Chaos, Solitons and Fractals 140 (2020) 110208

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

Frontiers

"Strange term" in homogenization of attractors of reaction–diffusion equation in perforated domain[☆]

Kuanysh A. Bekmaganbetov[a,b], Gregory A. Chechkin[c,d,*], Vladimir V. Chepyzhov[e,f]

[a] *M.V. Lomonosov Moscow State University, Kazakhstan Branch, Kazhymukan st. 11, Nur-Sultan, 010010, Kazakhstan*
[b] *Institute of Mathematics and Mathematical Modeling, Pushkin st. 125, Almaty, 050010, Kazakhstan*
[c] *M.V. Lomonosov Moscow State University, Leninskie Gory, 1, Moscow, 119991, Russia*
[d] *Institute of Mathematics with Computing Center - Subdivision of the Ufa Federal Research Center of Russian Academy of Science, Chernyshevskogo st., 112, Ufa, 450008, Russia*
[e] *Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetniy 19, Moscow 127994, Russia*
[f] *National Research University Higher School of Economics, Moscow 101000, Russia*

A R T I C L E I N F O

Article history:
Received 23 April 2020
Revised 31 July 2020
Accepted 12 August 2020

MSC:
35B40
35B41
35Q80

Keywords:
Attractors
Homogenization
Reaction-diffusion
Equation
Nonlinear equations
Weak convergence
Perforated domain
Strange term

A B S T R A C T

We consider reaction–diffusion equation in perforated domain, with rapidly oscillating coefficient in boundary conditions. We do not assume any Lipschitz condition for the nonlinear function in the equation, so, the uniqueness theorem for the corresponding initial boundary value problem may not hold for the considered reaction-diffusion equation. We prove that the trajectory attractors of this equation tend in a weak sense to the trajectory attractors of the homogenized reaction-diffusion equation with a "strange term" (potential).

Bibliography: 48 titles.

Introduction

The study of processes in porous and perforated media (see Fig. 1[1] for example) is an important task of physics, chemistry, biology, material science, and especially modern engineering and technology. The presence of small pores (holes) greatly complicates studying problems in such domains, especially with the use of numerical analysis, since this leads to a very large number of equations and unknowns. To describe the effective behavior of strongly inhomogeneous media, asymptotic analysis and homogenization theory are used. Such an approach leads to studying simpler problems in homogeneous domains, the solutions of which are close to the solution of the given problems, in the corresponding norms. In this case, the homogenized equation and boundary conditions can differ from the initial equation and boundary conditions. In particular, we study the case when an additional potential appears in the homogenized equation. One can study the asymptotic behavior of solutions, eigenfunctions and eigenvalues, as the small parameter tends to zero, but in the case of an incorrect boundary value problem (for example, the presence of nonuniqueness), this becomes ambiguous. We have chosen a different way. It is more convenient to talk about the asymptotic behavior of attractors and their Hausdorff convergence as the small parameter tends to zero. Thus, we construct the homogenized attractor and prove the convergence of the initial attractors to the attractor of the limit (homogenized) equation with additional potential.

☆ Work of VVC is partially supported by the Russian Foundation of Basic Researches (project 17-01-00515). Work of GAC is partially supported by Russian Science Foundation (project 20-11-20272). Work of KAB is supported in part by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (project AP05132071).

* Corresponding author.
E-mail addresses: bekmaganbetov-ka@yandex.kz (K.A. Bekmaganbetov), chechkin@mech.math.msu.su (G.A. Chechkin), chep@iitp.ru (V.V. Chepyzhov).

[1] The image of metal foam was taken from the internet https://www.advancedsciencenews.com/composite-metal-foams/

https://doi.org/10.1016/j.chaos.2020.110208
0960-0779/

Fig. 1. Periodically perforated domain.

In this paper we investigate the asymptotic behavior of attractors to an initial boundary value problem for nonlinear differential equations in perforated domains. Recent years many mathematical works were devoted to the asymptotic analysis of problems in perforated domains (see for instance [3,20–23,34]). Various homogenization results have been achieved for periodic, almost periodic and random structures. We mention here the general frameworks [30,31,35,37,40], [9,39], where the detail bibliography can be found.

We study a weak convergence and effective behavior of attractors as a small parameter characterizing the perforation, tends to zero. To study such phenomenon we apply the homogenization method (cf., for example, [2,5,9,31,34,35,40] as well as a delicate analysis of trajectory and global attractors.

It is well known that attractors describe the behaviour of solutions of dissipative nonlinear evolution equations as time tends to infinity. It is also convenient to study, using attractors, the stability and instability of the limiting structures of the corresponding dynamical systems. Attractors single out the most essential limit sets of trajectories, which characterize the whole dynamics of the considered model described by evolution equations (see, for examples, monographes [1,15,41] and the references therein).

More precisely, our interest is the asymptotic behavior of trajectory and global attractors of reaction-diffusion equations in perforated domain.

The Bogolyubov averaging principle [7] was used in the first papers [29,32,33] on homogenization of attractors of evolution equations with rapidly oscillating terms. The homogenization of global attractors for parabolic equations with oscillating parameters has been considered in [15,25–28]. Some problems related to the homogenization of uniform global attractors for dissipative wave equations has been considered in [10,19,32,43,47], in presence of time oscillations, and in [15,38,42,46], in presence of oscillations in space. Similar problems for autonomous and non-autonomous 2D Navier–Stokes equations has been studied in [15,16,18,42]. Papers [11,12,17,18,44] deal with partial differential equations containing singular oscillating terms.

The theory of trajectory attractors for dissipative partial differential equations was developed in [14,15,45]. This approach is very powerful in the study of the long time behaviour of solutions of evolution equations for which the uniqueness theorem of the corresponding initial–value problem is not proved yet (e.g., for the inhomogeneous 3D Navier–Stokes system) or does not hold (for the reaction-diffusion equations considered in the present paper).

In the paper we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the reaction-diffusion equation in perforated domain converges in a weak sense as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in an appropriate functional space. Here, ε characterizes the diameter of cavities and the distance between them in perforated medium.

In Section 1 we define the main notions and formulate theorems concerning the trajectory attractors of autonomous evolution equations. In Section 2 we define the geometric structure of the domain, formulate the problem to study and describe necessary functional spaces. Section 3 is devoted to the study of the homogenization of attractors of autonomous reaction-diffusion equation in perforated domain as well as demonstrate the appearance of a “strange term” (the potential term) in the homogenized equation (see pioneering works [21,34]).

1. Trajectory attractors of evolution equations

In this section we give a scheme for the construction of trajectory attractors of autonomous evolution equations. In the next section we shall apply this scheme to the study of trajectory attractors of the concrete evolution equations with rapidly oscillating coefficients and the corresponding averaged equations.

To begin with we consider an abstract autonomous evolution equation

$$\frac{\partial u}{\partial t} = A(u), \quad t \geq 0. \tag{1}$$

Here $A(\cdot)$: $E_1 \to E_0$ s a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. For instance, $A(u) = \Delta u - f(u) + g$ (see Section 2).

We are going to study solutions $u(s)$ of Eq. (1) as functions of $s \in \mathbb{R}_+$ as a whole. Here $s \equiv t$ denote the time variable. The set of solutions of (1) is said to be *a trajectory space* $\mathcal{K}^+$ of Eq. (1). Let us describe the trajectory space $\mathcal{K}^+$ in greater detail.

At first, we consider solutions $u(s)$ of (1) defined on a fixed time interval $[t_1, t_2]$ from $\mathbb{R}$. We study solutions of (1) in a Banach space $\mathcal{F}_{t_1,t_2}$ that depends on t_1 and t_2. The space $\mathcal{F}_{t_1,t_2}$ consists of functions $f(s), s \in [t_1, t_2]$ such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, $\mathcal{F}_{t_1,t_2}$ can be the space $C([t_1, t_2]; E)$, or $L_p(t_1, t_2; E)$, for $p \in [1, \infty]$, or the intersection of such spaces (see Section 2). We assume that $\Pi_{t_1,t_2}\mathcal{F}_{\tau_1,\tau_2} \subseteq \mathcal{F}_{t_1,t_2}$ and[1]

$$\|\Pi_{t_1,t_2} f\|_{\mathcal{F}_{t_1,t_2}} \leq C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathcal{F}_{\tau_1,\tau_2}}, \quad \forall f \in \mathcal{F}_{\tau_1,\tau_2}, \tag{2}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} denotes the restriction operator onto the interval $[t_1, t_2]$. The constant $C(t_1, t_2, \tau_1, \tau_2)$ is independent of f.

Let $S(h)$ for $h \in \mathbb{R}$ denote the translation operator

$$S(h)f(s) = f(h + s).$$

Evidently, if the argument s of $f(\,\cdot\,)$ belongs to $[t_1, t_2]$, then the argument s of $S(h)f(\,\cdot\,)$ can be taken form $[t_1 - h, t_2 - h]$ for $h \in \mathbb{R}$. We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathcal{F}_{t_1-h,t_2-h}} = \|f\|_{\mathcal{F}t_1,t_2}, \quad \forall f \in \mathcal{F}_{t_1,t_2}. \tag{3}$$

This assumption is fairly natural.

We assume that if $f(s) \in \mathcal{F}_{t_1,t_2}$, then $A(f(s)) \in \mathcal{D}_{t_1,t_2}$, where $\mathcal{D}_{t_1,t_2}$ is a larger Banach space, $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$. The derivative $\frac{\partial f(t)}{\partial t}$ is a distribution with values in E_0, $\frac{\partial f}{\partial t} \in D'((t_1, t_2); E_0)$ and we assume that $\mathcal{D}_{t_1,t_2} \subseteq D'((t_1, t_2); E_0)$ for all $(t_1, t_2) \subset \mathbb{R}$. A function $u(s) \in \mathcal{F}_{t_1,t_2}$ is said to be a *solution* of (1) from the space $\mathcal{F}_{t_1,t_2}$ (on the interval (t_1, t_2)) if $\frac{\partial u}{\partial t}(s) = A(u(s))$ in the distributional sense of the space $D'((t_1, t_2); E_0)$.

We also define the space

$$\mathcal{F}_+^{loc} = \{f(s),\ s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathcal{F}_{t_1,t_2}, \quad \forall\, [t_1, t_2] \subset \mathbb{R}_+\}. \tag{4}$$

For example, if $\mathcal{F}_{t_1,t_2} = C([t_1, t_2]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{t_1,t_2} = L_p(t_1, t_2; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is called a solution of (1) from $\mathcal{F}_+^{loc}$ if $\Pi_{t_1,t_2} u(s) \in \mathcal{F}_{t_1,t_2}$ and this function is a solution of (1) for every $[t_1, t_2] \subset \mathbb{R}_+$.

We denote by $\mathcal{K}^+$ a set of solutions of (1) from $\mathcal{F}_+^{loc}$. Notice, that $\mathcal{K}^+$ is not necessarily the set of *all* solutions from $\mathcal{F}_+^{loc}$. The elements of $\mathcal{K}^+$ are called *trajectories* and the set $\mathcal{K}^+$ is called the *trajectory space* of the Eq. (1).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for every $h \geq 0$. This is a very natural assumption for solutions of autonomous equations.

We now consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s+h), \quad h \geq 0.$$

It is clear that the mappings $\{S(h),\ h \geq 0\}$ form a semigroup in $\mathcal{F}_+^{loc}$: $S(h_1)S(h_2) = S(h_1+h_2)$ for $h_1,\ h_2 \geq 0$ and $S(0)$ is the identity operator. We change the variable h into the time variable t. The semigroup $\{S(t),\ t \geq 0\}$ is called the *translation semigroup.* By our assumption the translation semigroup maps the trajectory space $\mathcal{K}^+$ to itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+, \quad \forall t \geq 0. \tag{5}$$

We shall study attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We define a topology in the space $\mathcal{F}_+^{loc}$.

Let a metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ be defined on $\mathcal{F}_{t_1,t_2}$ for every $[t_1,t_2] \subset \mathbb{R}$. Similar to (2) and (3) we assume that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2}f,\ \Pi_{t_1,t_2}g) \leq D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}(f,g), \quad \forall f,g \in \mathcal{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$
$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g), \quad \forall f,g \in \mathcal{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

Denote by Θ_{t_1,t_2} the corresponding metric spaces on $\mathcal{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric associated with the norm $\|\cdot\|_{\mathcal{F}_{t_1,t_2}}$ of the Banach space $\mathcal{F}_{t_1,t_2}$. However, usually in application ρ_{t_1,t_2} generates the topology Θ_{t_1,t_2} that is weaker than the strong convergence topology of the Banach space $\mathcal{F}_{t_1,t_2}$.

The *projective limit* of the spaces Θ_{t_1,t_2} defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, i.e., by definition, a sequence $\{f_k(s)\} \subset \mathcal{F}_+^{loc}$ converges to $f(s) \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\rho_{t_1,t_2}(\Pi_{t_1,t_2}f_k, \Pi_{t_1,t_2}f) \to 0$ as $k \to \infty$ for each $[t_1,t_2] \subset \mathbb{R}_+$. It is not hard to prove that the topology Θ_+^{loc} is metrizable using, for example, the Frechet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m}\frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{6}$$

We claim that the translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This assertion follows directly from the definition of the topological space Θ_+^{loc}.

We also consider the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}, \tag{7}$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h\geq 0}\|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{8}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+;E)$, then the space $\mathcal{F}_+^b = C^b(\mathbb{R}_+;E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0}\|f(h)\|_E$ and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+;E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0}\int_h^{h+1}\|f(s)\|_E^p ds\right)^{1/p}$.

Recall that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We require the Banach space $\mathcal{F}_+^b$ only to define bounded subsets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we do not consider the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$. Instead, we utilize the local convergence topology Θ_+^{loc} which is much weaker.

We suppose that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e., every trajectory $u(s) \in \mathcal{K}^+$ of Eq. (1) has a finite norm (8). Let us define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 1.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ε-neighbourhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $t \geq t_1$.

It is clear that the attracting property of $\mathcal{P}$ can be formulated in the following equivalent form: for any set $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B},\ \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where

$$\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X}\mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X}\inf_{y\in Y}\rho_{\mathcal{M}}(x,y)$$

is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$. Recall that the Hausdorff semidistance is not symmetric.

Definition 1.2 ([15]). A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}, **(ii)** the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M}S(t)\mathcal{B},\ \Pi_{0,M}\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 1.1. Comparing with [1] one can say that the trajectory attractor $\mathfrak{A}$ is the *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$*-attractor* of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$, that is, $\mathfrak{A}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc} for any bounded (in $\mathcal{F}_+^b$) set $\mathcal{B}$ from $\mathcal{K}^+$:

$$\mathrm{dist}_{\Theta_+^{loc}}(S(t)\mathcal{B},\mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We now formulate the main result on the trajectory attractor for Eq. (1).

Theorem 1.1 ([1,14,15]). *Assume that the trajectory space $\mathcal{K}^+$ corresponding to Eq. (1) is contained in $\mathcal{F}_+^b$ and (5) holds. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t),\ t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.*

We now describe the structure of the trajectory attractor $\mathfrak{A}$ of equation (1) in terms of complete trajectories of this equation. Consider the equation (1) on the entire time axis

$$\frac{\partial u}{\partial t} = A(u),\ t \in \mathbb{R}. \tag{9}$$

We have defined the trajectory space $\mathcal{K}^+$ of equation (9) on $\mathbb{R}_+$. We now extend this definition on the entire $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of equation (9) if $\Pi_+u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to the semiaxis $\mathbb{R}_+$.

We have introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. We now define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2}f(s) \in \mathcal{F}_{t_1,t_2}\ \forall\ [t_1,t_2] \subseteq \mathbb{R}\};$$

$$\mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h\in\mathbb{R}}\|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_k(s) \to f(s) (k \to \infty)$ in Θ^{loc} if $\Pi_{t_1,t_2}f_k(s) \to$

$\Pi_{t_1,t_2} f(s)$ $(k \to \infty)$ in Θ_{t_1,t_2} for each $[t_1, t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc} is a metric space as well as Θ_+^{loc}.

Definition 1.3. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of Eq. (9) is the union of all complete trajectories $u(s), s \in \mathbb{R},$ of Eq. (9) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \le C_u, \quad \forall h \in \mathbb{R}.$$

Theorem 1.2. *Assume that the hypotheses of Theorem 1.1 holds. Then*

$$\mathfrak{A} = \Pi_+ \mathcal{K},$$

the set $\mathcal{K}$ *is compact in* Θ^{loc} *and bounded in* $\mathcal{F}^b$.

The complete proof can be found in [14,15].

Remark 1.2. In applications to concrete dissipative evolution equations (e.g. see the next sections) the required attracting set $\mathcal{P}$ can be constructed as a ball $B_R(0)$ in the space $\mathcal{F}_+^b$ with sufficiently large radius R. In this case, since by the assumption of Theorem 1.1, the set $\mathcal{P} = B_R(0)$ is compact in the topology Θ_+^{loc}, then this set $\mathcal{P}$ considered as a metric subspace is itself a complete metric space.

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0, M; E_1, E_0) = \Big\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{p_1}(0, M; E_1),\ \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\Big\},$$
$$W_{\infty,p_0}(0, M; E_1, E_0) = \Big\{\psi(s), s \in 0, M \mid \psi(\cdot) \in L_{\infty}(0, M; E_1),\ \ \psi'(\cdot) \in L_{p_0}(0, M; E_0)\Big\},$$

(where $p_1 \ge 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s \in [0, M]\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0}.$$

Lemma 1.1 (Aubin-Lions-Simon, [6]). *Assume that* $E_1 \Subset E \subset E_0$.*Then the following embeddings are compact:*

$$W_{p_1,p_0}(0, T; E_1, E_0) \Subset L_{p_1}(0, T; E), \tag{11}$$

$$W_{\infty,p_0}(0, T; E_1, E_0) \Subset C([0, T]; E). \tag{12}$$

In the next section we study evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 1.4. We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\varepsilon_1 \ge 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0).$$

2. Notation and settings

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 3$ with a piecewise smooth boundary $\partial\Omega$. Let G_0 be a domain in $Y = (-\frac{1}{2}, \frac{1}{2})^n$, such that $\overline{G}_0$ is a compact set diffeomorphic to a ball.

For $\delta > 0$ and B we denote $\delta B = \{x : \delta^{-1} x \in B\}$. Assume that ε is small enough so that

$$\varepsilon^{n/(n-2)} G_0 \subset \varepsilon Y.$$

For $j \in \mathbb{Z}^n$ we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + \varepsilon^{n/(n-2)} G_0.$$

We define the domain $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > \sqrt{n}\,\varepsilon\}$ and the set of admissible indexes as

$$\Upsilon_\varepsilon = \Big\{ j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \ne \emptyset \Big\}.$$

Notice that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ where $d > 0$ is a constant. Consider the following domain:

$$\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon \ \text{ where } \ G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j.$$

Denote

$$Q_\varepsilon = \Omega_\varepsilon \times (0, +\infty), \qquad Q = \Omega \times (0, +\infty).$$

We study the asymptotic behavior of attractors of the problem

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon - f(u_\varepsilon) + g(x), & x \in \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^{n/(2-n)} b_\varepsilon^j(x) u_\varepsilon = 0, & x \in \partial G_\varepsilon^j,\ j \in \Upsilon_\varepsilon,\ t \in (0, +\infty), \\ u_\varepsilon = 0, & x \in \partial\Omega \\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon,\ t = 0. \end{cases} \tag{13}$$

Here ν is the outward unit vector to the boundary, $g(x) \in L_2(\Omega)$,

$$b_\varepsilon^j(x) = b\left(x, \frac{x - P_\varepsilon^j}{\varepsilon^{n/(n-2)}}\right),$$

where $b(x, y) \in C(\Omega \times \mathbb{R}^n)$, such that $0 < b_0 \le b(x, y) \le B_0$ for some constants b_0, B_0, $b(x, y)$ is one-periodic in y and $f(v) \in C(\mathbb{R})$ satisfies the following inequalities

$$f(v) \cdot v \ge K|v|^p - C, \qquad |f(v)| \le C_1(|v|^{p-1} + 1), \qquad p \ge 2. \tag{14}$$

Notice that we *do not assume* that the nonlinear function $f(v)$ satisfies the Lipschitz condition with respect to v.

We denote the spaces $\mathbf{H} := L_2(\Omega)$, $\mathbf{H}_\varepsilon := L_2(\Omega_\varepsilon)$, $\mathbf{V} := H_0^1(\Omega)$, $\mathbf{V}_\varepsilon := H^1(\Omega_\varepsilon;\, \partial\Omega)$ – set of functions from $H^1(\Omega_\varepsilon)$ with zero trace on $\partial\Omega$, and $\mathbf{L}_p := L_p(\Omega)$, $\mathbf{L}_{p,\varepsilon} := L_p(\Omega_\varepsilon)$. The norms in these spaces are denoted, respectively, by

$\|v\|^2 := \int_\Omega |v(x)|^2 dx$, $\|v\|_\varepsilon^2 := \int_{\Omega_\varepsilon} |v(x)|^2 dx$, $\|v\|_1^2 := \int_\Omega |\nabla v(x)|^2 dx$,
$\|v\|_{1\varepsilon}^2 := \int_{\Omega_\varepsilon} |\nabla v(x)|^2 dx$, $\|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p dx$, $\|v\|_{\mathbf{L}_{p\,\varepsilon}}^p := \int_{\Omega_\varepsilon} |v(x)|^p dx$.

Recall that $\mathbf{V}' := H^{-1}(\Omega)$ and $\mathbf{L}_q$ are the dual spaces of $\mathbf{V}$ and $\mathbf{L}_p$ respectively, where $q = p/(p-1)$, moreover, $\mathbf{V}_\varepsilon'$ and $\mathbf{L}_{q,\varepsilon}$ are the dual spaces for $\mathbf{V}_\varepsilon$ and $\mathbf{L}_{p,\varepsilon}$.

As in [13,15] we study weak solutions of the initial boundary value problem (13), that is, the functions

$$u_\varepsilon(x, s) \in L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap L_p^{loc}(\mathbb{R}_+; \mathbf{L}_{p,\varepsilon})$$

which satisfy the problem (13) in the distributional sense, i.e.

$$\begin{aligned} &\int_{Q_\varepsilon} \frac{\partial u_\varepsilon}{\partial t} \psi \ dxdt + \int_{Q_\varepsilon} \nabla u_\varepsilon \nabla \psi \ dxdt + \int_{Q_\varepsilon} f(u_\varepsilon) \psi \ dxdt + \\ &+ \varepsilon^{n/(2-n)} \sum_{j \in \Upsilon_\varepsilon} \int_0^{+\infty} \int_{\partial G_\varepsilon^j} b_\varepsilon^j u_\varepsilon \psi \ dxdt = \int_{Q_\varepsilon} g(x) \psi \ dxdt \end{aligned} \tag{15}$$

for any function $\psi \in C_0^\infty(\mathbb{R}_+; \mathbf{H}_\varepsilon)$.

If $u_\varepsilon(x, t) \in L_p(0, M; \mathbf{L}_{p,\varepsilon})$, then it follows from the condition (14) that $f(u(x, t)) \in L_q(0, M; \mathbf{L}_{q,\varepsilon})$. At the same time, if $u_\varepsilon(x, t) \in L_2(0, M; \mathbf{V}_\varepsilon)$, then $\Delta u_\varepsilon(x, t) + g(x) \in L_2(0, M; \mathbf{V}_\varepsilon')$. Therefore, for an arbitrary weak solution $u_\varepsilon(x, s)$ of the problem (13) we have

$$\frac{\partial u_\varepsilon(x, t)}{\partial t} \in L_q(0, M; \mathbf{L}_{q,\varepsilon}) + L_2(0, M; \mathbf{V}_\varepsilon').$$

The Sobolev embedding theorem implies that

$$L_q(0,M;\mathbf{L}_{q,\varepsilon})+L_2(0,M;\mathbf{V}'_\varepsilon)\subset L_q\left(0,M;\mathbf{H}_\varepsilon^{-r}\right),$$

where the space $\mathbf{H}_\varepsilon^{-r}:=H^{-r}(\Omega_\varepsilon)$ and $r=\max\{1,n(1/2-1/p)\}$. Hence, for any weak solution $u_\varepsilon(x,t)$ of (13) we have $\frac{\partial u_\varepsilon(x,t)}{\partial t}\in L_q\left(0,M;\mathbf{H}_\varepsilon^{-r}\right)$.

Remark 2.1. The existence of weak solution $u(x, s)$ to the problem (13) for every $U\in\mathbf{H}_\varepsilon$ and fixed ε, such that $u(x,0)=U(x)$ can be proved by standard approach (see for instance [1,13]). This solution is not necessarily unique because we do not assume the Lipschitz condition for $f(v)$ with respect to v.

The following key Lemma can be proved similar to Proposition XV.3.1 from [15].

Lemma 2.1. *Let $u_\varepsilon(x,t)\in L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon)\cap L_p^{loc}(\mathbb{R}_+;\mathbf{L}_{p,\varepsilon})$ be a weak solution of the problem (13). Then*

(i) $u\in C(\mathbb{R}_+;\mathbf{H}_\varepsilon)$;
(ii) *the function $\|u_\varepsilon(\cdot,t)\|_\varepsilon^2$ is absolutely continuous on $\mathbb{R}_+$ and, moreover,*

$$\tfrac{1}{2}\tfrac{d}{dt}\|u_\varepsilon(\cdot,t)\|_\varepsilon^2+\int_{\Omega_\varepsilon}|\nabla u_\varepsilon(x,t)|^2dx+\int_{\Omega_\varepsilon}f(u_\varepsilon)u_\varepsilon dx+ \\ +\varepsilon^{\frac{n}{2-n}}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}b_\varepsilon^j|u_\varepsilon(x,t)|^2dx=\int_{\Omega_\varepsilon}g(x)u_\varepsilon dx,$$

for almost every $t\in\mathbb{R}_+$.

We now fix ε. In further analysis we shall omit the index ε in the notation of the spaces, where it is natural. We now apply the scheme described in Section 1 to construct the trajectory attractor for the problem (13), which has the form (1) if we set $E_1=\mathbf{L}_p\cap\mathbf{V}$, $E_0=\mathbf{H}^{-r}$, $E=\mathbf{H}$ and $A(u)=\Delta u-f(u)+g(\cdot)$.

To describe the trajectory space $\mathcal{K}_\varepsilon^+$ for the problem (13), we follow the general framework of Section 1 and define the Banach spaces for every $[t_1,t_2]\in\mathbb{R}$

$$\mathcal{F}_{t_1,t_2}:=L_p(t_1,t_2;\mathbf{L}_p)\cap L_2(t_1,t_2;\mathbf{V})\cap L_\infty(t_1,t_2;\mathbf{H})\cap \\ \times\left\{v\;\middle|\;\frac{\partial v}{\partial t}\in L_q\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\} \tag{16}$$

with norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}}:=\|v\|_{L_p(t_1,t_2;\mathbf{L}_p)}+\|v\|_{L_2(t_1,t_2;\mathbf{V})}+\|v\|_{L_\infty(0,M;\mathbf{H})} \\ +\left\|\frac{\partial v}{\partial t}\right\|_{L_q(t_1,t_2;\mathbf{H}^{-r})}. \tag{17}$$

It is clear that the condition (2) holds for the norm (17) and the translation semigroup $\{S(h)\}$ satisfies (3).

Setting $\mathcal{D}_{t_1,t_2}=L_q(t_1,t_2;\mathbf{H}^{-r})$ we have that $\mathcal{F}_{t_1,t_2}\subseteq\mathcal{D}_{t_1,t_2}$ and if $u(s)\in\mathcal{F}_{t_1,t_2}$, then $A(u(s))\in\mathcal{D}_{t_1,t_2}$. We can consider a weak solutions of the problem (13) as a solution of equation in the general scheme of Section 1.

Defining the space (4) we obtain that

$$\mathcal{F}_+^{loc}=L_p^{loc}(\mathbb{R}_+;\mathbf{L}_p)\cap L_2^{loc}(\mathbb{R}_+;\mathbf{V})\cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H})\cap\left\{v\;\middle|\;\tfrac{\partial v}{\partial t}\in L_q^{loc}(\mathbb{R}_+;\mathbf{H}^{-r})\right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc}=L_p^{loc}(\mathbb{R}_+;\mathbf{L}_{p,\varepsilon})\cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon)\cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)\cap\left\{v\;\middle|\;\tfrac{\partial v}{\partial t}\in L_q^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon{}^{-r})\right\}.$$

We denote by $\mathcal{K}_\varepsilon^+$ the set of all weak solutions of the problem (13). Recall that for any $U\in\mathbf{H}$ there exist at least one trajectory $u(\cdot)\in\mathcal{K}_\varepsilon^+$ such that $u(0)=U(x)$. Therefore, the trajectory space $\mathcal{K}_\varepsilon^+$ of the problem (13) is not empty and is sufficiently large.

It is clear that $\mathcal{K}_\varepsilon^+\subset\mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, that is, if $u(s)\in\mathcal{K}_\varepsilon^+$, then $u(h+s)\in\mathcal{K}_\varepsilon^+$ for all $h\geq 0$. Therefore,

$$S(h)\mathcal{K}_\varepsilon^+\subseteq\mathcal{K}_\varepsilon^+,\quad\forall h\geq 0.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathcal{F}_{t_1,t_2}$ using the norms of the spaces $L_2(t_1,t_2;\mathbf{H})$:

$$\rho_{0,M}(u,v)=\left(\int_0^M\|u(s)-v(s)\|^2ds\right)^{1/2},\quad\forall u(\cdot),v(\cdot)\in\mathcal{F}_{0,M}.$$

These metrics generates the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\varepsilon,+}^{loc}$ in $\mathcal{F}_{\varepsilon,+}^{loc}$). Recall that a sequence $\{v_k\}\subset\mathcal{F}_+^{loc}$ converges to $v\in\mathcal{F}_+^{loc}$ as $k\to\infty$ in Θ_+^{loc} if $\|v_k(\cdot)-v(\cdot)\|_{L_2(0,M;\mathbf{H})}\to 0$ $(k\to\infty)$ for each $M>0$. The topology Θ_+^{loc} is metrizable (see (6)). We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (13). The translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\varepsilon^+$ is continuous in the topology Θ_+^{loc}.

Following the general scheme of Section 1, we define bounded sets in $\mathcal{K}_\varepsilon^+$ using the Banach space $\mathcal{F}_+^b$ (see (7)). We clearly have

$$\mathcal{F}_+^b=L_p^b(\mathbb{R}_+;\mathbf{L}_p)\cap L_2^b(\mathbb{R}_+;\mathbf{V})\cap L_\infty(\mathbb{R}_+;\mathbf{H})\cap\left\{v\;\middle|\;\frac{\partial v}{\partial t}\in L_q^b(\mathbb{R}_+;\mathbf{H}^{-r})\right\} \tag{18}$$

and $\mathcal{F}_+^b$ is a subspace of $\mathcal{F}_+^{loc}$.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\varepsilon^+$, $S(t):\mathcal{K}_\varepsilon^+\to\mathcal{K}_\varepsilon^+$, $t\geq 0$.

Let $\mathcal{K}_\varepsilon$ be the kernel of the problem (13) that consists of all weak complete solutions $u(s),\in\mathbb{R}$, of the system bounded in the space

$$\mathcal{F}^b=L_p^b(\mathbb{R};\mathbf{L}_p)\cap L_2^b(\mathbb{R};\mathbf{V})\cap L_\infty(\mathbb{R};\mathbf{H})\cap\left\{v\;\middle|\;\frac{\partial v}{\partial t}\in L_q^b(\mathbb{R};\mathbf{H}^{-r})\right\}$$

Proposition 2.1. *Under the hypotheses (14) the problem (13) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\varepsilon$ is uniformly (w.r.t. $\varepsilon\in(0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon=\Pi_+\mathcal{K}_\varepsilon,$$

the kernel $\mathcal{K}_\varepsilon$ is non-empty and uniformly (w.r.t. $\varepsilon\in(0,1)$) bounded in $\mathcal{F}^b$. Recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on ε.

The proof of this proposition almost coincides with the proof given in [15] for a particular case. The existence of an absorbing set that is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc} is proved using Lemma 2.1 similar to [15].

We note that

$$\mathfrak{A}_\varepsilon\subset\mathcal{B}_0(R),\quad\forall\varepsilon\in(0,1),$$

where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Lemma 1.1 implies that

$$\mathcal{B}_0(R)\Subset L_2^{loc}(\mathbb{R}_+;\mathbf{H}^{1-\delta}), \tag{19}$$

$$\mathcal{B}_0(R)\Subset C^{loc}(\mathbb{R}_+;\mathbf{H}^{-\delta}),\quad 0<\delta\leq 1. \tag{20}$$

Inclusion (19) follows from (11), where we set $E_0=\mathbf{H}^{-r}$, $E=\mathbf{H}^{1-\delta}$, $E_1=\mathbf{H}^1=\mathbf{V}$, and $p_1=2$, $p_0=q$, and from the compact embedding $\mathbf{V}\Subset\mathbf{H}^{1-\delta}$. Inclusion (19) follows from (12) and from the compact embeddings $\mathbf{H}\Subset\mathbf{H}^{-\delta}$, if we set $E_0=\mathbf{H}^{-\mathbf{r}}(D)$, $E=\mathbf{H}^{-\delta}$, $E_1=\mathbf{H}^1=\mathbf{V}$, and $p_0=q$.

Using compact inclusions (19) and (20), we strengthen the attraction to the constructed trajectory attractor.

Corollary 2.1. *For any set $\mathcal{B}\subset\mathcal{K}_\varepsilon^+$ bounded in $\mathcal{F}_+^b$ we have*

$$\mathrm{dist}_{L_2(0,M;H^{1-\delta})}(\Pi_{0,M}S(t)\mathcal{B},\Pi_{0,M}\mathcal{K}_\varepsilon)\to 0\ (t\to\infty),$$

$$\operatorname{dist}_{C([0,M];H^{-\delta})}(\Pi_{0,M}S(t)\mathcal{B},\Pi_{0,M}\mathcal{K}_\varepsilon)\to 0\ (t\to\infty),$$

where M is an arbitrary positive number.

3. Homogenization of attractors to a problem for reaction-diffusion equations in perforated domain

In this section, we study the limit behaviour of trajectory attractors $\mathfrak{A}_\varepsilon$ of reaction-diffusion equations (13) as $\varepsilon\to 0+$ and their relation to the trajectory attractor of the corresponding homogenized equation.

In order to define the "strange term" (the potential in the limit equation) we consider the following problem:

$$\begin{cases} -\Delta_y v = 0, & y\in\mathbb{R}^n\setminus G_0,\\ \frac{\partial v}{\partial \nu_y}+b(x,y)v=b(x,y), & y\in\partial G_0,\\ v\to 0, & |y|\to\infty.\end{cases}$$

In this problem the variable x plays the role of slow parameter. The limit potential $V(x)$ can be determined by the formula

$$V(x)=\int\limits_{\partial G_0}\frac{\partial}{\partial \nu_y}v(x,y)\,d\sigma_y. \tag{21}$$

The homogenized (limit) problem reads as follows

$$\begin{cases} \frac{\partial u}{\partial t}=\Delta u - f(u)-V(x)u+g(x), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\\ u=U(x), & t=0.\end{cases} \tag{22}$$

Clearly the problem (22) also has trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the problem (22) and

$\overline{\mathfrak{A}}=\Pi_+\overline{\mathcal{K}}$

where $\overline{\mathcal{K}}$ is the kernel of problem (22) in $\mathcal{F}^b$.

Let us formulate the main theorem concerning the initial boundary value problem for reaction–diffusion equation.

Theorem 3.1. *The following limit holds in the topological space* Θ_+^{loc}

$$\mathfrak{A}_\varepsilon\to\overline{\mathfrak{A}}\ \text{ as } \varepsilon\to 0+. \tag{23}$$

Moreover,

$$\mathcal{K}_\varepsilon\to\overline{\mathcal{K}} \text{ as } \varepsilon\to 0+ \text{ in } \Theta^{loc}. \tag{24}$$

Remark 3.1. Recall that the functions from the sets $\mathfrak{A}_\varepsilon$ and $\mathcal{K}_\varepsilon$ are defined in the perforated domains Ω_ε. However, all these functions can be prolonged insides the holes in such a way, that their norms in the spaces **H, V**, and $\mathbf{L}_p$ (without perforation) remain almost the same (are equivalent with the constants independent of the small parameter) as in the perforated spaces $\mathbf{H}_\varepsilon$, $\mathbf{V}_\varepsilon$, and $\mathbf{L}_{p,\varepsilon}$ (the prolongation of functions defined in perforated domains, see, for instance, in [31, Ch.VIII] and [35, Ch. IV]). So, in Theorem 3.1, we measure all the distances in the spaces without perforation.

Proof. It is clear that (24) implies (23). Therefore it is sufficient to prove (24), that is, for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there are exists $\varepsilon_1=\varepsilon_1(\mathcal{O})>0$ such that

$$\mathcal{K}_\varepsilon\subset\mathcal{O}(\overline{\mathcal{K}}) \text{ for } \varepsilon<\varepsilon_1. \tag{25}$$

Suppose that (25) is not true. Then there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k\to 0+$ $(k\to\infty)$, and a sequence $u_{\varepsilon_k}(\cdot)=u_{\varepsilon_k}(s)\in\mathcal{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k}\notin\mathcal{O}'(\overline{\mathcal{K}}) \text{ for all } k\in\mathbb{N}. \tag{26}$$

The function $u_{\varepsilon_k}(s)$, $s\in\mathbb{R}$ is the solutions to the problem

$$\begin{cases} \frac{\partial u_{\varepsilon_k}}{\partial t}=\Delta u_{\varepsilon_k}-f(u_{\varepsilon_k})+g(x), & x\in\Omega_{\varepsilon_k},\\ \frac{\partial u_{\varepsilon_k}}{\partial \nu}+\varepsilon_k^{n/(2-n)}b^j_{\varepsilon_k}(x)u_{\varepsilon_k}=0, & x\in\partial G^j_{\varepsilon_k},\ j\in\Upsilon_{\varepsilon_k},\\ u_{\varepsilon_k}=0, & x\in\partial\Omega,\\ u_{\varepsilon_k}=U(x), & x\in\Omega_{\varepsilon_k},\ t=0,\end{cases} \tag{27}$$

on the entire time axis $t\in\mathbb{R}$. To obtain the uniform in ε estimate of the solution we use the following Lemmae (see [36, Ch. III, §5] and [4] respectively).

Lemma 3.1. *Suppose that*

$$W(f,g)=\int\limits_{\Omega_\varepsilon}\nabla f\,\nabla g\,dx+\int\limits_{\Omega_\varepsilon}qfg\,dx+\int\limits_{\partial\Omega_\varepsilon}rfg\,ds \tag{28}$$

is a bilinear form on $\mathbf{V}_\varepsilon$ *and let* $q(x)\geq 0$ *and* $r(x)\geq 0$ $(q\not\equiv 0$ *or* $r\not\equiv 0)$. *Then the bilinear form W(f, g) define an inner product on* $\mathbf{V}_\varepsilon$ *which is equivalent to the inner product*

$$(f,g)_{H^1(\Omega_\varepsilon)}=\int\limits_{\Omega_\varepsilon}(\nabla f\,\nabla g+fg)\,dx.$$

Lemma 3.2. *The coercivity of the problem (22) leads to the coercivity of the initial problem (13).*

Then, using the integral identity (15), by means of Lemmae 2.1, 3.1, 3.2, we obtain the estimate. More precise the sequence $\{u_{\varepsilon_k}(s)\}$ is bounded in $\mathcal{F}^b$, that is,

$$\|u_{\varepsilon_k}\|_{\mathcal{F}^b}=\sup_{t\in\mathbb{R}}\|u_{\varepsilon_k}(t)\|+ \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|u_{\varepsilon_k}(s)\|_1^2ds\right)^{1/2}+\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\|u_{\varepsilon_k}(s)\|_{\mathbf{L}_p}^p ds\right)^{1/p}+ \sup_{t\in\mathbb{R}}\left(\int_t^{t+1}\left\|\frac{\partial u_{\varepsilon_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^q ds\right)^{1/q}\leq C \text{ for all } k\in\mathbb{N}. \tag{29}$$

It is important that the constant C does not depend on ε.

Hence there exists a subsequence $\{u_{\varepsilon'_k}(s)\}\subset\{u_{\varepsilon_k}(s)\}$ which we label the same such that

$$u_{\varepsilon_k}(s)\to\bar u(s) \text{ as } n\to\infty \text{ in } \Theta^{loc}, \tag{30}$$

where $\bar u(s)\in\mathcal{F}^b$ and $\bar u(s)$ satisfies (29) with the same constant C. Due to (29) we have $u_{\varepsilon_k}(s)\rightharpoonup\bar u(s)$ $(n\to\infty)$ weakly in $L_2^{loc}(\mathbb{R};\mathbf{V}_\varepsilon)$, weakly in $L_p^{loc}(\mathbb{R};\mathbf{L}_{p,\varepsilon})$, $*$-weakly in $L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)$ and $\frac{\partial u_{\varepsilon_k}(s)}{\partial t}\rightharpoonup\frac{\partial\bar u(s)}{\partial t}$ $(k\to\infty)$ weakly in $L^{loc}_{q,w}(\mathbb{R};\mathbf{H}_\varepsilon^{-r})$. We claim that $\bar u(s)\in\overline{\mathcal{K}}$. We have already proved that $\|\bar u\|_{\mathcal{F}^b}\leq C$. So we have to establish that $\bar u(s)$ is a weak solution of (22). Using (29), we obtain that

$$\frac{\partial u_{\varepsilon_k}}{\partial t}-\Delta u_{\varepsilon_k}-g(x)\longrightarrow\frac{\partial\bar u}{\partial t}-\Delta\bar u-g(x) \text{ as } k\to\infty \tag{31}$$

in the space $D'(\mathbb{R};\mathbf{H}_\varepsilon^{-r})$ because the derivative operators are continuous in the space of distributions.

Since the function $f(v)$ is continuous with respect to $v\in\mathbb{R}$ we conclude that

$$f(u_{\varepsilon_k}(x,s))\to f(\bar u(x,s)) \text{ as } k\to\infty \text{ a.e. in } (x,s)\in\Omega\times(-M,M). \tag{32}$$

Following [24] and [8], one can prove the following statement.

Lemma 3.3. *We have*

$$\left|\varepsilon^{\frac{n}{n-2}}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G^j_\varepsilon}b^j_\varepsilon(x)\varphi\,ds-\int\limits_\Omega V(x)\bar\varphi\,dx\right|\leq M\varepsilon\|\varphi\|_{\mathbf{H}_\varepsilon} \tag{33}$$

for $\varphi\in\mathbf{H}_\varepsilon$ *and for all t*

$$\varepsilon^{\frac{n}{n-2}}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G^j_\varepsilon}b^j_\varepsilon(x)u_\varepsilon\psi\,ds\longrightarrow\int\limits_\Omega V(x)\bar u\psi\,dx \tag{34}$$

as $\varepsilon\to 0$ *for any* $\psi\in\mathcal{F}^b$, *where V(x) is defined in (21), the constant M is independent of* ε.

Proof. To prove the inequality (33) we repeat the steps of the proof from [8] (see Lemma 2, inequality (21)).

To obtain the convergence (34) first of all, substituting u_ε as a test–function in (15), we can get the uniform boundedness

$$\|\nabla u_\varepsilon\|_{\mathbf{H}_\varepsilon} \le K, \qquad \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j} b_\varepsilon^j u_\varepsilon \psi \, dx \le K\varepsilon^{-n/(2-n)},$$

where K is independent of ε.

We consider the family of extension operators

$$P_\varepsilon : \mathbf{V}_\varepsilon \to \mathbf{V},$$

such that $P_\varepsilon v = v$ a.e. in Ω_ε and

$$\|\nabla P_\varepsilon v\|_{\mathbf{H}} \le \|\nabla v\|_{\mathbf{H}_\varepsilon} \qquad \forall v \in \mathbf{V}_\varepsilon .$$

See details of the construction in [37].

Due to the previous statement the sequence $\widetilde{u}_\varepsilon = P_\varepsilon u_\varepsilon$ is a bounded sequence in $\mathbf{V}$. Therefore, it is weakly convergent in $\mathbf{V}$. There exists $u \in \mathbf{V}$ such that

$$\widetilde{u}_\varepsilon \rightharpoonup u \qquad \text{in } \mathbf{V} \qquad \text{as } \varepsilon \to 0.$$

From now on, we simply use u_ε instead of $\widetilde{u}_\varepsilon$.

Denote $T_r^j = \{x \in \mathbb{R}^n : |x - P_\varepsilon^j| \le r\}$. Consider the following auxiliary function v_ε^j which satisfy the problem

$$\begin{cases} \Delta v_\varepsilon^j = 0, & x \in T^j_{\frac{\varepsilon}{4}} \setminus \overline{G_\varepsilon^j}, \\ \frac{\partial v_\varepsilon^j}{\partial \nu} + \varepsilon^{n/(2-n)} b_\varepsilon^j(x) v_\varepsilon^j = \varepsilon^{n/(2-n)} b_\varepsilon^j(x), & x \in \partial G_\varepsilon^j, \\ v_\varepsilon^j = 0, & x \in \partial T^j_{\frac{\varepsilon}{4}}. \end{cases} \tag{35}$$

It is easy to see that

$$\varepsilon^{n/(2-n)} \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j} b_\varepsilon^j(x) u_\varepsilon \phi ds = -\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}} \frac{\partial v_\varepsilon^j}{\partial \nu} u_\varepsilon \phi \, ds.$$

We set

$$V_\varepsilon(x) = \begin{cases} v_\varepsilon^j(x), & x \in T^j_{\frac{\varepsilon}{4}} \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{T^j_{\frac{\varepsilon}{4}}}. \end{cases} \tag{36}$$

In [24] it is proved that

$$\|V_\varepsilon\|^2_{\mathbf{V}_\varepsilon} \le K\varepsilon^2$$

and

$$\widetilde{V}_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad \mathbf{V}, \qquad \widetilde{V}_\varepsilon \to 0 \quad \text{strongly in} \quad \mathbf{H} \qquad \text{as} \quad \varepsilon \to 0,$$

where $\widetilde{V}_\varepsilon = P_\varepsilon V_\varepsilon$.

By means of Lemma 4.1 and Lemma 4.2 from [24] we derive that

$$\left| \sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}} \frac{\partial v_\varepsilon^j}{\partial \nu} h_\varepsilon \, ds + \int_\Omega V(x) h \, dx \right| \to 0 \tag{37}$$

as $\varepsilon \to 0$ for h_ε, $h \in \mathbf{V}$ such that $h_\varepsilon \rightharpoonup h$ in $\mathbf{V}$.

Finally, from (37) we derive the convergence (34). Lemma is proved. □

Using (31), (32) and (34), passing to the limit in the equation of problem (27) as $k \to \infty$ in the space $D'(\mathbb{R}_+; \mathbf{H}^{-r})$, we obtain that the function $\bar{u}(x, s)$ satisfies the problem

$$\begin{cases} \frac{\partial \bar{u}}{\partial t} = \Delta \bar{u} - f(\bar{u}) - V(x)\,\bar{u} + g(x), & x \in \Omega, \\ \bar{u} = 0, & x \in \partial\Omega, \\ \bar{u} = U(x), & t = 0. \end{cases} \tag{38}$$

Consequently, $\bar{u} \in \overline{\mathcal{K}}$. We have proved above that $u_{\varepsilon_k}(s) \to \bar{u}(s)$ as $k \to \infty$ in Θ^{loc}. The hypotheses $u_{\varepsilon_k}(s) \notin \mathcal{O}'(\overline{\mathcal{K}})$ implies that $\bar{u} \notin \mathcal{O}'(\overline{\mathcal{K}})$ and moreover $\bar{u} \notin \overline{\mathcal{K}}$. We came to the contradiction. The theorem is proved. □

Using the compact inclusions (19) and (20), we can strengthen the convergence (23).

Corollary 3.1. *For every $0 < \delta \le 1$ and for any $M > 0$*

$$\text{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0, \tag{39}$$

$$\text{dist}_{C([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0+). \tag{40}$$

To prove (39) and (40), we just repeat the proof of Theorem 3.1 replacing the topology Θ^{loc} with $L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ or $C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$.

Finally we consider the reaction diffusion equations for which the uniqueness theorem of the Cauchy problem takes place. It is sufficient to assume that the nonlinear term $f(u)$ in the Eq. (13) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \ge -C|v_1 - v_2|^2 \quad \text{for } v_1, v_2 \in \mathbb{R}. \tag{41}$$

(see [13,15]). In [13] it was proved that if (41) holds, then equations (13) and (22) generate the dynamical semigroups in $\mathbf{H}$ which have the global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ bounded in the space $\mathbf{V} = H_0^1(\Omega)$ (see also [1], [41]). We have

$$\mathcal{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Convergence (40) implies

Corollary 3.2. *Under the assumptions of Theorem 3.1, the following limit holds:*

$$\text{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon, \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+).$$

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Kuanysh A. Bekmaganbetov: Writing - review & editing. **Gregory A. Chechkin:** Writing - review & editing. **Vladimir V. Chepyzhov:** Writing - review & editing.

References

[1] Babin A. V., Vishik M. I.. Attractors of evolution equations, amsterdam, north–holland. 1992. (Moscow: Nauka; 1989.).

[2] Bakhvalov NS, Panasenko GP. Averaging processes in periodic media. Dordrecht: Kluwer Academic Publ; 1989.

[3] Belyaev AG, Piatnitski AL, Chechkin GA. Asymptotic behavior of solution for boundary–value problem in a perforated domain with oscillating boundary. Siberian Math J 1998;39(4):730–54.

[4] Belyaev A.G., Piatnitski A.L., Chechkin G.A.. Averaging in a perforated domain with an oscillating third boundary condition. Russian Academy of Sciences Sbornik Mathematics 192 (7, 2001): 933–949 (Translated from Matematicheskii Sbornik 192 (7, 2001): 3–20).

[5] Bensoussan A., Lions J.-L., Papanicolau G.. Asymptotic analysis for periodic structures, amsterdam, north–holland. 1978.

[6] Boyer F, Fabrie P. Mathematical tools for the study of the incompressible navier-stokes equations and related models. Applied mathematical sciences, 183. New York (NY): Springer; 2013.

[7] Bogolyubov NN, Mitropolski YA. Asymptotic methods in the theory of non-linear oscillations. New York (NY): Gordon & Breach; 1961.

[8] Chechkin G.A., Piatnitski A.L.. Homogenization of boundary–value problem in a locally periodic perforated domain // applicable analysis. - 1999.- v. 71, No 1–4.- p. 215–235.

[9] Chechkin GA, Piatnitski AL, Shamaev AS. Homogenization: methods and applications. Providence (RI): Am Math Soc 2007.

[10] Chepyzhov VV, Goritsky AY, Vishik MI. Integral manifolds and attractors with exponential rate for nonautonomous hyperbolic equations with dissipation. Russ J Math Phys 2005;12:17–39.

[11] Chepyzhov VV, Pata V, Vishik MI. Averaging of nonautonomous damped wave equations with singularly oscillating external forces. J Math Pures Appl 2008;90:469–91.

[12] Chepyzhov VV, Pata V, Vishik MI. Averaging of 2d navier-stokes equations with singularly oscillating forces. Nonlinearity 2009;22:351–70.
[13] Chepyzhov VV, Vishik MI. Trajectory attractors for reaction-diffusion systems. TopMethNonlinAnal JJulius Schauder Center 1996;7(1):49–76.
[14] Chepyzhov VV, Vishik MI. Evolution equations and their trajectory attractors. JMathPures Appl 1997;76(10):913–64.
[15] Chepyzhov VV, Vishik MI. Attractors for equations of mathematical physics. Providence (RI): Amer Math Soc 2002.
[16] Chepyzhov VV, Vishik MI. Non-autonomous 2d navier-stokes system with a simple global attractor and some averaging problems. ESAIM Control Optim Calc Var 2002;8:467–87.
[17] Chepyzhov VV, Vishik MI. Global attractors for non-autonomous ginzburg-landau equation with singularly oscillating terms. Rend Accad Naz Sci XL Mem Mat Appl 2005;29:123–48.
[18] Chepyzhov VV, Vishik MI. Non-autonomous 2d navier-stokes system with singularly oscillating external force and its global attractor. J Dynam Diff Eq 2007;19:655–84.
[19] Chepyzhov VV, Vishik MI, Wendland WL. On non-autonomous sine-gordon type equations with a simple global attractor and some averaging. Disc Contin Dyn Syst 2005;12:27–38.
[20] Cioranescu D, Saint Jean Paulin J. Homogenization in open sets with holes. JMathAnalAppl 1979;71:590–607.
[21] Cioranescu D, Murat F. Un terme étrange venu d'ailleurs i & II. Nonlinear partial differential equations and their applications Collège de France seminar, Volume II & III. Berzis H, Lions JL, editors; 1982. Research Notes in Mathematics, 60 & 70, London: Pitman, 98–138 & 154–178
[22] Cioranescu D, Donato P. On a robin problem in perforated domains. Homogenization and applications to material sciences. Cioranescu D, Damlamian A, Donato P, editors; 1997. GAKUTO International Series. Mathematical Sciences and Applications. Tokyo: Gakkōtosho, 9: 123–136
[23] Conca C, Donato P. Non-homogeneous neumann problems in domains with small holes. Modélisation Mathématique et Analyse Numérique (M^2AN) 1988;22(4):561–607.
[24] Diaz JI, Gomez-Castro D, Shaposhnikova TA, Zubova MN. Classification of homogenized limits of diffusion problems with spatially dependent reaction over critical-size particles. Appl Anal 2018;98(1–2):232–55.
[25] Efendiev M, Zelik S. Attractors of the reaction-diffusion systems with rapidly oscillating coefficients and their homogenization. Ann Inst H Poincaré Anal Non Linéaire 2002;19:961–89.
[26] Efendiev M, Zelik S. The regular attractor for the reaction-diffusion system with a nonlinearity rapidly oscillating in time and its averaging. Adv Diff Eq 2003;8:673–732.
[27] Fiedler B, Vishik MI. Quantitative homogenization of analytic semigroups and reaction-diffusion equations with diophantine spatial sequences. Adv Diff Eq 2001;6:1377–408.
[28] Fiedler B, Vishik MI. Quantitative homogenization of global attractors for reaction–diffusion systems with rapidly oscillating terms. Asymptotic Anal 2003;34:159–85.
[29] Hale JK, Verduyn Lunel SM. Averaging in infinite dimensions. J Int Eq Appl 1990;2:463–94.
[30] Hornung U. Homogenization and porous media. IAM, Volume 6. Berlin, New York: Springer–Verlag; 1997.
[31] Jikov VV, Kozlov SM, Oleinik OA. Homogenization of differential operators and integral functionals. Berlin: Springer–Verlag; 1994.
[32] Ilyin AA. Averaging principle for dissipative dynamical systems with rapidly oscillating right-hand sides. Sb Math 1996;187:635–77.
[33] Ilyin AA. Global averaging of dissipative dynamical systems. Rend Accad Naz Sci XL Mem Mat Appl 1998;22:165–91.
[34] Marchenko VA, Khruslov EY. Boundary value problems in domains with fine–grain boundary. Kiev: Naukova Dumka; 1974. Russian
[35] Marchenko V.A., Khruslov E.Y.. Homogenization of partial differential equations. 2006. Boston (MA): Birkhäuser.
[36] Mikhailov VP. Partial differential equations. Translated from the Russian by P C Sinha "Mir", Moscow; distributed by Imported Publications, Inc, Chicago, Ill 1978:397.
[37] Oleinik O.A., Shamaev A.S., Yosifian G.A.. Mathematical problems in elasticity and homogenization. 1992. Amsterdam: North–Holland.
[38] Pankratov LS, Chueshov ID. Averaging of attractors of nonlinear hyperbolic equations with asymptotically degenerate coefficients. Sb Math 1999;190:1325–52.
[39] Sanchez-Palencia E. Non homogeneous media and vibration theory. Berlin, New York: Springer-Verlag; 1980.
[40] Sanchez-Palencia E. Homogenization techniques for composite media. Berlin: Springer–Verlag; 1987.
[41] Temam R. Infinite-dimensional dynamical systems in mechanics and physics. Applied mathematics series. V. 68. New York (NY): Springer-Verlag; 1988.
[42] Vishik MI, Chepyzhov VV. Averaging of trajectory attractors of evolution equations with rapidly oscillating terms. Sb Math 2001;192:11–47.
[43] Vishik MI, Chepyzhov VV. Approximation of trajectories lying on a global attractor of a hyperbolic equation with an exterior force that oscillates rapidly over time. Sb Math 2003;194:1273–300.
[44] Vishik MI, Chepyzhov VV. Attractors of dissipative hyperbolic equations with singularly oscillating external forces. Math Notes 2006;79:483–504.
[45] Vishik MI, Chepyzhov VV. Trajectory attractors of equations of mathematical physics. Russian Math Surv 2011;66:3–102.
[46] Vishik MI, Fiedler B. Quantitative averaging of global attractors of hyperbolic wave equations with rapidly oscillating coefficients. Russian Math Surv 2002;57:709–28.
[47] Zelik S. Global averaging and parametric resonances in damped semilinear wave equations. Proc Roy Soc Edinburgh SectA 2006;136:1053–97.