Large Deviation Limits of Invariant Measures

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The setup

- Suppose $X_n = (X_n(t)\,, t \geq 0)$ are $\mathbb{S}$–valued stochastic processes on $(\Omega, \mathcal{F}, \mathbf{P})$ with invariant measures P_n , i.e., $\mathbf{P}(X_n(t) \in \Gamma) = P_n(\Gamma)$
- Suppose the X_n satisfy a trajectorial LDP as random elements of $\mathbb{D}(\mathbb{R}_+, \mathbb{S})$
- What about an LDP for the P_n ?

The fundamental example: diffusions in $\mathbb{R}^d$

$$dX_n(t) = b(X_n(t))\,dt + \frac{1}{\sqrt{n}}\,dW(t)\,, X_n(0) = x \in \mathbb{R}^d$$

The LDP with $\mathbf{I}$ such that, for absolutely continuous $X = (X(t)\,, t \geq 0)$ with $X(0) = x\,,$

$$\mathbf{I}(X) = \frac{1}{2}\int_0^\infty |\dot{X}(t) - b(X(t))|^2\,dt$$

If the differential equation

$$\dot{X}(t) = b(X(t)) \qquad (1)$$

has a unique equilibrium O which is asymptotically stable, "then" (Freidlin and Wentzell, 1979), the P_n satisfy the LDP in $\mathbb{R}^d$ with

$$V(x) = \inf_{t\geq 0}\inf_{X:\,X(0)=O\,,X(t)=x}\mathbf{I}(X)$$

LDP and LD convergence

Measures Q_n on $(\mathbb{M}, \mathcal{M})$ satisfy the LDP with deviation function (a.k.a. action functional, a.k.a. rate function) $\mathbf{I}$ (for rate n) provided

$$\liminf_{n\to\infty} \frac{1}{n} \ln Q_n(H) \geq - \inf_{X\in \text{ int } H} \mathbf{I}(X)$$

$$\limsup_{n\to\infty} \frac{1}{n} \ln Q_n(H) \leq - \inf_{X\in \text{ cl } H} \mathbf{I}(X)$$

$\mathbf{I} : \mathbb{M} \to [0, \infty]$ with $\mathbf{I}^{-1}[0, a]$ compact, $a \geq 0$
For "nice" sets H,

$$\lim_{n\to\infty} Q_n(H)^{1/n} = \underbrace{\sup_{x\in H} \underbrace{e^{-\mathbf{I}(x)}}_{\mathbf{\Pi}(x)}}_{\mathbf{\Pi}(H)}$$

The Q_n LD converge to (deviability, or max measure) $\mathbf{\Pi}$

Weak Convergence of Invariant Measures

Suppose $Y_n \to Y$ in distribution for certain initial conditions and the Y_n have invariant measures μ_n

If $\mu_n \to \mu$ weakly along a subsequence, "then" μ is invariant for Y

If the μ_n are tight and Y has a unique invariant measure μ then $\mu_n \to \mu$ weakly

Prohorov's theorem is the key

LD Relative Compactness

A sequence Q_n is said to be exponentially tight if, for arbitrary $\epsilon > 0$, there exists compact $K \subset \mathbb{M}$ such that

$$\limsup_{n\to\infty} Q_n(K^c)^{1/n} < \epsilon$$

If Q_n is exponentially tight, then Q_n is LD relatively compact, i.e., for any subsequence n', there exists a further subsequence n'' such that the $Q_{n''}$ LD converge at rate n''
Any such limit is dubbed an LD limit point

Fluxes Across Cuts and Max Balance Equations

Suppose that $\mathcal{L}(X_n|X_n(0)=x_n)$ LD converge to $\mathbf{\Pi}_x$ whenever $x_n \to x$ and $\mathbf{\Pi}_x(X)=0$ unless X is continuous. Then $\mathcal{L}(X_n(t)|X_n(0)=x_n)$ LD converge to $\mathbf{\Pi}_{x,t}$, where

$$\mathbf{\Pi}_{x,t}(y) = \sup_{\substack{X\in\mathbb{D}(\mathbb{R}_+,\mathbb{S}):\\ X(0)=x,\, X(t)=y}} \mathbf{\Pi}_x(X)$$

For $\Gamma \in \mathcal{B}(\mathbb{S})$,

$$\boxed{\begin{array}{c}\displaystyle\int_\Gamma \mathbf{P}(X_n(t)\in\Gamma^c|X_n(0)=x)\,P_n(dx)\\ =\\ \displaystyle\int_{\Gamma^c} \mathbf{P}(X_n(t)\in\Gamma|X_n(0)=x)\,P_n(dx)\end{array}} \tag{2}$$

If $\mathbf{\Pi}$ is an LD limit point of the P_n , then, for "nice" sets Γ ,

$$\boxed{\sup_{x\in\Gamma}\sup_{y\in\Gamma^c}\mathbf{\Pi}_{x,t}(y)\mathbf{\Pi}(x) = \sup_{x\in\Gamma^c}\sup_{y\in\Gamma}\mathbf{\Pi}_{x,t}(y)\mathbf{\Pi}(x)} \tag{3}$$

Attractor and Continuity Hypotheses

There exists set A, which is locally finite in the sense that compact subsets of $\mathbb{S}$ contain at most finitely many of the elements of A, such that the following hold:

(1) if $\mathbf{\Pi}_x(X) = 1$, then $\inf_{t \geq 0} d(X(t), A) = 0$

(2) if $X(t) = a$, for all $t \geq 0$, then $\mathbf{\Pi}_a(X) = 1$, where $a \in A$

(3) if $a, a' \in A$, then $\mathbf{\Pi}_{a,t}(a') > 0$, for some $t \geq 0$

(4) for any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, A) < \delta$, then $\mathbf{\Pi}_{x,s}(a) > 1 - \epsilon$ and $\mathbf{\Pi}_{a,s'}(x) > 1 - \epsilon$, for some $s \geq 0$, $s' \geq 0$, and $a \in A$

(5) for arbitrary $x \in \mathbb{S}$ and $\epsilon > 0$, there exist $\delta > 0$, $t \geq 0$ and $t' \geq 0$ such that $\mathbf{\Pi}_{x,t}(x') > 1 - \epsilon$ and $\mathbf{\Pi}_{x',t'}(x) > 1 - \epsilon$ whenever $d(x, x') < \delta$

Limits of Transition Deviabilities

Suppose, in addition, that

- $\mathbf{\Pi}_x(X) = \mathbf{\Pi}_x(\pi_s^{-1}(\pi_s X))\mathbf{\Pi}_{X_s}(\theta_s X)$, where $\pi_s X = (X(t)\,, t \in [0, s])$ and $\theta_s X = (X(s+t)\,, t \geq 0)$
- The net $(\mathbf{\Pi}_{x,t}(y)\,, y \in \mathbb{S})\,, t \geq 0$, is tight uniformly over x from compact sets: for arbitrary $\epsilon > 0$ and compact K_1, there exists compact K_2 such that

$$\limsup_{t\to\infty} \sup_{x\in K_1} \mathbf{\Pi}_{x,t}(K_2^c) < \epsilon$$

Then, there exist the limits

$$\mathbf{\Pi}(x, y) = \lim_{t\to\infty} \mathbf{\Pi}_{x,t}(y)$$

LD Limits

Let $\mathbf{\Pi}$ represent an LD limit point of P_n. Then we have the max balance equations that, for arbitrary partitions $\{A', A''\}$ of A,

$$\sup_{x \in A'} \sup_{y \in A''} \mathbf{\Pi}(x, y)\mathbf{\Pi}(x) = \sup_{x \in A''} \sup_{y \in A'} \mathbf{\Pi}(x, y)\mathbf{\Pi}(x) \qquad (4)$$

and we have that $\mathbf{\Pi}(A) = 1$

These equations specify the restriction of $\mathbf{\Pi}$ to A uniquely

Also, for all $x \in \mathbb{S}$,

$$\mathbf{\Pi}(x) = \sup_{y \in A} \mathbf{\Pi}(y, x)\mathbf{\Pi}(y) \qquad (5)$$

Theorem 1

If the sequence P_n is exponentially tight and the above hypotheses hold, then the sequence P_n is LD convergent

The Solution to the Max Balance Equations (Freidlin and Wentzell (1979))

Given $a \in A$, let $G_A(a)$ denote the set of directed graphs that are in-trees with root a on the vertex set A. Thus, if $g \in G_A(a)$, then, for every $a' \in A$, there exists a unique directed path from a' to a in g. Let $E(g)$ denote the set of edges of g. Each edge $e = (a', a'') \in E(g)$ is assigned the weight $v(e) = \mathbf{\Pi}(a', a'')$. Let $w(g) = \prod_{e \in E(g)} v(e)$
For $a \in A$,

$$\mathbf{\Pi}(a) = \frac{\sup_{g \in G_A(a)} w(g)}{\sup_{a' \in A} \sup_{g \in G_A(a')} w(g)}$$

LDPs for Invariant Measures of Processes in $\mathbb{R}^d$

Suppose that

$$\mathbf{I}_x(X) = \int_0^\infty L(X(t), \dot{X}(t))\, dt$$

provided X is absolutely continuous and $X(0) = x$, and $\mathbf{I}_x(X) = \infty$, otherwise
Examples:

- jump diffusions
- slow–fast diffusions

Let A represent a set of equilibiria of $L(X(t), \dot{X}(t)) = 0$, which has a nonempty intersection with the set of the ω–limit points of each $x \in \mathbb{R}^d$. Assume also that the function $L(x, y)$ is bounded when y belongs to an arbitrary bounded set and when x belongs either to an arbitrary bounded set or to some neighbourhood of A. Then the attractor and continuity hypotheses hold