

## Research Article

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# On recurrent properties of Fisher–Wright’s diffusion on $(0, 1)$ with mutation

<https://doi.org/10.1515/rose-2021-2061>

Received May 5, 2021; accepted May 10, 2021

**Abstract:** A one-dimensional Fisher–Wright diffusion process on the interval  $(0, 1)$  with mutations is considered. This is a widely known model in population genetics. The goal of this paper is an exponential recurrence of the process, which also implies an exponential rate of convergence towards the invariant measure.

**Keywords:** Wright–Fisher diffusion, exponential recurrence

**MSC 2010:** 60J60, 37A25

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**Communicated by:** Nikolai Leonenko

## 1 Introduction

Consider a one-dimensional stochastic differential equation

$$dX_t = [a(1 - X_t) - bX_t] dt + \epsilon \sqrt{X_t(1 - X_t)} dW_t, \quad (1.1)$$

where  $a, b, \epsilon > 0$ ,  $X_0 = x \in (0, 1)$  is a nonrandom initial value of the process, and  $W_t$  is a one-dimensional Wiener process,  $t \geq 0$ .

Such equations were introduced in the population genetics by Wright and independently by Fisher, and remain a topical area of investigations until now. For recent sources see, in particular, [1, 3] and the references therein. Conditions for the existence of the invariant measure and its explicit form in the case without mutations can be found in [1]. The paper [2] gives conditions for the existence of the invariant measure (for more general models than (1.1)) and for an exponential rate of convergence; it does not study explicitly recurrence properties, so that the methods are quite different from ours. All issues related to recurrence and convergence rates are interesting not only for population biology (see [8]), but also because this is related to the CIR model in stochastic finance; see, for example, the same source [2]. For CIR there is a more extensive literature, but we do not use it here, so we do not propose further references except for [5], which contains a short proof of sufficiency of Feller’s condition for the CIR model, which is exploited here for the case of Wright–Fisher’s diffusion.

Equation (1.1) has a pathwise (and hence also weakly) unique strong solution which is pathwise unique according to [11]. As it is known – it follows from uniqueness [7] – this solution is a strong Markov process.

It should be highlighted that, although the state space  $[0, 1]$  is a bounded subset of  $\mathbb{R}^1$ , it is still natural to ask about its recurrence properties. Firstly, conditions will be imposed under which the endpoints 0 and 1 are not attained. Hence, if the initial point  $x$  belongs to the open interval  $(0, 1)$ , then the state space is, actually, not  $[0, 1]$ , but this open interval, which is equivalent to the whole  $\mathbb{R}^1$  topologically. Therefore, it makes sense to talk about recurrence “from  $\pm\infty$ ”, where the role of  $-\infty$  and  $+\infty$  are played by 0 and 1,

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respectively. Recurrence is then understood as a return from any small neighborhood of 0 or 1 to a compact set in  $(0, 1)$ . As is well known, very important features of recurrence are the moments of corresponding hitting times. It will be shown that for the intervals  $[a, b]$  with  $a$  close to 0 and  $b$  close to 1 such hitting times have exponential moments.

The paper is arranged as follows: Section 1 is this introduction. Section 2 contains the main results. Section 3 is devoted to the proof of Theorem 2.1, which is the principal result of this paper. Section 4 gives a hint about the proof of Corollary 2.2. For the convenience of the readers, Section 5 provides the proof of Proposition 2.3.

## 2 Main results

For any  $\alpha \in (0, \frac{1}{2})$  let

$$\tau = \tau_\alpha := \inf(t \geq 0 : X_t \in [\alpha, 1 - \alpha]).$$

**Theorem 2.1.** *Let Feller’s condition*

$$a \wedge b \equiv \min(a, b) > \frac{\epsilon^2}{2} \quad (2.1)$$

*be satisfied. Then for any  $c > 0$  there exist  $\alpha > 0$  and  $m > 0$  such that for any  $x \in (0, 1)$ ,*

$$\mathbb{E}_x e^{c\tau_\alpha} \leq C(m)c\alpha^{m+1}((1-x)^{-m} + x^{-m}) + 1, \quad (2.2)$$

where

$$C(m) = \max\left(\frac{2}{bm - \epsilon^2 m(m+1)/2}, \frac{2}{am - \epsilon^2 m(m+1)/2}\right) \equiv \frac{2}{(a \wedge b)m - \epsilon^2 m(m+1)/2}.$$

Moreover,

$$\mathbb{E}_x \int_0^{\tau_\alpha} X_s^{-m-1} ds \leq C(m)c\alpha^{m+1}(x^{-m} + (1-x)^{-m}). \quad (2.3)$$

It will be shown in the proof how to choose  $\alpha$  and  $m$  given the values of  $a, b, c, \epsilon > 0$ .

**Corollary 2.2.** *Under assumption (2.1) there exists a unique invariant measure  $\pi$  for the family of Markov diffusions (1.1), and there exist constants  $C, c, m > 0$  such that the distribution of  $X_t$ , denoted by  $\mu_t^x$ , converges for any  $x \in (0, 1)$  to this invariant measure in the total variation metric with the rate admitting the bound*

$$\|\mu_t^x - \pi\|_{TV} \leq C \exp(-ct)(x^{-m} + (1-x)^{-m} + 2). \quad (2.4)$$

We emphasize that, unlike in Theorem 2.1, here the value  $c > 0$  may not be arbitrary. The reason for this is that the smaller  $\alpha > 0$ , the greater  $c$  can be taken in (2.2). However, on the other hand, the smaller  $\alpha$ , the poorer the nondegeneracy of the diffusion coefficient, which, in turn, decreases the local mixing properties of the process inside the compact  $[\alpha, 1 - \alpha]$ . Yet, overall the bound (2.2) does lead to some exponential estimate for the convergence rate (2.4) with respect to the time variable, non-uniform with respect to the initial state  $x$  from the open interval  $(0, 1)$ .

Recall that the existence and uniqueness of the invariant measure for equation (1.1) are well known. However, both these claims follow again independently and automatically from Theorem 2.1 by virtue of the Harris–Khasminskii principle; see [6]. Yet, the main news of this corollary is the bound (2.4).

The upper bound (2.2) leaves open the question of recurrence – and hence of convergence rate – if the process starts at zero or one, or if it touches one of these endpoints. (Note that if this happens, it still does not spoil the statement about a strong solution.) So, for the convenience of the reader and having in mind further applications in the next studies, let us state the fact of the unattainability of the endpoints of the interval  $(0, 1)$  under Feller’s condition in a proposition; in what follows, a brief simple proof will be provided not repeating the original paper [4] (where it is shown rigorously exactly for this model by a different method), but based on the nice idea from [5] with some slight changes.

**Proposition 2.3.** *Under the assumptions of Theorem 2.1 including Feller’s condition (2.1),*

$$\mathbb{P}_x(\exists t < \infty : X_t(1 - X_t) = 0) = 0 \quad \text{for all } x \in (0, 1). \quad (2.5)$$

### 3 Proof of Theorem 2.1

For definiteness, suppose  $x \in (0, \alpha)$ , where  $\alpha \in (0, \frac{1}{2})$  is a constant to be chosen in what follows. The case  $x \in (1 - \alpha, 1)$  will be commented at the end of this section. Let  $c > 0$  and  $m > 0$ ; here  $c > 0$  is arbitrary, while  $m > 0$  will also be chosen. Let us consider the function  $V(x, t) = \exp(ct)x^{-m}$  and let us compute the stochastic differential of  $V(X_t, t)$ : by Itô’s formula, we have

$$\begin{aligned} de^{ct}X_t^{-m} &= e^{ct}\left\{cdt - mX_t^{-m-1}dX_t + \frac{m(m+1)}{2}(dX_t)^2\right\} \\ &= e^{ct}\left\{cX_t^{-m}dt - mX_t^{-m-1}([a - (a+b)X_t]dt + \epsilon\sqrt{X_t(1-X_t)}dW_t) + \frac{\epsilon^2m(m+1)}{2}X_t^{-m-1}(1-X_t)dt\right\} \\ &= e^{ct}X_t^{-m-1}\left\{cX_t - ma + m(a+b)X_t + \frac{\epsilon^2m(m+1)}{2}(1-X_t)\right\}dt - e^{ct}m\epsilon\sqrt{X_t(1-X_t)}X_t^{-m-1}dW_t \\ &= e^{ct}X_t^{-m-1}\left\{-ma + \frac{\epsilon^2m(m+1)}{2} + \left(c + m(a+b) - \frac{\epsilon^2m(m+1)}{2}\right)X_t\right\}dt \\ &\quad - e^{ct}m\epsilon\sqrt{X_t(1-X_t)}X_t^{-m-1}dW_t. \end{aligned}$$

Let us define stopping times

$$\tau_{t,N} := \min\left(\tau, t, \inf\left\{s \geq 0 : X_s \wedge (1-X_s) \leq \frac{1}{N}\right\}\right), \quad t \geq 0, N = 3, 4, \dots$$

Integrating and taking expectations, we obtain

$$\mathbb{E}_x e^{c\tau_{t,N}} X_{\tau_{t,N}}^{-m} - x^{-m} = \mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left[ \left( c + (a+b)m - \frac{\epsilon^2}{2}m(m+1) \right) X_s^{-m} + \left( \frac{\epsilon^2}{2}m(m+1) - am \right) X_s^{-m-1} \right] ds.$$

Since  $\mathbb{E}_x e^{c\tau_{t,N}} X_{\tau_{t,N}}^{-m} > 0$ , we may conclude that

$$x^{-m} \geq \mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left[ \left( am - \frac{\epsilon^2}{2}m(m+1) \right) X_s^{-m-1} - \left( c + (a+b)m - \frac{\epsilon^2}{2}m(m+1) \right) X_s^{-m} \right] ds.$$

Now let us choose  $\alpha > 0$  so that

$$\left( am - \frac{\epsilon^2}{2}m(m+1) \right) \geq 2(c + (a+b)m)\alpha. \quad (3.1)$$

(Here the multiplier 2 on the right-hand side can be replaced by any constant strictly greater than one; any such replacement would change the constants in the resulting inequality, but not the principal conclusion.)

In turn, for this it suffices that

$$\alpha < \frac{(am - \frac{\epsilon^2}{2}m(m+1))}{2(c + (a+b)m)} \quad (3.2)$$

and

$$am - \frac{\epsilon^2}{2}m(m+1) > 0. \quad (3.3)$$

Note that, with the multiplier 2 in (3.1), the right-hand side of (3.2) is automatically less than  $\frac{1}{2}$ , so that the interval  $[\alpha, 1 - \alpha]$  is not empty. If some other multiplier close to 1 were used in (3.1), it would be natural to take

$$0 < \alpha < \frac{(am - \frac{\epsilon^2}{2}m(m+1))}{2(c + (a+b)m)} \wedge \left( \frac{1}{2} - \delta \right)$$

with any small  $\delta \in (0, \frac{1}{2})$ . The bound (3.3) is equivalent to

$$\frac{2a}{\epsilon^2} > m + 1,$$

and exactly Feller’s condition (2.1) guarantees that there exists a positive  $m$  such that

$$0 < m < \frac{2a}{\epsilon^2} - 1.$$

With this choice of  $m > 0$ , we obtain

$$\begin{aligned} & \mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left[ \left( am - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m-1} - \left( c + (a+b)m - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m} \right] ds \\ & \geq \mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left[ \left( am - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m-1} - \frac{1}{2} \left( am - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m-1} \right] ds \\ & = \frac{1}{2} \mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left( am - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m-1} ds. \end{aligned}$$

Note that the integrand in the last equality is positive due to the condition (3.3). Hence, we get

$$\mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} \left[ \frac{1}{2} \left( am - \frac{\epsilon^2}{2} m(m+1) \right) X_s^{-m-1} \right] ds \leq x^{-m}.$$

Since  $X_s \leq \alpha$  and  $e^{cs} \geq 1$  for any  $s \in [0, \tau_{t,N}]$ ,  $c > 0$ , the following two inequalities hold:

$$\mathbb{E}_x \int_0^{\tau_{t,N}} e^{cs} ds \leq \frac{2}{am - \epsilon^2 m(m+1)/2} \alpha^{m+1} x^{-m} \quad (3.4)$$

and

$$\mathbb{E}_x \int_0^{\tau_{t,N}} X_s^{-m-1} ds \leq \frac{2}{am - \epsilon^2 m(m+1)/2} x^{-m}. \quad (3.5)$$

Set

$$C(m) := \frac{2}{am - \epsilon^2 m(m+1)/2}.$$

Then, by integration of the left-hand side of (3.4), for  $0 < x < \alpha$  we obtain

$$\mathbb{E}_x e^{c\tau_{t,N}} \leq C(m) \alpha^{m+1} x^{-m} + 1.$$

By virtue of Fatou’s lemma (or by the monotone convergence theorem), we get, as  $N \uparrow \infty$  and  $t \uparrow \infty$ ,

$$\mathbb{E}_x e^{c\tau} \leq C(m) \alpha^{m+1} x^{-m} + 1.$$

Similarly, applying Fatou’s lemma to inequality (3.5), we get, as  $N \uparrow \infty$  and  $t \uparrow \infty$ ,

$$\mathbb{E}_x \int_0^{\tau} X_s^{-m-1} ds \leq C(m) x^{-m}.$$

This proves the statement of the theorem in the case of  $x \in (0, \alpha)$ .

The case  $x \in (1 - \alpha, 1)$  can be treated similarly via the Lyapunov function  $V(t, x) = \exp(ct)(1 - x)^{-m}$ . Otherwise, the change of variables  $y = 1 - x$  may be used. Then the transformation

$$Y_t := 1 - X_t$$

reduces the situation  $x > 1 - \alpha$  to the previous case  $x < \alpha$ . Here the choice of  $m > 0$  and  $\alpha > 0$  has to satisfy the inequalities

$$bm - \frac{\epsilon^2}{2} m(m+1) > 0$$

and

$$\alpha < \frac{(bm - \frac{\epsilon^2}{2} m(m+1))}{2(c + (a+b)m)}$$

instead of (3.2) and (3.3), respectively. This means that overall we may take any value of  $m$  in the interval

$$0 < m < \frac{2(a \wedge b)}{\epsilon^2} - 1,$$

and then any value of  $\alpha$  in the interval

$$0 < \alpha < \frac{((a \wedge b)m - \frac{\epsilon^2}{2}m(m+1))}{2(c + (a+b)m)}.$$

The theorem is proved.

## 4 Proof of Corollary 2.2, sketch

The statement follows from inequalities (2.2) and (2.3) of Theorem 2.1 via one of the standard techniques of the Harris–Khasminskii principle and coupling, as can be seen, for example, in [9, 10]. The details will be presented in the next publications for a more general model.

## 5 Proof of Proposition 2.3

Note that this proof is fully independent of the statement of Theorem 2.1 and of its proof.

Let us study the case of the endpoint 0. Set  $B(x) := a - (a+b)x$  and  $\sigma(x) := \sqrt{x(1-x)}$ . Let

$$0 < \kappa < \frac{a - \epsilon^2/2}{a+b}, \quad b_0 := a - (a+b)\kappa, \quad n \in \left(0, \frac{2b_0}{\epsilon^2} - 1\right]$$

(this value  $n$  does not necessarily need to be integer). Note that  $b_0 > \epsilon^2/2$ , so that the choice of such a positive  $n$  is possible, and  $b_0 - (n+1)\epsilon^2/2 \geq 0$ . Consider a Lyapunov function  $V(x) = x^{-n}$ . By Itô’s formula,

$$dX_t^{-n} = [-nB(X_t)X_t^{-n-1} + \frac{n(n+1)\epsilon^2}{2}\sigma(X_t)^2X_t^{-n-2}] dt - n\epsilon\sigma(X_t)X_t^{-n-1} dW_t.$$

Let any  $\beta \in (0, \kappa)$ , and define the stopping times

$$\begin{aligned} \gamma_\beta &:= \min\{t \geq 0 : X_t \leq \beta\}, \\ \gamma_0 &:= \min\{t \geq 0 : X_t = 0\}, \\ T_\kappa &:= \min\{t \geq 0 : X_t \geq \kappa\}, \\ \tau_{t,\beta,\kappa} &:= \min(\gamma_\beta, T_\kappa, t) \equiv \min(\gamma_\beta \wedge t, T_\kappa). \end{aligned}$$

It suffices to show that  $\mathbb{P}_x(\gamma_0 < T_\kappa) = 0$ . We have

$$\begin{aligned} \mathbb{E}_x X_{\tau_{t,\beta,\kappa}}^{-n} &= x^{-n} + \mathbb{E}_x \int_0^{\tau_{t,\beta,\kappa}} \left[ -nB(X_s)X_s^{-n-1} + \frac{n(n+1)\epsilon^2}{2}\sigma(X_s)^2X_s^{-n-2} \right] ds \\ &\leq x^{-n} + \mathbb{E}_x \int_0^{\tau_{t,\beta,\kappa}} \left[ -nb_0X_s^{-n-1} + \frac{n(n+1)\epsilon^2}{2}X_s(1-X_s)X_s^{-n-2} \right] ds \\ &= x^{-n} + \mathbb{E}_x \int_0^{\tau_{t,\beta,\kappa}} \left[ -nX_s^{-n-1} \left( b_0 - \frac{(n+1)\epsilon^2}{2} \right) - \frac{n(n+1)\epsilon^2}{2}X_s^{-n} \right] ds \\ &\leq x^{-n} - \mathbb{E}_x \int_0^{\tau_{t,\beta,\kappa}} \frac{n(n+1)\epsilon^2}{2}X_s^{-n} ds \leq x^{-n}, \end{aligned}$$

where the second to last inequality follows from  $b_0 - \frac{(n+1)e^2}{2} \geq 0$  by the choice of  $n$ . Now by the Bienaymé–Chebyshev–Markov inequality,

$$\mathbb{P}_x(X_{\tau_{t,\beta,x}} \leq \beta) = \mathbb{P}_x(X_{\tau_{t,\beta,x}}^{-n} \geq \beta^{-n}) \leq \beta^n x^{-n} \rightarrow 0, \quad \beta \rightarrow 0.$$

So for any  $t \geq 0$ ,

$$\mathbb{P}_x(\gamma_\beta \wedge t < T_\kappa, \gamma_\beta \leq t) = \mathbb{P}_x(X_{\tau_{t,\beta,x}} \leq \beta, \gamma_\beta \wedge t < T_\kappa, \gamma_\beta \leq t) \leq \mathbb{P}_x(X_{\tau_{t,\beta,x}} \leq \beta) \rightarrow 0, \quad \beta \rightarrow 0.$$

Since  $\gamma_0 > \gamma_\beta$  for any  $\beta$ , we obtain for each  $t$ ,

$$\mathbb{P}_x(\gamma_0 \wedge t < T_\kappa, \gamma_0 \leq t) \leq \mathbb{P}_x(\gamma_\beta \wedge t < T_\kappa, \gamma_\beta \leq t) \rightarrow 0, \quad \beta \rightarrow 0.$$

Hence  $\mathbb{P}_x(\gamma_0 < T_\kappa) = 0$ , and so also

$$\mathbb{P}_x(\exists t < \infty : X_t = 0) = 0 \quad \text{for all } x \in (0, 1),$$

as required. The case of the endpoint 1 is considered similarly; these two cases combined finally lead to equation (2.5).

**Funding:** For the second author of this study, part of Proposition 2.3 was prepared within the framework of the HSE University Basic Research Program, and part of Corollary 2.2 it was funded by the Russian Science Foundation grant 17-11-0198 (extended).

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