weight w codewords of the code $\mathcal{C}$;
$S(\mathcal{C})$	the set of non-zero weights in $\mathcal{C}$; $S(\mathcal{C}) = \{w > 0 \mid A_w(\mathcal{C}) \neq 0\}$;
$s(\mathcal{C}) = \#S(\mathcal{C})$	the number of non-zero weights in $\mathcal{C}$;
$\mathcal{C}^{\perp}$	the $[n, n-k, d^{\perp}]_q$ code dual to $\mathcal{C}$;
$\mathbf{c}_w$	weight w codeword of $\mathcal{C}$; $wt(\mathbf{c}_w) = w$;
$B_w(\mathcal{V})$	the number of weight w vectors in the coset $\mathcal{V}$;
a coset leader	a vector in the coset having the smallest Hamming weight;
the weight of a coset	the smallest Hamming weight of any vector in the coset;
$\mathcal{V}^{(W)}$	a coset of weight W; $B_w(\mathcal{V}^{(W)}) = 0$ if $w < W$;
$\mathbb{N}_{\Sigma}^{(W)}(\mathcal{C})$	the total number of the cosets of weight W of the code $\mathcal{C}$;
$\mathscr{B}_w^{\Sigma}(\mathcal{V}^{(W)})$	the overall number of weight w vectors in all the cosets of weight W;
$\mathbf{v}_w$	weight w vector of $\mathbb{F}_q^n$; $wt(\mathbf{v}_w) = w$;

$\mathbf{v}+\mathcal{C}$	the coset of $\mathcal{C}$ of the form (1.1);
$H(\mathcal{C})$	an $(n-k)\times n$ parity check matrix of $\mathcal{C}$;
tr	sign of the transposition;
$H(\mathcal{C})\mathbf{x}^{tr}$	the *syndrome* of a vector $\mathbf{x}\in\mathbb{F}_q^n$, $H(\mathcal{C})\mathbf{x}^{tr}\in\mathbb{F}_q^{n-k}$;
$d(\mathbf{x},\mathbf{c})$	the Hamming distance between vectors $\mathbf{x}$ and $\mathbf{c}$ of $\mathbb{F}_q^n$;
$d(\mathbf{x},\mathcal{C})=\min\limits_{\mathbf{c}\in\mathcal{C}} d(\mathbf{x},\mathbf{c})$	the Hamming distance between vector $\mathbf{x}\in\mathbb{F}_q^n$ and $\mathcal{C}$;
$f_\delta(\mathbf{x},\mathcal{C})$	for vector $\mathbf{x}\in\mathbb{F}_q^n$, the number of codewords $\mathbf{c}\in\mathcal{C}$ such that $d(\mathbf{x},\mathbf{c})=\delta$.

For a coset $\mathcal{V}^{(W)}$ of weight W the number of all coset leaders is $B_W(\mathcal{V}^{(W)})$. If $W\le t(\mathcal{C})$ we have a unique leader and $B_W(\mathcal{V}^{(W)})=1$. Also, for an $[n,k,d]_q$ code $\mathcal{C}$ the following holds:

$$\mathbb{N}_\Sigma^{(W)}(\mathcal{C})=\binom{n}{W}(q-1)^W \text{ if } W\le t(\mathcal{C})=\left\lfloor\frac{d-1}{2}\right\rfloor. \tag{2.1}$$

If $W>t(\mathcal{C})$, then $B_W(\mathcal{V}^{(W)})\ge 1$, i.e. a vector of minimal weight is not necessarily unique. The code $\mathcal{C}$ is the coset of weight zero. The leader of $\mathcal{C}$ is the zero vector of $\mathbb{F}_q^n$. The distinct cosets of $\mathcal{C}$ partition $\mathbb{F}_q^n$ into q^{n-k} subsets of size q^k.

All vectors in a code coset have the same syndrome; it is called the *coset syndrome*. Thus, there is a one-to-one correspondence between cosets and syndromes. The syndrome of $\mathcal{C}$ is the zero vector of $\mathbb{F}_q^{n-k}$.

The covering radius of a linear $[n,k,d]_q$ code $\mathcal{C}$ is the least integer R such that the space $\mathbb{F}_q^n$ is covered by Hamming spheres of radius R centered at the codewords. Every column of $\mathbb{F}_q^{n-k}$ is equal to a linear combination of at most R columns of a parity check matrix of $\mathcal{C}$. The covering radius R of the code $\mathcal{C}$ is equal to the maximum weight of a coset of $\mathcal{C}$.

Theorem 2.1. (i) [27, Lemma 7.5.1] *For a code $\mathcal{C}$, the weight distributions of all cosets $\alpha\mathbf{v}+\mathcal{C}$, $\alpha\in\mathbb{F}_q^*$, are identical.*

(ii) [16], [27, Theorem 7.5.2] *For the covering radius R of a code $\mathcal{C}$, we have $R\le s(\mathcal{C}^\perp)$.*

(iii) [27, Theorem 7.5.2] *Each weight $s(\mathcal{C}^\perp)$ coset of $\mathcal{C}$ has the same weight distribution.*

(iv) [27, Theorem 7.5.2], [34, Theorem 6.20] *For a code $\mathcal{C}$, the weight distribution of any coset of weight $<s(\mathcal{C}^\perp)$ is uniquely determined if, in the coset, the numbers of vectors of weights $1,2,\ldots,s(\mathcal{C}^\perp)-1$ are known.*

(v) [10,16] *For an MDS code of distance d, the weight distribution of a coset is uniquely determined if, in the coset, the numbers of vectors of weights $1,2,\ldots,d-2$ are known.*

Note that using Theorem 2.2 of Section 2.2, one can show that, in Theorem 2.1, the point (v) is a special case of (iv).

2.2 MDS, GDRS, and GTRS codes; normal rational curves

Theorem 2.2. [20, Theorems 6 and 10] *Any $[n \le q, k, n-k+1]_q$ MDS code has k nonzero weights. Any $[q+1, k, q+2-k]_q$ MDS code has k nonzero weights, except when $k = 2$.*

Theorem 2.3. (Bonneau formula) [10] *Let $\mathcal{C}$ be an $[n, k, d = n-k+1]_q$ MDS code. Let $\mathcal{V}$ be one of its coset. Assume that all the values of $B_v(\mathcal{V})$ with $0 \le v \le d-2$ are known. Then, for $w \ge d-1$, the weight distribution of $\mathcal{V}$ is as follows:*

$$B_w(\mathcal{V}) = \mathfrak{B}_{w,1}(\mathcal{V}) + \mathfrak{B}_{w,2}(\mathcal{V}),\ \ w \ge d-1, \tag{2.2}$$

where

$$\mathfrak{B}_{w,1}(\mathcal{V}) = \binom{n}{w} \sum_{j=0}^{w-d+1} (-1)^j \binom{w}{j} q^{w-d+1-j}, \tag{2.3}$$

$$\mathfrak{B}_{w,2}(\mathcal{V}) = \sum_{j=w-d+2}^{w} (-1)^j \sum_{v=0}^{w-j} \binom{j+n-w}{j} \binom{n-v}{w-j-v} B_v(\mathcal{V}). \tag{2.4}$$

For an $[n, k, d]_q R$ MDS code we have $R \le d-1$, see e.g. [8, 21].

For $w \ge d$, the weight distribution of an $[n, k, d = n-k+1]_q$ MDS code $\mathcal{C}$ has the following form, see e.g. [27, Theorem 7.4.1], [34, Theorem 11.3.6]:

$$A_w(\mathcal{C}) = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1). \tag{2.5}$$

For q odd and even, a $(d-1) \times (q+1)$ parity check matrix H_d of the $[q+1, q+2-d, d]_q$ GDRS code with $d \ge 3$ can be represented [36, Sect. 5.1] as

$$H_d = \begin{bmatrix} 1 & 1 & \dots & 1 & 1 & 0 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{q-1} & 0 & 0 \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_{q-1}^2 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_1^{d-3} & \alpha_2^{d-3} & \dots & \alpha_{q-1}^{d-3} & 0 & 0 \\ \alpha_1^{d-2} & \alpha_2^{d-2} & \dots & \alpha_{q-1}^{d-2} & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 & 0 & \dots & 0 & 0 \\ 0 & v_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v_q & 0 \\ 0 & 0 & \dots & 0 & v_{q+1} \end{bmatrix}, \tag{2.6}$$

where $\alpha_i \in \mathbb{F}_q^*$, $\alpha_i \ne \alpha_j$ if $i \ne j$, $v_i \in \mathbb{F}_q^*$, the elements v_i do not have to be distinct.

If, from H_d, we remove the column $[0,0,\ldots,v_{q+1}]^{tr}$, we obtain a parity check matrix of the *singly-extended* $[q,q+1-d,d]_qR$ GRS code of covering radius $R=d-1$. If, from H_d, we remove the columns $[0,0,\ldots,v_{q+1}]^{tr}$, $[v_q,0,\ldots,0]^{tr}$, and also $\delta\geq 0$ other columns, we obtain a parity check matrix of a $[q-1-\delta,q-\delta-d,d]_qR$ GRS code with $R=d-1$.

For q even, a $3\times(q+2)$ parity check matrix $\widetilde{H_4}$ of the $[q+2,q-1,4]_q2$ GTRS code can be represented as

$$\widetilde{H_4}=\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 & 0 & 0\\ \alpha_1 & \alpha_2 & \ldots & \alpha_{q-1} & 0 & 0 & 1\\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_{q-1}^2 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 & 0 & 0\\ 0 & v_2 & 0\\ 0 & 0 & v_3\end{bmatrix}, \tag{2.7}$$

where $\alpha_i\in\mathbb{F}_q^*$, $\alpha_i\neq\alpha_j$, if $i\neq j$, $v_i\in\mathbb{F}_q^*$, the elements v_i do not have to be distinct.

In $\mathrm{PG}(N,q)$, $2\leq N\leq q-2$, a *normal rational curve* is any $(q+1)$-arc projectively equivalent to the arc $\{(1,t,t^2,\ldots,t^N):t\in\mathbb{F}_q\}\cup\{(0,\ldots,0,1)\}$ where $(1,t,\ldots,t^N)$ and $(0,\ldots,0,1)$ are points in homogeneous coordinates. The points of a normal rational curve in $\mathrm{PG}(N,q)$ treated as columns define a parity check matrix of a $[q+1,q-N,d=N+2]_qR$ GDRS code [19, 32, 36].

If in $\mathrm{PG}(N,q)$, the normal rational curve is a *complete* $(q+1)$*-arc*, then the corresponding $[q+1,q-N,d=N+2]_qR$ GDRS code $\mathcal{C}$ cannot be extended to a $[q+2,q-N+1,d=N+2]_q$ MDS code, i.e. $\mathcal{C}$ has covering radius $R=d-2$. In this case, we know the exact number $\mathbb{N}(\mathcal{V}^{(d-1)})=(q-1)(q+1-n)$ of weight $d-1$ cosets $\mathcal{V}^{(d-1)}$ of an $[n<q+1,n-N-1,N+2]_qR$ GRS code with a parity check matrix consisting of n columns of H_d (2.6), see Theorem 7.6(i).

The following conjectures are well known.

Conjecture 2.4. *Let* $2\leq N\leq q-2$. *In* $\mathrm{PG}(N,q)$, *every normal rational curve is a complete* $(q+1)$*-arc except for the cases when* q *is even and* $N\in\{2,q-2\}$, *in which one point can be added to the curve.*

Conjecture 2.5. (MDS conjecture) *Let* $2\leq N\leq q-2$. *An* $[n,n-N-1,N+2]_q$ *MDS code (or, equivalently, an* n*-arc in* $\mathrm{PG}(N,q)$*) has length* $n\leq q+1$ *except for the cases when* q *is even and* $N\in\{2,q-2\}$, *in which* $n\leq q+2$.

If the MDS conjecture holds for some pair (N,q) then Conjecture 2.4 holds too, but in general, the reverse is not true. For the pairs (N,q) for which MDS conjecture is proved, see [3] and the references therein.

For the pairs (N,q) for which Conjecture 2.4 is proved, see [6] and the references therein. Note that the results of [6] develop the approaches and results of [38].

3 Transformation of the Bonneau formula for the weight distribution of MDS code cosets

Throughout the paper we put that a sum $\sum_{i=A}^{B}\ldots$ is equal to zero if $B < A$.

We use the following combinatorial identities [35, Section 1, Equations (I), (III), (IV), Problem 9(a),(b)]:

$$\binom{h}{\ell} = \binom{h}{h-\ell}; \tag{3.1}$$

$$\binom{h}{\ell} = \binom{h-1}{\ell} + \binom{h-1}{\ell-1}; \tag{3.2}$$

$$\binom{h}{m}\binom{m}{p} = \binom{h}{p}\binom{h-p}{m-p} = \binom{h}{m-p}\binom{h-m+p}{p}; \tag{3.3}$$

$$\sum_{j=0}^{m}(-1)^j\binom{h}{j} = (-1)^m\binom{h-1}{m} = (-1)^m\binom{h-1}{h-1-m} = \binom{m-h}{m}; \tag{3.4}$$

$$\sum_{j=0}^{h-m}(-1)^j\binom{h}{m+j} = \binom{h-1}{m-1}. \tag{3.5}$$

We remind the notation (1.3); for an $[n, k, d]_q$ code $\mathcal{C}$, we denote

$$\Omega_w^{(j)}(\mathcal{C}) = (-1)^{w-d}\binom{n-j}{w-j}\binom{w-1-j}{d-2-j},\ 0 \le w \le n,\ 0 \le j \le \min\{w-1, d-2\}. \tag{3.6}$$

We note some particular cases of (3.6):

$$\Omega_w^{(j)}(\mathcal{C}) = 0 \text{ if } w \le d-2; \tag{3.7}$$

$$\Omega_w^{(0)}(\mathcal{C}) = (-1)^{w-d}\binom{n}{w}\binom{w-1}{d-2}; \tag{3.8}$$

$$\Omega_{d-1}^{(j)}(\mathcal{C}) = -\binom{n-j}{d-1-j},\ 0 \le j \le d-2; \tag{3.9}$$

$$\Omega_w^{(d-2)}(\mathcal{C}) = (-1)^{w-d}\binom{n-d+2}{w-d+2} = (-1)^{w-d}\binom{n-d+2}{n-w} \text{ if } w \ge d-1. \tag{3.10}$$

Lemma 3.1. *Let $\mathcal{C}$ be an $[n, k, d]_q$ MDS code. Let $\mathcal{V}$ be one of its cosets. Let $\mathfrak{B}_{w,1}(\mathcal{V})$ be as in (2.3). Then*

$$\mathfrak{B}_{w,1}(\mathcal{V}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}),\ w \ge d-1. \tag{3.11}$$

Proof. We write $\mathfrak{B}_{w,1}(\mathcal{V})$ of (2.3) as

$$\binom{n}{w}\sum_{j=0}^{w-d}(-1)^j\binom{w}{j}\left(q^{w+1-d-j} - 1 + 1\right) + \binom{n}{w}\cdot(-1)^{w-d+1}\binom{w}{w-d+1}$$

$$= \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} \left(q^{w+1-d-j} - 1\right) + \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j}$$
$$- \binom{n}{w} \cdot (-1)^{w-d} \binom{w}{d-1}$$

where we apply (3.1) for the last summand. Now we use (2.5) and apply the 2-nd equality of (3.4) to $\sum_{j=0}^{w-d}(-1)^j \binom{w}{j}$. As a result, we have

$$\mathfrak{B}_{w,1}(\mathcal{V}) = A_w(\mathcal{C}) + \binom{n}{w} \cdot (-1)^{w-d} \binom{w-1}{d-1} - \binom{n}{w} \cdot (-1)^{w-d} \binom{w}{d-1}$$
$$= A_w(\mathcal{C}) - (-1)^{w-d} \binom{n}{w} \left(\binom{w}{d-1} - \binom{w-1}{d-1} \right)$$
$$= A_w(\mathcal{C}) - (-1)^{w-d} \binom{n}{w} \binom{w-1}{d-2}$$

where we apply (3.2) to $\binom{w}{d-1} - \binom{w-1}{d-1}$. Now the assertion follows by (3.8). ☐

Lemma 3.2. *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code. Let $\mathcal{V}$ be one of its cosets. Let $\mathfrak{B}_{w,2}(\mathcal{V})$ be as in (2.4). Assume that all the values of $B_v(\mathcal{V})$ with $0 \le v \le d-2$ are known. Then*

$$\mathfrak{B}_{w,2}(\mathcal{V}) = \sum_{v=0}^{d-2} \Omega_w^{(v)}(\mathcal{C}) B_v(\mathcal{V}), \;\; w \ge d-1. \tag{3.12}$$

Proof. By (3.1) and the 2-nd equality of (3.3), in $\mathfrak{B}_{w,2}(\mathcal{V})$ of (2.4), we have

$$\binom{j+n-w}{j} \binom{n-v}{w-j-v} = \binom{n-v}{j+n-w} \binom{j+n-w}{j} = \binom{n-v}{n-w} \binom{w-v}{j}$$

whence

$$\mathfrak{B}_{w,2}(\mathcal{V}) = \sum_{j=w-d+2}^{w} (-1)^j \sum_{v=0}^{w-j} \binom{n-v}{n-w} \binom{w-v}{j} B_v(\mathcal{V}). \tag{3.13}$$

In (3.13), the restriction for v is $w - v \ge j$ and the maximal possible value of v is $d-2$. Changing the order of summation in (3.13), we obtain

$$\mathfrak{B}_{w,2}(\mathcal{V}) = \sum_{v=0}^{d-2} \binom{n-v}{n-w} B_v(\mathcal{V}) \sum_{j=w-d+2}^{w} (-1)^j \binom{w-v}{j}$$

$$= \sum_{v=0}^{d-2} \binom{n-v}{n-w} B_v(\mathcal{V}) \sum_{j=w-d+2}^{w-v} (-1)^j \binom{w-v}{j} \tag{3.14}$$

where we take into account that $\binom{w-v}{j} = 0$ if $j > w - v$. Note, that in (3.13), for every $j = w - d + 2 + \delta$ the value of v runs over $0, \ldots, d - 2 - \delta$, and vice versa, in (3.14), for every $v = d - 2 - \delta$ the value of j runs over $w - d + 2, \ldots, w - d + 2 + \delta$. Thus, the sets of pairs (j, v) in (3.13) and (3.14) are the same.

Now, in (3.14), we change j by $i + w - d + 2$ and obtain

$$\begin{aligned}
\mathfrak{B}_{w,2}(\mathcal{V}) &= \sum_{v=0}^{d-2} \binom{n-v}{n-w} B_v(\mathcal{V}) \sum_{i=0}^{d-2-v} (-1)^{i+w-d+2} \binom{w-v}{i+w-d+2} \\
&= (-1)^{w-d} \sum_{v=0}^{d-2} \binom{n-v}{w-v} B_v(\mathcal{V}) \sum_{i=0}^{d-2-v} (-1)^{i+2} \binom{w-v}{i+w-d+2} \\
&= (-1)^{w-d} \sum_{v=0}^{d-2} \binom{n-v}{w-v} B_v(\mathcal{V}) \binom{w-v-1}{w-d+1}
\end{aligned}$$

where for the sum $\sum_{i=0}^{d-2-v} (-1)^{i+2} \binom{w-v}{i+w-d+2}$ we use $(-1)^{i+2} = (-1)^i$ and then apply (3.5) taking $h = w - v$, $m = w - d + 2$. Finally, by (3.1), we have $\binom{w-v-1}{w-d+1} = \binom{w-1-v}{d-2-v}$ and the assertion follows by (3.6). □

Proof of Theorem 1.1. Substituting (3.11) and (3.12) to (2.2) we obtain the new version (1.2), (1.3) of the Bonneau formula and thereby prove Theorem 1.1.

4 Specific cases of the weight distribution of cosets of MDS codes

We consider specific applications of the new version (1.2) of the Bonneau formula.

4.1 A coset is the code itself

We consider the coset $\mathcal{V}^{(0)}$ of weight 0, i.e. the code itself. In this case $B_v(\mathcal{V}^{(0)}) = 0$ if $1 \le v \le d - 1$, $B_0(\mathcal{V}^{(0)}) = 1$. By (1.2), we have $B_w(\mathcal{V}^{(0)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(0)}(\mathcal{C}) = A_w(\mathcal{C})$. This expected answer says on universalism of the relation (1.2).

4.2 The cosets $\mathcal{V}^{(1)}$ of weight 1 for MDS codes of distance $d \geq 3$

Theorem 4.1. *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code of distance $d \geq 3$. Then all $n(q-1)$ its cosets $\mathcal{V}^{(1)}$ of weight 1 have the same weight distribution $B_w(\mathcal{V}^{(1)})$ of the form:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(1)}) = 0 \text{ if } w \in \{0,1,\ldots,d-2\} \setminus \{1\},\ B_1(\mathcal{V}^{(1)}) = 1, \\
&B_{d-1}(\mathcal{V}^{(1)}) = \binom{n-1}{d-1}, \\
&B_w(\mathcal{V}^{(1)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(1)}(\mathcal{C}) \text{ if } w = d, d+1, \ldots, n.
\end{aligned} \tag{4.1}$$

Proof. The total number $\mathbb{N}_\Sigma^{(1)}(\mathcal{C}) = n(q-1)$ of weight 1 cosets follows from (2.1). In (4.1), the 1-st two equalities are obvious. Due to them, in (1.2), we have

$$\sum_{v=0}^{d-2} \Omega_w^{(v)}(\mathcal{C}) B_v(\mathcal{V}^{(1)}) = \Omega_w^{(1)}(\mathcal{C}) B_1(\mathcal{V}^{(1)}) = \Omega_w^{(1)}(\mathcal{C})$$

that provides the last relation of (4.1) in the region $w = d-1, d, d+1, \ldots, n$. Now we put $w = d-1$, take $A_{d-1}(\mathcal{C}) = 0$ into account, use (3.9), and then apply (3.2). As a result,

$$B_{d-1}(\mathcal{V}^{(1)}) = -\Omega_{d-1}^{(0)}(\mathcal{C}) + \Omega_{d-1}^{(1)}(\mathcal{C}) = \binom{n}{d-1} - \binom{n-1}{d-2} = \binom{n-1}{d-1}. \qquad \square$$

4.3 The cosets $\mathcal{V}^{(W)}$ of weight $2 \leq W \leq \lfloor (d-1)/2 \rfloor$ for MDS codes of distance $d \geq 5$

Theorem 4.2. *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code of distance $d \geq 5$. Let $\mathcal{V}^{(W)}$ be one of its cosets of weight $2 \leq W \leq \lfloor (d-1)/2 \rfloor$. Assume that all the values of $B_v(\mathcal{V}^{(W)})$ with $d - W \leq v \leq d-2$ are known. Then the weight distribution $B_w(\mathcal{V}^{(W)})$ of $\mathcal{V}^{(W)}$ is as follows:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(W)}) = 0 \text{ if } w \in \{0,1,\ldots,d-W-1\} \setminus \{W\},\ B_W(\mathcal{V}^{(W)}) = 1, \\
&B_w(\mathcal{V}^{(W)}) \text{ are given if } w \in \{d-W, d-W+1, \ldots, d-2\}, \\
&B_w(\mathcal{V}^{(W)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(W)}(\mathcal{C}) + \sum_{v=d-W}^{d-2} \Omega_w^{(v)}(\mathcal{C}) B_v(\mathcal{V}^{(W)}) \\
&\text{if } w = d-1, d, \ldots, n.
\end{aligned} \tag{4.2}$$

Proof. The 1-st equality of (4.2) is obvious. The 2-nd one holds as the coset $\mathcal{V}^{(W)}$ has a unique leader of weight W. The 3-rd relation of (4.2) is by the hypothesis. Finally, the last equality follows from (1.2). Note, that $W \in \{2, 3, \ldots, d-W-1\}$. $\square$

4.4 The cosets $\mathcal{V}^{(W)}$ of weight $\lfloor (d+1)/2 \rfloor \le W \le d-3$ for MDS codes of distance $d \ge 6$

Theorem 4.3. *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code of distance $d \ge 6$. Let $\mathcal{V}^{(W)}$ be one of its cosets of weight $\lfloor (d+1)/2 \rfloor \le W \le d-3$. Assume that all the values of $B_v(\mathcal{V}^{(W)})$ with $d-W \le v \le d-2$ are known. Then the weight distribution $B_w(\mathcal{V}^{(W)})$ of $\mathcal{V}^{(W)}$ is as follows:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(W)}) = 0 \text{ if } w \in \{0,1,\ldots,d-W-1\}, \\
&B_w(\mathcal{V}^{(W)}) \text{ are given if } w \in \{d-W, d-W+1, \ldots, d-2\}, \\
&B_w(\mathcal{V}^{(W)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \sum_{v=d-W}^{d-2} \Omega_w^{(v)}(\mathcal{C}) B_v(\mathcal{V}^{(W)}) \text{ if } w = d-1, d, \ldots, n.
\end{aligned} \tag{4.3}$$

Proof. The 1-st equality of (4.3) is obvious. The 2-nd relation is by the hypothesis. Finally, the last equality follows from (1.2). Note, that $W \in \{d-W, d-W+1, \ldots, d-2\}$. □

4.5 The cosets $\mathcal{V}^{(d-2)}$ of weight $d-2$ for MDS codes of distance $d \ge 4$

Theorem 4.4. *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code of distance $d \ge 4$. Let $\mathcal{V}^{(d-2)}$ be one of its cosets of weight $d-2$. Assume that the value $B_{d-2}(\mathcal{V}^{(d-2)})$ is known. Then the weight distribution $B_w(\mathcal{V}^{(d-2)})$ of $\mathcal{V}^{(d-2)}$ is as follows:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(d-2)}) = 0 \text{ if } w = 0,1,\ldots,d-3, \; B_{d-2}(\mathcal{V}^{(d-2)}) \text{ is given}, \\
&B_w(\mathcal{V}^{(d-2)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(d-2)}(\mathcal{C}) B_{d-2}(\mathcal{V}^{(d-2)}) \\
&= A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + (-1)^{w-d} \binom{n-d+2}{n-w} B_{d-2}(\mathcal{V}^{(d-2)}) \text{ if } w = d-1, d, \ldots, n.
\end{aligned} \tag{4.4}$$

Proof. The 1-st relations of (4.4) follow from the hypothesis. The 2-nd and 3-rd equalities follows from (1.2), where the only non-zero term is $\Omega_w^{(d-2)}(\mathcal{C}) B_{d-2}(\mathcal{V}^{(d-2)})$, and from (3.10). □

Theorem 4.5. (symmetry of not identical weight distributions) *Let $\mathcal{C}$ be an $[n,k,d]_q$ MDS code with $d \ge 4$. Let $\mathcal{V}_a^{(d-2)}$ and $\mathcal{V}_b^{(d-2)}$ be its weight $d-2$ cosets with distinct weight distributions. Then independently of the values of $B_{d-2}(\mathcal{V}_a^{(d-2)})$ and $B_{d-2}(\mathcal{V}_b^{(d-2)})$, there is the following symmetry of the weight distributions:*

$$(-1)^{n+d} B_w(\mathcal{V}_a^{(d-2)}) - B_{n+d-2-w}(V_a^{(d-2)}) \tag{4.5}$$

$$= (-1)^{n+d} B_w(V_b^{(d-2)}) - B_{n+d-2-w}(V_b^{(d-2)}), \; w = d-1, d, \dots, n.$$

Proof. By (4.4) and by (3.1), we have

$$\begin{aligned}
&B_w(\mathcal{V}_a^{(d-2)}) - B_w(\mathcal{V}_b^{(d-2)}) \\
&= (-1)^{w-d}\binom{n-d+2}{n-w}\left[B_{d-2}(\mathcal{V}_a^{(d-2)}) - B_{d-2}(\mathcal{V}_b^{(d-2)})\right]; \\
&B_{n+d-2-w}(\mathcal{V}_a^{(d-2)}) - B_{n+d-2-w}(\mathcal{V}_b^{(d-2)}) \\
&= (-1)^{n-2-w}\binom{n-d+2}{-d+2+w}\left[B_{d-2}(\mathcal{V}_a^{(d-2)}) - B_{d-2}(\mathcal{V}_b^{(d-2)})\right] \\
&= (-1)^{n+d}(-1)^{w-d}\binom{n-d+2}{n-w}\left[B_{d-2}(\mathcal{V}_a^{(d-2)}) - B_{d-2}(\mathcal{V}_b^{(d-2)})\right].
\end{aligned}$$

□

4.6 The cosets $\mathcal{V}^{(d-1)}$ of weight $d-1$ for MDS codes of distance $d \geq 3$ and covering radius $R = d-1$

Theorem 4.6. *Let $\mathcal{C}$ be an $[n,k,d]_q R$ MDS code of distance $d \geq 3$ and covering radius $R = d-1$. Then all its cosets $\mathcal{V}^{(d-1)}$ of weight $d-1$ have the same weight distribution $B_w(\mathcal{V}^{(d-1)})$ of the form:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(d-1)}) = 0 \text{ if } w = 0,1,\dots,d-2; \; B_{d-1}(\mathcal{V}^{(d-1)}) = \binom{n}{d-1}; \qquad (4.6)\\
&B_w(\mathcal{V}^{(d-1)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) = A_w(\mathcal{C}) - (-1)^{w-d}\binom{n}{w}\binom{w-1}{d-2} \\
&\text{if } w = d, d+1, \dots, n.
\end{aligned}$$

Proof. For $\mathcal{V}^{(d-1)}$, in (1.2), we have $B_v(\mathcal{V}^{(d-1)}) = 0$ if $0 \leq v \leq d-2$ that provides the last relation of (4.6) in the region $w = d-1, d, d+1, \dots, n$. If $w = d-1$, we use (3.9) and take $A_{d-1}(\mathcal{C}) = 0$ into account. □

5 Arcs in the plane $\mathrm{PG}(2,q)$ and the weight distribution of cosets of MDS codes of distance $d = 3, 4$

Theorem 5.1. *Let $\mathcal{C}$ be an $[n < q+1, n-2, 3]_q 2$ shortened Hamming code with the parity check matrix consisting of n columns of H_3 (2.6). Then there are $q^2 - 1 - n(q-1)$ cosets $\mathcal{V}^{(2)}$ of weight 2; they all have the same weight distribution $B_w(\mathcal{V}^{(2)})$ of the following form:*

$$B_w(\mathcal{V}^{(2)}) = 0 \text{ if } w = 0,1; \; B_2(\mathcal{V}^{(2)}) = \binom{n}{2}; \qquad (5.1)$$

$$B_w(\mathcal{V}^{(2)}) = A_w(\mathcal{C}) + (-1)^w \binom{n}{w}(w-1) \text{ if } w = 3, 4, \dots, n.$$

Proof. As $d-1=2=R$, we may use Theorem 4.6 and relation (4.6) with $d=3$. Also, the total number of the weight 1 cosets is $\mathbb{N}_{\Sigma}^{(1)}(\mathcal{C}) = n(q-1)$, see (2.1). □

The points, in homogenous coordinates, of a complete (resp. incomplete) n-arc in $\mathrm{PG}(2,q)$ can be treated as columns of a $3 \times n$ parity check matrix of an $[n, n-3, 4]_q 2$ (resp. $[n, n-3, 4]_q 3$) code and vice versa.

In $\mathrm{PG}(2,q)$, a bisecant (resp. unisecant) of a point set is a line having two common points (resp. one common point) with the set. Every n-arc has, in total, $\binom{n}{2}$ bisecants and $n(q+2-n)$ unisecants; there are $q+2-n$ unisecants in each point of the n-arc.

The $q+1$ columns of H_4 (2.6) (as well as the 1-st $q+1$ columns of $\widetilde{H_4}$ (2.7)) can be treated as the points of a *conic* $\mathcal{K} \subset \mathrm{PG}(2,q)$ in homogenous coordinates. The $q+2$ columns of $\widetilde{H_4}$ form a *regular hyperoval* $\mathcal{H} \subset \mathrm{PG}(2,q)$. The conic $\mathcal{K}$ and the hyperoval $\mathcal{H}$ have the following properties [23]:

• Let q be odd. The conic $\mathcal{K}$ is a complete $(q+1)$-arc. Outside $\mathcal{K}$, there are $\mathcal{N}_{\text{int}} := \frac{1}{2}(q^2-q)$ internal points and $\mathcal{N}_{\text{ext}} := \frac{1}{2}(q^2+q)$ external points. Every internal and external point lies, respectively, on $\mathbb{B}_{\text{int}} := \frac{1}{2}(q+1)$ and $\mathbb{B}_{\text{ext}} := \frac{1}{2}(q-1)$ bisecants of $\mathcal{K}$. In addition, every external point lies on two unisecants of $\mathcal{K}$. Thus, for q odd, H_4 (as well as the 1-st $q+1$ columns of $\widetilde{H_4}$) is a $3 \times (q+1)$ parity check matrix of the $[q+1, q-2, 4]_q 2$ GDRS code.

• Let q be even. The conic $\mathcal{K}$ is an incomplete $(q+1)$-arc. Outside $\mathcal{K}$, there are $\mathcal{N}_{\text{ev}} := q^2-1$ points every of which lies on $\mathbb{B}_{\text{ev}} := \frac{1}{2}q$ bisecants of $\mathcal{K}$ and one unisecant. The so-called nucleus $\mathcal{O} = (0,1,0)^{tr}$ does not lie on any bisecant of $\mathcal{K}$; it is the intersection of all $q+1$ unisecants of $\mathcal{K}$. So, for q even, H_4 (as well as the 1-st $q+1$ columns of $\widetilde{H_4}$) is a parity check matrix of the $[q+1, q-2, 4]_q 3$ GDRS code. The hyperoval $\mathcal{H}$ is a complete $(q+2)$-arc. Every point outside $\mathcal{H}$ lies on $\frac{1}{2}(q+2)$ bisecants of $\mathcal{H}$. So, $\widetilde{H_4}$ is a parity check matrix of the $[q+2, q-1, 4]_q 2$ GTRS code.

Remark 1. Every point P of $\mathrm{PG}(2,q)$ gives rise to $q-1$ nonzero syndromes of cosets of an $[n, n-3, 4]_q R$ code corresponding to an n-arc, say $\mathcal{A}$; the syndromes can be generated by multiplying P (in homogeneous coordinates) by elements of $\mathbb{F}_q^*$. The points of $\mathcal{A}$ (i.e. the columns of the parity check matrices) generate the syndromes of weight 1 cosets. The points off $\mathcal{A}$ lying on its bisecants give the syndromes of weight 2 cosets. Finally, if $\mathcal{A}$ is incomplete, then $R=3$ and the points off $\mathcal{A}$, that do not lie on any bisecant of $\mathcal{A}$, give the syndromes of weight 3 cosets. For instance, if q is even, the nucleus of $\mathcal{K}$ (not lying on any bisecant) gives $q-1$ syndromes of the weight 3 cosets of the $[q+1, q-2, 4]_q 3$ GDRS code.

Theorem 5.2. *Let $\mathcal{C}$ be the $[q+1, q-2, 4]_q R$ GDRS code with the parity check matrix H_4 (2.6) corresponding to the conic $\mathcal{K}$ in the projective plane* $\mathrm{PG}(2,q)$.

(i) *Let q be odd. Then $R=2$ and there are $(q-1)\mathcal{N}_{\rm int}=\frac{1}{2}q(q-1)^2$ weight 2 cosets $\mathcal{V}_a^{(2)}$ with $B_2(\mathcal{V}_a^{(2)})=\mathbb{B}_{\rm int}=\frac{1}{2}(q+1)$ and $(q-1)\mathcal{N}_{\rm ext}=\frac{1}{2}(q^3-q)$ weight 2 cosets $\mathcal{V}_b^{(2)}$ with $B_2(\mathcal{V}_b^{(2)})=\mathbb{B}_{\rm ext}=\frac{1}{2}(q-1)$. The weight distribution of a coset of each class is as in (4.4) with $d=4$, $n=q+1$, and the value $B_{d-2}(\mathcal{V}^{(d-2)})$ equal to either $B_2(\mathcal{V}_a^{(2)})$ or $B_2(\mathcal{V}_b^{(2)})$.*

(ii) *Let q be even. Then $R=3$. There are $(q-1)\mathcal{N}_{\rm ev}=(q-1)(q^2-1)$ weight 2 cosets $\mathcal{V}^{(2)}$. The weight distribution of each weight 2 coset is as in (4.4) with $d=4$, $n=q+1$, $B_2(\mathcal{V}^{(2)})=\mathbb{B}_{\rm ev}=\frac{1}{2}q$. Also, there are $q-1$ weight 3 cosets $\mathcal{V}^{(3)}$ with the following weight distribution:*

$$B_w(\mathcal{V}^{(3)})=0 \text{ if } w=0,1,2;\ B_3(\mathcal{V}^{(3)})=\binom{q+1}{3}; \tag{5.2}$$
$$B_w(\mathcal{V}^{(3)})=A_w(\mathcal{C})-(-1)^w\binom{q+1}{w}\binom{w-1}{2} \text{ if } w=4,5,\dots,q+1.$$

Proof. **(i)** The weight 2 cosets $\mathcal{V}_a^{(2)}$ and $\mathcal{V}_b^{(2)}$ correspond to the internal and external points, respectively.

(ii) The weight 2 cosets $\mathcal{V}^{(2)}$ correspond to points off $\mathcal{K}$ apart $\mathcal{O}$. The weight 3 cosets $\mathcal{V}^{(3)}$ correspond to $\mathcal{O}$. We use Theorem 4.6 with (4.6). □

Theorem 5.3. *Let q be even. Let $\mathcal{C}$ be the $[q+2,q-1,4]_q2$ GTRS code of distance $d=4$ with the parity check matrix $\widetilde{H_4}$ (2.7) corresponding to the hyperoval $\mathcal{H}$ in the plane $\mathrm{PG}(2,q)$. Then there are $(q-1)(q^2-1)$ cosets $\mathcal{V}^{(2)}$ of weight 2 with $B_2(\mathcal{V}^{(2)})=\frac{1}{2}(q+2)$. The weight distribution of each weight 2 coset is as in (4.4) with $d=4$, $n=q+2$, $B_{d-2}(\mathcal{V}^{(d-2)})=\frac{1}{2}(q+2)$.*

Proof. Cosets $\mathcal{V}^{(2)}$ correspond to q^2-1 points outside the hyperoval. □

Proposition 5.4. *Let $q\ge 5$. In $\mathrm{PG}(2,q)$, let $P\in\mathcal{K}$ be a point of the conic $\mathcal{K}$. Let $\mathcal{K}^*=\mathcal{K}\setminus\{P\}$ be the "shortened" conic obtained from $\mathcal{K}$ by removing P. Then the q^2+1 points off $\mathcal{K}^*$ can be partitioned into 3 classes $\mathcal{A}_j$, $j=1,2,3$, of cardinality $\#\mathcal{A}_j=\mathcal{N}_j$ so that every point of $\mathcal{A}_j$ lies on $\mathbb{B}_j$ bisecants of $\mathcal{K}^*$.*

(i) *Let q be odd. Then $\mathcal{N}_1=\frac{1}{2}(q^2+q)$, $\mathbb{B}_1=\frac{1}{2}(q-1)$, $\mathcal{N}_2=\frac{1}{2}(q^2-q)$, $\mathbb{B}_2=\frac{1}{2}(q-3)$, $\mathcal{N}_3=1$, $\mathbb{B}_3=0$.*

(ii) *Let q be even. Then $\mathcal{N}_1=q-1$, $\mathbb{B}_1=\frac{1}{2}q$, $\mathcal{N}_2=q^2-q$, $\mathbb{B}_2=\frac{1}{2}(q-2)$, $\mathcal{N}_3=2$, $\mathbb{B}_3=0$.*

Proof. **(i)** The point P does not lie on any bisecant of $\mathcal{K}^*$. The unisecant of $\mathcal{K}$ in P contains q external points [23, Table 8.1] lying on $\mathbb{B}_{\text{ext}}$ bisecants of $\mathcal{K}^*$ as well as for $\mathcal{K}$. Every of the remaining $\mathcal{N}_{\text{ext}} - q$ external points loses one bisecant of $\mathcal{K}$ after removing P, i.e. it lies on $\mathbb{B}_{\text{ext}} - 1$ bisecants of $\mathcal{K}^*$. Every internal point also loses one bisecant of $\mathcal{K}$, i.e. it lies on $\mathbb{B}_{\text{int}} - 1$ bisecants of $\mathcal{K}^*$. Based on the above, the assertion follows. Note that $\mathbb{B}_2 \geq 1$.

(ii) The points $\mathcal{O}$ and P do not lie on any bisecant of $\mathcal{K}^*$. The unisecant $\mathscr{U}$ of $\mathcal{K}$ in P contains $\mathcal{O}$ and $q-1$ points lying on $\mathbb{B}_{\text{ev}}$ bisecants of $\mathcal{K}^*$ as well as for $\mathcal{K}$. Every point of $\mathrm{PG}(2,q) \setminus (\mathcal{K}^* \cup \mathscr{U})$ loses one bisecant of $\mathcal{K}$ after removing P, i.e. it lies on $\mathbb{B}_{\text{ev}} - 1$ bisecants of $\mathcal{K}^*$. Based on the above, the assertion follows. □

Theorem 5.5. *Let $\mathcal{C}$ be a $[q, q-3, 4]_q R$ GRS code with the parity check matrix obtained from the matrix H_4 (2.6) by removing any one column. Let $q \geq 5$. Let the shortened conic $\mathcal{K}^* \subset \mathrm{PG}(2,q)$ and values $\mathcal{N}_j$ and $\mathbb{B}_j$ be as in Proposition 5.4. Then covering radius $R = 3$, the code $\mathcal{C}$ corresponds to $\mathcal{K}^*$, and the following holds:*

There are $(q-1)\mathcal{N}_1$ weight 2 cosets $\mathcal{V}_a^{(2)}$ with $B_2(\mathcal{V}_a^{(2)}) = \mathbb{B}_1$ and $(q-1)\mathcal{N}_2$ weight 2 cosets $\mathcal{V}_b^{(2)}$ with $B_2(\mathcal{V}_b^{(2)}) = \mathbb{B}_2$. The weight distribution of a coset of each class is as in (4.4) with $d = 4$, $n = q$, and $B_{d-2}(\mathcal{V}^{(d-2)})$ equal to $B_2(\mathcal{V}_a^{(2)})$ or $B_2(\mathcal{V}_b^{(2)})$. Also, there are $(q-1)\mathcal{N}_3$ weight 3 cosets $\mathcal{V}^{(3)}$ with the following weight distribution:

$$B_w(\mathcal{V}^{(3)}) = 0 \text{ if } w = 0,1,2;\; B_3(\mathcal{V}^{(3)}) = \binom{q}{3}; \tag{5.3}$$
$$B_w(\mathcal{V}^{(3)}) = A_w(\mathcal{C}) - (-1)^w \binom{q}{w}\binom{w-1}{2} \text{ if } w = 4,5,\ldots,q+1.$$

Proof. The classes of the weight 2 cosets $\mathcal{V}_a^{(2)}$ and $\mathcal{V}_b^{(2)}$ correspond to the classes of points $\mathcal{A}_1$ and $\mathcal{A}_2$ lying on $\mathbb{B}_1$ and $\mathbb{B}_2$ bisecants of $\mathcal{K}^*$, respectively, see Proposition 5.4 and Remark 1. The weight 3 cosets $\mathcal{V}^{(3)}$ correspond to points that do not lie on any bisecant of $\mathcal{K}^*$. We use Theorem 4.6 with (4.6). □

Proposition 5.6. *Let $q \geq 7$. In $\mathrm{PG}(2,q)$, let $P_1, P_2 \in \mathcal{K}$ be points of the conic $\mathcal{K}$. Let $\mathcal{K}^{**} = \mathcal{K} \setminus \{P_1, P_2\}$ be the "double shortened" conic obtained from $\mathcal{K}$ by removing $\{P_1, P_2\}$. Then the q^2+2 points off $\mathcal{K}^{**}$ can be partitioned into $\mathfrak{N}_q$ classes $\mathcal{A}_j$, $j = 1, \ldots, \mathfrak{N}_q$, so that every point of $\mathcal{A}_j$ lies on $\mathbb{B}_j$ bisecants of $\mathcal{K}^{**}$.*

(i) *Let q be odd. Then $\mathfrak{N}_q = 4$ and $\mathcal{N}_1 = \frac{1}{2}(q+1)$, $\mathbb{B}_1 = \frac{1}{2}(q-1)$, $\mathcal{N}_2 = \frac{1}{2}(q-1)(q+4)$, $\mathbb{B}_2 = \frac{1}{2}(q-3)$, $\mathcal{N}_3 = \frac{1}{2}(q-1)(q-3)$, $\mathbb{B}_3 = \frac{1}{2}(q-5)$, $\mathcal{N}_4 = 2$, $\mathbb{B}_4 = 0$.*

(ii) *Let q be even. Then $\mathfrak{N}_q = 3$ and $\mathcal{N}_1 = 3(q-1)$, $\mathbb{B}_1 = \frac{1}{2}(q-2)$, $\mathcal{N}_2 = (q-1)(q-2)$, $\mathbb{B}_2 = \frac{1}{2}(q-4)$, $\mathcal{N}_3 = 3$, $\mathbb{B}_3 = 0$.*

Proof. **(i)** The points P_1 and P_2 do not lie on any bisecant of $\mathcal{K}^{**}$. Each unisecant $\mathscr{U}_1$ and $\mathscr{U}_2$ of $\mathcal{K}$ in P_1 and P_2, respectively, contains q external points [23, Table 8.1]; $2(q-1)$ of these points lie on one unisecant and lose one bisecant of $\mathcal{K}$ after removing $\{P_1, P_2\}$, i.e. every of them lies on $\mathbb{B}_{\text{ext}} - 1$ bisecants of $\mathcal{K}^{**}$. The point $\mathscr{U}_1 \cap \mathscr{U}_2$ lies on $\mathbb{B}_{\text{ext}}$ bisecants of $\mathcal{K}^{**}$. The bisecant $\overline{P_1, P_2}$ of $\mathcal{K}$ through P_1 and P_2 contains $\frac{1}{2}(q-1)$ external points as well as internal ones [23, Table 8.1]. Every of these points loses one bisecant of $\mathcal{K}$ after removing $\{P_1, P_2\}$, i.e. they lie on $\mathbb{B}_{\text{ext}} - 1$ and $\mathbb{B}_{\text{int}} - 1$ bisecants of $\mathcal{K}^{**}$, respectively. Every of the remaining $\mathcal{N}_{\text{ext}} - 2(q-1) - 1 - \frac{1}{2}(q-1)$ external points and $\mathcal{N}_{\text{int}} - \frac{1}{2}(q-1)$ internal ones loses two bisecants, i.e. they lie on $\mathbb{B}_{\text{ext}} - 2$ and $\mathbb{B}_{\text{int}} - 2$ bisecants of $\mathcal{K}^{**}$, respectively. Based on the above, the assertion follows. Note that $\mathbb{B}_3 \geq 1$.

(ii) The points $\mathcal{O}, P_1, P_2$ do not lie on any bisecant of $\mathcal{K}^{**}$. The unisecants $\mathscr{U}_1$ and $\mathscr{U}_2$ of $\mathcal{K}$ in P_1 and P_2 contain $\mathcal{O}$ and $2(q-1)$ points every of which loses one bisecant after removing $\{P_1, P_2\}$ and lies on $\mathbb{B}_{\text{ev}} - 1$ bisecants of $\mathcal{K}^{**}$. The same properties hold for $q-1$ points of the bisecant $\overline{P_1, P_2}$. The remaining $\mathcal{N}_{\text{ev}} - 2(q-1) - (q-1)$ points lose two bisecants after removing $\{P_1, P_2\}$, i.e. every of them lies on $\mathbb{B}_{\text{ev}} - 2$ bisecants of $\mathcal{K}^{**}$. Based on the above, the assertion follows. □

Theorem 5.7. *Let $\mathcal{C}$ be the $[q-1, q-4, 4]_q R$ GRS code with the parity check matrix obtained from the matrix H_4 (2.6) by removing any two columns. Let $q \geq 7$. Let the double shortened conic $\mathcal{K}^{**} \subset \mathrm{PG}(2,q)$ and values $\mathfrak{N}_q$, $\mathcal{N}_j$, and $\mathbb{B}_j$ be as in Proposition 5.6. Then covering radius $R = 3$, the code $\mathcal{C}$ corresponds to $\mathcal{K}^{**}$, and the following holds:*

There are $(q-1)\mathcal{N}_1$ weight 2 cosets $\mathcal{V}_a^{(2)}$ with $B_2(\mathcal{V}_a^{(2)}) = \mathbb{B}_1$ and $(q-1)\mathcal{N}_2$ weight 2 cosets $\mathcal{V}_b^{(2)}$ with $B_2(\mathcal{V}_b^{(2)}) = \mathbb{B}_2$. Also, for odd q, there are $(q-1)\mathcal{N}_3$ weight 2 cosets $\mathcal{V}_c^{(2)}$ with $B_2(\mathcal{V}_c^{(2)}) = \mathbb{B}_3$. The weight distribution of a coset of each class is as in (4.4) with $d = 4$, $n = q-1$, and $B_{d-2}(\mathcal{V}^{(d-2)})$ equal to $B_2(\mathcal{V}_a^{(2)})$, or $B_2(\mathcal{V}_b^{(2)})$, or $B_2(\mathcal{V}_c^{(2)})$. In addition, there are $(q-1)\mathcal{N}_{\mathfrak{N}_q}$ weight 3 cosets $\mathcal{V}^{(3)}$ with the the following weight distribution:

$$
\begin{aligned}
&B_w(\mathcal{V}^{(3)}) = 0 \text{ if } w = 0,1,2;\; B_3(\mathcal{V}^{(3)}) = \binom{q-1}{3}; \\
&B_w(\mathcal{V}^{(3)}) = A_w(\mathcal{C}) - (-1)^w \binom{q-1}{w}\binom{w-1}{2} \text{ if } w = 4,5,\dots,q+1.
\end{aligned} \tag{5.4}
$$

Proof. The classes of the weight 2 cosets $\mathcal{V}_a^{(2)}$, $\mathcal{V}_b^{(2)}$, and $\mathcal{V}_c^{(2)}$ correspond to the classes of points $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ lying on $\mathbb{B}_1$, $\mathbb{B}_2$, and $\mathbb{B}_3$ bisecants of $\mathcal{K}^{**}$, respectively, see Proposition 5.6 and Remark 1. The weight 3 cosets $\mathcal{V}^{(3)}$ correspond to points that do not lie on any bisecant of $\mathcal{K}^{**}$. We use Theorem 4.6 with (4.6). □

Remark 2. If the parity check matrix of the code $\mathcal{C}$ of Theorem 5.7 is obtained from the matrix H_4 (2.6) by removing two the last columns $(v_q, 0, 0)^{tr}$ and $(0, 0, v_{q+1})^{tr}$ then $\mathcal{C}$ is a *non-extended* $[q-1, q-4, 4]_q R$ GRS code.

6 The weight distribution of the weight 2 cosets of MDS codes of distance $d \geq 5$

Theorem 6.1. *Let $\mathcal{C}$ be an $[n, k, d]_q$ MDS code of distance $d \geq 5$. Let $\mathcal{V}^{(2)}$ be one of its cosets of weight 2. Assume that the value $B_{d-2}(\mathcal{V}^{(2)})$ is known. Then the number $B_w(\mathcal{V}^{(2)})$ of weight w vectors in the weight 2 coset $\mathcal{V}^{(2)}$ of $\mathcal{C}$ has the form:*

$$\begin{aligned}
&B_w(\mathcal{V}^{(2)}) = 0 \text{ if } w \in \{0, 1, \ldots, d-3\} \setminus \{2\},\; B_2(\mathcal{V}^{(2)}) = 1, \qquad (6.1)\\
&B_{d-2}(\mathcal{V}^{(2)}) \text{ is known by the assumption,}\\
&B_w(\mathcal{V}^{(2)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(2)}(\mathcal{C}) + \Omega_w^{(d-2)}(\mathcal{C}) B_{d-2}(\mathcal{V}^{(2)})\\
&= A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(2)}(\mathcal{C}) + (-1)^{w-d} \binom{n-d+2}{n-w} B_{d-2}(\mathcal{V}^{(2)})\\
&\text{if } w = d-1, d, \ldots, n.
\end{aligned}$$

Proof. We use (4.2) with $W = 2$ and apply (3.10). □

Lemma 6.2. *Let $\mathcal{C}$ be an $[n, k, d]_q$ MDS code with $d \geq 5$. Then the overall number $\mathscr{B}_{d-2}^{\Sigma}(\mathcal{V}^{(2)})$ of weight $d-2$ vectors in all weight 2 cosets of $\mathcal{C}$ is as follows:*

$$\mathscr{B}_{d-2}^{\Sigma}(\mathcal{V}^{(2)}) = (q-1)\binom{n}{2}\binom{n-2}{d-2}. \qquad (6.2)$$

Proof. As $d \geq 5$, each weight 2 coset $\mathcal{V}^{(2)}$ has a unique leader. For each weight d codeword $\mathbf{c}_d$ of $\mathcal{C}$, there are $\binom{d}{2}$ coset leaders $\mathbf{v}_2$ such that $\mathbf{c}_d + \mathbf{v}_2 = \mathbf{v}_{d-2}$. Therefore, using (2.5) and (3.3), we have

$$\mathscr{B}_{d-2}^{\Sigma}(\mathcal{V}^{(2)}) = A_d(\mathcal{C})\binom{d}{2} = (q-1)\binom{n}{d}\binom{d}{2} = (q-1)\binom{n}{2}\binom{n-2}{d-2}. \qquad □$$

Theorem 6.3. *Let $\mathcal{C}$ be an $[n, k, d]_q$ MDS code of distance $d \geq 5$. Assume that all $\binom{n}{2}(q-1)^2$ the weight 2 cosets $\mathcal{V}^{(2)}$ of $\mathcal{C}$ have the same weight distribution. Then, the weight distribution of any weight 2 coset $\mathcal{V}^{(2)}$ is as follows:*

$$B_w(\mathcal{V}^{(2)}) = 0 \text{ if } w \in \{0, 1, \ldots, d-3\} \setminus \{2\},\; B_2(\mathcal{V}^{(2)}) = 1, \qquad (6.3)$$

$$B_{d-2}(\mathcal{V}^{(2)}) = \frac{\mathscr{B}_{d-2}^{\Sigma}(\mathcal{V}^{(2)})}{\binom{n}{2}(q-1)^2} = \frac{1}{q-1}\binom{n-2}{d-2}, \qquad (6.4)$$

$$B_w(\mathcal{V}^{(2)}) = A_w(\mathcal{C}) - \Omega_w^{(0)}(\mathcal{C}) + \Omega_w^{(2)}(\mathcal{C}) + (-1)^{w-d}\frac{1}{q-1}\binom{n-d+2}{n-w}\binom{n-2}{d-2}$$

if $w = d-1, d, \ldots, n$.

Proof. By (2.1), the total number of the weight 2 cosets $\mathcal{V}^{(2)}$ of $\mathcal{C}$ is $\binom{n}{2}(q-1)^2$ that together with (6.2) gives rise to (6.4). For the rest of relations we apply (6.1). □

Theorem 6.4. (symmetry of not identical weight distributions) *Let* $\mathcal{C}$ *be an* $[n, k, d]_q$ *MDS code with* $d \geq 5$*. Let* $\mathcal{V}_a^{(2)}$ *and* $\mathcal{V}_b^{(2)}$ *be its weight* 2 *cosets with distinct weight distributions. Then independently of the values of* $B_{d-2}(\mathcal{V}_a^{(2)})$ *and* $B_{d-2}(\mathcal{V}_b^{(2)})$*, there is the following symmetry of the weight distributions:*

$$(-1)^{n+d} B_w(\mathcal{V}_a^{(2)}) - B_{n+d-2-w}(\mathcal{V}_a^{(2)}) = (-1)^{n+d} B_w(\mathcal{V}_b^{(2)}) - B_{n+d-2-w}(\mathcal{V}_b^{(2)}).$$

Proof. The proof is similar to one of Theorem 4.5. Instead of (4.4) we use (6.1). □

Theorem 6.5. (necessary condition for identity of weight distributions) *Let* $\mathcal{C}$ *be an* $[n, k, d]_q$ *MDS code of distance* $d \geq 5$*. The necessary condition for identity of the weight distributions of all the weight* 2 *cosets of* $\mathcal{C}$ *is as follows:*

$$\frac{1}{q-1}\binom{n-2}{d-2} \text{ is an integer.} \tag{6.5}$$

Proof. All values of $B_w(\mathcal{V}^{(2)})$ must be integer. The assertion follows from (6.4). □

Lemma 6.6. *Let* $n = q+1$*,* $d \geq 5$*, and* $q-1$ *is co-prime with* $d-2$*. Then the necessary condition* (6.5) *holds.*

Proof. We have $n - 2 = q - 1$ and $(d-2) | (q-1)(q-2)(q-3)\ldots(q-d+2)$. As $q-1$ is co-prime with $d-2$ we have also $(d-2)|(q-2)(q-3)\ldots(q-d+2)$. □

Remark 3. It can be shown that for a $[q+1, q+2-d, d \geq 5]_q$ GDRS code with co-primes $q-1$ and $d-2$, all the weight 2 cosets have identical weight distributions. For $d = 5$, it is proved in [15, Theorem 26(ii)]. On the case $d > 5$, we will write in our future papers.

7 MDS codes as multiple coverings of deep holes and multiple saturating sets in $\mathrm{PG}(N, q)$

7.1 Preliminaries on multiple coverings

We give some definitions and propositions of [4, 5, 7, 11], see also the references therein.

Definition 7.1. (i) An $[n, k, d]_q R$ code $\mathcal{C}$ is said to be an (R, μ) *multiple covering of the farthest-off points* ((R, μ)-MCF code for short) if for all $\mathbf{x} \in \mathbb{F}_q^n$ with $d(\mathbf{x}, \mathcal{C}) = R$ we have $f_R(\mathbf{x}, \mathcal{C}) \geq \mu$.

(ii) An $[n, k, d(\mathcal{C})]_q R$ code $\mathcal{C}$ is said to be an (R, μ) *almost perfect* multiple covering of the farthest-off points ((R, μ)-APMCF code for short) if for all $\mathbf{x} \in \mathbb{F}_q^n$ with $d(\mathbf{x}, \mathcal{C}) = R$ we have $f_R(\mathbf{x}, \mathcal{C}) = \mu$. If, in addition, $d(\mathcal{C}) \geq 2R$, then the code is called (R, μ) *perfect* multiple covering of the farthest-off points ((R, μ)-PMCF code for short).

In the literature, MCF codes are also called *multiple coverings of deep holes*.

The covering quality of an $[n, k, d(\mathcal{C})]_q R$ (R, μ)-MCF code $\mathcal{C}$ is characterized by its *μ-density* $\gamma_\mu(\mathcal{C}, R, q) \geq 1$ that is the average value of $f_R(\mathbf{x}, \mathcal{C})$ divided by μ where the average is calculated over all $\mathbf{x} \in \mathbb{F}_q^n$ with $d(\mathbf{x}, \mathcal{C}) = R$. For APMCF and PMCF codes we have $\gamma_\mu(\mathcal{C}, R, q) = 1$. Also,

$$\gamma_\mu(\mathcal{C}, 2, q) = \frac{\binom{n}{2}(q-1)^2}{\mu(q^{n-k} - 1 - n(q-1))} \text{ if } d(\mathcal{C}) > 3. \tag{7.1}$$

From the covering problem point of view, the best codes are those with small μ-density.

If the μ-density $\gamma_\mu(\mathcal{C}, R, q)$ tends to 1 when q tends to infinity we have an *asymptotically optimal collection of MCF codes* or, in another words, an *asymptotically optimal multiple covering*.

Definition 7.2. Let S be a subset of points of $\mathrm{PG}(N, q)$. Then S is said to be (ρ, μ)-saturating if the following holds: S generates $\mathrm{PG}(N, q)$; there exists a point Q in $\mathrm{PG}(N, q)$ which does not belong to any subspace of dimension $\rho - 1$ generated by the points of S; and, finally, every point $Q \in \mathrm{PG}(N, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S, is such that the number of subspaces of dimension ρ generated by the points of S and containing Q, counted with multiplicity, is at least μ. The multiplicity m_T of a subspace T is computed as the number of distinct sets of $\rho + 1$ independent points contained in $T \cap S$. If any $\rho + 1$ points of S are linearly independent, then $m_T = \binom{\#(T \cap S)}{\rho+1}$.

Definition 7.3. Let S be a (ρ, μ)-saturating set in $\mathrm{PG}(N, q)$. The set S is called *optimal (ρ, μ)-saturating set* ((ρ, μ)-OS set for short) if every point Q in $\mathrm{PG}(N, q)$ not belonging to any subspace of dimension $\rho - 1$ generated by the points of S, is such that the number of subspaces of dimension ρ generated by the points of S and containing Q, counted with multiplicity, is exactly μ.

Proposition 7.4. *Let $\mathcal{C}$ be an $[n, k, d = n - k + 1]_q R$ MDS code with an $(n - k) \times n$ parity check matrix H. Let $\mathcal{C}$ be an (R, μ)-MCF (resp. (R, μ)-APMCF) code. Let S be the n-set of points in $\mathrm{PG}(n - k - 1, q)$ such that its points in homogeneous coordinates are columns of H. Then S is an $(R - 1, \mu)$-saturating (resp. $(R - 1, \mu)$-OS) set.*

Proposition 7.4 allows us to consider linear MDS (R, μ)-MCF (resp. (R, μ)-APMCF) codes as $(R - 1, \mu)$-saturating (resp. $(R - 1, \mu)$-OS) sets and vice versa. Thus, there is the one-to-one correspondence between MCF codes and multiple saturating sets in the projective spaces $\mathrm{PG}(N, q)$.

7.2 New results

Lemma 7.5. *Let $\mathcal{C}$ be an $[n,k,d(\mathcal{C})]_q$ code and $\mathcal{V}^{(W)}$ be one of its weight W coset.*

(i) *For any vector $\mathbf{x} \in \mathcal{V}^{(W)}$ we have $d(\mathbf{x},\mathcal{C}) = W$ and $f_W(\mathbf{x},\mathcal{C}) = B_W(\mathcal{V}^{(W)})$.*

(ii) *If for all cosets $\mathcal{V}^{(R)}$ we have $B_R(\mathcal{V}^{(R)}) \geq \mu$ then $\mathcal{C}$ is an (R,μ)-MCF code; it gives rise to an $(R-1,\mu)$-saturating set in $\mathrm{PG}(n-k-1,q)$.*

(iii) *If for all cosets $\mathcal{V}^{(R)}$ we have $B_R(\mathcal{V}^{(R)}) = \mu$ then $\mathcal{C}$ is an (R,μ)-APMCF code; it gives rise to an $(R-1,\mu)$-OS set in $\mathrm{PG}(n-k-1,q)$. If, in addition, $d(\mathcal{C}) \geq 2R$, then $\mathcal{C}$ is an (R,μ)-PMCF code.*

Proof. The assertions can be obtained from the definitions of the objects considered. □

Theorem 7.6. **(i)** *Let $\mathcal{C}$ be an $[n,k,d]_qR$ MDS code of length $d-1 < n < q+1$, distance $d \geq 3$, with a parity check matrix consisting of n columns of H_d (2.6). Then $R = d-1$ and, in total, there are $\mathbb{N}(\mathcal{V}^{(d-1)}) \geq (q-1)(q+1-n)$ weight $d-1$ cosets $\mathcal{V}^{(d-1)}$ of $\mathcal{C}$. All the weight $d-1$ cosets $\mathcal{V}^{(d-1)}$ of $\mathcal{C}$ have the same weight distribution $B_w(\mathcal{V}^{(d-1)})$ of the form (4.6). In addition, if H_d gives a $[q+1,q+2-d,d]_qR$ MDS code with $R = d-2$ then $\mathbb{N}(\mathcal{V}^{(d-1)}) = (q-1)(q+1-n)$.*

(ii) *Let $\mathcal{C}$ be an $[n,k,d]_qR$ MDS code of distance $d \geq 3$ and covering radius $R = d-2$. If we remove $1 \leq \Delta \leq n-d$ columns from a parity check matrix of $\mathcal{C}$ we obtain a parity check matrix of an $[n-\Delta,k-\delta,d]_qR$ MDS code $\mathcal{C}_\Delta$ with $R = d-1$. In total, there are $\mathbb{N}(\mathcal{V}^{(d-1)}) = (q-1)\Delta$ weight $d-1$ cosets $\mathcal{V}^{(d-1)}$ of $\mathcal{C}_\Delta$; all these cosets have the same weight distribution $B_w(\mathcal{V}^{(d-1)})$ of the form (4.6).*

Proof. **(i)** From H_d, we remove $\delta = q+1-n$ columns. The removed columns are not a linear combination of $\leq d-2$ columns of the parity check matrix, but they can be obtained as a linear combination of $d-1$ columns. Therefore, $R = d-1$ and we may use Theorem 4.6. Each of the removed columns gives rise to $q-1$ syndromes of distinct weight $d-1$ cosets. If H_d gives a $[q+1,q+2-d,d]_qR$ MDS code with $R = d-2$ then we have no other weight $d-1$ cosets.

(ii) The proof is similar to the case (i). □

Theorem 7.7. *Let $\mathcal{C}$ be an $[n,k,d]_qR$ MDS code of distance $d \geq 3$ and covering radius $R = d-1$. Let S be the n-set of points in $\mathrm{PG}(n-k-1,q)$ such that its points in homogeneous coordinates are columns of the parity check matrix of $\mathcal{C}$. Let $\mu = \binom{n}{d-1}$. Then $\mathcal{C}$ is a $(d-1,\mu)$-APMCF code and S is an $(d-2,\mu)$-OS set in $\mathrm{PG}(n-k-1,q)$.*

Proof. We use Theorem 4.6, Lemma 7.5(iii) and Proposition 7.4. □

Corollary 7.8. *Let q be even. Let $\mathcal{C}$ be an $[n < q+2, n-3, 4]_q 3$ MDS code with a parity check matrix consisting of n columns of $\widetilde{H_4}$ (2.7). Then $\mathcal{C}$ is a $(3, \mu)$-APMCF code with $\mu = \binom{n}{3}$; it gives rise to an $\left(R-1, \binom{n}{3}\right)$-OS set in* $\mathrm{PG}(n-k-1, q)$.

Proof. We use Theorems 7.6, 7.7, and Proposition 7.4. □

Proposition 7.9. *Let q be even. Let $\mathcal{C}$ be the $[q+2, q-1, 4]_q 2$ GTRS code with the parity check matrix $\widetilde{H_4}$ (2.7). Then $\mathcal{C}$ is a $(2, \mu)$-PMCF code with $\mu = \frac{1}{2}(q+2)$.*

Proof. By Theorem 5.3, for all the weight 2 cosets we have $B_2(\mathcal{V}^{(2)}) = \frac{1}{2}(q+2)$. Then we use Lemma 7.5(iii). □

Proposition 7.10. *Let q be odd. Let $\mathcal{C}$ be the $[q+1, q-2, 4]_q 2$ GDRS code with the parity check matrix H_4 (2.6). Then $\mathcal{C}$ is a $(2, \mu)$-MCF code with $\mu = \frac{1}{2}(q-1)$ and μ-density $\gamma_\mu(\mathcal{C}, 2, q) = 1 + \frac{1}{q}$. The μ-density tends to 1 when q tends to infinity. Thereby, we have an asymptotical optimal collection of MCF codes.*

Proof. By Theorem 5.2(i), for weight 2 cosets we have $B_2(\mathcal{V}^{(2)}) \geq \frac{1}{2}(q-1)$. The μ-density is calculated by (7.1). □

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References

[1] E. F. Assmus, Jr. and H. F. Mattson, Jr., The weight-distribution of a coset of a linear code, *IEEE Trans. Inform. Theory* **24** (1978), 497. doi:10.1109/tit.1978.1055903.

[2] S. Ball, *Finite Geometry and Combinatorial Applications*, London Math. Soc. Student Texts **82**, Cambridge Univ. Press, Cambridge, UK, 2015. doi:10.1017/CBO9781316257449.

[3] S. Ball and M. Lavrauw, Arcs in finite projective spaces, *EMS Surveys in Math. Sciences*, **6** (2019), 133–172. doi:10.4171/EMSS/33.

[4] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco, Multiple coverings of the farthest-off points with small density from projective geometry, *Adv. Math. Commun.*, **9** (2015), 63–85. doi:10.3934/amc.2015.9.63.

[5] D. Bartoli, A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco, Further results on multiple coverings of the farthest-off points, *Adv. Math. Commun.*, **10** (2016), 613–632. doi:10.3934/amc.2016030.

[6] D. Bartoli, A. A. Davydov, S. Marcugini and F. Pambianco, On the smallest size of an almost complete subset of a conic in $\mathrm{PG}(2, q)$ and extendability of Reed–Solomon codes, *Problems Inform. Transmiss.*, **54** (2018), 101–115. doi:10.1134/S0032946018020011.

[7] D. Bartoli, A. A. Davydov, S. Marcugini and F. Pambianco, On planes through points off the twisted cubic in $\mathrm{PG}(3, q)$ and multiple covering codes, *Finite Fields Appl.*, **67** (2020), paper 101710, 25 pp. doi:10.1016/j.ffa.2020.101710.

[8] D. Bartoli, M. Giulietti, I. Platoni, On the covering radius of MDS codes, *IEEE Trans. Inform. Theory*, **61**(2) (2015), 801–811. doi:10.1109/TIT.2014.2385084.

[9] R. E. Blahut, *Theory and Practice of Error Control Codes*, Addison Wesley, Reading, 1984.

[10] P. G. Bonneau, Weight distribution of translates of MDS codes, *Combinatorica*, **10** (1990), 103–105. doi:10.1007/BF02122700.

[11] G. Cohen, I. Honkala, S. Litsyn and A. Lobstein, *Covering codes*, North-Holland Math. Library, **54**, Elsevier, Amsterdam, The Netherlands, 1997.

[12] P. Charpin, T. Helleseth and V. Zinoviev, The coset distribution of triple-error-correcting binary primitive BCH codes, *IEEE Trans. Inform. Theory*, **52** (2006), 1727–1732. doi:10.1109/TIT.2006.871605.

[13] K.-M. Cheung, More on the decoder error probability for Reed-Solomon codes, *IEEE Trans. Inform. Theory*, **35** (1989), 895-–900. doi:10.1109/18.32169.

[14] A. A. Davydov, S. Marcugini and F. Pambianco, On integral weight spectra of the MDS codes cosets of weight 1, 2, and 3, preprint, arXiv:2007.02405 [cs.IT] (2020) https : //arxiv.org/abs/2007.02405

[15] A. A. Davydov, S. Marcugini and F. Pambianco, On cosets weight distributions of the doubly-extended Reed-Solomon codes of codimension 4, preprint, arXiv:2007.08798 [cs.IT] (2020) https : //arxiv.org/abs/2007.08798

[16] P. Delsarte, Four fundamental parameters of a code and their combinatorial significance, *Inform. Control*, **23** (1973), 407–438. doi:10.1016/s0019-9958(73)80007-5.

[17] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Repts. Suppl.* **10** (1973).

[18] P. Delsarte and V. I. Levenshtein, Association schemes and coding theory, *IEEE Trans. Inform. Theory*, **44** (1998), 2477–2504. doi:10.1109/18.720545.

[19] T. Etzion and L. Storme, Galois geometries and coding theory, *Des. Codes Cryptogr.* **78** (2016), 311–350. doi:10.1007/s10623-015-0156-5.

[20] M. F. Ezerman, M. Grassl and P. Sole, The weights in MDS codes, *IEEE Trans. Inform. Theory*, **57** (2011), 392–396. doi:10.1109/TIT.2010.2090246.

[21] E. M. Gabidulin and T. Klove, The Newton radius of MDS codes, in *Proc. Inf. Theory Workshop (ITW 1998) (Cat. No.98EX131)*, Killarney, Ireland, Jun. 1998, 50–51. doi:10.1109/ITW.1998.706412.

[22] T. Helleseth, The weight distribution of the coset leaders of some classes of codes with related parity-check matrices, *Discrete Math.*, **28** (1979), 161–171. doi:10.1016/0012-365X(79)90093-1.

[23] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, 2^{nd} edition, Oxford Univ. Press, Oxford, 1999.

[24] J. W. P. Hirschfeld and L. Storme, The packing problem in statistics, coding theory and finite projective spaces: Update 2001, in (eds. A. Blokhuis, J. W. P. Hirschfeld, D. Jungnickel and J. A. Thas), *Finite Geometries (Proc. 4th Isle of Thorns Conf., July 16-21, 2000)*, Develop. Math., **3**, Kluwer, Dordrecht, 2001, 201–246. doi:10.1007/978-1-4613-0283-4_13.

[25] J. W. P. Hirschfeld and J. A. Thas, Open problems in finite projective spaces, *Finite Fields Appl.*, **32**(1) (2015), 44–81. doi:10.1016/j.ffa.2014.10.006.

[26] S. Hong and R. Wu, On deep holes of generalized Reed-Solomon codes, *AIMS Math.* **1** (2016), 96–101. doi:10.3934/Math.2016.2.96.

[27] W. C. Huffman and V. S. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge Univ. Press, 2003.

[28] R. Jurrius and R. Pellikaan, The coset leader and list weight enumerator, in (eds. G. Kyureghyan, G. L. Mullen, A. Pott), *Contemporary Math.*, **632**, *Topics in Finite Fields* (2015) 229–251. doi:10.1090/conm/632/12631. corrected version (2019) https : //www.win.tue.nl/~ruudp/paper/71.pdf

[29] J. Justesen and T. Høholdt, Bounds on list decoding of MDS codes, *IEEE Trans. Inform. Theory*, **47** (2001), 1604–1609. doi:10.1109/18.923744.

[30] K. Kaipa, Deep holes and MDS extensions of Reed-Solomon codes, *IEEE Trans. Inform. Theory*, **63** (2017), 4940–4948. doi:10.1109/TIT.2017.2706677.

[31] T. Kasami and S. Lin, On the probability of undetected error for the maximum distance separable codes, *IEEE Trans. Commun.*, **32** (1984), 998–1006. doi:10.1109/TCOM.1984.1096175.

[32] I. Landjev and L. Storme, Galois geometry and coding theory, in *Current Research Topics in Galois geometry,* (eds. J. De Beule and L. Storme), Chapter 8, NOVA Academic, New York, 2011, 187–214.

[33] F. J. MacWilliams, A theorem on the distribution of weights in a systematic code, *Bell Syst. Tech. J.*, **42** (1963), 79–94. doi:10.1002/j.1538-7305.1963.tb04003.x.

[34] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, 3^{rd} edition, North-Holland, Amsterdam, The Netherlands, 1981.

[35] J. Riordan, *Combinatorial Identities,* Willey, New York, 1968.

[36] R. M. Roth, *Introduction to Coding Theory*, Cambridge, Cambridge Univ. Press, 2006.

[37] J. R. Schatz, On the weight distributions of cosets of a linear code, *American Math. Month.*, **87** (1980), 548–551. doi:10.1080/00029890.1980.11995087.

[38] L. Storme, Completeness of normal rational curves, *J. Algebraic Combin.*, **1** (1992), 197–202. doi:10.1023/A:1022428405084.

[39] X. Xu and Y. Xu, Some results on deep holes of generalized projective Reed-Solomon codes, *AIMS Math.*, **4** (2019), 176–192. doi:10.3934/math.2019.2.176.

[40] J. Zhang, D. Wan and K. Kaipa, Deep holes of projective Reed-Solomon codes, *IEEE Trans. Inform. Theory*, **66** (2020), 2392–2401. doi:10.1109/TIT.2019.2940962.