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STRONG CONVERGENCE OF TRAJECTORY ATTRACTORS FOR REACTION–DIFFUSION SYSTEMS WITH RANDOM RAPIDLY OSCILLATING TERMS

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Abstract. We consider reaction–diffusion systems with random terms that oscillate rapidly in space variables. Under the assumption that the random functions are ergodic and statistically homogeneous we prove that the random trajectory attractors of these systems tend to the deterministic trajectory attractors of the averaged reaction-diffusion system whose terms are the average of the corresponding terms of the original system. Special attention is given to the case when the convergence of random trajectory attractors holds in the strong topology.

1. **Introduction.** In this paper, we consider the autonomous reaction–diffusion systems in a bounded domain $D \Subset \mathbb{R}^d$ with random inhomogeneous terms of the form

$$\partial_t u = a\Delta u - b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u) + g\left(x, \frac{x}{\varepsilon}, \omega\right),\ u|_{\partial D} = 0,\ x \in D,\ t \geq 0, \qquad (1)$$

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where $u = (u^1(x,t), \dots, u^N(x,t))$ is an unknown vector function, $f = (f^1, \dots, f^N)$, and $g = (g^1, \dots, g^N)$. Here a is an $N \times N$ matrix with positive symmetric part and the nonlinear vector function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfies a dissipative condition (see Sec. 4).

We consider a random rapidly oscillation coefficient $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and a vector function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$, where $\omega \in \Omega$ is a random parameter and $\varepsilon > 0$ is a small parameter. We assume that terms $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ are statistically homogeneous (see Sec. 4). We study the limit behaviour of random trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ of the system (1) as $\varepsilon \to 0+$.

Systems of the form (1) model, for example, complex composite chemical reactions in media with microinhomogeneous structure, where diffusion and cross-diffusion of chemical components may occur (the matrix a can be non-diagonal and non-symmetric). To study such phenomenon we apply the homogenization methods (cf., for example, [44, 8, 4, 49, 38, 45, 14]). These methods enable to explore media with periodic or quasiperiodic microstructure as well as with random one (see, for instance, [15, 16, 17, 1, 18, 19]).

Attractors describe the behaviour of solutions of dissipative nonlinear evolution equations on large time intervals and in the limit as time tends to infinity. Attractors help to single out the most essential limit sets of trajectories, which characterize the whole dynamics of the complicated model described by evolution PDEs (see, for examples, monographs [3, 26, 50] and the references therein). Using the attractors, it is convenient to study perturbations of trajectories (solutions) of evolution equations.

Following the Bogolyubov averaging principle [11], the first results related to attractors of evolution equations with rapidly, but non-randomly oscillating periodic or almost periodic terms, were obtained in the papers [37, 39, 40]. The homogenization of global attractors of parabolic equations with oscillating parameters has been studied in [26, 32, 33, 34, 35]. The similar problems have been considered in [26, 27, 29, 51] for autonomous and non-autonomous 2D Navier–Stokes systems. Some problems related to the homogenization of uniform global attractors for dissipative wave equations has been considered in [21, 30, 39, 52, 59] in presence of time oscillations and in [26, 47, 51, 56] for oscillations in space. Papers [22, 23, 28, 29, 54] deal with PDEs containing singular oscillating terms.

In the paper [31], the authors study random attractors and random inertial manifolds for scalar parabolic equations with random terms on fast time scale. Under some spectral gap condition, it was shown that the inertial manifolds of the fast time scale equations tend to the inertial manifold of the averaged system when the scaling parameter tends to zero.

The theory of trajectory attractors for evolution PDEs were suggested in [25, 26] (see also the review [55]). This approach is very fruitful in the study of the long time behaviour of solutions of evolution equations for which the uniqueness theorem of the corresponding initial-value problem is not proved yet (e.g., for the inhomogeneous 3D Navier–Stokes system) or does not hold (for example, reaction-diffusion systems with nonlinear terms that do not satisfy the Lipschitz condition, the case considered in this paper). The trajectory attractors were constructed for a number of important evolution equations and systems of mathematical physics, e.g. for the 3D Navier-Stokes system, various reaction-diffusion systems, the dissipative hyperbolic equation with arbitrary polynomial growth of nonlinear terms, for nonlinear elliptic systems and for other equations (see [24, 26, 57, 55]). Trajectory attractors for non-autonomous Ginzburg-Landau complex equations have been constructed in [53]. Some homogenization problem for trajectory attractors of evolution equation with (non-randomly) rapidly oscillating terms were studied in [26, 51].

Paper [6] deals with homogenization of trajectory attractors for autonomous and non-autonomous 3D Navier–Stokes systems with random external forces that oscillate rapidly in space variables or in time (see also [7] for weak homogenization of reaction–diffusion systems with random terms). Paper [20] is devoted to investigations of trajectory attractors of Ginzburg–Landau equation.

In the present paper, we study reaction-diffusion systems of the form (1) with coefficient $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and with right-hand side $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ that are random functions which oscillate rapidly with respect to the space variable, $\varepsilon > 0$ is a small parameter, and $1/\varepsilon$ is the oscillation frequency. Here ω is an element of a standard probability space $(\Omega, \mathcal{B}, \mu)$. We assume that the random functions $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ are statistically homogeneous and ergodic with smooth realizations (see Sec. 4).

Along with the random system (1), we also consider the corresponding averaged reaction-diffusion system with terms $b^{\text{hom}}(x)$ and $g^{\text{hom}}(x)$ that are mathematical expectations of $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$. It follows from the Birkhoff ergodic theorem that functions $b^{\text{hom}}(x)$ and $g^{\text{hom}}(x)$ coincides almost surely in $\omega \in \Omega$ with space means of the functions $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ as $\varepsilon \to 0$.

In this paper we prove that the trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ of the reaction-diffusion system (1) with random rapidly oscillating terms converge ω-almost surely as $\varepsilon \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the averaged reaction-diffusion system with deterministic terms $b^{\text{hom}}(x)$ and $g^{\text{hom}}(x)$ in an appropriate functional spaces.

The paper is organized as follows. In Sec. 2 we give necessary definitions of randomness and formulate the Birkhoff ergodic theorem and related assertions. In Sec. 3 we give the main notions and theorems concerning the trajectory attractors of autonomous evolution equations. Sec. 4 is devoted to the study of solutions for reaction-diffusion system. We formulate the main assumptions concerning all the terms in (1). It is important that we do not assume any Lipschitz condition for the nonlinear function $f(u)$ and, thus, the uniqueness theorem for the corresponding Cauchy problem of the system (1) may not hold. We also formulate the so-called energy identity for solutions of the system (1) which is the main tool in the construction of the strong trajectory attractors for these systems and in the study of strong convergence of random trajectory attractors.

In Sec. 5, we construct the trajectory attractor $\mathfrak{A}_\varepsilon(\omega)$ for the reaction-diffusion system (1) in the strong topology $\Theta_+^{s,loc}$. In Sec. 6, we prove the main theorems concerning the homogenization of trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ of reaction–diffusion systems with randomly rapidly oscillating terms. The first theorem states that the trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ converges with probability 1 to $\overline{\mathfrak{A}}$ as $\varepsilon \to 0+$ in the weak topology Θ_+^{loc}. The second theorem states that this convergence holds in the strong topology $\Theta_+^{s,loc}$ for the case when the coefficient $b = b(x)$ is not random and $g = g\left(x, \frac{x}{\varepsilon}, \omega\right)$ is random.

To study trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ for (1) and their convergence in the strong topology $\Theta_+^{s,loc}$ we apply the energy identity method developed in [5, 36, 46, 48] which is very effective in the study of global and trajectory attractors for dissipative evolution equations in unbounded, non-smooth domains, and for PDEs without uniqueness. This method uses the energy balance for the trajectories (solutions) and provides the strong compactness of bounded sets of trajectories.

2. **Preliminaries.** Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, i.e., the set Ω is endowed with σ-algebra $\mathcal{B}$ of its subsets and with σ-additive nonnegative measure μ on $\mathcal{B}$ such that $\mu(\Omega) = 1$.

Definition 2.1. A family of measurable maps $\mathcal{T}_\xi : \Omega \to \Omega$, $\xi \in \mathbb{R}^d$, is called a *space dynamical system* if the following properties hold:

1) *the group property*: $\mathcal{T}_{\xi_1+\xi_2} = \mathcal{T}_{\xi_1}\mathcal{T}_{\xi_2}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^d$; $\mathcal{T}_0 = Id$, where Id is the identity mapping on Ω;

2) *the isometry property* (the mappings $\mathcal{T}_\xi$ preserve the measure μ on Ω) $\forall B \in \mathcal{B}$, $\mathcal{T}_\xi B \in \mathcal{B}$, $\mu(\mathcal{T}_\xi B) = \mu(B)$, $\forall \xi \in \mathbb{R}^d$;

3) *the measurability*: for any measurable function $\psi(\omega)$ on Ω, the function $\psi(\mathcal{T}_\xi \omega)$ is measurable on $\Omega \times \mathbb{R}^d$ and continuous in ξ.

Let $L_q(\Omega, \mu)$ $(q \geq 1)$ be the space of measurable functions on Ω whose absolute value at the power q is integrable with respect to the measure μ. If $\mathcal{T}_\xi : \Omega \to \Omega$ is a space dynamical system, then, on the space $L_q(\Omega, \mu)$, we define a group of operators $\{\mathcal{T}_\xi\}$ depending on the parameter $\xi \in \mathbb{R}^d$ by the formula $(\mathcal{T}_\xi \psi)(\omega) := \psi(\mathcal{T}_\xi \omega)$, $\psi \in L_q(\Omega, \mu)$.

Condition 3) in Definition 2.1 implies that the group $\{\mathcal{T}_\xi\}$ is strongly continuous, i.e., we have $\lim_{\xi \to 0} \|\mathcal{T}_\xi \psi - \psi\|_{L_q(\Omega,\mu)} = 0$ for any $\psi \in L_q(\Omega, \mu)$.

Definition 2.2. Suppose that $\psi(\omega)$ is a measurable function on Ω. The function $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$ $(\xi \in \mathbb{R}^d)$ for a fixed $\omega \in \Omega$ is called a *realization of the function* ψ.

The following assertion is proved, for instance, in [14, 38].

Proposition 1. *If $\psi \in L_q(\Omega, \mu)$, then ω-almost all realizations $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$ belong to $L_q^{loc}(\mathbb{R}^d)$.*

If the sequence $\{\psi_k\} \subset L_q(\Omega, \mu)$ converges in $L_q(\Omega, \mu)$ to the function ψ, then there exists a subsequence $\{\psi_{k'}\}$ such that ω-almost all realizations $\xi \mapsto \psi_{k'}(\mathcal{T}_\xi \omega)$ converge in $L_q^{loc}(\mathbb{R}^d)$ to the realization $\xi \mapsto \psi(\mathcal{T}_\xi \omega)$.

Definition 2.3. A measurable function $\psi(\omega)$ on Ω is called *invariant*, if $\psi(\mathcal{T}_\xi \omega) = \psi(\omega)$ for any $\xi \in \mathbb{R}^d$ and almost all $\omega \in \Omega$.

Definition 2.4. The space dynamical system $\mathcal{T}_\xi$ is called *ergodic*, if any invariant function is a constant ω-almost everywhere.

We denote by $P_{a_1\cdots a_d}^{b_1\cdots b_d} = [a_1, b_1] \times \cdots \times [a_d, b_d] = P$ $(a_i < b_i, i = 1, \dots, d)$, a parallelepiped in $\mathbb{R}^d$ with volume $|P| = \prod_{i=1}^d (b_i - a_i)$.

Definition 2.5. Let $F(\xi)$ be an arbitrary function from the space $L_1^{loc}(\mathbb{R}^d)$. We say that $F(\xi)$ has a *space average*, if the following limit

$$M(F) := \lim_{\lambda \to +\infty} \frac{1}{|\lambda P|} \int \cdots \int_{\lambda P} F(\xi)\, d\xi.$$

exists for any parallelepiped P and does not depend on the choice of P. The number $M(F)$ is called the SPACE MEAN VALUE of the function F.

Equivalently, the space average is defined by

$$M(F) := \lim_{\varepsilon \to 0} \frac{1}{|P|} \int \cdots \int_P F\left(\frac{\chi}{\varepsilon}\right)\, d\chi.$$

Throughout the paper we use the Birkhoff theorem (see [9, 2]) in the following form (see, for instance, [14, 38]):

Theorem 2.6 (Birkhoff ergodic theorem)**.** *Let $P \subset \mathbb{R}^d$. Let the space dynamical system $\mathcal{T}_\xi$ satisfy Definition 2.1. Consider a measurable real function $\psi = \psi(x,\omega)$, $x \in P$, $\omega \in \Omega$, such that, for every $x \in P$, the function $\psi(x,\cdot) \in L_q(\Omega,\mu)$.*

Then, for every $x \in P$ and for almost all $\omega \in \Omega$, the realization $\psi(x,\mathcal{T}_\xi\omega)$ has the space mean value $M(\psi(x,\mathcal{T}_\xi\omega))$. Moreover, $M(\psi(x,\mathcal{T}_\xi\omega))$ is an invariant function and

$$\mathbb{E}(\psi)(x) \equiv \int_\Omega \psi(x,\omega)\ d\mu = \int_\Omega M(\psi(x,\mathcal{T}_\xi\omega))\ d\mu.$$

In particular, if the space dynamical system $\mathcal{T}_\xi$ is ergodic then, for almost all $\omega \in \Omega$, we have the identity

$$\mathbb{E}(\psi)(x) = M(\psi)(x).$$

Definition 2.7. Let $P \subset \mathbb{R}^d$. A random function $\psi(x,\xi,\omega)$, $\in P$, $\xi \in \mathbb{R}^d$, $\omega \in \Omega$, is called *statistically homogeneous* for any x, if the representation

$$\psi(x,\xi,\omega) = \Psi(x,\mathcal{T}_\xi\omega),$$

is valid for some measurable function $\Psi :\ P\times\Omega \to \mathbb{R}$, where $\mathcal{T}_\xi$ is a space dynamical system in Ω.

Consider some examples.

Example. Let Ω be the unit cube $\{\omega \in \mathbb{R}^m, 0 \le \omega_j \le 1, j = 1,\dots,m\}$. We have a space dynamical system on Ω:

$$\mathcal{T}_\xi\omega = \omega + \xi \,(\mathrm{mod}\, 1),\ \xi \in \mathbb{R}^d.$$

The Lebesgue measure is invariant and this space dynamical system is ergodic. The realization of the function $f(\omega) \in L_q(\Omega)$ has the form $f(\xi+\omega)$.

Example. Let Ω be a unit cube in $\mathbb{R}^m$, μ be a Lebesgue measure on it. For $\xi \in \mathbb{R}^d$ we set $\mathcal{T}_\xi\omega = \omega + \lambda\xi \,(\mathrm{mod}\, 1)$, where $\lambda = \{\lambda_{ij}\}$ is a $m \times d$-matrix. Obviously the mapping $\mathcal{T}_\xi$ preserve the measure μ on Ω. The space dynamical system is ergodic if and only if $\lambda_{ij}k_j \neq 0$ for any integer vector $k \neq 0$.

Thus, $L_q(\Omega)$ is the space of periodic functions of m variables, and the realizations have the form $f(\omega+\lambda\xi)$. These realizations are called *quasi-periodic functions*, if $f(\omega)$ is continuous on Ω.

The following statement can be found, for instance, in [14].

Proposition 2. *Let P be a measurable subset of $\mathbb{R}^d$. Let $p \geqslant 1$ or $p = \infty$. Suppose that a measurable function $\digamma(x,\xi)$, $x \in P$, $\xi \in \mathbb{R}^d$, has a space mean value $M(\digamma)(x)$ in $\mathbb{R}^d_\xi$ for every $x \in P$ and the family $\{\digamma\left(x,\frac{x}{\varepsilon}\right) \mid 0 < \varepsilon \leqslant 1\}, x \in \mathcal{K}$, is bounded in $L_p(\mathcal{K})$, where $\mathcal{K}$ is an arbitrary compact subset in P.*

Then $M(\digamma)(\cdot) \in L_p^{loc}(P)$ and, for $p \geqslant 1$, we have

$$\digamma\left(x,\tfrac{x}{\varepsilon}\right) \rightharpoonup M(\digamma)(x) \text{ weakly in } L_p^{loc}(P) \text{ as } \varepsilon \to 0$$

and, for $p = \infty$, we have

$$\digamma\left(x,\tfrac{x}{\varepsilon}\right) \rightharpoonup M(\digamma)(x) \text{ $*$-weakly in } L_\infty^{loc}(P) \text{ as } \varepsilon \to 0.$$

3. **Trajectory attractors of evolution equations.** In this section we give a scheme for the construction of trajectory attractors of evolution equations. In the next sections we shall apply this scheme to the study of trajectory attractors of random reaction-diffusion systems with rapidly oscillating coefficients and the corresponding averaged systems.

Consider an abstract autonomous evolution equation

$$\partial_t u = A(u), \quad t \geq 0. \tag{2}$$

Here $A(\cdot) : E_1 \to E_0$ is a nonlinear operator, E_1, E_0 are Banach spaces and $E_1 \subseteq E_0$. For instance, $A(u) = a\Delta u - bf(u) + g$ (see also Sec. 4).

We are going to study solutions $u(s)$ of equation (2) as functions of $s \in \mathbb{R}_+$ as a whole. Here $s \equiv t$ denote the time variable. A set of solutions of (2) is said to be *a trajectory space* $\mathcal{K}^+$ of equation (2). Let us describe a trajectory space $\mathcal{K}^+$ in greater detail.

At first, we consider solutions $u(s)$ of (2) defined on a fixed time segment $[0, M]$ from $\mathbb{R}_+$. We study solutions of (2) in a Banach space $\mathcal{F}_{0,M}$ that depends on M. The space $\mathcal{F}_{0,M}$ consists of functions $f(s), s \in [0, M]$ such that $f(s) \in E$ for almost all $s \in [0, M]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, $\mathcal{F}_{0,M}$ can be the space $C([0, M]; E)$, or $L_p(0, M; E)$, for $p \in [1, \infty]$, or the intersection of such spaces (see Sec. 4). We assume that $\Pi_{0,m}\mathcal{F}_{0,M} \subseteq \mathcal{F}_{0,m}$ and

$$\|\Pi_{0,m} f\|_{\mathcal{F}_{0,m}} \leq \|f\|_{\mathcal{F}_{0,M}}, \quad \forall f \in \mathcal{F}_{0,M}, \tag{3}$$

where $m < M$ and $\Pi_{0,M}$ denotes the restriction operator onto the segment $[0, M]$.

Let $S(h)$ denote the translation operator for every $h \in \mathbb{R}_+$:

$$S(h)f(s) = f(h + s).$$

Evidently, if the argument s of a function $f(\cdot)$ belongs $[0, M]$, then the argument s of $S(h)f(\cdot)$ can be taken form $[0, M - h]$ for $0 \leq h < M$. We assume that $S(h)$ is a continuous map from $F_{0,M}$ to $F_{0,M-h}$ and, moreover,

$$\|S(h)f\|_{\mathcal{F}_{0,M-h}} \leq \|f\|_{\mathcal{F}_{0,M}}, \quad \forall f \in \mathcal{F}_{0,M}. \tag{4}$$

This assumption is fairly natural.

We assume that if $f(s) \in \mathcal{F}_{0,M}$, then $A(f(s)) \in \mathcal{D}_{0,M}$, where $\mathcal{D}_{0,M}$ is a larger Banach space, $\mathcal{F}_{0,M} \subseteq \mathcal{D}_{0,M}$. The derivative $\partial_t f(t)$ is a distribution with values in $E_0, \partial_t f(\cdot) \in D'((0, M); E_0)$ and we assume that $\mathcal{D}_{0,M} \subseteq D'((0, M); E_0)$ for all $(0, M) \subset \mathbb{R}_+$. A function $u(s) \in \mathcal{F}_{0,M}$ is said to be a *solution* of (2) from the space $\mathcal{F}_{0,M}$ (on the interval $(0, M)$) if $\partial_t u(t) = A(u(t))$ in the distributional sense of the space $D'((0, M); E_0)$.

We also define the space

$$\mathcal{F}_+^{loc} = \{f(s),\, s \in \mathbb{R}_+ \mid \Pi_{0,M} f(s) \in \mathcal{F}_{0,M}, \quad \forall\, [0, M] \subset \mathbb{R}_+\}. \tag{5}$$

For example, if $\mathcal{F}_{0,M} = C([0, M]; E)$, then $\mathcal{F}_+^{loc} = C(\mathbb{R}_+; E)$ and if $\mathcal{F}_{0,M} = L_p(0, M; E)$, then $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+; E)$.

A function $u(s) \in \mathcal{F}_+^{loc}$ is called a solution of (2) from $\mathcal{F}_+^{loc}$ if $\Pi_{0,M} u(s) \in \mathcal{F}_{0,M}$ and this function is a solution of (2) for every segment $[0, M] \subset \mathbb{R}_+$.

We denote by $\mathcal{K}^+$ a given fixed set of solutions of (2) from $\mathcal{F}_+^{loc}$. Notice, that $\mathcal{K}^+$ is not necessarily the set of *all* solutions from $\mathcal{F}_+^{loc}$. The elements of $\mathcal{K}^+$ are called *trajectories* and the set $\mathcal{K}^+$ is called the *trajectory space* of the equation (2).

We assume that the trajectory space $\mathcal{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathcal{K}^+$, then $u(h+s) \in \mathcal{K}^+$ for every $h \geq 0$. This is a very natural assumption for solutions of autonomous equations.

We now consider the translation operators $S(h)$ in $\mathcal{F}_+^{loc}$:

$$S(h)f(s) = f(s+h), \quad h \geq 0.$$

It is clear that the mappings $\{S(h), h \geq 0\}$ form a semigroup in $\mathcal{F}_+^{loc}$, that is,

$$S(h_1)S(h_2) = S(h_1 + h_2) \quad \forall h_1, h_2 \geq 0$$

and $S(0)$ is the identity operator. We change the variable h into the time variable t. The semigroup $\{S(t), t \geq 0\}$ is called the *translation semigroup.* By our assumption the translation semigroup maps the trajectory space $\mathcal{K}^+$ to itself:

$$S(t)\mathcal{K}^+ \subseteq \mathcal{K}^+, \quad \forall t \geq 0. \tag{6}$$

We shall study attracting properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+ \subset \mathcal{F}_+^{loc}$. We define a topology in the space $\mathcal{F}_+^{loc}$.

Let a metrics $\rho_{0,M}(\cdot,\cdot)$ be defined on $\mathcal{F}_{0,M}$ for all $[0,M] \subset \mathbb{R}_+$. Similar to (3) and (4) we assume that

$$\rho_{0,m}\left(\Pi_{0,m}f, \Pi_{0,m}g\right) \leq \rho_{0,M}\left(f,g\right), \quad \forall f,g \in \mathcal{F}_{0,M},\ m \leq M,$$
$$\rho_{0,M-h}(S(h)f, S(h)g) \leq \rho_{0,M}(f,g), \quad \forall f,g \in \mathcal{F}_{0,M},\ 0 \leq h \leq M.$$

Denote by $\Theta_{0,M}$ the corresponding metric spaces on $\mathcal{F}_{0,M}$. For example, $\rho_{0,M}$ can be the metrics associated with the norm $\|\cdot\|_{\mathcal{F}_{0,M}}$ of the Banach space $\mathcal{F}_{0,M}$. However, frequently in application $\rho_{0,M}$ generate the topologies $\Theta_{0,M}$ that are weaker than the strong convergence topology of the Banach spaces $\mathcal{F}_{0,M}$.

The inductive limit of the spaces $\Theta_{0,M}$ defines the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$, i.e., by definition, a sequence $\{f_n(s)\} \subset \mathcal{F}_+^{loc}$ converges to $f(s) \in \mathcal{F}_+^{loc}$ as $n \to \infty$ in Θ_+^{loc} if $\rho_{0,M}(\Pi_{0,M}f_n, \Pi_{0,M}f) \to 0$ as $n \to \infty$ for each $M > 0$. It is not hard to prove that the topology Θ_+^{loc} is metrizable using, for example, the Fréchet metrics

$$\rho_+(f_1,f_2) := \sum_{m \in \mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{7}$$

If it is known that all metric spaces $\Theta_{0,M}$ are complete, then the space Θ_+^{loc} is also complete.

We claim that the translation semigroup $\{S(t)\}$ is continuous in Θ_+^{loc}. This assertion follows directly from the definition of the topological space Θ_+^{loc}.

We also consider the following Banach space

$$\mathcal{F}_+^b := \{f(s) \in \mathcal{F}_+^{loc} \mid \|f\|_{\mathcal{F}_+^b} < +\infty\}, \tag{8}$$

where the norm

$$\|f\|_{\mathcal{F}_+^b} := \sup_{h \geq 0} \|\Pi_{0,1}f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{9}$$

For example, if $\mathcal{F}_+^{loc} = C(\mathbb{R}_+;E)$, then the space $\mathcal{F}_+^b = C^b(\mathbb{R}_+;E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \sup_{h\geq 0}\|f(h)\|_E$ and if $\mathcal{F}_+^{loc} = L_p^{loc}(\mathbb{R}_+;E)$, then $\mathcal{F}_+^b = L_p^b(\mathbb{R}_+;E)$ with norm $\|f\|_{\mathcal{F}_+^b} = \left(\sup_{h\geq 0}\int_h^{h+1}\|f(s)\|_E^p ds\right)^{1/p}$.

Recall that $\mathcal{F}_+^b \subseteq \Theta_+^{loc}$. We require the Banach space $\mathcal{F}_+^b$ only to define bounded subsets in the trajectory space $\mathcal{K}^+$. To construct a trajectory attractor in $\mathcal{K}^+$, we do not consider the corresponding uniform convergence topology of the Banach space $\mathcal{F}_+^b$. Instead, we utilize the local convergence topology Θ_+^{loc} which is much weaker.

We suppose that $\mathcal{K}^+ \subseteq \mathcal{F}_+^b$, i.e., every trajectory $u(s) \in \mathcal{K}^+$ of equation (2) has a finite norm (9). Let us define an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$.

Definition 3.1. A set $\mathcal{P} \subseteq \Theta_+^{loc}$ is called an *attracting set* of the semigroup $\{S(t)\}$ acting on $\mathcal{K}^+$ in the topology Θ_+^{loc} if for any bounded in $\mathcal{F}_+^b$ set $\mathcal{B} \subseteq \mathcal{K}^+$ the set $\mathcal{P}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc}, i.e., for any ε-neighbourhood $O_\varepsilon(\mathcal{P})$ in Θ_+^{loc} there exists $t_1 \geq 0$ such that $S(t)\mathcal{B} \subseteq O_\varepsilon(\mathcal{P})$ for all $t \geq t_1$.

It is clear that the attracting property of $\mathcal{P}$ can be formulated in the following equivalent form: for any set $\mathcal{B} \subseteq \mathcal{K}^+$ bounded in $\mathcal{F}_+^b$ and for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{P}) \to 0 \quad (t \to +\infty),$$

where $\mathrm{dist}_{\mathcal{M}}(X,Y) := \sup_{x\in X} \mathrm{dist}_{\mathcal{M}}(x,Y) = \sup_{x\in X} \inf_{y\in Y} \rho_{\mathcal{M}}(x,y)$ is the Hausdorff semidistance from a set X to a set Y in a metric space $\mathcal{M}$.

Definition 3.2 (see [26])**.** A set $\mathfrak{A} \subseteq \mathcal{K}^+$ is called the *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, if **(i)** $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}, **(ii)** the set $\mathfrak{A}$ is strictly invariant with respect to the semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geq 0$, and **(iii)** $\mathfrak{A}$ is an attracting set for $\{S(t)\}$ on $\mathcal{K}^+$ in the topology Θ_+^{loc}, that is, for each $M > 0$

$$\mathrm{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\ \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

Remark 1. Using the terminology from [3] one can say that the trajectory attractor $\mathfrak{A}$ is the *global* $(\mathcal{F}_+^b, \Theta_+^{loc})$*-attractor* of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathcal{K}^+$, that is, $\mathfrak{A}$ attracts $S(t)\mathcal{B}$ as $t \to +\infty$ in the topology Θ_+^{loc} for any bounded (in $\mathcal{F}_+^b$) set of trajectories $\mathcal{B}$ from $\mathcal{K}^+$:

$$\mathrm{dist}_{\Theta_+^{loc}}(S(t)\mathcal{B}, \mathfrak{A}) \to 0 \quad (t \to +\infty).$$

We now formulate the central result on the trajectory attractor for equation (2).

Theorem 3.3. *Assume that the trajectory space $\mathcal{K}^+$ corresponding to equation (2) is contained in $\mathcal{F}_+^b$ and (6) holds. Suppose that the translation semigroup $\{S(t)\}$ has an attracting set $\mathcal{P} \subseteq \mathcal{K}^+$ which is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geq 0\}$ acting on $\mathcal{K}^+$ has the trajectory attractor $\mathfrak{A} \subseteq \mathcal{P}$. The set $\mathfrak{A}$ is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}.*

Sketch of the proof. Indeed, the semigroup $\{S(t)\}$ is continuous on $\mathcal{K}^+$ in the metric space Θ_+^{loc}. The set $\mathcal{P}$ is $(\mathcal{F}_+^b, \Theta_+^{loc})$-attracting, compact in the space Θ_+^{loc}, and bounded in $\mathcal{F}_+^b$. Then the semigroup $\{S(t)\}$ has the global $(\mathcal{F}_+^b, \Theta_+^{loc})$-attractor $\mathfrak{A}$ which is evidently the trajectory attractor (see [3, 25, 26] for the complete proof). This attractor can be constructed from the set $\mathcal{P}$ by the standard formula

$$\mathfrak{A} = \omega(\mathcal{P}) := \bigcap_{h\geq 0} \left[\bigcup_{t \geq h} S(t)\mathcal{P} \right]_{\Theta_+^{loc}}.$$

We now describe the structure of the trajectory attractor $\mathfrak{A}$ of equation (2) in terms of complete trajectories of this equation.

Consider the equation (2) on the entire time axis

$$\partial_t u = A(u),\ t \in \mathbb{R}. \tag{10}$$

We have defined the trajectory space $\mathcal{K}^+$ of equation (10) on $\mathbb{R}_+$. We now extend this definition on $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is defined on the entire time axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s), s \in \mathbb{R}$ is called a *complete trajectory* of equation (10) if $\Pi_+ S(h)f(s) = \Pi_+ u(s+h) \in \mathcal{K}^+$ for all $h \in \mathbb{R}$. Here $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator to the semiaxis $\mathbb{R}_+$.

We have introduced the spaces $\mathcal{F}_+^{loc}, \mathcal{F}_+^b$, and Θ_+^{loc}. We now define spaces $\mathcal{F}^{loc}, \mathcal{F}^b$, and Θ^{loc} in the same way:

$$\mathcal{F}^{loc} := \{f(s), s \in \mathbb{R} \mid \Pi_{0,M} S(h) f(s) \in \ \mathcal{F}_{0,M} \ \forall h \in \mathbb{R},\ M > 0\};$$
$$\mathcal{F}^b := \{f(s) \in \mathcal{F}^{loc} \mid \|f\|_{\ \mathcal{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathcal{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathcal{F}_{0,1}}. \tag{11}$$

The topological space Θ^{loc} coincides (as a set) with $\mathcal{F}^{loc}$ and, by definition, $f_n(s) \to f(s)$ in Θ^{loc} if $\Pi_{0,M} S(h) f_n(s) \to \Pi_{0,M} S(h) f(s)$ $(n \to \infty)$ in $\Theta_{0,M}$ for every $M > 0$ and for all $h \in \mathbb{R}$. It is clear that Θ^{loc} is a complete metric space as well as Θ_+^{loc}.

Definition 3.4. The *kernel* $\mathcal{K}$ in the space $\mathcal{F}^b$ of equation (10) is the union of all complete trajectories $u(s), s \in \mathbb{R}$, of equation (10) that are bounded in the space $\mathcal{F}^b$ with respect to the norm (11):

$$\|\Pi_{0,1} u(h+s)\|_{\mathcal{F}_{0,1}} \le C_u, \quad \forall h \in \mathbb{R}. \tag{12}$$

Theorem 3.5. *Assume that the hypotheses of Theorem 3.3 holds. Then*

$$\mathfrak{A} = \Pi_+ \mathcal{K}, \tag{13}$$

the set $\mathcal{K}$ is compact in Θ^{loc} and bounded in $\mathcal{F}^b$.

The complete proof can be found in [25, 26].

Theorem 3.3 shows that for the construction of the trajectory attractor we require an attracting set $\mathcal{P}$ compact in Θ_+^{loc} and bounded in $\mathcal{F}_+^b$. Usually in application, a large ball $B_R = \{\|f\|_{\mathcal{F}_+^b} \le R\}$ in $\mathcal{F}_+^b$ $(R \gg 1)$ can be taken as such an attracting (or even absorbing) set and the existence of such ball B_R follows from the inequality of the form

$$\|S(t)u\|_{\mathcal{F}_+^b} \le Q(\|u\|_{\mathcal{F}_+^b}) e^{-\gamma t} + R_0, \quad \forall t \ge 0, \quad (\gamma > 0) \tag{14}$$

holding for each trajectory $u(\cdot)$ of equation (2). Here, $Q(y)$ depends on y, while R_0 and γ do not depend on a trajectory u. Inequality (14) follows usually from a priori estimates for solutions of equations (2).

In various applications, to prove that a ball in $\mathcal{F}_+^b$ is compact in Θ_+^{loc} the following lemma is useful. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the following Banach spaces

$$W_{p_1,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s \in (0,M) \mid \psi \in L_{p_1}(0,M;E_1), \psi' \in L_{p_0}(0,M;E_0)\right\},$$
$$W_{\infty,p_0}(0,M;E_1,E_0) = \left\{\psi(s), s \in (0,M) \mid \psi \in L_{\infty}(0,M;E_1), \psi' \in L_{p_0}(0,M;E_0)\right\},$$

(where $p_1 \geq 1$ and $p_0 > 1$) with norms

$$\|\psi\|_{W_{p_1,p_0}} := \left(\int_0^M \|\psi(s)\|_{E_1}^{p_1} ds\right)^{1/p_1} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s \in [0,M]\} + \left(\int_0^M \|\psi'(s)\|_{E_0}^{p_0} ds\right)^{1/p_0}.$$

Lemma 3.6 (Aubin–Lions–Simon, see [10]). *Assume that $E_1 \Subset E \subset E_0$. Then the following embeddings are compact:*

$$W_{p_1,p_0}(0,M;E_1,E_0) \Subset L_{p_1}(0,M;E), \tag{15}$$

$$W_{\infty,p_0}(0,M;E_1,E_0) \Subset C([0,M];E). \tag{16}$$

In the next section we study evolution equations and their trajectory attractors depending on a small parameter $\varepsilon > 0$.

Definition 3.7. We say that the trajectory attractors $\mathfrak{A}_\varepsilon$ converge to the trajectory attractor $\overline{\mathfrak{A}}$ as $\varepsilon \to 0$ in the topological space Θ_+^{loc} if for any neighborhood $\mathcal{O}(\mathfrak{A})$ in Θ_+^{loc} there is an $\varepsilon_1 > 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathcal{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, that is, for each $M > 0$

$$\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0 \ (\varepsilon \to 0).$$

4. Reaction-diffusion systems with random terms.

We consider the reaction-diffusion system (RD-system) in a bounded domain $D \Subset \mathbb{R}^d$ with rapidly oscillating random terms of the form

$$\partial_t u = a\Delta u - b\left(x, \frac{x}{\varepsilon}, \omega\right) f(u) + g\left(x, \frac{x}{\varepsilon}, \omega\right), \ u|_{\partial D} = 0, \tag{17}$$

where $x \in D \Subset \mathbb{R}^d$, $u = u(x,t)$, $u = (u^1, \dots, u^N)$, $f = (f^1, \dots, f^N)$, and $g = (g^1, \dots, g^N)$. Here a is an $N \times N$ matrix with positive symmetric part: $\frac{1}{2}(a + a^*) \geq \beta I$, where $\beta > 0$, and the real function $b(x, \xi, \omega) \in C(D \times \mathbb{R}^d)$ is positive for almost every $\omega \in \Omega$. The Laplace operator $\Delta := \partial_{x_1}^2 + \ldots + \partial_{x_d}^2$ acts in the x-space.

Remark 2. In application usually, nonlinear interaction functions $f^j(u)$ are polynomials with respect to u. The functions g^j model inhomogeneous external actions and can be interpreted as flux of light, radiation, etc. Note that the complex Ginzburg–Landau equation can also be written in the form (17) (see [28, 53]).

We note that all the results can also be applied to the systems with nonlinear terms of the form $\sum_{j=1}^m b_j\left(x, \frac{x}{\varepsilon}, \omega\right) f_j(u)$,where b_j are positively defined matrices and $f_j(u)$ are vector functions. For brevity, we consider the case $m = 1$ and $b_1\left(x, \frac{x}{\varepsilon}, \omega\right) = b\left(x, \frac{x}{\varepsilon}, \omega\right) I$, where I is the identity matrix and b is a positive real function.

We assume that the vector function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfies the following inequalities:

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i-1}} \leq C_0\left(\sum_{i=1}^N |v^i|^{p_i} + 1\right), \tag{18}$$

$$\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C \leq \sum_{i=1}^N f^i(v) v^i, \ \forall v \in \mathbb{R}^N, \tag{19}$$

where $\gamma_i > 0$ for $i = 1, \dots, N$. For definiteness, we assume that $p_N \geq p_{N-1} \geq \dots \geq p_1 \geq 2$. Inequality (18) is related to the fact that, in real-life RD-systems, the functions $f^i(u)$ are polynomials with possibly different degrees. Inequality (19) is called the *dissipativity condition* for the RD-system (17). The simple case $p_i \equiv p$ for $i = 1, \dots, N$ reduces (18) and (19) to the following inequalities:

$$|f(v)| \leq C_0 \left(|v|^{p-1} + 1\right), \ \gamma|v|^p - C \leq f(v)v, \ \forall v \in \mathbb{R}^N. \tag{20}$$

Notice that we *do not assume* that the function $f(v)$ satisfies the Lipschitz condition with respect to v.

We introduce the spaces $\mathbf{H} := [L_2(D)]^N$ and $\mathbf{V} := [H_0^1(D)]^N$. The norms in these spaces are denoted, respectively, by

$$\|v\|^2 := \int_D \sum_{i=1}^N |v^i(x)|^2 dx \quad \text{and} \quad \|v\|_1^2 := \int_D \sum_{i=1}^N |\nabla v^i(x)|^2 dx.$$

As usual, $\mathbf{V}' := [H^{-1}(D)]^N$ denotes the dual space of $\mathbf{V}$.

Let $\mathcal{T}_\xi, \xi \in \mathbb{R}^d$, be an ergodic space dynamical system in Ω.

We assume that the random function $b(x, \frac{x}{\varepsilon}, \omega)$ is statistically homogeneous, that is,

$$b(x, \xi, \omega) = \mathbf{B}(x, \mathcal{T}_\xi \omega),$$

where $\mathbf{B} : D \times \Omega \to \mathbb{R}$ is a measurable function.

We also assume that $\mathbf{B}(x, \omega) \in C_b(\overline{D})$ for almost all $\omega \in \Omega$ and

$$0 < \beta_0 \leq \mathbf{B}(x, \omega) \leq \beta_1, \ \forall x \in D. \tag{21}$$

Due to Birkhoff ergodic theorem the function $b\,(x, \xi, \omega) = \mathbf{B}(x, \mathcal{T}_\xi \omega)$ has the space mean value

$$b^{\text{hom}}(x) := M(b)(x) = \mathbb{E}(\mathbf{B})(x)$$

for every $x \in D$. It is clear that the function $b^{\text{hom}}(x)$ also satisfies the inequality

$$0 < \beta_0 \leq b^{\text{hom}}(x) \leq \beta_1, \ \forall x \in D.$$

It follows from Proposition 2 ($p = \infty$), that almost surely in $\omega \in \Omega$

$$\int\limits_D b\left(x, \frac{x}{\varepsilon}, \omega\right) \varphi(x) dx \to \int\limits_D b^{\text{hom}}\,(x)\, \varphi(x) dx \ (\varepsilon \to 0+), \quad \forall \varphi \in L_1(D). \tag{22}$$

For the random vector function $g(x, \frac{x}{\varepsilon}, \omega)$, we also assume that it is statistically homogeneous, i.e.

$$g(x, \xi, \omega) = \mathbf{G}(x, \mathcal{T}_\xi \omega),$$

where $\mathbf{G} : D \times \Omega \to \mathbb{R}^N$ is a measurable function and the following inequality holds almost surely in Ω:

$$|\mathbf{G}\,(x, \omega)\,| \leqslant \phi(x) \quad \text{for almost all } x \in D \tag{23}$$

and the positive majorant $\phi(\cdot) \in L_2(D)$.

Birkhoff ergodic theorem implies that the function $g(x, \xi, \omega) = \mathbf{G}(x, \mathcal{T}_\xi \omega)$, has the space mean value

$$g^{\text{hom}}(x) := M(g)(x) = \mathbb{E}(\mathbf{G})(x)$$

for every $x \in D$. It follows from (23) that

$$|g^{\text{hom}}(x)| \leqslant \phi(x) \quad \text{for almost all } x \in D$$

and therefore $g^{\text{hom}}(\cdot) \in \mathbf{H}$.

Inequalities (23) imply that the vector function $g\left(x, \frac{x}{\varepsilon}, \omega\right)$ belongs to the space $\mathbf{H}$ and is uniformly (w.r.t. $\varepsilon \in (0,1]$) bounded in this space.

Therefore, Proposition 2 is applicable with $P = D$ and $p = 2$ and we obtain that, almost surely in $\omega \in \Omega$ for any $\varphi \in \mathbf{H}$,

$$\int\limits_D \left\langle g\left(x, \frac{x}{\varepsilon}, \omega\right), \varphi(x)\right\rangle dx \longrightarrow \int\limits_D \left\langle g^{\text{hom}}(x), \varphi(x)\right\rangle dx \quad (\varepsilon \to 0+). \tag{24}$$

Let $q_i = p_i/(p_i - 1),\ 1/p_i + 1/q_i = 1,\ 1 < q_i \le 2,\ i = 1, \dots, N$. We shall use the following vector notations $\mathbf{p} = (p_1, \dots, p_N)$ and $\mathbf{q} = (q_1, \dots, q_N)$ and we consider the spaces

$$\mathbf{L}_\mathbf{p}(D) := L_{p_1}(D) \times \ldots \times L_{p_N}(D),$$
$$\mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(D)) := L_{p_1}(0, M; L_{p_1}(D)) \times \ldots \times L_{p_N}(0, M; L_{p_N}(D)).$$

We also consider the Banach space

$$\mathcal{L}(0, M) := \mathbf{L}_2(0, M; \mathbf{V}') + \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_\mathbf{q}(D)) \quad (\text{see [43]}).$$

If $u(x,t) \in \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(D))$, then it follows from the condition (18) that the function $f(u(x,t)) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_\mathbf{q}(D))$ and

$$\sum_{i=1}^N \|f^i(u(\cdot))\|^{q_i}_{L_{q_i}(0,M;L_{q_i})} \le C_0 \left(\sum_{i=1}^N \|u(\cdot)\|^{p_i}_{L_{p_i}(0,M;L_{p_i})} + |D|\right). \tag{25}$$

Moreover, due to (21), $b\left(x, \frac{x}{\varepsilon}, \omega\right)(f(u(x,t)) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{L}_\mathbf{q}(D))$ and the estimate (25) implies

$$\sum_{i=1}^N \|b(\cdot)f^i(u(\cdot))\|^{q_i}_{L_{q_i}(0,M;L_{q_i})} \le C_1 \left(\sum_{i=1}^N \|u(\cdot)\|^{p_i}_{L_{p_i}(0,M;L_{p_i})} + |D|\right), \tag{26}$$

where C_1 is independent of ε. At the same time, if $u(x,t) \in \mathbf{L}_2(0, M; \mathbf{V})$, then $a\Delta u(x,t) + g\left(x, \frac{x}{\varepsilon}, \omega\right) \in \mathbf{L}_2(0, M; \mathbf{V}')$.

Definition 4.1. A function $u(x,t),\ x \in \Omega,\ t \in (0, M)$ is called a weak solution of the system (17) if $u(x,t) \in \mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(D)) \cap \mathbf{L}_2(0, M; \mathbf{V})$ and for any test function $\varphi \in \mathbf{L}_\mathbf{p}(D) \cap \mathbf{V}$ the following identity holds

$$\frac{d}{dt}\int_D u(x,t) \cdot \varphi(x) dx + \int_D \left\{a \nabla u(x,t) \cdot \nabla \varphi(x) + b_\varepsilon(x)\, f(u(x,t)) \cdot \varphi(x)\right\} dx$$
$$= \int_D g_\varepsilon(x) \cdot \varphi(x) dx, \tag{27}$$

where $b_\varepsilon(x) = b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g_\varepsilon(x) = g\left(x, \frac{x}{\varepsilon}, \omega\right)$. Here $y_1 \cdot y_2$ denotes the scalar products of vectors $y_1, y_2 \in \mathbb{R}^N$.

It follows from (27) that a weak solution of (17) satisfies (in a distribution sense) the belonging $\partial_t u(x,t) \in \mathcal{L}(0, M)$.

By the Sobolev embedding theorem, we have

$$\mathcal{L}(0, M) \subset \mathbf{L}_\mathbf{q}\left(0, M; \mathbf{H}^{-\mathbf{r}}(D)\right), \tag{28}$$

where the space $\mathbf{H}^{-\mathbf{r}}(D) := H^{-r_1}(D) \times \ldots \times H^{-r_N}(D)$, $\mathbf{r} = (r_1, \dots, r_N)$, and the values $r_i := \max\{1, d(1/q_i - 1/2)\}$ for $i = 1, \dots, N$. Therefore, for a weak solution $u(x,t)$ of (17) we have $\partial_t u(x,t) \in \mathbf{L}_\mathbf{q}(0, M; \mathbf{H}^{-\mathbf{r}}(D))$.

For brevity we shall write sometimes $u(t)$ instead $u(x,t)$.

If $u(t)$ is a weak solution of (17), then $u(\cdot) \in C([0,M];\mathbf{H}^{-\mathbf{r}}(D))$. Moreover, if $u(\cdot) \in \mathbf{L}_\infty([0,M];\mathbf{H})$, then, according to the Lions–Magenes lemma (see, e.g., [42, 43]), we have that $u(\cdot) \in \mathbf{C}_w([0,M];\mathbf{H})$. Therefore, we can consider the following initial data for the system (17):

$$u|_{t=0} = u_0, \text{ where } u_0 \in \mathbf{H}. \tag{29}$$

Using the standard Galerkin method (see, e.g., [43, 3, 50, 26]), we prove fee following assertion.

Proposition 3. *For any $u_0 \in \mathbf{H}$, the system (17) possesses a weak solution*

$$u(t) \in \mathbf{L}_\mathbf{p}(0,M;\mathbf{L}_\mathbf{p}(D)) \cap \mathbf{L}_2(0,M;\mathbf{V}) \cap \mathbf{L}_\infty([0,M];\mathbf{H}),$$

satisfying the initial data (29).

Remark 3. A weak solution of the problem (17), (29) is not necessarily unique because the function $f(v)$ satisfies conditions (18), (19) and is not assumed to be Lipschitz. The uniqueness theorem holds under, e.g., the additional assumption that $f(v) \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ and the Jacobian $\mathbf{J}(v) = \frac{\partial f}{\partial v}(v)$ satisfies the inequality

$$\mathbf{J}(v) + \alpha \mathbf{I} \geq \mathbf{0}, \ \forall v \in \mathbb{R}^N,$$

for some $\alpha > 0$ (see [3, 50, 26]). However, this condition is very restrictive, we do not impose it in this work, and we do not need it to construct the strong trajectory attractor for the system (17).

In [26], it is proved that under the assumptions (18), (19) any weak solution $u(t), t \in (0,M)$, of the system (17) is strongly continuous in the space $\mathbf{C}([0,M];\mathbf{H})$. Moreover, the real function $\|u(t)\|^2$, $t \in [0,M]$, is absolutely continuous and the following differential equality holds:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \int_D \{a\nabla u(x,t)\cdot\nabla u(x,t) + b_\varepsilon(x) f(u(x,t))\cdot u(x,t)\}\,dx \\ = \int_D g_\varepsilon(x)\cdot u(x,t)dx. \tag{30}$$

We call this equality the *energy identity* for the system (17).

The identity (30) implies that any weak solution $u(t)$ of (17) satisfies the following inequalities:

$$\|u(t)\|^2 \leq \|u(0)\|^2 e^{-\lambda_1 \beta t} + R_1^2, \tag{31}$$

$$\beta \int_t^{t+1} \|u(s)\|_1^2 ds + 2\sum_{i=1}^N \gamma_i \int_t^{t+1} \|u(s)\|_{L_{p_i}}^{p_i} ds \leq \|u(t)\|^2 + R_2^2, \tag{32}$$

where λ_1 is the first eigenvalue of the scalar operator $-\Delta$ with Dirichlet boundary conditions. The positive values R_1 and R_2 depend on $\|\phi\|$ (see (23)) and is independent of $u(0)$ and ε. For the proof see [26].

5. **Trajectory attractors for RD-systems.** To construct the trajectory attractor for the RD-system (17), we apply the scheme described in Sec. 3. The system (17) has the form (2) if we set $E_1 = \mathbf{L}_\mathbf{p}(D) \cap \mathbf{V}$, $E_0 = \mathbf{H}^{-\mathbf{r}}(D)$, $E = \mathbf{H}$ and $A(u) = a\Delta u - b_\varepsilon(\cdot) f(u) + g_\varepsilon(\cdot)$.

We now describe the trajectory space $\mathcal{K}_\varepsilon^+ = \mathcal{K}_\varepsilon^+(\omega)$ for the system (17). We fixe $\omega \in \Omega$. In this section for brevity, we shall omit random parameter ω.

Following the general framework of Sec. 3, we define the Banach spaces for every $M > 0$

$$\mathcal{F}_{0,M} := \mathbf{L_p}(0, M; \mathbf{L_p}(D)) \cap \mathbf{L_2}(0, M; \mathbf{V}) \cap \mathbf{L_\infty}([0, M]; \mathbf{H}) \cap \{v \mid \partial_t v \in \mathcal{L}(0, M)\} \quad (33)$$

with norm

$$\|v\|_{\mathcal{F}_{0,M}} := \|v\|_{\mathbf{L_p}(0,M;\mathbf{L_p}(D))} + \|v\|_{\mathbf{L_2}(0,M;\mathbf{V})} + \|v\|_{\mathbf{L_\infty}([0,M];\mathbf{H})} + \|\partial_t v\|_{\mathcal{L}(0,M)}. \quad (34)$$

It is clear that the condition (3) holds for the norm (34) and the translation semigroup $\{S(h)\}$ satisfies (4).

Setting $\mathcal{D}_{0,M} = \mathbf{L_q}(0, M; \mathbf{H^{-r}}(D))$ we have that $\mathcal{F}_{0,M} \subseteq \mathcal{D}_{0,M}$ and if $u(s) \in \mathcal{F}_{0,M}$, then $A(u(s)) \in \mathcal{D}_{0,M}$. We can consider weak solutions of the system (17) (see Definition 4.1) as solutions of equation in the general scheme of Sec. 3.

Defining the space (5) we see that

$$\mathcal{F}_+^{loc} = \mathbf{L_p}^{loc}(\mathbb{R}_+; \mathbf{L_p}(D)) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{v \mid \partial_t v \in \mathcal{L}^{loc}(\mathbb{R}_+)\right\}. \quad (35)$$

Definition 5.1. The trajectory space $\mathcal{K}_\varepsilon^+$ of the system (17) consists of all functions $u(s), s \geq 0$, such that $u(t), t \in (0, M)$ is a weak solution of (17) for every $M > 0$.

It follows from Proposition 3 that for any $u_0 \in \mathbf{H}$ there exist at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = u_0$. Therefore, the trajectory space $\mathcal{K}_\varepsilon^+$ of the system (17) is not empty and is wide enough.

It is obvious that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\varepsilon^+$ is translation invariant, that is, if $u(s) \in \mathcal{K}_\varepsilon^+$, then $u(h + s) \in \mathcal{K}_\varepsilon^+$ for all $h \geq 0$. Therefore,

$$S(h)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+, \quad \forall h \geq 0.$$

We now define metrics $\rho_{0,M}(\cdot, \cdot)$ on the spaces $\mathcal{F}_{0,M}$ using the norms of the spaces $L_2(0, M; \mathbf{H})$, that is,

$$\rho_{0,M}(u, v) = \left(\int_0^M \|u(s) - v(s)\|^2 ds\right)^{1/2}, \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

These metrics generates the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$. Recall that a sequence $\{v_n\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $n \to \infty$ in Θ_+^{loc} if $\|v_n(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(n \to \infty)$ for each $M > 0$. The topology Θ_+^{loc} is metrizable (see (7)) and the corresponding metric space is complete. We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (17). The translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\varepsilon^+$ is continuous in the considering topology Θ_+^{loc}.

Following the general scheme of Sec. 3, we define bounded sets in $\mathcal{K}_\varepsilon^+$ using the Banach space $\mathcal{F}_+^b$ (see (8)). We clearly have

$$\mathcal{F}_+^b = \mathbf{L_p}^b(\mathbb{R}_+; \mathbf{L_p}(D)) \cap \mathbf{L}_2^b(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{v \mid \partial_t v \in \mathcal{L}^b(\mathbb{R}_+)\right\} \quad (36)$$

and $\mathcal{F}_+^b$ is a subspace of $\mathcal{F}_+^{loc}$. Recall that

$$\|v\|_{L_p^b(\mathbb{R}_+;E)} = \sup_{h\geq 0} \|v\|_{L_p(h,h+1;E)}.$$

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\varepsilon^+$, $S(t) : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+,\ t \geq 0$.

Inequalities (31) and (32) imply

Proposition 4. *The space $\mathcal{K}_\varepsilon^+$ belongs to $\mathcal{F}_+^b$ and for any trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$*

$$\|S(t)u(\cdot)\|_{\mathcal{F}_+^b}^2 \leq C_2\|u(0)\|^2 e^{-\sigma t} + R_3^2,\ \forall t \geq 0, \quad (37)$$

where $\sigma = \beta\lambda_1$ and R_3 depend on $\|\phi\|$ (see (23)) and is independent of $u(0)$ and ε.

For the complete proof see [26]. It follows from inequality (37) that the ball

$$\mathcal{B}_0 = \left\{ u \in \mathcal{F}_+^b \mid \|u(\cdot)\|_{\mathcal{F}_+^b} \le 2R_3 \right\}$$

is absorbing for the translation semigroup $\{S(t)\}$ in $\mathcal{K}_\varepsilon^+$, that is, for any set $\mathcal{B} \subset \mathcal{K}_\varepsilon^+$ bounded in $\mathcal{F}_+^b$, there exists a number $t_1 = t_1(\mathcal{B})$ such that $S(t)\mathcal{B} \subseteq \mathcal{B}_0$ for all $t \ge t_1$. Consider the set

$$\mathcal{P}_\varepsilon = \mathcal{B}_0 \cap \mathcal{K}_\varepsilon^+.$$

It is clear that $\mathcal{P}_\varepsilon \subseteq \mathcal{K}_\varepsilon^+$ is also absorbing, that is

$$S(t)\mathcal{P}_\varepsilon \subseteq \mathcal{P}_\varepsilon,\ \forall t \ge 0.$$

and the sets $\mathcal{P}_\varepsilon$ are uniformly (w.r.t. ε) bounded in $\mathcal{F}_+^b$.

Using Lemma 3.6 for $E_1 = \mathbf{V},\ E_0 = \mathbf{H}^{-\mathbf{r}}(D),\ E = \mathbf{H}$ and $p_1 = 2,\ p_0 = q_N$, we obtain

Proposition 5. *The set $\mathcal{P}_\varepsilon$ is compact in the topology Θ_+^{loc} and uniformly bounded in the norm $\mathcal{F}_+^b$.*

Consider the kernel $\mathcal{K}_\varepsilon$ of the system (17) that consists of all weak complete solutions $u(s), \in \mathbb{R}$, of the system that are bounded in the space

$$\mathcal{F}^b = \mathbf{L}_\mathbf{p}^b(\mathbb{R}; \mathbf{L}_\mathbf{p}(D)) \cap \mathbf{L}_2^b(\mathbb{R}; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \mid \partial_t v \in \mathcal{L}^b(\mathbb{R}) \right\}. \tag{38}$$

Due to Propositions 4 and 5, Theorems 3.3 and 3.5 are applicable and we have

Theorem 5.2. *The RD-system (17) has the trajectory attractor $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_+^{loc} = L_2^{loc}(\mathbb{R}_+; \mathbf{H})$. The set $\mathfrak{A}_\varepsilon$ is ω-almost surely uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,*

$$\mathfrak{A}_\varepsilon = \Pi_+ \mathcal{K}_\varepsilon, \tag{39}$$

the kernel $\mathcal{K}_\varepsilon$ is non-empty, uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}^b$, and compact in the space Θ^{loc}.

Notice that the trajectory attractors $\mathfrak{A}_\varepsilon \subset \mathcal{F}_+^{loc}$ are uniformly (w.r.t. $\varepsilon \in (0,1)$) bounded in $\mathcal{F}_+^b$. We now prove that the constructed above trajectory attractor $\mathfrak{A}_\varepsilon$ in the "weak" topology $\Theta_+^{loc} = L_2^{loc}(\mathbb{R}_+; \mathbf{H})$ in fact is the trajectory attractor in the stronger local topology generated by the spaces $\mathcal{F}_{0,M}$.

Let $\Theta_+^{s,loc}$ denotes the topology in $\mathcal{F}_+^{loc}$ generated by the metrics of the spaces $\mathcal{F}_{0,M}$. That is, by definition, a sequence $\{v_n\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $n \to \infty$ in $\Theta_+^{s,loc}$ if $\|v_n(\cdot) - v(\cdot)\|_{\mathcal{F}_{0,M}} \to 0$ for each $M > 0$ (see (34)).

Theorem 5.3. *The trajectory attractor $\mathfrak{A}_\varepsilon$ is compact in the topology $\Theta_+^{s,loc}$ and attracts bounded sets of trajectories from $\mathcal{K}_\varepsilon^+$ in this topology, that is, $\mathfrak{A}_\varepsilon$ is the trajectory attractor in the strong topology $\Theta_+^{s,loc}$.*

Proof. We fix $\varepsilon > 0$ and $\omega \in \Omega$. It is sufficient to prove that the set $S(1)\mathcal{P}_\varepsilon$ is compact in the strong topology of the space $\mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(D)) \cap \mathbf{L}_2(0, M; \mathbf{V})$ for any $M > 0$. We note that $\mathbf{L}_\mathbf{p}(0, M; \mathbf{L}_\mathbf{p}(D)) = \mathbf{L}_\mathbf{p}(D \times [0, M])$.

Thus, let us show that any sequence $\{u_n(s)\} \subset \mathcal{P}_\varepsilon$ has a subsequence that strongly converges in the space $\mathbf{L}_\mathbf{p}(D \times [1, M]) \cap \mathbf{L}_2(1, M; \mathbf{V})$ for each $M > 0$.

The set $\mathcal{P}_\varepsilon$ is bounded in the space $\mathcal{F}_+^b$, therefore, $\{u_n(s)\}$ is bounded in the spaces $\mathbf{L}_\mathbf{p}(D \times [0, M])$ and $\mathbf{L}_2(0, M; \mathbf{V})$. Passing to a subsequence that we denote the same, we can assume that $u_n(\cdot) \rightharpoonup \hat{u}(\cdot)$ as $n \to \infty$ weakly in the spaces $\mathbf{L}_\mathbf{p}(D \times [0, M])$ and $\mathbf{L}_2(0, M; \mathbf{V})$, where $\hat{u}(s)$ is a solution of the system (17) belonging to

$\mathcal{P}_\varepsilon$. The Lions–Magenes lemma (see, e.g., [42, 43]) implies that $u_n(t) \rightharpoonup \hat{u}(t)$ weakly in $\mathbf{H}$ for each $t \in [0, M]$. Moreover, according to the standard embedding theorem, $u_n(\cdot) \to \hat{u}(\cdot)$ strongly in the space $\mathbf{L}_2(D \times [0, M])$ and $u_n(x, s) \to \hat{u}(x, s)$ for almost all $(x, t) \in D \times [0, M]$.

We note that, if a sequence $\chi_n \rightharpoonup \hat{\chi}$ weakly in a Banach space X, then

$$\|\hat{\chi}\|_X \le \liminf_{n\to\infty} \|\chi_n\|_X$$

(see, e.g., [58]). Hence, for a weakly convergent sequence $\{u_n(\cdot)\}$, we have the following limit relations:

$$\|\hat{u}(M)\| \le \liminf_{n\to\infty} \|u_n(M)\|, \tag{40}$$

$$\int_0^M \int_D s(a\nabla\hat{u} \cdot \nabla\hat{u})dxds \le \liminf_{n\to\infty} \int_0^M \int_D s(a\nabla u_n \cdot \nabla u_n)dxds, \tag{41}$$

$$\int_0^M \int_D sb_\varepsilon(x)|\hat{u}^i|^{p_i}dxds \le \liminf_{n\to\infty} \int_0^M \int_D sb_\varepsilon(x)|u_n^i|^{p_i}dxds,\ i = 1, 2, \dots, N, \tag{42}$$

where, for brevity, we denote $u_n = u_n(x, s)$ and $\hat{u} = \hat{u}(x, s)$. The norms in (41) and (42) correspond to the weight spaces $\mathbf{L}_{2,s}(0, M; \mathbf{V})$ and $\mathbf{L}_{\mathbf{p},sb_\varepsilon(x)}(D \times [0, M])$ with weights s and $sb_\varepsilon(x)$, respectively. Here, the quadratic form $ay \cdot y$ with $y \in \mathbb{R}^N$ is equivalent to the standard norm of the vector y in $\mathbb{R}^N$ since the matrix a has a positive symmetric part. Therefore, the quadratic form $\int_D (a\nabla v(x) \cdot \nabla v(x))dx$ is equivalent to the norm of $v(\cdot)$ in the space $\mathbf{V}$. It is clear, that weak convergence $u_n(\cdot) \rightharpoonup \hat{u}(\cdot)$ holds in the weight spaces as well.

Consider the continuous scalar function

$$F(v) = \sum_{i=1}^N f^i(v)v^i - \sum_{i=1}^N \gamma_i |v^i|^{p_i},\ v \in \mathbb{R}^N.$$

Then, clearly $sb_\varepsilon(x) F(u_n(x, s)) \to sb_\varepsilon(x) F(\hat{u}(x, s))$ as $n \to \infty$ for almost all $(x, t) \in D \times [0, M]$ since the function $F(v)$ is continuous. We claim that

$$\int_0^M \int_D sb_\varepsilon(x) F(\hat{u}(x, t))dxds \le \liminf_{n\to\infty} \int_0^M \int_D sb_\varepsilon(x) F(u_n(x, t))dxds. \tag{43}$$

The prove of this inequality uses the inequalities $F(v) + C_1 \ge 0$, $b_\varepsilon(\cdot) \ge 0$ (see (19) and (21)), the convergence of the sequence $\{u_n(x, s)\}$ for almost all $(x, s) \in D \times [0, M]$ and the Fatou lemma on bounds for integrals of convergent sequences of non-negative functions (see, e.g. [58]).

Recall that weak solutions $u_n(\cdot)$ and $\hat{u}(\cdot)$ of the system (17) satisfy the energy identity (30). Multiplying this identity by t, integrating the result over $[0, M]$, and using the definition of the function $F(\cdot)$, we obtain the equalities

$$\begin{aligned} &\frac{1}{2}\|u_n(M)\|^2 + \int_0^M \int_D s(a\nabla u_n \cdot \nabla u_n)dxds + \sum_{i=1}^N \gamma_i \int_0^M \int_D sb_\varepsilon(x) |u_n^i|^{p_i}dxds \\ &+ \int_0^M \int_D sb_\varepsilon(x) F(u_n)dxds = \frac{1}{2}\int_0^M \int_D |u_n|^2 dxds + \int_0^M \int_D g_\varepsilon(x) \cdot u_n dxds, \end{aligned} \tag{44}$$

$$\frac{1}{2}\|\hat{u}(M)\|^2+\int_0^M\int_D s(a\nabla\hat{u}\cdot\nabla\hat{u})dxds+\sum_{i=1}^N\gamma_i\int_0^M\int_D sb_\varepsilon(x)\,|\hat{u}^i|^{p_i}dxds$$
$$+\int_0^M\int_D sb_\varepsilon(x)\,F(\hat{u})dxds=\frac{1}{2}\int_0^M\int_D|\hat{u}|^2dxds+\int_0^M\int_D g_\varepsilon(x)\cdot\hat{u}dxds. \quad (45)$$

Recall that $u_n(\cdot)\to\hat{u}(\cdot)$ strongly in the space $\mathbf{L}_2(D\times[0,M])$, therefore, the right-hand side of equation (44) tends to that of equation (45). Hence, the left-hand side of (44) also converges to the left-hand side of (45).

It follows from the inequalities (40) – (43) that the four real number sequences on the right-hand sides have limits as $n\to\infty$ and these limits coincides with the corresponding quantities on the left-hand sides of (40) – (43). In particular we have

$$\lim_{n\to\infty}\int_0^M\int_D s(a\nabla u_n\cdot\nabla u_n)dxds=\int_0^M\int_D s(a\nabla\hat{u}\cdot\nabla\hat{u})dxds,$$
$$\lim_{n\to\infty}\int_0^M\int_D sb_\varepsilon(x)|u_n^i|^{p_i}dxds=\int_0^M\int_D sb_\varepsilon(x)|\hat{u}^i|^{p_i}dxds,\ i=1,2,\ldots,N.$$

It is known, in a uniformly convex Banach space X, the weak convergence $\chi_n\rightharpoonup\hat{\chi}$ of vectors and convergence $\|\chi_n\|_X\to\|\hat{\chi}\|_X$ of their norms implies the strong convergence $\|\chi_n-\hat{\chi}\|_X\to 0$ as $n\to\infty$ (this assertion follows the Mazur theorem, see [58]). The weight spaces $\mathbf{L}_{2,s}(0,M;\mathbf{V})$ and $\mathbf{L}_{\mathbf{p},sb_\varepsilon(x)}(D\times[0,M])$ are uniformly convex. Therefore, the weak convergence of the sequences of functions $u_n^i(\cdot)$ to $\hat{u}(\cdot)$ and the convergence of their norms in the space $\mathbf{L}_{2,s}(0,M;\mathbf{V})\cap\mathbf{L}_{\mathbf{p},sb_\varepsilon(x)}(D\times[0,M])$ implies the strong convergence $u_n(\cdot)\to\hat{u}(\cdot)$ in the space $\mathbf{L}_{2,s}(1,M;\mathbf{V})\cap\mathbf{L}_{\mathbf{p},sb_\varepsilon(x)}(D\times[1,M])$ which is, obviously, equivalent to the space $\mathbf{L}_2(1,M;\mathbf{V})\cap\mathbf{L}_{\mathbf{p}}(D\times[1,M])$ (without weights).

Thus, we have proved the compactness of the set $S(1)\mathcal{P}_\varepsilon$ in the strong topology of the space $\mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p}}(D))\cap\mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V})$.

The compactness of the corresponding set of derivatives $\partial_t u(\cdot)$ in the strong topology of the space $\mathcal{L}^{loc}(\mathbb{R}_+)=\mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}')+\mathbf{L}_{\mathbf{q}}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{q}}(D))$ follows directly from the equation (17) and from the continuity of the Nemytskii operator $u\mapsto f(u)$, which, by virtue of (18) acts from $\mathbf{L}_{\mathbf{p}}(D\times[0,M])$ to $\mathbf{L}_{\mathbf{q}}(D\times[0,M])$ (see [41]) and, hence, $b_\varepsilon(\cdot)\,f(u_n(x,s))\to b_\varepsilon(\cdot)\,f(\hat{u}(\cdot))$ strongly in $\mathbf{L}_{\mathbf{q}}^{loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{q}}(D))$. At the same time, it is clear that $a\nabla u_n(\cdot)\to a\nabla\hat{u}(\cdot)$ strongly in $\mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{V}')$. Therefore, from the equation (17) we conclude that $\partial_t u_n(\cdot)\to\partial_t\hat{u}(\cdot)$ strongly in $\mathcal{L}^{loc}(\mathbb{R}_+)$.

It is remains to note that $\mathcal{P}_\varepsilon$ belongs to $C^{loc}(\mathbb{R}_+;\mathbf{H})$ (this fact follows from the energy identity (30)) and the set $S(1)\mathcal{P}_\varepsilon$ is compact in the space $C^{loc}(\mathbb{R}_+;\mathbf{H})$. The last assertion follows from the continuity of the embedding

$$\mathbf{L}_2(0,M;\mathbf{V})\cap\mathbf{L}_{\mathbf{p}}(0,M;\mathbf{L}_{\mathbf{p}}(D))\cap\{\partial_t v\in\mathcal{L}(0,M)\}\subset C([0,M];\mathbf{H})$$

which was proved in, e.g. [26]. This completes the proof of Theorem 5.3. □

Along with random system (17) we consider the averaged deterministic system

$$\partial_t\bar{u}=a\Delta\bar{u}-b^{\hom}(x)\,f(\bar{u})+g^{\hom}(x)\,,\ \bar{u}|_{\partial D}=0, \quad (46)$$

where the deterministic functions $b^{\mathrm{hom}}(x)$ and $g^{\mathrm{hom}}(x)$ are averages of the random functions $b\left(x, \frac{x}{\varepsilon}, \omega\right)$ and $g(x, \frac{x}{\varepsilon}, \omega)$ as $\varepsilon \to 0+$. Recall that almost surely in $\omega \in \Omega$

$$b\left(x, \frac{x}{\varepsilon}, \omega\right) = \mathbf{B}\left(x, \mathcal{T}_{\frac{x}{\varepsilon}}\omega\right) \rightharpoonup b^{\mathrm{hom}}(x) \ (\varepsilon \to 0+) \ \ast\text{-weakly in } L_\infty(D),$$
$$g\left(x, \frac{x}{\varepsilon}, \omega\right) = \mathbf{G}\left(x, \mathcal{T}_{\frac{x}{\varepsilon}}\omega\right) \rightharpoonup g^{\mathrm{hom}}(x) \ (\varepsilon \to 0+) \text{ weakly in } \mathbf{H}.$$

Clearly system (46) also has trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathcal{K}}^+$ corresponding to the system (46) and

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}}$$

where $\overline{\mathcal{K}}$ is the kernel of system (46) in $\mathcal{F}^b$. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}_+^b$ and compact in the space $\Theta_+^{s,loc}$.

6. **Homogenization of trajectory attractors for RD-systems.** In this section, we study the limit behaviour of trajectory attractors $\mathfrak{A}_\varepsilon(\omega)$ of random RD-systems (17) as $\varepsilon \to 0+$ and their relation to the trajectory attractor $\overline{\mathfrak{A}}$ of the deterministic averaged system (46).

To begin with, we consider the "weak" topology $\Theta_+^{loc} = L_2^{loc}(\mathbb{R}_+; \mathbf{H})$.

Theorem 6.1. *The following limit holds ω-almost surely in the topological space Θ_+^{loc}*

$$\mathfrak{A}_\varepsilon(\omega) \to \overline{\mathfrak{A}} \ \text{ as } \varepsilon \to 0+. \tag{47}$$

Moreover, ω-almost surely

$$\mathcal{K}_\varepsilon(\omega) \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{ in } \Theta^{loc}. \tag{48}$$

Proof. It is clear that (48) implies (47). Therefore it is sufficient to prove (48), that is, for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there are exists $\varepsilon_1 = \varepsilon_1(\mathcal{O}) > 0$ such that almost surely

$$\mathcal{K}_\varepsilon(\omega) \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for } \varepsilon < \varepsilon_1. \tag{49}$$

Suppose that (49) is not true. Consider the corresponding subset $\Omega' \subset \Omega$ with $\mu(\Omega') > 0$ and (49) does not hold for all $\omega \in \Omega'$. Then, for each $\omega \in \Omega'$, there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_n \to 0+$, and a sequence $u_{\varepsilon_n}(\cdot) = u_{\varepsilon_n}(\omega, s) \in \mathcal{K}_{\varepsilon_n}(\omega)$ such that

$$u_{\varepsilon_n} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } n \in \mathbb{N}, \omega \in \Omega'. \tag{50}$$

For each $\omega \in \Omega'$, the function $u_{\varepsilon_n}(s), s \in \mathbb{R}$ is a weak solution of the system

$$\partial_t u_{\varepsilon_n} = a\Delta u_{\varepsilon_n} - b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) + g\left(x, \frac{x}{\varepsilon_n}, \omega\right), \ u_{\varepsilon_n}|_{\partial D} = 0 \tag{51}$$

on the entire time axis $t \in \mathbb{R}$. Moreover the sequence $\{u_{\varepsilon_n}(s)\}$ is bounded in $\mathcal{F}^b$, that is,

$$\begin{aligned}\|u_{\varepsilon_n}\|_{\mathcal{F}^b} =& \sup_{t\in\mathbb{R}} \|u_{\varepsilon_n}(t)\| + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\varepsilon_n}(s)\|_1^2 ds\right)^{1/2} + \sup_{t\in\mathbb{R}} \|u_{\varepsilon_n}(\cdot)\|_{\mathbf{L_p}(t,t+1;\mathbf{L_p}(D))} \\ &+ \sup_{t\in\mathbb{R}} \|\partial_t u_{\varepsilon_n}(\cdot)\|_{\mathcal{L}(t,t+1)} \le C \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{52}$$

Hence there exists a subsequence $\{u_{\varepsilon_n'}(s)\} \subset \{u_{\varepsilon_n}(s)\}$ which we label the same such that

$$u_{\varepsilon_n}(s) \to \bar{u}(s) \text{ as } n \to \infty \ \text{ in } \Theta^{loc}, \tag{53}$$

where $\bar{u}(\cdot) \in \mathcal{F}^b$ and $\bar{u}(s)$ satisfies (52) with the same constant C. Due to (52) we can also assume that $u_{\varepsilon_n}(s) \rightharpoonup \bar{u}(s)$ $(n \to \infty)$ weakly in $L^{loc}_{2,w}(\mathbb{R}; \mathbf{V})$, weakly in $L^{loc}_{\mathbf{p},w}(\mathbb{R}; \mathbf{L_p}(D))$, $*$-weakly in $L^{loc}_{\infty,*w}(\mathbb{R}_+; \mathbf{H})$ and $\partial_t u_{\varepsilon_n}(s) \rightharpoonup \partial_t \bar{u}(s)$ $(n \to \infty)$ weakly in $\mathcal{L}^{loc}(\mathbb{R}_+)$. We claim that $\bar{u}(s) \in \overline{\mathcal{K}}$. We have already proved that $\|\bar{u}\|_{\mathcal{F}^b} \le C$. So we have to establish that $\bar{u}(\cdot)$ is a weak solution of (46). Using (52), (24), and (28) we obtain that

$$\partial_t u_{\varepsilon_n} - a\Delta u_{\varepsilon_n} - g\left(x, \frac{x}{\varepsilon_n}, \omega\right) \to \partial_t \bar{u} - a\Delta \bar{u} - g^{\text{hom}}(x) \text{ as } n \to \infty \tag{54}$$

in the space $D'(\mathbb{R}; \mathbf{H^{-r}}(D))$ because the derivative operators are continuous in the space of distributions. Let us prove that

$$b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) \rightharpoonup b^{\text{hom}}(x) f(\bar{u}) \text{ as } n \to \infty \tag{55}$$

weakly in $L^{loc}_{\mathbf{q},w}(\mathbb{R}; L_{\mathbf{q}}(D))$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_n}(s)\}$ is bounded in $\mathbf{L_p}(]-M, M[; \mathbf{L_p}(D))$ (see (52)). Hence by (18) the sequence $\{f(u_{\varepsilon_n}(s))\}$ is bounded in the space $\mathbf{L_q}(]-M, M[; \mathbf{L_q}(D))$. Since $\{u_{\varepsilon_n}(s)\}$ is bounded in the space $\mathbf{L}_2\left(]-M, M[; \left(H^1_0(D)\right)^N\right)$ and $\{\partial_t u_{\varepsilon_n}(s)\}$ is bounded in $L_{\mathbf{q}}(]-M, M[; \mathbf{H^{-r}}(D))$ we can assume that

$$u_{\varepsilon_n}(s) \to \bar{u}(s) \text{ strongly in } \mathbf{L_2}\left(]-M, M[; (L_2(D))^N\right) = (L_2(D\times]-M, M[))^N$$

(see lemma 3.6) and therefore

$$u_{\varepsilon_n}(x,s) \to \bar{u}(x,s) \text{ as } n \to \infty \text{ a.e. in } (x,s) \in D\times]-M, M[.$$

Since the function $f(v)$ is continuous with respect to $v \in \mathbb{R}^N$ we conclude that

$$f(u_{\varepsilon_n}(x,s)) \to f(\bar{u}(x,s)) \text{ as } n \to \infty \text{ a.e. in } (x,s) \in D\times]-M, M[. \tag{56}$$

We have

$$\begin{aligned} &b\left(x, \frac{x}{\varepsilon_n}, \omega\right) f(u_{\varepsilon_n}) - b^{\text{hom}}(x) f(\bar{u}) \\ &= b\left(x, \frac{x}{\varepsilon_n}, \omega\right)(f(u_{\varepsilon_n}) - f(\bar{u})) + \left(b\left(x, \frac{x}{\varepsilon_n}, \omega\right) - b^{\text{hom}}(x)\right) f(\bar{u}). \end{aligned} \tag{57}$$

Let us show that both summand in the right-hand side of (57) converges to zero as $n \to \infty$ weakly in the space $\mathbf{L_q}(]-M, M[; \mathbf{L_q}(D)) = \mathbf{L_q}(D \times]-M, M[)$. The sequence $b\left(x, \frac{x}{\varepsilon_n}, \omega\right)(f(u_{\varepsilon_n}) - f(\bar{u}))$ tends to zero as $n \to \infty$ almost everywhere in $(x,s) \in D \times]-M, M[$ (see (56)) and is bounded in $\mathbf{L_q}(D \times]-M, M[)$ (see (21)). Therefore Lemma 1.3 from [43, Chap. 1, Sec. 1] implies that

$$b\left(x, \frac{x}{\varepsilon_n}, \omega\right)(f(u_{\varepsilon_n}) - f(\bar{u})) \rightharpoonup 0 \text{ as } n \to \infty$$

weakly in the space $\mathbf{L_q}(D \times]-M, M[)$. The sequence $\left(b\left(x, \frac{x}{\varepsilon_n}, \omega\right) - b^{\text{hom}}(x)\right) f(\bar{u})$ approaches zero as $n \to \infty$ weakly in $\mathbf{L_q}(D \times]-M, M[)$ because, due to (22), $b\left(x, \frac{x}{\varepsilon_n}, \omega\right) \rightharpoonup b^{\text{hom}}(x)$ as $n \to \infty$ $*$-weakly in $L_{\infty,*w}(]-M, M[; L_2(D))$ and $f(\bar{u}) \in \mathbf{L_q}(D \times]-M, M[)$. We have proved (55). Using (54) and (55) we pass to the limit in the equation (51) as $n \to \infty$ in the space $D'(\mathbb{R}_+; \mathbf{H^{-r}}(D))$ and we obtain that the function $\bar{u}(x,s)$ satisfies the equation

$$\partial_t \bar{u} = a\Delta \bar{u} - b^{\text{hom}}(x) f(\bar{u}) + g^{\text{hom}}(x), \ \bar{u}|_{\partial D} = 0, \ t \in \mathbb{R}.$$

Consequently, $\bar{u} \in \overline{\mathcal{K}}$. We proved above that $u_{\varepsilon_n}(s) \to \bar{u}(s)$ as $n \to \infty$ in Θ^{loc}. The hypotheses $u_{\varepsilon_n}(s) \notin \mathcal{O}'(\overline{\mathcal{K}})$ implies that $\bar{u} \notin \mathcal{O}'(\overline{\mathcal{K}})$ and moreover $\bar{u} \notin \overline{\mathcal{K}}$ for all $\omega \in \Omega'$. We came to the contradiction. The theorem is proved. □

We note that Lemma 3.6 implies that

$$\mathcal{B}_0 \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{58}$$

$$\mathcal{B}_0 \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{59}$$

Inclusion (58) follows from (15), where we set $E_0 = \mathbf{H}^{-\mathbf{r}}(D)$, $E = \mathbf{H}^{1-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_1 = 2$, $p_0 = q_N$, and from the compact embedding $\mathbf{V} \Subset \mathbf{H}^{1-\delta}$. Inclusion (59) follows from (16) and from the compact embeddings $\mathbf{H} \Subset \mathbf{H}^{-\delta}$, if we set $E_0 = \mathbf{H}^{-\mathbf{r}}(D)$, $E = \mathbf{H}^{-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_0 = q_N$.

Using compactness of inclusions (40) and (41), we can strengthen the convergence (47).

Corollary 1. *For every $0 < \delta \leq 1$ and for any $M > 0$ with probability 1 in $\omega \in \Omega$*

$$\mathrm{dist}_{L_2([0,M];\mathbf{H}^{1-\delta})} \left(\Pi_{0,M}\mathfrak{A}_\varepsilon(\omega), \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0, \tag{60}$$

$$\mathrm{dist}_{C([0,M];\mathbf{H}^{-\delta})} \left(\Pi_{0,M}\mathfrak{A}_\varepsilon(\omega), \Pi_{0,M}\overline{\mathfrak{A}}\right) \to 0 \quad (\varepsilon \to 0+). \tag{61}$$

To prove (60) and (61), we just repeat the proof of Theorem 6.1 replacing the topology Θ^{loc} with $L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ or $C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$.

A natural question arise: is it possible to take $\delta = 0$ in the limit relations (60) and (61)? The answer is: YES, at least for the case of RD-systems (17) with deterministic coefficient $b(\cdot)$ independent of ω and ε and with random function $g_\varepsilon(\cdot)$ that depends on ω and ε.

In fact, we have the following result on convergence of trajectory attractors of the system (17) in the strong topology $\Theta_+^{s,loc}$, where we just construct the trajectory attractors (see Theorem 5.3).

Theorem 6.2. *Let the coefficient $b = b(x)$ is deterministic, i.e., it does not depend on $\omega \in \Omega$ and ε, while the function $g = g(x, \frac{x}{\varepsilon}, \omega)$ is random and statistically homogeneous. Then, ω-almost surely we have the following convergence in the strong topology $\Theta_+^{s,loc}$*

$$\mathfrak{A}_\varepsilon(\omega) \to \overline{\mathfrak{A}} \ \text{ as } \varepsilon \to 0+. \tag{62}$$

Moreover, ω-almost surely

$$\mathcal{K}_\varepsilon(\omega) \to \overline{\mathcal{K}} \text{ as } \varepsilon \to 0+ \ \text{ in } \Theta^{s,loc}. \tag{63}$$

Proof. Repeating the reasoning from the proof of Theorem 6.1 we find a bounded in $\mathcal{F}^b$ sequence $\{u_{\varepsilon_n}(s), s \in \mathbb{R}\}$ of complete solutions of systems (51) that converges in the topology Θ^{loc} as $\varepsilon_n \to 0+$ to a function $\bar{u}(s)$ which is a bounded complete solution of the averaged equation (46).

We claim that $u_{\varepsilon_n}(s)$ converges to $\bar{u}(s)$ in the strong topology $\Theta_+^{s,loc}$. To prove this, we use the method of energy identities from the proof of Theorem 5.3. It is sufficient to show that the sequence $\{u_{\varepsilon_n}(s)\}$ has a subsequence that strongly converges to $\bar{u}(s)$ in the space $\mathbf{L_p}(D \times [-M+1, M]) \cap \mathbf{L}_2(-M+1, M; \mathbf{V})$ for each $M > 0$. For an arbitrary fixed M, shifting the time $s = -M + s'$, we can assume that the functions $\{u_{\varepsilon_n}(s')\}$ and $\bar{u}(s')$ are defined on the interval $[0, M']$, $M' = 2M$ and we seek a subsequence that converges strongly in $\mathbf{L_p}(D \times [1, M']) \cap \mathbf{L}_2(1, M'; \mathbf{V})$. For brevity, we omit the primes in s' and M'.

Since $\{u_{\varepsilon_n}(s)\}$ is bounded in the spaces $\mathbf{L_p}(D\times[0,M])$ and $\mathbf{L}_2(0,M;\mathbf{V})$, we can assume that $u_{\varepsilon_n}(\cdot)\rightharpoonup\bar{u}(\cdot)$ as $\varepsilon_n\to 0+$ weakly in the spaces $\mathbf{L_p}(D\times[0,M])$ and $\mathbf{L}_2(0,M;\mathbf{V})$. We can also assume that $u_{\varepsilon_n}(M)\rightharpoonup\bar{u}(M)$ as $\varepsilon_n\to 0+$ weakly in $\mathbf{H}$.

Similar to (40)–(42), we have

$$\|u(M)\|\le\liminf_{n\to\infty}\|u_{\varepsilon_n}(M)\|, \tag{64}$$

$$\int_0^M\int_D s(a\nabla\bar{u}\cdot\nabla\bar{u})dxds\le\liminf_{n\to\infty}\int_0^M\int_D s(a\nabla u_{\varepsilon_n}\cdot\nabla u_{\varepsilon_n})dxds, \tag{65}$$

$$\int_0^M\int_D sb(x)|\hat{u}^i|^{p_i}dxds\le\liminf_{n\to\infty}\int_0^M\int_D sb(x)|u^i_{\varepsilon_n}|^{p_i}dxds,\ i=1,2,\dots,N, \tag{66}$$

where, for brevity, we denote $u_{\varepsilon_n}=u_{\varepsilon_n}(x,s)$ and $\bar{u}=\bar{u}(x,s)$.

Similar to (43), we obtain

$$\int_0^M\int_D sb\,(x)\,F(\bar{u}(x,t))dxds\le\liminf_{n\to\infty}\int_0^M\int_D sb\,(x)\,F(u_{\varepsilon_n}(x,t))dxds \tag{67}$$

(recall that in the considered case, the coefficient $b\,(x)$ is independent of ε).

We now apply the energy identities for the functions $u_{\varepsilon_n}(s)$ and $\bar{u}(\cdot)$ and obtain similar to (44) and (45) the following equalities:

$$\frac{1}{2}\|u_{\varepsilon_n}(M)\|^2+\int_0^M\int_D s(a\nabla u_{\varepsilon_n}\cdot\nabla u_{\varepsilon_n})dxds+\sum_{i=1}^N\gamma_i\int_0^M\int_D sb\,(x)\,|u^i_{\varepsilon_n}|^{p_i}dxds$$
$$+\int_0^M\int_D sb(x)F(u_{\varepsilon_n})dxds=\frac{1}{2}\int_0^M\int_D|u_{\varepsilon_n}|^2dxds+\int_0^M\int_D g_{\varepsilon_n}(x)\cdot u_{\varepsilon_n}\ dxds, \tag{68}$$

$$\frac{1}{2}\|\bar{u}(M)\|^2+\int_0^M\int_D s(a\nabla\bar{u}\cdot\nabla\bar{u})dxds+\sum_{i=1}^N\gamma_i\int_0^M\int_D sb\,(x)\,|\bar{u}^i|^{p_i}dxds$$
$$+\int_0^M\int_D sb\,(x)\,F(\bar{u})dxds=\frac{1}{2}\int_0^M\int_D|\bar{u}|^2dxds+\int_0^M\int_D g^{\mathrm{hom}}(x)\cdot\bar{u}\ dxds. \tag{69}$$

Consider the difference

$$\left|\int_0^M\int_D g_{\varepsilon_n}(x)\cdot u_{\varepsilon_n}\ dxds-\int_0^M\int_D g^{\mathrm{hom}}(x)\cdot\bar{u}\ dxds\right|$$
$$=\left|\int_0^M\int_D g_{\varepsilon_n}(x)\cdot(u_{\varepsilon_n}-\bar{u})\ dxds+\int_0^M\int_D\big(g_{\varepsilon_n}(x)-g^{\mathrm{hom}}(x)\big)\cdot\bar{u}\ dxds\right|$$
$$\le\|g_{\varepsilon_n}(\cdot)\|_{\mathbf{L}_2}\|u_{\varepsilon_n}-\bar{u}\|_{\mathbf{L}_2}+\left|\int_0^M\int_D\big(g_{\varepsilon_n}(x)-g^{\mathrm{hom}}(x)\big)\cdot\bar{u}\ dxds\right| \tag{70}$$

Recall that $u_{\varepsilon_n}(\cdot)\to\bar{u}(\cdot)$ strongly in the space $\mathbf{L}_2(D\times[0,M])$ and $g_{\varepsilon_n}(\cdot)\rightharpoonup g^{\mathrm{hom}}(\cdot)$ weakly in $\mathbf{L}_2(D\times[0,M])$ (see (24)) and, hence, $g_{\varepsilon_n}(\cdot)$ is uniformly bounded in $\mathbf{L}_2(D\times[0,M])$. Therefore, both summands in (70) approach zero and, consequently,

$$\int_0^M\int_D g_{\varepsilon_n}(x)\cdot u_{\varepsilon_n}\ dxds\to\int_0^M\int_D g^{\mathrm{hom}}(x)\cdot\bar{u}\ dxds\text{ as }\varepsilon_n\to 0+. \tag{71}$$

Thus, the right-hand side of equation (68) tends to that of equation (69), that is, the left-hand side of (68) also converges to the left-hand side of (69). Combining this observation with inequalities (64)–(67) we conclude that

$$\lim_{n\to\infty} \|u_{\varepsilon_n}(M)\|^2 = \|\bar{u}(M)\|^2,$$

$$\lim_{n\to\infty} \int_0^M \int_D s(a\nabla u_{\varepsilon_n} \cdot \nabla u_{\varepsilon_n}) dxds = \int_0^M \int_D s(a\nabla \bar{u} \cdot \nabla \bar{u}) dxds,$$

$$\lim_{n\to\infty} \int_0^M \int_D sb(x)|u^i_{\varepsilon_n}|^{p_i} dxds = \int_0^M \int_D sb(x)|\bar{u}^i|^{p_i} dxds, \ i = 1, 2, \dots, N,$$

$$\lim_{n\to\infty} \int_0^M \int_D sb(x)F(u_{\varepsilon_n}) dxds = \int_0^M \int_D sb\,(x)\,F(\bar{u}) dxds.$$

To complete the proof, we use the reasonings as in the end of the proof of Theorem 5.3 and obtain that $u_{\varepsilon_n}(\cdot) \to \bar{u}(\cdot)$ strongly in the space $\mathbf{L_p}(D \times [0, M]) \cap \mathbf{L}_2(0, M; \mathbf{V}) \cap C([0, M]; \mathbf{H})$ and $\partial_t u_{\varepsilon_n}(\cdot) \to \partial_t \bar{u}(\cdot)$ strongly in $\mathcal{L}(0, M)$ as $\varepsilon_n \to 0+$. Thus, we have proved (63) and (62). □

Finally we consider the reaction-diffusion systems for which the uniqueness theorem of the Cauchy problem takes place. It is sufficient to assume that the nonlinear term $f(u)$ in the equation (17) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \geq -C|v_1 - v_2|^2 \ \text{ for } v_1, v_2 \in \mathbb{R}^N. \tag{72}$$

(see [24, 26]). In [24] it was proved that if (72) holds, then equations (17) and (46) generate the dynamical semigroups in $\mathbf{H} = (L_2(D))^N$, which have the global attractors $\mathcal{A}_\varepsilon(\omega)$ and $\overline{\mathcal{A}}$ bounded in the space $\mathbf{V} = \left(H^1_0(D)\right)^N$ (see also [3], [50]). We have

$$\mathcal{A}_\varepsilon(\omega) = \{u(0) \mid u \in \mathfrak{A}_\varepsilon(\omega)\}, \ \overline{\mathcal{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}.$$

Convergence (62) implies

Corollary 2. *Under the assumptions of Theorem 6.2, the following limit holds ω-almost surely:*

$$\mathrm{dist}_{\mathbf{H}}\left(\mathcal{A}_\varepsilon(\omega), \overline{\mathcal{A}}\right) \to 0 \ (\varepsilon \to 0+). \tag{73}$$

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