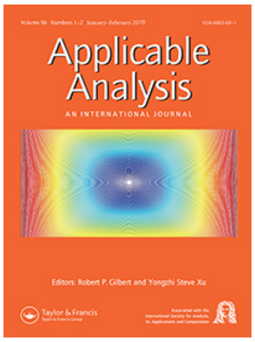

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Weak convergence of attractors of reaction–diffusion systems with randomly oscillating coefficients

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ABSTRACT
We consider reaction–diffusion systems with random rapidly oscillating coefficient. We do not assume any Lipschitz condition for the nonlinear function in the system, so, the uniqueness theorem for the corresponding initial-value problem may not hold for the considered reaction–diffusion system. Under the assumption that the random function is ergodic and statistically homogeneous in space variables we prove that the trajectory attractors of these systems tend in a weak sense to the trajectory attractors of the homogenized reaction–diffusion systems whose coefficient is the average of the corresponding term of the original systems.

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1. Introduction

This paper is devoted to investigations of the asymptotic behavior of attractors to a system of nonlinear differential equations in domains with microinhomogeneous structure. We study a weak convergence and effective behavior of attractors as a small parameter tends to zero. To study such phenomenon we apply the homogenization method (cf. e.g. [1–7], for random case cf. for instance, [8–13]) as well as a delicate analysis of trajectory and global attractors.

Attractors describe the behavior of solutions of dissipative nonlinear evolution equations as time tends to infinity. It is also convenient to study, using attractors, the stability and instability of the limiting structures of the corresponding dynamical systems. Attractors single out the most essential limit sets of trajectories, which characterize the whole dynamics of the considered model described by evolution equations (see, e.g. monographes [14–16] and the references therein).

More precisely, our interest is the asymptotic behavior of trajectory and global attractors of reaction–diffusion systems with random terms that oscillate rapidly in space variable.

The Bogolyubov averaging principle [17] was used in the first papers [18–20] on averaging of attractors of evolution equations with rapidly, but non-randomly oscillating terms. The averaging of global attractors for parabolic equations with oscillating parameters has been considered in [15,21–24]. Some problems related to the homogenization and the averaging of uniform global attractors for dissipative wave equations has been considered in [19,25–28], in presence of time oscillations,

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To the blessed memory of V. V. Zhikov.

and in [15,29–31], in presence of oscillations in space. Similar problems for autonomous and non-autonomous 2D Navier–Stokes equations has been studied in [15,30,32,33]. Papers [33–37] deal with partial differential equations containing singular oscillating terms.

In the paper [38], the authors study random attractors and inertial manifolds for scalar parabolic equations with random terms on fast time scale. Under some spectral gap condition, it was shown that the inertial manifolds of the fast time scale equations tend to the inertial manifold of the averaged system when the scaling parameter tends to zero.

The method of trajectory attractors (see Figure 1[1] for example) for dissipative partial differential equations was developed in [15,39,40]. This approach is very powerful in the study of the long time behavior of solutions of evolution equations for which the uniqueness theorem of the corresponding initial-value problem is not proved yet (e.g. for the inhomogeneous 3D Navier–Stokes system) or does not hold (for the reaction–diffusion systems considered in the present paper). Some averaging problem for trajectory attractors of autonomous and non-autonomous evolution equations with rapidly oscillating terms were studied in [15,30].

In the paper we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the reaction–diffusion system with randomly oscillating coefficient converges in a weak sense almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized system in an appropriate functional space. Here, $1/\varepsilon$ is proportional to the rate of oscillations.

In Section 2 we give necessary definitions of randomness and formulate the Birkhoff ergodic theorem and related assertions. In Section 3 we define the main notions and formulate theorems concerning the trajectory attractors of autonomous evolution equations. In Section 4 we construct the trajectory attractor for the considered reaction–diffusion systems. Section 5 is devoted to the study of the averaging of attractors of autonomous reaction–diffusion with randomly rapidly oscillating terms in space variable.

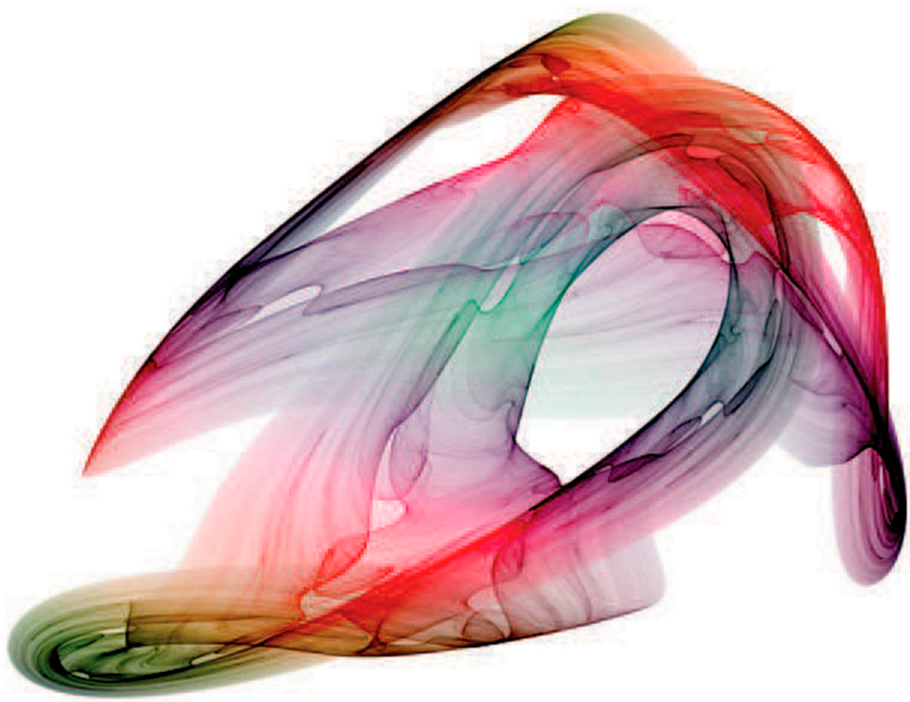

Figure 1. Attractor.

2. Preliminaries

Assume that $(\Omega, \mathcal{B}, \mu)$ is a probability space, i.e. the set Ω is endowed with a σ-algebra $\mathcal{B}$ of its subsets and a σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

Definition 2.1: A family of measurable maps $\mathcal{T}_\xi : \Omega \to \Omega$, $\xi \in \mathbb{R}^n$ is called a *space dynamical system* if the following properties hold:

1. group property: $\mathcal{T}_{\xi_1+\xi_2} = \mathcal{T}_{\xi_1}\mathcal{T}_{\xi_2}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^n$; $\mathcal{T}_0 = Id$ Id is the identity mapping on Ω;
2. isometry property (the mappings $\mathcal{T}_\xi$ preserve the measure μ on Ω): $\mathcal{T}_\xi B \in \mathcal{B}$, $\mu(\mathcal{T}_\xi B) = \mu(B)$, $\forall \xi \in \mathbb{R}^n$, $\forall B \in \mathcal{B}$;
3. measurability: for any measurable function $\psi(\omega)$ on Ω, the function $\psi(\mathcal{T}_\xi \omega)$ is measurable on $\Omega \times \mathbb{R}^n$ and continuous in ξ.

Let $L_q(\Omega, \mu)$ $(q \geq 1)$ be the space of measurable functions on Ω whose absolute value at the power q is integrable with respect to the measure μ. For $q = \infty$ we consider the space $L_\infty(\Omega, \mu)$. If $\mathcal{T}_\xi : \Omega \to \Omega$ is a space dynamical system, then on the space $L_q(\Omega, \mu)$ we define a parameter dependent group of operators $\{\mathcal{T}_\xi\}, \xi \in \mathbb{R}^n$ (we keep the same notation), by the formula $(\mathcal{T}_\xi \psi)(\omega) := \psi(\mathcal{T}_\xi \omega)$, $\psi \in L_q(\Omega, \mu)$.

Condition (3) in Definition 2.1 implies that the group $\mathcal{T}_\xi$ is strongly continuous, i.e. we have $\lim_{\xi \to 0} \|\mathcal{T}_\xi \psi - \psi\|_{L_q(\Omega,\mu)} = 0$ for any $\psi \in L_q(\Omega, \mu)$.

Definition 2.2: Suppose that $\psi(\omega)$ is a measurable function on Ω. The real function $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$, $\xi \in \mathbb{R}^n$, for fixed $\omega \in \Omega$ is called the *realization of the function* ψ.

The following assertion is proved, for instance, in [5,7].

Proposition 2.1: *If $\psi \in L_q(\Omega, \mu)$, then ω-almost all realizations $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$ belong to $L_q^{loc}(\mathbb{R}^n)$. If the sequence $\{\psi_k\} \subset L_q(\Omega, \mu)$ converges in $L_q(\Omega, \mu)$ to the function ψ, then there exists a subsequence $\{\psi_{k'}\}$ such that ω-almost all realizations $\xi \mapsto \psi_{k'}(\mathcal{T}_\xi \omega)$ converge in $L_q^{loc}(\mathbb{R}^n)$ to the realization $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$.*

Definition 2.3: A measurable function $\psi(\omega)$ on Ω is called *invariant*, if $\psi(\mathcal{T}_\xi \omega) = \psi(\omega)$ for any $\xi \in \mathbb{R}^n$ and almost all $\omega \in \Omega$.

Definition 2.4: A space dynamical system $\mathcal{T}_\xi$ is called *ergodic*, if any invariant function is ω-almost everywhere a constant.

We denote by $\mathcal{R}$ the natural Borel σ-algebra of subsets of $\mathbb{R}^n$. Suppose that a function $\mathcal{F}(\xi) \in L_1^{loc}(\mathbb{R}^n)$.

Definition 2.5: We say that the function $\mathcal{F}(\xi)$ has a SPACE AVERAGE, if the limit

$$M(\mathcal{F}) := \lim_{\varepsilon \to 0} \frac{1}{|R|} \int \dots \int_R \mathcal{F}\left(\frac{x}{\varepsilon}\right) \, dx$$

exists for any bounded Borel set $R \in \mathcal{R}$ with positive measure and does not depend on the choice of R. The number $M(\mathcal{F})$ is called the SPATIAL MEAN VALUE of the function $\mathcal{F}$.

Equivalently, the space average is defined by

$$M(\mathcal{F}) := \lim_{s \to +\infty} \frac{1}{|B_s|} \int \dots \int_{B_s} \mathcal{F}(\xi) \, d\xi, \quad \text{where} \quad B_s = \left\{ \xi \in \mathbb{R}^n \;\middle|\; \frac{\xi}{s} \in B \right\}.$$

From now on we make use of a generalization of the well-known Birkhoff theorem (see [41,42] and also [5,7]). Namely, following the lines of [43, Chapter VIII, §7] it can be obtained in the form (see, e.g. [44]):

Theorem 2.1 (Birkhoff ergodic theorem): *Let $P \subset \mathbb{R}^n$. Assume that the space dynamical system $\mathcal{T}_\xi$ ($\xi \in \mathbb{R}^n$) satisfy Definition 2.1. Consider a measurable real function $\psi = \psi(x, \omega)$, $x \in P$, $\omega \in \Omega$,* *such that, for every $x \in P$, the function $\psi(x,\cdot) \in L_q(\Omega,\mu)$ with $q \geqslant 1$. Then, for every $x \in P$ and for almost all $\omega \in \Omega$, the realization $\psi(x,\mathcal{T}_\xi\omega)$ has the space mean value $M(\psi(x,\mathcal{T}_\xi\omega))$. Moreover, the space mean value $M(\psi(x,\mathcal{T}_\xi\omega))$ is a conditional mathematical expectation of the function $\psi(x,\omega)$ with respect to the σ -algebra of invariant subsets. Hence, $M(\psi(x,\mathcal{T}_\xi\omega))$ is an invariant function and*

$$\mathbb{E}(\psi)(x) \equiv \int_\Omega \psi(x,\omega)\,d\mu = \int_\Omega M(\psi(x,\mathcal{T}_\xi\omega))\,d\mu.$$

In particular, if the space dynamical system $\mathcal{T}_\xi$ is ergodic then, for almost all $\omega \in \Omega$, we have the identity

$$\mathbb{E}(\psi)(x) = M(\psi)(x).$$

Note that in the formulation of Theorem 2.1 the variable $x \in P$ plays the role of the parameter. In the next sections, we consider $P = \mathbb{T}^n$.

Definition 2.6: Let $P \subset \mathbb{R}^n$. A random function $\psi(x,\xi,\omega)$, $x \in P$, $\xi \in \mathbb{R}^n$, $\omega \in \Omega$, is called *statistically homogeneous* for any x, if the representation $\psi(x,\xi,\omega) = \Psi(x,\mathcal{T}_\xi\omega)$ is valid for some measurable function $\Psi : P \times \Omega \to \mathbb{R}$, where $\mathcal{T}_\xi$ is a space dynamical system in Ω.

The following statement can be found, for instance, in [7].

Proposition 2.2: *Let P be a measurable subset of $\mathbb{R}^n$. Suppose that a measurable function $\digamma(x,\xi)$, $x \in P$, $\xi \in \mathbb{R}^n$, has a space mean value $M(\digamma)(x)$ in $\mathbb{R}^n_\xi$ for every $x \in P$ and the family $\{\digamma\left(x,\frac{x}{\varepsilon}\right), : 0 < \varepsilon \leqslant 1\}, x \in \mathcal{K}$, is bounded in $L_\infty(\mathcal{K})$, where $\mathcal{K}$ is an arbitrary compact subset in P.*

Then $M(\digamma)(\,\cdot\,) \in L^{loc}_\infty(P)$ and we have

$$\digamma\left(x,\frac{x}{\varepsilon}\right) \rightharpoonup M(\digamma)(x) \ast\text{-weakly in } L^{loc}_\infty(P) \text{ as } \varepsilon \to 0.$$

3. Trajectory attractors of evolution equations

In this section we give a scheme for the construction of trajectory attractors of autonomous evolution equations. In the next section we shall apply this scheme to the study of trajectory attractors of the concrete evolution equations with rapidly oscillating coefficients and the corresponding averaged equations.

To begin with we consider an abstract autonomous evolution equation

$$\partial_t u = A(u), \quad t \geq 0. \tag{1}$$

Here $A(\,\cdot\,) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. For instance, $A(u) = a\Delta u - bf(u) + g$ (see Section 4).

We are going to study solutions $u(s)$ of Equation (1) as functions of $s \in \mathbb{R}_+$ as a whole. Here $s \equiv t$ denote the time variable. The set of solutions of (1) is said to be *a trajectory space* $\mathcal{K}^+$ of Equation (1). Let us describe the trajectory space $\mathcal{K}^+$ in greater detail.

At first, we consider solutions $u(s)$ of (1) defined on a fixed time interval $[t_1, t_2]$ from $\mathbb{R}$. We study solutions of (1) in a Banach space $\mathcal{F}_{t_1,t_2}$ that depends on t_1 and t_2. The space $\mathcal{F}_{t_1,t_2}$ consists of functions $f(s)$, $s \in [t_1,t_2]$ such that $f(s) \in E$ for almost all $s \in [t_1,t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, $\mathcal{F}_{t_1,t_2}$ can be the space $C([t_1,t_2]; E)$, or $L_p(t_1,t_2; E)$, for $p \in [1,\infty]$, or the intersection of such spaces (see Section 4). We assume that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1,t_2,\tau_1,\tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}, \quad \forall f \in \mathcal{F}_{\tau_1,\tau_2}, \tag{2}$$

where $[t_1,t_2] \subseteq [\tau_1,\tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1,t_2]$. The constant $C(t_1,t_2,\tau_1,\tau_2)$ is independent of f.

Let $S(h)$ for $h \in \mathbb{R}$ denote the translation operator

$$S(h)f(s) = f(h+s).$$

Evidently, if the argument s of $f(\cdot)$ belongs to $[t_1, t_2]$, then the argument s of $S(h)f(\cdot)$ can be taken form $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}_{t_1,t_2}}, \quad \forall f \in \mathcal{F}_{t_1,t_2}. \tag{3}$$

This assumption is fairly natural.

We assume that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a larger Banach space, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\partial_t f(t)$ is a distribution with values in E_0, $\partial_t f(s) \in D'((t_1,t_2); E_0)$ and we assume that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1,t_2); E_0)$ for all $(t_1,t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is said to be a *solution* of (1) from the space $\mathcal{F}_{t_1,t_2}$ (on the interval (t_1,t_2)) if $\partial_t u(s) = A(u(s))$ in the distributional sense of the space $D'((t_1,t_2); E_0)$.

We also define the space

$$\mathcal{F}_+^{loc} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1,t_2] \subset \mathbb{R}_+\}. \tag{4}$$

For example, if $\mathcal{F}_{t_1,t_2} = C([t_1,t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1,t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is called a solution of (1) from $\mathcal{F}_+^{loc}$ if $\Pi_{t_1,t_2} u(s) \in \mathcal{F}_{t_1,t_2}$ and this function is a solution of (1) for every $[t_1,t_2] \subset \mathbb{R}_+$.

We denote by $\mathcal{K}^+$ a set of solutions of (1) from $\mathcal{F}_+^{loc}$. Notice, that $\mathcal{K}^+$ is not necessarily the set of *all* solutions from $\mathcal{F}_+^{loc}$. The elements of $\mathcal{K}^+$ are called *trajectories* and the set $\mathcal{K}^+$ is called the *trajectory space* of the Equation (1).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for every $h \geq 0$. This is a very natural assumption for solutions of autonomous equations.

We now consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s+h), \quad h \geq 0.$$

It is clear that the mappings $\{S(h), h \geq 0\}$ form a semigroup in $\mathcal{F}_+^{loc}$: $S(h_1)S(h_2) = S(h_1 + h_2)$ for $h_1, h_2 \geq 0$ and $S(0)$ is the identity operator. We change the variable h into the time variable t. The semigroup $\{S(t), t \geq 0\}$ is called the *translation semigroup*. By our assumption the translation semigroup maps the trajectory space $\mathcal{K}^+$ to itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+, \quad \forall t \geq 0. \tag{5}$$

We shall study attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We define a topology in the space $\mathcal{F}_+^{loc}$.

Let a metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ be defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1,t_2] \subset \mathbb{R}$. Similar to (2) and (3) we assume that

$$\rho_{t_1,t_2}\left(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g\right) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}\left(f,g\right), \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$
$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g), \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

Denote by Θ_{t_1,t_2} the corresponding metric spaces on $\mathcal{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of the Banach space $\mathcal{F}_{t_1,t_2}$. However, usually in application ρ_{t_1,t_2}

generates the topology Θ_{t_1,t_2} that is weaker than the strong convergence topology of the Banach space $\mathcal{F}_{t_1,t_2}$.

The inductive limit of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, i.e. by definition, a sequence $\{f_n(s)\} \subset \mathcal{F}_+^{loc}$ converges to $f(s) \in \mathcal{F}_+^{loc}$ as $n \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_n, \Pi_{t_1,t_2}f) \to 0$ as $n \to \infty$ for each $[t_1, t_2] \subset \mathbb{R}_+$. It is not hard to prove that the topology Θ_+^{loc} is metrizable using, for example, the Frechet metric

$$\rho_+(f_1, f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1, f_2)}{1 + \rho_{0,m}(f_1, f_2)}. \tag{6}$$

If it is known that all metric spaces Θ_{t_1,t_2} are complete, then clearly the metric space Θ_+^{loc} is also complete.

We claim that the translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This assertion follows directly from the definition of the topological space Θ_+^{loc}.

We also consider the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}, \tag{7}$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0} \|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{8}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$, then the space $\mathcal{F}_+^b = C^b(\mathbb{R}_+; E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0} \|f(h)\|_E$ and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+; E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0} \int_h^{h+1} \|f(s)\|_E^p \mathrm{d}s \right)^{1/p}$.

Recall that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We require the Banach space $\mathcal{F}_+^b$ only to define bounded subsets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we do not consider the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$. Instead, we utilize the local convergence topology Θ_+^{loc} which is much weaker.

We suppose that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e. every trajectory $u(s) \in \mathcal{K}^+$ of Equation (1) has a finite norm (8). Let us define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 3.1: A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e. for any ε-neighborhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $t \geq t_1$.

It is clear that the attracting property of $\mathcal{P}$ can be formulated in the following equivalent form: for any set $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where

$$\mathrm{dist}_{\mathcal{M}}(X, Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x, Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x, y)$$

is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Definition 3.2: (see [15]) A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc},

(ii) the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B}, \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 3.1: Comparing with [14] one can say that the trajectory attractor $\mathfrak{A}$ is the *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$-*attractor* of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$, that is, $\mathfrak{A}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc} for any bounded (in $\mathcal{F}_+^b$) set $\mathcal{B}$ from $\mathcal{K}^+$:

$$\mathrm{dist}_{\Theta_+^{loc}}(S(t)\mathcal{B}, \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We now formulate the main result on the trajectory attractor for Equation (1).

Theorem 3.1 (see [14,15,39]): *Assume that the trajectory space $\mathcal{K}^+$ corresponding to Equation (1) is contained in $\mathcal{F}_+^b$ and (5) holds. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.*

We now describe the structure of the trajectory attractor $\mathfrak{A}$ of Equation (1) in terms of complete trajectories of this equation. Consider the Equation (1) on the entire time axis

$$\partial_t u = A(u), \; t \in \mathbb{R}. \tag{9}$$

We have defined the trajectory space $\mathcal{K}^+$ of Equation (9) on $\mathbb{R}_+$. We now extend this definition on the entire $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of Equation (9) if $\Pi_+ u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to the semiaxis $\mathbb{R}_+$.

We have introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. We now define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\begin{aligned}\mathcal{F}^{loc} &:= \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2} \; \forall \, [t_1, t_2] \subseteq \mathbb{R}\};\\ \mathcal{F}^b &:= \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},\end{aligned}$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_n(s) \to f(s)$ $(n \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2} f_n(s) \to \Pi_{t_1,t_2} f(s)$ $(n \to \infty)$ in Θ_{t_1,t_2} for each $[t_1, t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 3.3: The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of Equation (9) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of Equation (9) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \leq C_u, \quad \forall h \in \mathbb{R}.$$

Theorem 3.2: *Assume that the hypotheses of Theorem 3.1 holds. Then*

$$\mathfrak{A} = \Pi_+ \mathcal{K},$$

the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

The complete proof can be found in [15,39].

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s \in 0,M \mid \psi(\cdot) \in L_{p_1}(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\right\},$$
$$W_{\infty,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s \in 0,M \mid \psi(\cdot) \in L_{\infty}(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\right\},$$

(where $p_1 \geq 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} \mathrm{d}s\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} \mathrm{d}s\right)^{1/p_0},$$
$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\left\{\|\psi(s)\|_{E_1} \mid s \in [0,M]\right\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} \mathrm{d}s\right)^{1/p_0}.$$

Lemma 3.1 (Aubin–Lions–Simon, [45]): *Assume that $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:*

$$W_{p_1,p_0}(0,T;E_1,E_0) \Subset L_{p_1}(0,T;E), \tag{11}$$
$$W_{\infty,p_0}(0,T;E_1,E_0) \Subset C([0,T];E). \tag{12}$$

In the next section we study evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 3.4: We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\varepsilon_1 \geq 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0\ (\varepsilon \to 0).$$

4. Reaction-diffusion systems and their trajectory attractors

We consider a reaction–diffusion system with randomly oscillating coefficient of the form

$$\partial_t u = a\Delta u - b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u) + g\left(x\right),\ u|_{\partial D} = 0, \tag{13}$$

where $x \in D \Subset \mathbb{R}^n$, $u = (u^1, \dots, u^N)$, $f = (f^1, \dots, f^N)$, and $g = (g^1, \dots, g^N)$. Here a is an $N \times N$ matrix with positive symmetric part: $\frac{1}{2}(a + a^*) \geq \beta I$, where $\beta > 0$, and the real function $b(x, \xi, \omega) \in C(D \times \mathbb{R}^n)$ is positive for almost every $\omega \in \Omega$. The Laplace operator $\Delta := \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ acts in the x-space.

We note that all the results can also be applied to the systems with nonlinear terms of the form $\sum_{j=1}^m b_j\left(x, \frac{x}{\varepsilon}, \omega\right) f_j(u)$,where b_j are matrices and $f_j(u)$ are polynomial vectors with respect to u. For brevity, we consider the case $m = 1$ and $b_1\left(x, \frac{x}{\varepsilon}, \omega\right) = b\left(x, \frac{x}{\varepsilon}, \omega\right) I$, where I is the identity matrix.

For the simplicity we assume that the vector function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfies the following inequalities:

$$f(v) \cdot v \geq \gamma |v|^p - C,\ |f(v)| \leq C_1\left(|v|^{p-1} + 1\right),\ p \geq 2. \tag{14}$$

Other conditions are also admissible, for example, the inequalities with different degrees $\mathbf{p} = (p_1, \ldots, p_2)$ of the form

$$f(v) \cdot v \geq \gamma \sum_{i=1}^{N} |v^i|^{p_i} - C,$$

$$\sum_{i=1}^{N} |f^i(v)|^{\frac{p_i}{p_i-1}} \leq C_1 \left(\sum_{i=1}^{N} |v^i|^{p_i} + 1 \right), \; p_k \geq 2, \; \forall v \in \mathbb{R}^N.$$

Notice that we *do not assume* that the nonlinear vector function $f(v)$ satisfies the Lipschitz condition with respect to v.

Let $\mathcal{T}_\xi, \xi \in \mathbb{R}^n$, be an ergodic space dynamical system in a probability space $(\Omega, \mathcal{B}, \mu)$.

We assume that the function $b(x, \frac{x}{\varepsilon}, \omega)$ is statistically homogeneous, that is

$$b(x, \xi, \omega) = \mathbf{B}(x, \mathcal{T}_\xi \omega),$$

where $\mathbf{B} : D \times \Omega \to \mathbb{R}$ is measurable.

We also assume that $\mathbf{B}(x, \omega) \in C_b(\overline{D})$ for almost all $\omega \in \Omega$ and

$$0 < \beta_0 \leq \mathbf{B}(x, \omega) \leq \beta_1, \; \forall x \in D. \tag{15}$$

Birkhoff ergodic theorem implies that the function $b\left(x, \xi, \omega\right) = \mathbf{B}(x, \mathcal{T}_\xi \omega)$ has the space mean value

$$b^{\text{hom}}(x) := M(b)(x) = \mathbb{E}(\mathbf{B})(x)$$

for every $x \in D$. It is clear that the function $b^{\text{hom}}(x)$ also satisfies the inequality

$$0 < \beta_0 \leq b^{\text{hom}}(x) \leq \beta_1, \; \forall x \in D.$$

It follows from Proposition 2.2, that almost surely in $\omega \in \Omega$

$$\int_D b\left(x, \frac{x}{\varepsilon}, \omega\right) \varphi(x) \mathrm{d}x \to \int_D b^{\text{hom}}\left(x\right) \varphi(x) \mathrm{d}x \; (\varepsilon \to 0+) \; \forall \varphi \in L_1(D). \tag{16}$$

We introduce the spaces $\mathbf{H} := [L_2(D)]^N$, $\mathbf{V} := [H_0^1(D)]^N$, and $\mathbf{L}_p := [L_p(D)]^N$. The norms in these spaces are denoted, respectively, by

$$\|v\|^2 := \int_D \sum_{i=1}^{N} |v^i(x)|^2 \mathrm{d}x, \; \|v\|_1^2 := \int_D \sum_{i=1}^{N} |\nabla v^i(x)|^2 \mathrm{d}x, \; \|v\|_{\mathbf{L}_p}^p := \int_D \sum_{i=1}^{N} |v^i(x)|^p \mathrm{d}x.$$

Recall that $\mathbf{V}' := [H^{-1}(D)]^N$ and $\mathbf{L}_q$ are the dual spaces of $\mathbf{V}$ and $\mathbf{L}_p$, where $q = p/(p-1)$.

As in [15,46] we study weak solutions of the system (13), that is, the functions

$$u(x, s) \in L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap L_p^{loc}\left(\mathbb{R}_+; \mathbf{L}_p\right)$$

which satisfy the system (13) in the distributions sense.

If $u(x, t) \in L_p(0, M; \mathbf{L}_p)$, then it follows from the condition (14) that $f(u(x, t)) \in L_q(0, M; \mathbf{L}_q)$. Inequality (15) implies that $b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u(x, t)) \in L_q(0, M; \mathbf{L}_q)$ as well. At the same time, if $u(x, t) \in L_2(0, M; \mathbf{V})$, then $a\Delta u(x, t) + g\left(x, \frac{x}{\varepsilon}, \omega\right) \in L_2(0, M; \mathbf{V}')$. Therefore, for an arbitrary weak solution $u(x, s)$ of the system (13) we have

$$\partial_t u(x, t) \in L_q(0, M; \mathbf{L}_q) + L_2(0, M; \mathbf{V}').$$

The Sobolev embedding theorem implies that

$$L_q(0,M;\mathbf{L}_q)+L_2(0,M;\mathbf{V}')\subset L_q\left(0,M;\mathbf{H}^{-r}\right),$$

where the space $\mathbf{H}^{-r}:=[H^{-r}(D)]^N$ and $r=\max\left\{1,n(1/2-1/p)\right\}$. Hence, for any weak solution $u(x,t)$ of (13) we have $\partial_t u(x,t)\in L_q\left(0,M;\mathbf{H}^{-r}\right)$.

For every $u_0\in\mathbf{H}$ there exists at least one weak solution $u(x,s)$ of the system (13) such that $u(0)=u_0$ (see [14,46]). This solution is not necessarily unique because we do not assume the Lipschitz condition for $f(v)$ with respect to v.

We now apply the scheme described in Section 3 to construct the trajectory attractor for the system (13), which has the form (1) if we set $E_1=\mathbf{L}_p\cap\mathbf{V}$, $E_0=\mathbf{H}^{-r}$, $E=\mathbf{H}$ and $A(u)=a\Delta u-b_\varepsilon(\,\cdot\,)f(u)+g_\varepsilon(\,\cdot\,)$.

To describe the trajectory space $\mathcal{K}_\varepsilon^+$ for the system (13), we follow the general framework of Section 3 and define the Banach spaces for every $[t_1,t_2]\in\mathbb{R}$

$$\mathcal{F}_{t_1,t_2}:=L_p(t_1,t_2;\mathbf{L}_p)\cap L_2(t_1,t_2;\mathbf{V})\cap L_\infty(t_1,t_2;\mathbf{H})\cap\left\{v\mid\partial_t v\in L_q\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\}\tag{17}$$

with norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}}:=\|v\|_{L_p(t_1,t_2;\mathbf{L}_p)}+\|v\|_{L_2(t_1,t_2;\mathbf{V})}+\|v\|_{L_\infty(0,M;\mathbf{H})}+\|\partial_t v\|_{L_q\left(t_1,t_2;\mathbf{H}^{-r}\right)}.\tag{18}$$

It is clear that the condition (2) holds for the norm (18) and the translation semigroup $\{S(h)\}$ satisfies (3).

Setting $\mathcal{D}_{t_1,t_2}=L_q\left(t_1,t_2;\mathbf{H}^{-r}\right)$ we have that $\mathcal{F}_{t_1,t_2}\subseteq\mathcal{D}_{t_1,t_2}$ and if $u(s)\in\mathcal{F}_{t_1,t_2}$, then $A(u(s))\in\mathcal{D}_{t_1,t_2}$. We can consider a weak solutions of the system (13) as a solution of equation in the general scheme of Section 3.

Defining the space (4) we obtain that

$$\mathcal{F}_+^{loc}=L_p^{loc}(\mathbb{R}_+;\mathbf{L}_p)\cap L_2^{loc}(\mathbb{R}_+;\mathbf{V})\cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H})\cap\left\{v\mid\partial_t v\in L_q^{loc}(\mathbb{R}_+;\mathbf{H}^{-r})\right\}.$$

We denote by $\mathcal{K}_\varepsilon^+$ the set of all weak solutions of the system (13). Recall that for any $u_0\in\mathbf{H}$ there exist at least one trajectory $u(\,\cdot\,)\in\mathcal{K}_\varepsilon^+$ such that $u(0)=u_0$. Therefore, the trajectory space $\mathcal{K}_\varepsilon^+$ of the system (13) is not empty and is sufficiently large.

It is clear that $\mathcal{K}_\varepsilon^+\subset\mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, that is, if $u(s)\in\mathcal{K}_\varepsilon^+$, then $u(h+s)\in\mathcal{K}_\varepsilon^+$ for all $h\geq 0$. Therefore,

$$S(h)\mathcal{K}_\varepsilon^+\subseteq\mathcal{K}_\varepsilon^+,\quad\forall h\geq 0.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathcal{F}_{t_1,t_2}$ using the norms of the spaces $L_2(t_1,t_2;\mathbf{H})$:

$$\rho_{0,M}(u,v)=\left(\int_0^M\|u(s)-v(s)\|^2\mathrm{d}s\right)^{1/2},\quad\forall u(\,\cdot\,),v(\,\cdot\,)\in\mathcal{F}_{0,M}.$$

These metrics generates the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$. Recall that a sequence $\{v_n\}\subset\mathcal{F}_+^{loc}$ converges to $v\in\mathcal{F}_+^{loc}$ as $n\to\infty$ in Θ_+^{loc} if $\|v_n(\,\cdot\,)-v(\,\cdot\,)\|_{L_2(0,M;\mathbf{H})}\to 0$ $(n\to\infty)$ for each $M>0$. The topology Θ_+^{loc} is metrizable (see (6)) and the corresponding metric space is complete. We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (13). The translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\varepsilon^+$ is continuous in the considering topology Θ_+^{loc}.

Following the general scheme of Section 3, we define bounded sets in $\mathcal{K}_\varepsilon^+$ using the Banach space $\mathcal{F}_+^b$ (see (7)). We clearly have

$$\mathcal{F}_+^b = L_p^b(\mathbb{R}_+; \mathbf{L}_p) \cap L_2^b(\mathbb{R}_+; \mathbf{V}) \cap L_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \mid \partial_t v \in L_q^b(\mathbb{R}_+; \mathbf{H}^{-r}) \right\} \tag{19}$$

and $\mathcal{F}_+^b$ is a subspace of $\mathcal{F}_+^{loc}$.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\varepsilon^+$, $S(t) : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+$, $t \geq 0$.

Let $\mathcal{K}_\varepsilon$ be the kernel of the system (13) that consists of all weak complete solutions $u(s), \in \mathbb{R}$, of the system bounded in the space

$$\mathcal{F}^b = L_p^b(\mathbb{R}; \mathbf{L}_p) \cap L_2^b(\mathbb{R}; \mathbf{V}) \cap L_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \mid \partial_t v \in L_q^b(\mathbb{R}; \mathbf{H}^{-r}) \right\}$$

Proposition 4.1: *Under the hypotheses* (14) *and* (15) *the system (13) has the trajectory attractors* $\mathfrak{A}_\varepsilon$ *in the topological space* Θ_+^{loc}. *The set* $\mathfrak{A}_\varepsilon$ *is almost surely uniformly (w.r.t.* $\varepsilon \in (0,1)$*) bounded in* $\mathcal{F}_+^b$ *and compact in* Θ_+^{loc}. *Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon,$$

the kernel $\mathcal{K}_\varepsilon$ *is non-empty and uniformly (w.r.t.* $\varepsilon \in (0,1)$*) bounded in* $\mathcal{F}^b$.

The proof of this proposition almost coincides with the proof given in [15] for a particular case.

We note that

$$\mathfrak{A}_\varepsilon \subset \mathcal{B}_0(R), \quad \forall \varepsilon \in (0,1),$$

where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Lemma 3.1 implies that

$$\mathcal{B}_0(R) \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{20}$$

$$\mathcal{B}_0(R) \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{21}$$

Inclusion (20) follows from (11), where we set $E_0 = \mathbf{H}^{-r}$, $E = \mathbf{H}^{1-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_1 = 2$, $p_0 = q$, and from the compact embedding $\mathbf{V} \Subset \mathbf{H}^{1-\delta}$. Inclusion (20) follows from (12) and from the compact embeddings $\mathbf{H} \Subset \mathbf{H}^{-\delta}$, if we set $E_0 = \mathbf{H}^{-\mathbf{r}}(D)$, $E = \mathbf{H}^{-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_0 = q$.

Using compact inclusions (20) and (21), we strengthen the attraction to the constructed trajectory attractor.

Corollary 4.1: *For any set* $\mathcal{B} \subset \mathcal{K}_\varepsilon^+$ *bounded in* $\mathcal{F}_+^b$ *we have*

$$\operatorname{dist}_{L_2(0,M;H^{1-\delta})} \left(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon \right) \to 0 \ (t \to \infty),$$
$$\operatorname{dist}_{C([0,M];H^{-\delta})} \left(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon \right) \to 0 \ (t \to \infty),$$

where M *is an arbitrary positive number.*

5. Homogenization of attractors for reaction–diffusion systems

In this section, we study the limit behavior of trajectory attractors $\mathfrak{A}_\varepsilon$ of random reaction–diffusion systems (13) as $\varepsilon \to 0+$ and their relation to the trajectory attractor of the corresponding homogenized system.

Along with the system (13) we consider the homogenized system

$$\partial_t \bar{u} = a\Delta \bar{u} - b^{hom}(x) f(\bar{u}) + g(x), \ \bar{u}|_{\partial D} = 0. \tag{22}$$

Clearly the system (22) also has trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the system (22) and

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$$

where $\overline{\mathcal{K}}$ is the kernel of system (22) in $\mathcal{F}^b$.

Let us formulate the main theorem concerning the reaction–diffusion system.

Theorem 5.1: *The following limit holds almost surely in the topological space* Θ_+^{loc}

$$\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}} \ \text{as } \varepsilon \to 0+ . \tag{23}$$

Moreover, almost surely

$$\mathcal{K}_\varepsilon \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{in } \Theta^{loc}. \tag{24}$$

***Proof*:** It is clear that (24) implies (23). Therefore it is sufficient to prove (24), that is, for every neighborhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there are exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that almost surely

$$\mathcal{K}_\varepsilon \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for } \varepsilon < \varepsilon_1. \tag{25}$$

Suppose that (25) is not true. Consider the corresponding subset $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and (25) does not hold for all $\omega \in \Omega'$. Then, for each $\omega \in \Omega'$, there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_n \to 0+ \ (n \to \infty)$, and a sequence $u_{\varepsilon_n}(\,\cdot\,) = u_{\varepsilon_n}(\omega, s) \in \mathcal{K}_{\varepsilon_n}$ such that

$$u_{\varepsilon_n} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } n \in \mathbb{N}, \omega \in \Omega'. \tag{26}$$

For each $\omega \in \Omega'$, the function $u_{\varepsilon_n}(s), s \in \mathbb{R}$ is the solutions to the system

$$\partial_t u_{\varepsilon_n} = a\Delta u_{\varepsilon_n} - b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) + g\left(x\right), \ u_{\varepsilon_n}|_{\partial D} = 0 \tag{27}$$

on the entire time axis $t \in \mathbb{R}$. Moreover the sequence $\{u_{\varepsilon_n}(s)\}$ is bounded in $\mathcal{F}^b$ for each $\omega \in \Omega'$, that is,

$$\begin{aligned} \|u_{\varepsilon_n}\|_{\mathcal{F}^b} = {} & \sup_{t\in\mathbb{R}} \|u_{\varepsilon_n}(t)\| \\ & + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_n}(s)\|_1^2 \mathrm{d}s \right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_n}(s)\|_{\mathbf{L}_p}^p \mathrm{d}s \right)^{1/p} \\ & + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|\partial_t u_{\varepsilon_n}(s)\|_{\mathbf{H}^{-r}}^q \mathrm{d}s \right)^{1/q} \le C \ \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{28}$$

Hence there exists a subsequence $\{u_{\varepsilon_n'}(s)\} \subset \{u_{\varepsilon_n}(s)\}$ which we label the same such that

$$u_{\varepsilon_n}(s) \to \bar{u}(s) \text{ as } n \to \infty \ \text{ in } \Theta^{loc}, \tag{29}$$

where $\bar{u}(s) \in \mathcal{F}^b$ and $\bar{u}(s)$ satisfies (28) with the same constant C. Due to (28) we can also assume that $u_{\varepsilon_n}(s) \rightharpoonup \bar{u}(s) \ (n \to \infty)$ weakly in $L_{2,w}^{loc}(\mathbb{R}; \mathbf{V})$, weakly in $L_{p,w}^{loc}\left(\mathbb{R}; \mathbf{L}_p\right)$, $*$-weakly in $L_{\infty,*w}^{loc}(\mathbb{R}_+; \mathbf{H})$ and $\partial_t u_{\varepsilon_n}(s) \rightharpoonup \partial_t \bar{u}(s) \ (n \to \infty)$ weakly in $L_{q,w}^{loc}\left(\mathbb{R}; \mathbf{H}^{-r}\right)$. We claim that $\bar{u}(s) \in \overline{\mathcal{K}}$. We have already proved that $\|\bar{u}\|_{\mathcal{F}^b} \le C$. So we have to establish that $\bar{u}(s)$ is a weak solution of (22). Using (28), we obtain that

$$\partial_t u_{\varepsilon_n} - a\Delta u_{\varepsilon_n} - g\left(x\right) \longrightarrow \partial_t \bar{u} - a\Delta \bar{u} - g\left(x\right) \ \text{ as } n \to \infty \tag{30}$$

in the space $D'\left(\mathbb{R}; \mathbf{H}^{-r}\right)$ because the derivative operators are continuous in the space of distributions. Let us prove that

$$b\left(x, \frac{x}{\varepsilon_n}\right) f(u_{\varepsilon_n}) \rightharpoonup b^{hom}\left(x\right) f(\bar{u}) \ \text{ as } n \to \infty \tag{31}$$

weakly in $L^{loc}_{q,w}\left(\mathbb{R};\mathbf{L}_q\right)$. We fix an arbitrary number $M>0$. The sequence $\{u_{\varepsilon_n}(s)\}$ is bounded in $L_p\left(-M,M;\mathbf{L}_p\right)$ (see (28)). Hence by (14) the sequence $\{f(u_{\varepsilon_n}(s))\}$ is bounded in $L_q\left(-M,M;\mathbf{L}_q\right)$. Since $\{u_{\varepsilon_n}(s)\}$ is bounded in $L_2\left(-M,M;\mathbf{V}\right)$ and $\{\partial_t u_{\varepsilon_n}(s)\}$ is bounded in $L_q\left(-M,M;\mathbf{H}^{-r}\right)$ we can assume that $u_{\varepsilon_n}(s)\to\bar{u}(s)$ as $n\to\infty$ strongly in $L_2\left(-M,M;\mathbf{H}\right)=L_2\left(D\times(-M,M)\right)^N$ and therefore

$$u_{\varepsilon_n}(x,s)\to\bar{u}(x,s) \text{ as } n\to\infty \text{ a.e. in } (x,s)\in D\times(-M,M).$$

Since the function $f(v)$ is continuous with respect to $v\in\mathbb{R}^N$ we conclude that

$$f(u_{\varepsilon_n}(x,s))\to f(\bar{u}(x,s)) \text{ as } n\to\infty \text{ a.e. in } (x,s)\in D\times(-M,M). \tag{32}$$

We have

$$b\left(x,\frac{x}{\varepsilon_n},\omega\right)f(u_{\varepsilon_n})-b^{hom}\left(x\right)f(\bar{u})=b\left(x,\frac{x}{\varepsilon_n},\omega\right)\left(f(u_{\varepsilon_n})-f(\bar{u})\right)+\left(b\left(x,\frac{x}{\varepsilon_n},\omega\right)-b^{hom}\left(x\right)\right)f(\bar{u}). \tag{33}$$

Let us show that both summand in the right-hand side of (33) converges to zero as $n\to\infty$ weakly in the space $L_q\left(-M,M;\mathbf{L}_q\right)=L_q\left(D\times(-M,M)\right)^N$. The sequence $b\left(x,\frac{x}{\varepsilon_n},\omega\right)\left(f(u_{\varepsilon_n})-f(\bar{u})\right)$ tends to zero as $n\to\infty$ almost everywhere in $(x,s)\in D\times(-M,M)$ (see (32)) and is bounded in $L_q\left(D\times(-M,M)\right)^N$ (see (15)). Therefore Lemma 1.3 from [47, Chapter 1, Section 1] implies that

$$b\left(x,\frac{x}{\varepsilon_n},\omega\right)\left(f(u_{\varepsilon_n})-f(\bar{u})\right)\rightharpoonup 0 \text{ as } n\to\infty$$

weakly in the space $L_q\left(D\times(-M,M)\right)^N$. The sequence $\left(b\left(x,\frac{x}{\varepsilon_n},\omega\right)-b^{hom}\left(x\right)\right)f(\bar{u})$ also approaches zero as $n\to\infty$ weakly in $L_q\left(D\times(-M,M)\right)^N$ because, by the proved property (16) $b\left(x,\frac{x}{\varepsilon_n},\omega\right)\rightharpoonup b^{hom}\left(x\right)$ as $n\to\infty$ $*$-weakly in $L_{\infty,*w}\left(-M,M;\mathbf{L}_2\right)$ and $f(\bar{u})\in L_q\left(D\times(-M,M)\right)^N$. We have proved (31). Using (30) and (31) we pass to the limit in the Equation (27) as $n\to\infty$ in the space $D'\left(\mathbb{R}_+;\mathbf{H}^{-r}\right)$ and we obtain that the function $\bar{u}(x,s)$ satisfies the equation

$$\partial_t\bar{u}=a\Delta\bar{u}-b^{hom}\left(x\right)f(\bar{u})+g\left(x\right),\ \bar{u}|_{\partial D}=0,\ t\in\mathbb{R}.$$

Consequently, $\bar{u}\in\overline{\mathcal{K}}$. We have proved above that $u_{\varepsilon_n}(s)\to\bar{u}(s)$ as $n\to\infty$ in Θ^{loc} for each $\omega\in\Omega'$. The hypotheses $u_{\varepsilon_n}(s)\notin\mathcal{O}'(\overline{\mathcal{K}})$ implies that $\bar{u}\notin\mathcal{O}'(\overline{\mathcal{K}})$ and moreover $\bar{u}\notin\overline{\mathcal{K}}$ for all $\omega\in\Omega'$. We came to the contradiction. The theorem is proved. □

Using the compact inclusions (20) and (21), we can strengthen the convergence (23).

Corollary 5.1: *For every* $0<\delta\leq 1$ *and for any* $M>0$

$$\operatorname{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon,\Pi_{0,M}\overline{\mathfrak{A}}\right)\to 0, \tag{34}$$

$$\operatorname{dist}_{C([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon,\Pi_{0,M}\overline{\mathfrak{A}}\right)\to 0\quad(\varepsilon\to 0+). \tag{35}$$

To prove (34) and (35), we just repeat the proof of Theorem 5.1 replacing the topology Θ^{loc} with $L_2^{loc}(\mathbb{R}_+;\mathbf{H}^{1-\delta})$ or $C^{loc}(\mathbb{R}_+;\mathbf{H}^{-\delta})$.

Finally we consider the reaction diffusion systems for which the uniqueness theorem of the Cauchy problem takes place. It is sufficient to assume that the nonlinear term $f(u)$ in the Equation (13) satisfies the condition

$$\big(f(v_1) - f(v_2), v_1 - v_2\big) \geq -C|v_1 - v_2|^2 \ \text{ for } v_1, v_2 \in \mathbb{R}^N. \tag{36}$$

(see [15,46]). In [46] it was proved that if (36) holds, then Equations (13) and (22) generate the dynamical semigroups in $\mathbf{H}$ which have the global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ bounded in the space $\mathbf{V} = \big(H_0^1(D)\big)^N$ (see also [14,16]). We have

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Convergence (35) implies

Corollary 5.2: *Under the assumptions of Theorem 5.1, the following limit holds almost surely:*

$$\operatorname{dist}_{\mathbf{H}^{-\delta}}\big(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\big) \to 0 \ (\varepsilon \to 0+).$$

Note

1. The image of the attractor of evolutionary equation was taken from the internet https://fr.wikipedia.org/wiki/Attracteur.

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