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HOMOGENIZATION OF ATTRACTORS OF REACTION–DIFFUSION SYSTEM WITH RAPIDLY OSCILLATING TERMS IN AN ORTHOTROPIC POROUS MEDIUM

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In a perforated domain, we consider a reaction-diffusion system with rapidly oscillating terms in the equations and boundary conditions. No Lipschitz condition is imposed, so the uniqueness of a solution to the corresponding initial-boundary value problem is not guaranteed. We prove that the trajectory attractors of the system weakly converge to the trajectory attractors of the homogenized reaction-diffusion systems with a strange term (*potential*). *Bibliography*: 18 *titles. Illustrations*: 1 *figure.*

In this paper, we study the asymptotic behavior of attractors of the initial-boundary value problem for a system of nonlinear differential equations in perforated domains. We are interested in the weak convergence and limit behavior of attractors as the small parameter characterizing, in particular, perforation tends to zero. For this purpose we apply homogenization methods, as

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well as fine methods for analyzing trajectories and global attractors.

The homogenization of attractors was studied by the authors in their recent works [1]–[3], where the reader can also find an overview of the results, historical notes, and an extensive bibliography. In particular, the case of a periodic perforated domain and scalar evolution equations with dissipation was treated in [1] and [3].

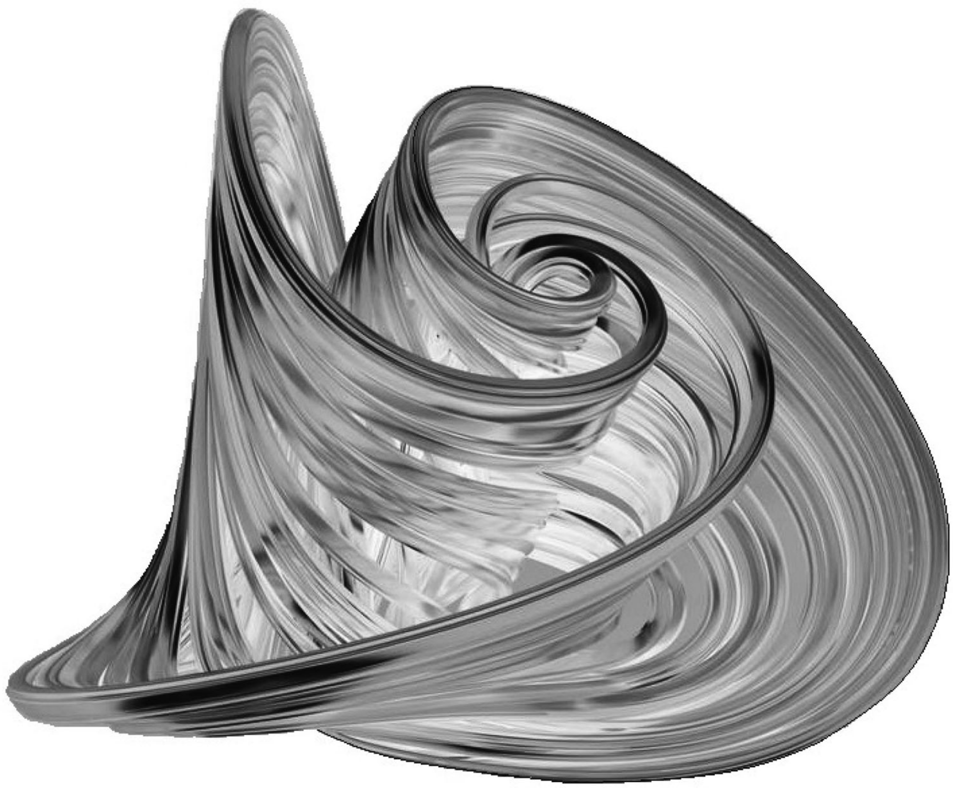

Figure. Attractor. More images of attractors can be found on the website https://oir.mobi/631015-fraktalnye-attraktory.html

As known, attractors describe the behavior of solutions to dissipative nonlinear evolution equations as the time tends to infinity. Using attractors, it is also convenient to study the stability and instability of limit structures of the corresponding dynamical systems. Attractors extract the most important object; namely, the limit sets of trajectories characterizing the entire dynamics of the model governed by evolution equations. In this paper, we study the asymptotic behavior of trajectory and global attractors of the system of reaction–diffusion equations with rapidly oscillating terms in a perforated domain.

The theory of trajectory attractors for dissipative partial differential equations was developed in [4]–[6]. This approach is espessially useful in the study of the long-time behavior of solutions to evolution equations for which the uniqueness result for the corresponding original problems has not been proved yet (for example, the inhomogeneous three-dimensional system of Navier–Stokes equations) or fails (for example, the reaction–diffusion equation considered in this paper).

In the present paper, we prove that the trajectory attractor $\mathfrak{A}_\varepsilon$ of the reaction–diffusion system in a perforated domain weakly converges to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized system in the corresponding function space as $\varepsilon \to 0$. Here, the parameter ε characterizes the cavity diameter, as well as the distance between cavities in the perforated medium.

The paper is organized as follows. In Section 1, we introduce the main notions and formulate the results on trajectory attractors of autonomous evolution equations. In Section 2, we describe the geometric structure of the perforated domain under consideration, formulate the problems, and introduce the necessary function spaces. Section 3 is devoted to the study of homogenization of attractors of autonomous reaction–diffusion systems with rapidly oscillating terms in a perforated domain. We also show how a strange term (potential) appears in the homogenized system.

1 Trajectory Attractors of Evolution Equations

In this section, we describe a general scheme for constructing trajectory attractors of autonomous evolution equations. This scheme will be used in Section 2 to study trajectory attractors of the reaction–diffusion system with rapidly oscillating terms in a perforated domain and the corresponding homogenized equation.

We consider the abstract autonomous evolution equation

$$\frac{\partial u}{\partial t} = A(u), \quad t \geqslant 0, \tag{1.1}$$

where $A(\cdot) : E_1 \to E_0$ is a given nonlinear operator and E_1, E_0 are Banach spaces such that $E_1 \subseteq E_0$. For example, $A(u) = \lambda\Delta u - af(u) + g$ (cf. Section 2).

We study the solution $u(s)$ to Equation (1.1) globally, as a function of the variable $s \in \mathbb{R}_+$. Here, $s \equiv t$ denotes the time variable. The set of solutions to Equation (1.1) is called the *trajectory space* $\mathscr{K}^+$ of Equation (1.1). Let us describe the trajectory space $\mathscr{K}^+$ in detail.

First of all, we consider solutions $u(s)$ to Equation (1.1) defined on a fixed time interval $[t_1, t_2] \subset \mathbb{R}$. We study solutions to Equation (1.1) in a Banach space $\mathscr{F}_{t_1,t_2}$ depending on t_1 and t_2. The space $\mathscr{F}_{t_1,t_2}$ consists of functions $f(s), s \in [t_1, t_2]$ such that $f(s) \in E$ for almost all $s \in [t_1, t_2]$, where E is a Banach space. It is assumed that $E_1 \subseteq E \subseteq E_0$.

For example, for the space $\mathscr{F}_{t_1,t_2}$ we can take $C([t_1, t_2]; E)$, $L_p(t_1, t_2; E)$, $p \in [1, \infty]$, or the intersection of such spaces (cf. Section 2). We assume that $\Pi_{t_1,t_2}\mathscr{F}_{\tau_1,\tau_2} \subseteq \mathscr{F}_{t_1,t_2}$ and

$$\|\Pi_{t_1,t_2} f\|_{\mathscr{F}_{t_1,t_2}} \leqslant C(t_1, t_2, \tau_1, \tau_2)\|f\|_{\mathscr{F}_{\tau_1,\tau_2}} \quad \forall\ f \in \mathscr{F}_{\tau_1,\tau_2}, \tag{1.2}$$

where $[t_1, t_2] \subseteq [\tau_1, \tau_2]$ and Π_{t_1,t_2} is the restriction operator on the segment $[t_1, t_2]$. The constant $C(t_1, t_2, \tau_1, \tau_2)$ is independent of f. Usually, one considers the homogeneous case of spaces, where $C(t_1, t_2, \tau_1, \tau_2) = C(t_2 - t_1, \tau_2 - \tau_1)$.

Let $S(h)$, $h \in \mathbb{R}$, denote the translation operator $S(h)f(s) = f(h + s)$. It is obvious that if the variable s of the function $f(\cdot)$ lies in the segment $[t_1, t_2]$, then the variable s of the function $S(h)f(\cdot)$ lies in the segment $[t_1 - h, t_2 - h]$, where $h \in \mathbb{R}$.

We assume that the mapping $S(h)$ is an isomorphism from F_{t_1,t_2} to F_{t_1-h,t_2-h} and

$$\|S(h)f\|_{\mathscr{F}_{t_1-h,t_2-h}} = \|f\|_{\mathscr{F}t_1,t_2} \quad \forall\ f \in \mathscr{F}_{t_1,t_2}. \tag{1.3}$$

This assumption is natural, for example, for homogeneous spaces.

We assume that $f(s) \in \mathscr{F}_{t_1,t_2}$ implies $A(f(s)) \in \mathscr{D}_{t_1,t_2}$, where $\mathscr{D}_{t_1,t_2}$ is a Banach space larger than $\mathscr{F}_{t_1,t_2}$, $\mathscr{F}_{t_1,t_2} \subseteq \mathscr{D}_{t_1,t_2}$. The derivative $\dfrac{\partial f(t)}{\partial t}$ is a distribution with the values in

$E_0, \dfrac{\partial f}{\partial t} \in D'((t_1,t_2);E_0)$. We assume that $\mathscr{D}_{t_1,t_2} \subseteq D'((t_1,t_2);E_0)$ for all $(t_1,t_2) \subset \mathbb{R}$. A function $u(s) \in \mathscr{F}_{t_1,t_2}$ is called a *solution* to Equation (1.1) in the space $\mathscr{F}_{t_1,t_2}$ (on the interval (t_1,t_2)) if

$$\frac{\partial u}{\partial t}(s) = A(u(s))$$

in the sense of distributions $D'((t_1,t_2);E_0)$.

We also introduce the space

$$\mathscr{F}_+^{\mathrm{loc}} = \{f(s),\ s \in \mathbb{R}_+ \mid \Pi_{t_1,t_2} f(s) \in \mathscr{F}_{t_1,t_2} \quad \forall\ [t_1,t_2] \subset \mathbb{R}_+\}. \tag{1.4}$$

For example, $\mathscr{F}_+^{\mathrm{loc}} = C(\mathbb{R}_+;E)$ if $\mathscr{F}_{t_1,t_2} = C([t_1,t_2];E)$ and $\mathscr{F}_+^{\mathrm{loc}} = L_p^{\mathrm{loc}}(\mathbb{R}_+;E)$ if $\mathscr{F}_{t_1,t_2} = L_p(t_1,t_2;E)$.

A function $u(s) \in \mathscr{F}_+^{\mathrm{loc}}$ is called a *solution* to Equation (1.1) in $\mathscr{F}_+^{\mathrm{loc}}$ if $\Pi_{t_1,t_2} u(s) \in \mathscr{F}_{t_1,t_2}$ and $\Pi_{t_1,t_2} u(s)$ is a solution to Equation (1.1) for any time segment $[t_1,t_2] \subset \mathbb{R}_+$.

We consider a set $\mathscr{K}^+$ of (not necessarily all) solutions to Equation (1.1) in $\mathscr{F}_+^{\mathrm{loc}}$. Elements of the set $\mathscr{K}^+$ are called *trajectories*, and $\mathscr{K}^+$ is the *trajectory space* of Equation (1.1).

We assume that the trajectory space $\mathscr{K}^+$ is *translation invariant* in the following sense: if $u(s) \in \mathscr{K}^+$, then $u(h+s) \in \mathscr{K}^+$ for any $h \geqslant 0$. This property is natural for solutions to autonomous equations in a homogeneous space.

We consider the translation operator $S(h)$ in $\mathscr{F}_+^{\mathrm{loc}} : S(h)f(s) = f(s+h)$, $h \geqslant 0$. It is clear that the set of mappings $\{S(h), h \geqslant 0\}$ forms a semigroup in $\mathscr{F}_+^{\mathrm{loc}}$: $S(h_1)S(h_2) = S(h_1+h_2)$ for $h_1, h_2 \geqslant 0$ and $S(0) = I$ is the identity mapping. We replace the variable h with the time t. The semigroup $\{S(t), t \geqslant 0\}$ is called the *translation semigroup*. By the above assumption, the translation semigroup maps the trajectory space $\mathscr{K}^+$ onto itself:

$$S(t)\mathscr{K}^+ \subseteq \mathscr{K}^+ \quad \forall\ t \geqslant 0. \tag{1.5}$$

Further we study the attraction properties of the translation semigroup $\{S(t)\}$ acting on the trajectory space $\mathscr{K}^+ \subset \mathscr{F}_+^{\mathrm{loc}}$. We introduce a topology in $\mathscr{F}_+^{\mathrm{loc}}$.

Let $\rho_{t_1,t_2}(\cdot,\cdot)$ be a metric defined on the space $\mathscr{F}_{t_1,t_2}$ for all segments $[t_1,t_2] \subset \mathbb{R}$. As in (1.2) and (1.3), we assume that

$$\rho_{t_1,t_2}(\Pi_{t_1,t_2} f, \Pi_{t_1,t_2} g) \leqslant D(t_1,t_2,\tau_1,\tau_2)\rho_{\tau_1,\tau_2}(f,g) \quad \forall\ f,g \in \mathscr{F}_{\tau_1,\tau_2},\ [t_1,t_2] \subseteq [\tau_1,\tau_2],$$

$$\rho_{t_1-h,t_2-h}(S(h)f, S(h)g) = \rho_{t_1,t_2}(f,g) \quad \forall\ f,g \in \mathscr{F}_{t_1,t_2},\ [t_1,t_2] \subset \mathbb{R},\ h \in \mathbb{R}.$$

(For homogeneous space we have $D(t_1,t_2,\tau_1,\tau_2) = D(t_2-t_1,\tau_2-\tau_1)$.)

We denote by Θ_{t_1,t_2} the corresponding metric space on $\mathscr{F}_{t_1,t_2}$. For example, ρ_{t_1,t_2} can be the metric generated by the norm $\|\cdot\|_{\mathscr{F}_{t_1,t_2}}$ in the Banach space $\mathscr{F}_{t_1,t_2}$. In applications, it can happen that the metric ρ_{t_1,t_2} generates a weaker topology in Θ_{t_1,t_2} than the topology of strong convergence in the Banach space $\mathscr{F}_{t_1,t_2}$.

We denote by Θ_+^{loc} the space $\mathscr{F}_+^{\mathrm{loc}}$ equipped with the topology of local convergence on Θ_{t_1,t_2} for any $[t_1,t_2] \subset \mathbb{R}_+$. More exactly, by definition, a sequence of functions $\{f_k(s)\} \subset \mathscr{F}_+^{\mathrm{loc}}$ converges to a function $f(s) \in \mathscr{F}_+^{\mathrm{loc}}$ in Θ_+^{loc} as $k \to \infty$ if $\rho_{t_1,t_2}(\Pi_{t_1,t_2} f_k, \Pi_{t_1,t_2} f) \to 0$ as $k \to \infty$ for any segment $[t_1,t_2] \subset \mathbb{R}_+$. It is easy to prove that the topology in Θ_+^{loc} is metrizable, for example, by using the Fréchet metric

$$\rho_+(f_1,f_2) := \sum_{m\in\mathbb{N}} 2^{-m} \frac{\rho_{0,m}(f_1,f_2)}{1+\rho_{0,m}(f_1,f_2)}. \tag{1.6}$$

If all metric spaces Θ_{t_1,t_2} are complete, then it is obvious that the metric space Θ_+^{loc} is also complete.

We note that the translation semigroup $\{S(t)\}$ is continuous in the topology of Θ_+^{loc}, which directly follows from the definition of the topological space Θ_+^{loc}.

We introduce the Banach space

$$\mathscr{F}_+^b := \{f(s) \in \mathscr{F}_+^{\text{loc}} \mid \|f\|_{\mathscr{F}_+^b} < +\infty\} \tag{1.7}$$

equipped with the norm

$$\|f\|_{\mathscr{F}_+^b} := \sup_{h \geqslant 0} \|\Pi_{0,1} f(h+s)\|_{\mathscr{F}_{0,1}}. \tag{1.8}$$

For example, $\mathscr{F}_+^b = C^b(\mathbb{R}_+; E)$ with the norm $\|f\|_{\mathscr{F}_+^b} = \sup_{h \geqslant 0} \|f(h)\|_E$ if $\mathscr{F}_+^{\text{loc}} = C(\mathbb{R}_+; E)$ and, if $\mathscr{F}_+^{\text{loc}} = L_p^{\text{loc}}(\mathbb{R}_+; E)$, then $\mathscr{F}_+^b = L_p^b(\mathbb{R}_+; E)$ with the norm

$$\|f\|_{\mathscr{F}_+^b} = \left(\sup_{h \geqslant 0} \int\limits_h^{h+1} \|f(s)\|_E^p ds \right)^{1/p}.$$

We note that $\mathscr{F}_+^b \subseteq \Theta_+^{\text{loc}}$. The Banach space $\mathscr{F}_+^b$ is used to define bounded sets in the trajectory space $\mathscr{K}^+$. Constructing a trajectory attractor in $\mathscr{K}^+$, we do not consider the corresponding uniform convergence in the topology of the Banach space $\mathscr{F}_+^b$, but use the weaker topology of local convergence in Θ_+^{loc}.

We assume that $\mathscr{K}^+ \subseteq \mathscr{F}_+^b$, i.e., any trajectory $u(s) \in \mathscr{K}^+$ of Equation (1.1) has finite norm (1.8). We recall the definitions of an attracting set and a trajectory attractor of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$.

Definition 1.1. A set $\mathscr{P} \subseteq \Theta_+^{\text{loc}}$ is called an *attracting set* of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$ in the topology of the space Θ_+^{loc} if for any bounded set $\mathscr{B} \subseteq \mathscr{K}^+$ in $\mathscr{F}_+^b$ the set $\mathscr{P}$ attracts $S(t)\mathscr{B}$ as $t \to +\infty$ in the topology of the space Θ_+^{loc}, i.e., for any ε-neighborhood $O_\varepsilon(\mathscr{P})$ in Θ_+^{loc} there exists $t_1 \geqslant 0$ such that $S(t)\mathscr{B} \subseteq O_\varepsilon(\mathscr{P})$ for any $t \geqslant t_1$.

The attraction property $\mathscr{P}$ can be formulated in the following equivalent form: for any bounded set $\mathscr{B} \subseteq \mathscr{K}^+$ in $\mathscr{F}_+^b$ and $M > 0$ we have $\text{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathscr{P}) \to 0$ as $t \to +\infty$, where $\text{dist}_{\mathscr{M}}(X,Y) := \sup\limits_{x \in X} \text{dist}_{\mathscr{M}}(x,Y) = \sup\limits_{x \in X} \inf\limits_{y \in Y} \rho_{\mathscr{M}}(x,y)$ denotes the semidistance in the sense of Hausdorff between the sets X and Y in the metric space $\mathscr{M}$.

Definition 1.2 (cf. [5])**.** A set $\mathfrak{A} \subseteq \mathscr{K}^+$ is called a *trajectory attractor* of the translation semigroup $\{S(t)\}$ on $\mathscr{K}^+$ in the topology of Θ_+^{loc} if the following conditions are satisfied:

(i) $\mathfrak{A}$ is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc},

(ii) $\mathfrak{A}$ is strictly invariant under the action of the translation semigroup: $S(t)\mathfrak{A} = \mathfrak{A}$ for all $t \geqslant 0$,

(iii) $\mathfrak{A}$ is an attracting set of the translation semigroup $\{S(t)\}$ for $\mathscr{K}^+$ in the topology Θ_+^{loc}, i.e., for any $M > 0$ we have $\text{dist}_{\Theta_{0,M}}(\Pi_{0,M} S(t)\mathscr{B}, \Pi_{0,M}\mathfrak{A}) \to 0$ as $t \to +\infty$.

Remark 1.1. Following the terminology of [7], we say that a trajectory attractor $\mathfrak{A}$ is a *global* $(\mathscr{F}_+^b, \Theta_+^{\text{loc}})$*-attractor* of the translation semigroup $\{S(t)\}$ acting on $\mathscr{K}^+$, i.e., $\mathfrak{A}$ attracts $S(t)\mathscr{B}$ as $t \to +\infty$ in the topology of Θ_+^{loc}, where $\mathscr{B}$ is any bounded (in $\mathscr{F}_+^b$) set in $\mathscr{K}^+$: $\text{dist}_{\Theta_+^{\text{loc}}}(S(t)\mathscr{B}, \mathfrak{A}) \to 0$ as $t \to +\infty$.

We formulate the main results on the existence and structure of a trajectory attractor of Equation (1.1).

Theorem 1.1 (cf. [4, 5, 7])**.** *Let the trajectory space $\mathscr{K}^+$ corresponding to Equation* (1.1) *be closed in $\mathscr{F}_+^b$, and let* (1.5) *hold. We assume that the translation semigroup $\{S(t)\}$ has an attracting set $\mathscr{P} \subseteq \mathscr{K}^+$ that is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}. Then the translation semigroup $\{S(t), t \geqslant 0\}$ acting on $\mathscr{K}^+$ has a trajectory attractor $\mathfrak{A} \subseteq \mathscr{P}$. The set $\mathfrak{A}$ is bounded in $\mathscr{F}_+^b$ and compact in Θ_+^{loc}.*

We describe the structure of the trajectory attractor $\mathfrak{A}$ of Equation (1.1) in terms of complete trajectories of this equation. We consider Equation (1.1) on the whole time-axis

$$\frac{\partial u}{\partial t} = A(u), \quad t \in \mathbb{R}. \tag{1.9}$$

We have already defined the trajectory space $\mathscr{K}^+$ of Equation (1.9) on $\mathbb{R}_+$. Now, we expand the definition to the whole axis $\mathbb{R}$. If a function $f(s)$, $s \in \mathbb{R}$, is given on the whole time-axis, then the translations $S(h)f(s) = f(s+h)$ are also defined for negative h. A function $u(s)$, $s \in \mathbb{R}$, is called a *complete trajectory* of Equation (1.9) if $\Pi_+ u(s+h) \in \mathscr{K}^+$ for any $h \in \mathbb{R}$. Here, $\Pi_+ = \Pi_{0,\infty}$ denotes the restriction operator on the half-axis $\mathbb{R}_+$.

We introduced the spaces $\mathscr{F}_+^{\text{loc}}$, $\mathscr{F}_+^b$, and Θ_+^{loc}. In a similar way, we can define the spaces $\mathscr{F}^{\text{loc}}$, $\mathscr{F}^b$, and Θ^{loc}:

$$\mathscr{F}^{\text{loc}} := \{f(s), s \in \mathbb{R} \mid \Pi_{t_1,t_2} f(s) \in \mathscr{F}_{t_1,t_2}\ \forall\ [t_1,t_2] \subseteq \mathbb{R}\},$$

$$\mathscr{F}^b := \{f(s) \in \mathscr{F}^{\text{loc}} \mid \|f\|_{\mathscr{F}^b} < +\infty\},$$

where

$$\|f\|_{\mathscr{F}^b} := \sup_{h \in \mathbb{R}} \|\Pi_{0,1} f(h+s)\|_{\mathscr{F}_{0,1}}. \tag{1.10}$$

The topological space Θ^{loc} coincides (as a set) with $\mathscr{F}^{\text{loc}}$, and, by definition, $f_k(s) \to f(s)$ in Θ^{loc} as $k \to \infty$ if $\Pi_{t_1,t_2} f_k(s) \to \Pi_{t_1,t_2} f(s)$ in Θ_{t_1,t_2} as $k \to \infty$ for any $[t_1,t_2] \subseteq \mathbb{R}$. It is clear that Θ^{loc}, as well as Θ_+^{loc}, is a metric space.

Definition 1.3. The *kernel* $\mathscr{K}$ of Equation (1.9) in the space $\mathscr{F}^b$ is the union of all complete trajectories $u(s)$, $s \in \mathbb{R}$, of Equation (1.9) that are bounded in $\mathscr{F}^b$ in the norm (1.10):

$$\|\Pi_{0,1} u(h+s)\|_{\mathscr{F}_{0,1}} \leqslant C_u \quad \forall\ h \in \mathbb{R}.$$

Theorem 1.2. *Let the assumptions of Theorem* 1.1 *hold. Then $\mathfrak{A} = \Pi_+ \mathscr{K}$. The set $\mathscr{K}$ is compact in Θ^{loc} and bounded in $\mathscr{F}^b$.*

The complete proof of Theorem 1.2 can be found in [4, 5].

To prove that some ball in $\mathscr{F}_+^b$ is compact in Θ_+^{loc}, we use the following assertion. Let E_0 and E_1 be Banach spaces such that $E_1 \subset E_0$. We consider the Banach spaces

$$W_{p_1,p_0}(0,M;E_1,E_0) = \{\psi(s), s \in 0,M \mid \psi(\cdot) \in L_{p_1}(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\},$$

$$W_{\infty,p_0}(0,M;E_1,E_0) = \{\psi(s), s \in 0,M \mid \psi(\cdot) \in L_\infty(0,M;E_1),\ \psi'(\cdot) \in L_{p_0}(0,M;E_0)\},$$

where $p_1 \geqslant 1$ and $p_0 > 1$, with the norms

$$\|\psi\|_{W_{p_1,p_0}} := \Bigg(\int\limits_0^M \|\psi(s)\|_{E_1}^{p_1} ds \Bigg)^{1/p_1} + \Bigg(\int\limits_0^M \|\psi'(s)\|_{E_0}^{p_0} ds \Bigg)^{1/p_0},$$

$$\|\psi\|_{W_{\infty,p_0}} := \operatorname{ess\,sup}\{\|\psi(s)\|_{E_1} \mid s \in [0, M]\} + \Bigg(\int\limits_0^M \|\psi'(s)\|_{E_0}^{p_0} ds \Bigg)^{1/p_0}.$$

Lemma 1.1 (cf. [8]). *Let $E_1 \Subset E \subset E_0$. Then the following embeddings are compact*:

$$W_{p_1,p_0}(0, T; E_1, E_0) \Subset L_{p_1}(0, T; E), \tag{1.11}$$

$$W_{\infty,p_0}(0, T; E_1, E_0) \Subset C([0, T]; E). \tag{1.12}$$

In Section 2, we study the systems of reaction–diffusion equations and their trajectory attractors depending on the small parameter $\varepsilon > 0$.

Definition 1.4. We say that trajectory attractors $\mathfrak{A}_\varepsilon$ *converge* to a trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ_+^{loc} as $\varepsilon \to 0$ if for any neighborhood $\mathscr{O}(\mathfrak{A})$ in Θ_+^{loc} there is $\varepsilon_1 \geqslant 0$ such that $\mathfrak{A}_\varepsilon \subseteq \mathscr{O}(\overline{\mathfrak{A}})$ for any $\varepsilon < \varepsilon_1$, i.e., for any $M > 0$ we have $\operatorname{dist}_{\Theta_{0,M}}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0$ as $\varepsilon \to 0$.

2 Notation and Statement of the Problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 3$, containing the origin with piecewise-smooth boundary $\partial\Omega$, and let G_0 be a domain in $Y = (-1/2, 1/2)^n$ such that $\overline{G}_0$ is a compact set diffeomorphic to a ball. We assume that $\delta > 0$ and M is a set. We denote $\delta M = \{x : \delta^{-1}x \in M\}$. We assume that $\varepsilon > 0$ is small enough so that $\varepsilon^{n/(n-2)}G_0 \subset \varepsilon Y$. For $j \in \mathbb{Z}^n$ we define $P_\varepsilon^j = \varepsilon j$, $Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y$, $G_\varepsilon^j = P_\varepsilon^j + \varepsilon^{n/(n-2)}G_0$. We consider the domain $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > \sqrt{n}\,\varepsilon\}$ and the set of admissible indices $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \varnothing\}$. We note that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, where $d > 0$ is a constant. We consider the domain $\Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon$ where $G_\varepsilon = \bigcup\limits_{j\in\Upsilon_\varepsilon} G_\varepsilon^j$. We set $Q_\varepsilon = \Omega_\varepsilon \times (0, +\infty)$ and $Q = \Omega \times (0, +\infty)$.

We study the asymptotic behavior of trajectory attractors of the initial-boundary value problem

$$\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial t} = \lambda\Delta u_\varepsilon - a\Big(x, \frac{x}{\varepsilon}\Big) f(u_\varepsilon) + g\Big(x, \frac{x}{\varepsilon}\Big), \quad x \in \Omega_\varepsilon,\\
&\frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^{n/(2-n)} B_\varepsilon^j(x) u_\varepsilon = 0, \quad x \in \partial G_\varepsilon^j, \quad j \in \Upsilon_\varepsilon, \quad t \in (0, +\infty),\\
&u_\varepsilon = 0, \quad x \in \partial\Omega,\\
&u_\varepsilon = U(x), \quad x \in \Omega_\varepsilon, \quad t = 0;
\end{aligned} \tag{2.1}$$

here, $u = (u^1, \dots, u^N)^\top$, $f = (f^1, \dots, f^N)^\top$, $g = (g^1, \dots, g^N)^\top$, and λ is an $N \times N$-matrix with constant entries and positive symmetric part $\frac{1}{2}(\lambda + \lambda^\top) \geqslant \beta I$, where $\beta > 0$, I is the identity matrix of order N, ν is the outward normal vector to the boundary, $a(x, y) \in C(\overline{\Omega} \times \mathbb{R}^n)$ is such that $0 < a_0 \leqslant a(x,y) \leqslant A_0$ with some constants a_0, A_0, and $a_\varepsilon(x) = a(x, x/\varepsilon)$ has the mean $\overline{a}(x)$ in the space $L_{\infty,*w}(\Omega)$ as $\varepsilon \to 0+$, i.e.,

$$\int\limits_\Omega a\Big(x, \frac{x}{\varepsilon}\Big)\varphi(x)dx \to \int\limits_\Omega \overline{a}(x)\varphi(x)dx \quad \forall\, \varphi \in L_1(\Omega), \quad \varepsilon \to 0+ . \tag{2.2}$$

For the vector-valued function $g(x,y)$ we assume that for any $\varepsilon > 0$ the functions $g^i_\varepsilon(x) = g^i(x, x/\varepsilon)$ belong to $H^{-1}(\Omega)$ and have the means $\overline{g}^i(x)$ in $V' = H^{-1}(\Omega)$ as $\varepsilon \to 0+$, i.e.,

$$g^i\Big(x, \frac{x}{\varepsilon}\Big) \rightharpoonup \overline{g}^i(x) \ \text{ weakly in } V, \quad \varepsilon \to 0+,$$

or

$$\int\limits_\Omega g^i\Big(x, \frac{x}{\varepsilon}\Big)\varphi(x)dx \to \int\limits_\Omega \overline{g}^i(x)\varphi(x)dx \quad \forall\, \varphi \in V = H^1_0(\Omega), \ \ i = 1, \ldots, N, \quad \varepsilon \to 0+ . \tag{2.3}$$

The matrix $B^j_\varepsilon(x)$ in the boundary condition in (2.1) is diagonal with bounded entries

$$b^{11}\Big(x, \frac{x - P^j_\varepsilon}{\varepsilon^{n/(n-2)}}\Big), \ldots, b^{NN}\Big(x, \frac{x - P^j_\varepsilon}{\varepsilon^{n/(n-2)}}\Big), \quad j \in \Upsilon_\varepsilon,$$

where $b^{kk}(x,y) \in C(\Omega \times \mathbb{R}^n)$ is a 1-periodic function of y such that

$$0 < b_0 \leqslant b^{kk}(x,y) \leqslant B_0 \tag{2.4}$$

with constants b_0, B_0 for all $k = 1, \ldots, N$. We set $\overline{B}(x,y) := (b^{11}(x,y), \ldots, b^{NN}(x,y))^\top$ and denote by $B(x,y)$ the diagonal matrix with entries $b^{11}(x,y), \ldots, b^{NN}(x,y)$.

Let a vector-valued function $f(v) \in C(\mathbb{R}^N; \mathbb{R}^N)$ satisfy the inequalities

$$\sum_{i=1}^N |f^i(v)|^{p_i/(p_i-1)} \leqslant C_0\bigg(\sum_{i=1}^N |v^i|^{p_i} + 1\bigg), \tag{2.5}$$

$$\sum_{i=1}^N \gamma_i |v^i|^{p_i} - C \leqslant \sum_{i=1}^N f^i(v)v^i \quad \forall\, v \in \mathbb{R}^N, \tag{2.6}$$

where $\gamma_i > 0$ for all $i = 1, \ldots, N$. For the sake of definiteness we assume that $p_N \geqslant p_{N-1} \geqslant \ldots \geqslant p_1 \geqslant 2$. The inequality (2.5) is connected with the fact that the functions $f^i(u)$ in the actual reaction–diffusion equations are polynomials, possibly, of different degrees. The inequality (2.6) is called the *condition of dissipativity* for the problem (2.1). In the simple case $p_i \equiv p$ for all $i = 1, \ldots, N$, the conditions (2.5) and (2.6) are reduced to the inequalities

$$|f(v)| \leqslant C_0(|v|^{p-1} + 1), \quad \gamma|v|^p - C \leqslant f(v)v \quad \forall\, v \in \mathbb{R}^N. \tag{2.7}$$

We emphasize that no Lipschitz condition on $f(v)$ with respect to v is assumed.

Remark 2.1. The methods we propose can be used to study systems with nonlinear terms of the form

$$\sum_{j=1}^m a_j\Big(x, \frac{x}{\varepsilon}\Big) f_j(u),$$

where the matrices a_j can be homogenized and $f_j(u)$ are vector-valued polynomials in u satisfying conditions of the form (2.5) and (2.6). For the sake of brevity we consider only the case $m = 1$ and $a_1(x, x/\varepsilon) = a(x, x/\varepsilon)I$, where I is the identity matrix.

We consider some examples of functions satisfying the limit conditions (2.2) and (2.3) and refer the reader to [9] for justification.

Example 2.1. Let $a(x, y) \in C(\overline{\Omega} \times \mathbb{R}^n)$ be periodic in each variable y_k, $k = 1, \ldots, n$, with period 1. Then it is obvious that (2.2) holds for $a(x, x/\varepsilon)$; moreover,

$$\overline{a}(x) = \int\limits_{\mathbb{T}^n} a(x, y) dy,$$

where $\mathbb{T}^n = \mathbb{R}^n \pmod 1$ is an n-dimensional torus.

Let a vector-valued function $g(x, y) \in C(\mathbb{R}^n; H^{-1}(\Omega))$ be 1-periodic in each variable y_k, $k = 1, \ldots, n$, with the values in $H^{-1}(\Omega)$. The property (2.3) is valid for $g(x, x/\varepsilon)$ with the mean

$$\overline{g}(x) = \int\limits_{\mathbb{T}^n} g(x, y) dy.$$

Example 2.2. Let functions $a(x, y)$ and $g(x, y)$ be quasiperiodic in the corresponding spaces. Hence, for example, for $a(x, y)$ there exists a continuous function

$$A(x, \omega_{11}, \ldots, \omega_{1k_1}, \ldots, \omega_{n1}, \ldots, \omega_{nk_n}) \in C(\overline{\Omega} \times \mathbb{T}^{k_1} \times \ldots \times \mathbb{T}^{k_n})$$

that is 1-periodic in each variable ω_{ij} and such that

$$a(x, y_1, \ldots, y_n) = A(x, \alpha_{11} y_1, \ldots, \alpha_{1k_1} y_1, \ldots, \alpha_{n1} y_n, \ldots, \alpha_{nk_n} y_n) \quad \forall\, y \in \mathbb{R}^n, \tag{2.8}$$

where $\{\alpha_{ij}\}_{j=1,\ldots,k_i}^{i=1,\ldots,n}$ are rationally independent real numbers. Similar formulas hold for components of the vector-valued function $g(x, y)$.

The mean function $\overline{a}(x)$ is obtained by homogenization of $A(x, \cdot\,)$ over all tori $\mathbb{T}^{k_1} \times \ldots \times \mathbb{T}^{k_n}$:

$$\overline{a}(x) = \int\limits_{\mathbb{T}^{k_1}} \cdots \int\limits_{\mathbb{T}^{k_n}} A(x, \overline{\omega}_1, \ldots, \overline{\omega}_n) d\overline{\omega}_1 \cdots d\overline{\omega}_n. \tag{2.9}$$

Since the set $\overline{\Omega} \times \mathbb{T}^{k_1} \times \ldots \times \mathbb{T}^{k_n}$ is compact, $a(x, y)$ is uniformly continuous with respect to x :

$$|a(x_1, y) - a(x_2, y)| \leqslant \alpha(|x_1 - x_2|) \quad \forall\, x_1, x_2 \in \overline{\Omega}, \quad \forall\, y \in \mathbb{R}^n, \tag{2.10}$$

where $\alpha(s) \to 0$ as $s \to 0+$ and $\alpha(s)$ is independent of y.

Example 2.3. We consider a function $a(x, y) \in C_b(\overline{\Omega} \times \mathbb{R}^n)$ satisfying (2.10). Let $a(x, y)$ be almost periodic in y in the sense of Bohr, i.e., there exist quasiperiodic functions $a_N(x, y) \in C_b(\overline{\Omega} \times \mathbb{R}^n)$ (cf. (2.8)) that satisfy (2.10) with the same function $\alpha(s)$ and

$$\lim_{N \to \infty} \|a(x, y) - a_N(x, y)\|_{C_b(\overline{\Omega} \times \mathbb{R}^n)} = 0$$

(cf. [10]). Under the above conditions, the function $a(x, x/\varepsilon)$ has the mean $\overline{a}(x)$ in $L_{\infty, *w}(\Omega)$ as $\varepsilon \to 0+$, where $\overline{a}(x) = \lim\limits_{N \to \infty} \overline{a}_N(x)$ and $\overline{a}_N(x)$ are the means of $a_N(x, z)$ in $L_{\infty, *w}(\Omega)$ (cf. (2.9)).

In a similar way, we can construct examples of vector-valued almost periodic functions $g(x, y)$ with the values in $H^{-1}(\Omega)$ such that $g(x, x/\varepsilon)$ can be homogenized over ε.

We note that the above examples admit vector-valued functions

$$g\Big(x,\frac{x}{\varepsilon}\Big)=G_0\Big(x,\frac{x}{\varepsilon}\Big)+\sum_{i=1}^{n}\partial_{x_i}G_i\Big(x,\frac{x}{\varepsilon}\Big),$$

where $G_i(x,y)\in C(\mathbb{R}^n;L_2(\Omega))$ are periodic, quasiperiodic, or almost periodic functions with the values in the space $L_2(\Omega)$ and the means $\overline{G}_i(x)\in L_2(\Omega)$, $i=0,1,\ldots,n$. Consequently, we can have infinite growth of the $L_2(\Omega)$-norms of the functions

$$\partial_{x_i}G_i\Big(x,\frac{x}{\varepsilon}\Big)=G_{ix_i}\Big(x,\frac{x}{\varepsilon}\Big)+\frac{1}{\varepsilon}G_{iy_i}\Big(x,\frac{x}{\varepsilon}\Big),\quad \varepsilon\to 0+\,.$$

These functions are bounded only in the space $H^{-1}(\Omega)$.

From (2.2), (2.3) and the absolute continuity of the Lebesgue integral it follows that

$$\int\limits_{\Omega_\varepsilon} a\Big(x,\frac{x}{\varepsilon}\Big)\varphi_1(x)dx\to\int\limits_{\Omega}\overline{a}(x)\varphi_1(x)dx, \tag{2.11}$$

$$\int\limits_{\Omega_\varepsilon} g\Big(x,\frac{x}{\varepsilon}\Big)\varphi_2(x)dx\to\int\limits_{\Omega}\overline{g}(x)\varphi_2(x)dx \tag{2.12}$$

for any $\varphi_1\in L_1(\Omega)$ and $\varphi_2\in V=H_0^1(\Omega)$ as $\varepsilon\to 0+$.

We set $\mathbf{H}:=[L_2(\Omega)]^N$, $\mathbf{H}_\varepsilon:=[L_2(\Omega_\varepsilon)]^N$, $\mathbf{V}:=[H_0^1(\Omega)]^N$, and let $\mathbf{V}_\varepsilon:=[H^1(\Omega_\varepsilon;\partial\Omega)]^N$ be the set of vector-valued functions in $[H^1(\Omega_\varepsilon)]^N$ with zero trace on $\partial\Omega$. The norms in these spaces are defined by

$$\|v\|^2:=\int\limits_{\Omega}\sum_{i=1}^{N}|v^i(x)|^2dx,\quad \|v\|_\varepsilon^2:=\int\limits_{\Omega_\varepsilon}\sum_{i=1}^{N}|v^i(x)|^2dx,$$

$$\|v\|_1^2:=\int\limits_{\Omega}\sum_{i=1}^{N}|\nabla v^i(x)|^2dx,\quad \|v\|_{1\varepsilon}^2:=\int\limits_{\Omega_\varepsilon}\sum_{i=1}^{N}|\nabla v^i(x)|^2dx.$$

We denote by $\mathbf{V'}:=[H^{-1}(\Omega)]^N$ the dual of $\mathbf{V}$ and by $\mathbf{V}'_\varepsilon$ the dual of $\mathbf{V}_\varepsilon$.

Let $q_i=p_i/(p_i-1)$ for all $i=1,\ldots,N$. We use the vector notation $\mathbf{p}=(p_1,\ldots,p_N)$ and $\mathbf{q}=(q_1,\ldots,q_N)$ and introduce the spaces

$$\mathbf{L_p}:=L_{p_1}(\Omega)\times\ldots\times L_{p_N}(\Omega),\quad \mathbf{L}_{\mathbf{p},\varepsilon}:=L_{p_1}(\Omega_\varepsilon)\times\ldots\times L_{p_N}(\Omega_\varepsilon),$$

$$\mathbf{L_p}(\mathbb{R}_+;\mathbf{L_p}):=L_{p_1}(\mathbb{R}_+;L_{p_1}(\Omega))\times\ldots\times L_{p_N}(\mathbb{R}_+;L_{p_N}(\Omega)),$$

$$\mathbf{L_p}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}):=L_{p_1}(\mathbb{R}_+;L_{p_1}(\Omega_\varepsilon))\times\ldots\times L_{p_N}(\mathbb{R}_+;L_{p_N}(\Omega_\varepsilon)).$$

As in [5, 11], we study the weak solution to the initial-boundary value problem (2.1), i.e., $u_\varepsilon(x,s)\in\mathbf{L}_\infty^{\rm loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon)\cap\mathbf{L}_2^{\rm loc}(\mathbb{R}_+;\mathbf{V}_\varepsilon)\cap\mathbf{L}_\mathbf{p}^{\rm loc}(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon})$, solving the problem (2.1) in the sense of distributions, i.e.,

$$\int\limits_{Q_\varepsilon} \frac{\partial u_\varepsilon}{\partial t} \cdot \psi \, dxdt + \int\limits_{Q_\varepsilon} \lambda \nabla u_\varepsilon \cdot \nabla \psi \, dxdt + \int\limits_{Q_\varepsilon} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi \, dxdt$$

$$+ \varepsilon^{n/(2-n)} \sum_{j \in \Upsilon_\varepsilon} \int\limits_0^{+\infty} \int\limits_{\partial G_\varepsilon^j} B_\varepsilon^j(x) u_\varepsilon \cdot \psi \, dxdt = \int\limits_{Q_\varepsilon} g_\varepsilon(x) \cdot \psi \, dxdt \tag{2.13}$$

for any $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{H}_\varepsilon)$. Here, $y_1 \cdot y_2$ denotes the inner product of vectors $y_1, y_2 \in \mathbb{R}^N$.

If $u_\varepsilon(x,t) \in \mathbf{L_p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then (2.5) implies $f(u(x,t)) \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon(x,t) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon(x,t) + g_\varepsilon(x) \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon')$. Therefore, for an arbitrary weak solution $u_\varepsilon(x,s)$ to the problem (2.1) we have

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon').$$

By the Sobolev embedding theorem, $\mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon') \subset \mathbf{L_q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$, where $\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(\Omega_\varepsilon) \times \ldots \times H^{-r_N}(\Omega_\varepsilon)$, $\mathbf{r} = (r_1, \ldots, r_N)$, $r_i = \max\{1, n(1/q_i - 1/2)\}$, $i = 1, \ldots, N$. Here, $H^{-r}(\Omega_\varepsilon)$ denotes the dual of the Sobolev space $H^r(\Omega_\varepsilon)$ with exponent $r > 0$ in the perforated domain Ω_ε.

Consequently, for any weak solution $u_\varepsilon(x,t)$ to the problem (2.1)

$$\frac{\partial u_\varepsilon(x,t)}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}).$$

Remark 2.2. The existence of a weak solution $u(x,s)$ to the problem (2.1) with any function $U \in \mathbf{H}_\varepsilon$ and fixed ε such that $u(x,0) = U(x)$ can be proved in a standard way (cf., for example, [7, 11]). Such a solution is not necessarily unique since the function $f(v)$ satisfies only the conditions (2.5), (2.6), whereas no Lipschitz condition on $f(v)$ with respect to v is imposed.

The following lemma can be proved in the same way as Proposition XV.3.1 in [5].

Lemma 2.1. *Let $u_\varepsilon(x,t) \in \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_{\mathbf{p}}^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$ is a weak solution to the problem* (2.1)*. Then the following assertions hold*:

(i) $u \in \mathbf{C}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$,

(ii) $\|u_\varepsilon(\cdot,t)\|^2$ *is absolutely continuous on $\mathbb{R}_+$ and for almost all $t \in \mathbb{R}_+$*

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\cdot,t)\|^2 + \int\limits_{\Omega_\varepsilon} \lambda \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) dx + \int\limits_{\Omega_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t)) \cdot u_\varepsilon(x,t) dx$$

$$+ \varepsilon^{\frac{n}{2-n}} \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} B_\varepsilon^j(x) u_\varepsilon(x,t) \cdot u_\varepsilon(x,t) dx = \int\limits_{\Omega_\varepsilon} g_\varepsilon(x) \cdot u_\varepsilon(x,t) dx.$$

We sometimes omit the subscript ε provided that no confusion arises.

Now, we use the scheme of Section 1 to construct a trajectory attractor of the problem (2.1) in the form (1.1) for $E_1 = \mathbf{L_p} \cap \mathbf{V}$, $E_0 = \mathbf{H}^{-\mathbf{r}}$, $E = \mathbf{H}$, and $A(u) = \lambda \Delta u - a(\cdot) f(u) + g(\cdot)$.

To describe the trajectory space $\mathscr{K}_\varepsilon^+$ of the problem (2.1), for each segment $[t_1, t_2] \in \mathbb{R}$ we consider the Banach space

$$\mathscr{F}_{t_1,t_2} := \mathbf{L_p}(t_1, t_2; \mathbf{L}_p) \cap \mathbf{L}_2(t_1, t_2; \mathbf{V}) \cap \mathbf{L}_\infty(t_1, t_2; \mathbf{H}) \cap \left\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L_q}(t_1, t_2; \mathbf{H^r}) \right\} \tag{2.14}$$

equipped with the norm

$$\|v\|_{\mathscr{F}_{t_1,t_2}} := \|v\|_{\mathbf{L_p}(t_1,t_2;\mathbf{L}_p)} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\| \frac{\partial v}{\partial t} \right\|_{\mathbf{L_q}(t_1,t_2;\mathbf{H^r})}. \tag{2.15}$$

It is obvious that the condition (1.2) is satisfied by the norm (2.15) and the translation semigroup $\{S(h)\}$ satisfies (1.3). Setting $\mathscr{D}_{t_1,t_2} = \mathbf{L_q}(t_1, t_2; \mathbf{H^{-r}})$, we find that $\mathscr{F}_{t_1,t_2} \subseteq \mathscr{D}_{t_1,t_2}$, whereas, $u(s) \in \mathscr{F}_{t_1,t_2}$ implies $A(u(s)) \in \mathscr{D}_{t_1,t_2}$. In what follows, we can regard a weak solution to the problem (2.1) as a solution to the system of equations from the general scheme in Section 1.

Having defined the space (1.4), we find

$$\mathscr{F}_+^{\mathrm{loc}} = {\mathbf{L_p}}^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L_p}) \cap \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L_q^{\mathrm{loc}}}(\mathbb{R}_+; \mathbf{H^{-r}}) \right\},$$

$$\mathscr{F}_{\varepsilon,+}^{\mathrm{loc}} = \mathbf{L_p^{\mathrm{loc}}}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \left\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L_q^{\mathrm{loc}}}(\mathbb{R}_+; \mathbf{H}_\varepsilon^{-\mathbf{r}}) \right\}.$$

We denote by $\mathscr{K}_\varepsilon^+$ the set of all weak solutions to the problem (2.1). We recall that for any function $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathscr{K}_\varepsilon^+$ such that $u(0) = U(x)$. Consequently, the trajectory space $\mathscr{K}_\varepsilon^+$ of the problem (2.1) is not empty and sufficiently large.

It is clear that $\mathscr{K}_\varepsilon^+ \subset \mathscr{F}_+^{\mathrm{loc}}$ and the trajectory space $\mathscr{K}_\varepsilon^+$ is translation invariant, i.e., if $u(s) \in \mathscr{K}_\varepsilon^+$, then $u(h+s) \in \mathscr{K}_\varepsilon^+$ for any $h \geqslant 0$. Consequently, $S(h)\mathscr{K}_\varepsilon^+ \subseteq \mathscr{K}_\varepsilon^+$ for all $h \geqslant 0$.

Further, using the $\mathbf{L}_2(t_1, t_2; \mathbf{H})$-norm, we introduce the metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in $\mathscr{F}_{t_1,t_2}$ as follows:

$$\rho_{0,M}(u, v) = \left(\int\limits_0^M \|u(s) - v(s)\|^2 ds \right)^{1/2} \quad \forall\ u(\cdot), v(\cdot) \in \mathscr{F}_{0,M}.$$

These metrics generate the topology of Θ_+^{loc} in the space $\mathscr{F}_+^{\mathrm{loc}}$ (respectively, $\Theta_{\varepsilon,+}^{\mathrm{loc}}$ in the space $\mathscr{F}_{\varepsilon,+}^{\mathrm{loc}}$). We recall that a sequence $\{v_k\} \subset \mathscr{F}_+^{\mathrm{loc}}$ converges to a function $v \in \mathscr{F}_+^{\mathrm{loc}}$ in Θ_+^{loc} as $k \to \infty$ if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(0,M;\mathbf{H})} \to 0$ as $k \to \infty$ for any $M > 0$. The topology in Θ_+^{loc} is metrizable (cf. (1.6)), and the corresponding metric space is complete. We consider the topology in the trajectory space $\mathscr{K}_\varepsilon^+$ of the problem (2.1). The translation semigroup $\{S(t)\}$ acting on $\mathscr{K}_\varepsilon^+$ is continuous in the topology of Θ_+^{loc}.

Following the general scheme of Section 1, we introduce a bounded set in $\mathscr{K}_\varepsilon^+$ using the Banach spaces $\mathscr{F}_+^b$ (cf. (1.7)). It is clear that

$$\mathscr{F}_+^b = \mathbf{L_p^b}(\mathbb{R}_+; \mathbf{L_p} \cap \mathbf{L}_2^b(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L_q^b}(\mathbb{R}_+; \mathbf{H^{-r}} \right\} \tag{2.16}$$

and $\mathscr{F}_+^b$ is a subspace of $\mathscr{F}_+^{\mathrm{loc}}$.

We consider the translation semigroup $\{S(t)\}$ on $\mathscr{K}_\varepsilon^+$, $S(t) : \mathscr{K}_\varepsilon^+ \to \mathscr{K}_\varepsilon^+$, $t \geqslant 0$. We assume that $\mathscr{K}_\varepsilon$ is the kernel of the problem (2.1) which consists of all weak solutions $u(s)$, $s \in \mathbb{R}$, bounded in the space

$$\mathscr{F}^b = \mathbf{L_p^b}(\mathbb{R}; \mathbf{L_p}) \cap \mathbf{L}_2^b(\mathbb{R}; \mathbf{V}) \cap \mathbf{L}_\infty(\mathbb{R}; \mathbf{H}) \cap \left\{ v \Big| \frac{\partial v}{\partial t} \in \mathbf{L_q^b}(\mathbb{R}; \mathbf{H^{-r}}) \right\}.$$

Proposition 2.1. *Let* (2.5) *and* (2.6) *hold. Then the problem* (2.1) *has trajectory attractors* $\mathfrak{A}_\varepsilon$ *in the topological space* $\Theta_+^{\rm loc}$. *The set of* $\mathfrak{A}_\varepsilon$ *is uniformly* (*with respect to* $\varepsilon \in (0,1)$) *bounded in* $\mathscr{F}_+^b$ *and compact in* $\Theta_+^{\rm loc}$. *Furthermore,* $\mathfrak{A}_\varepsilon = \Pi_+\mathscr{K}_\varepsilon$, *the kernel* $\mathscr{K}_\varepsilon$ *is nonempty and uniformly* (*with respect to* $\varepsilon \in (0,1)$) *bounded in* $\mathscr{F}^b$.

We recall that the spaces $\mathscr{F}_+^b$ and $\Theta_+^{\rm loc}$ depend on ε.

Proposition 2.1 is proved in the same way as the corresponding particular assertion in [5]. The existence of an absorbing set that is bounded in $\mathscr{F}_+^b$ and compact in $\Theta_+^{\rm loc}$ can be proved by using Lemma 2.1, in the same way as in [5].

We note that $\mathfrak{A}_\varepsilon \subset \mathscr{B}_0(R)$ for all $\varepsilon \in (0,1)$, where $\mathscr{B}_0(R)$ is a ball in $\mathscr{F}_+^b$ with sufficiently large radius R. From Lemma 1.1 it follows that

$$\mathscr{B}_0(R) \Subset \mathbf{L}_2^{\rm loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{2.17}$$

$$\mathscr{B}_0(R) \Subset \mathbf{C}^{\rm loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leqslant 1. \tag{2.18}$$

The inclusion (2.17) is obtained from (1.11) if we set $E_0 = \mathbf{H}^{-\mathbf{r}}$, $E = \mathbf{H}^{1-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$ and $p_1 = 2$, $p_0 = q_N$ and take into account the compact embedding $\mathbf{V} \Subset H^{1-\delta}$. The inclusion (2.18) is obtained from (1.12) and the compact embedding $\mathbf{H} \Subset \mathbf{H}^{-\delta}$ if we set $E_0 = \mathbf{H}^{-\mathbf{r}}(D)$, $E = \mathbf{H}^{-\delta}$, $E_1 = \mathbf{H}^1 = \mathbf{V}$, and $p_0 = q_N$.

Using the compact embeddings (2.17) and (2.18), we can see that the attraction to the constructed trajectory attractor is strengthened.

Corollary 2.1. *For any bounded set* $\mathscr{B} \subset \mathscr{K}_\varepsilon^+$ *in* $\mathscr{F}_+^b$

$$\operatorname{dist}_{\mathbf{L}_2(0,M;\mathbf{H}^{1-\delta})}(\Pi_{0,M}S(t)\mathscr{B}, \Pi_{0,M}\mathscr{K}_\varepsilon) \to 0, \quad t \to \infty,$$

$$\operatorname{dist}_{\mathbf{C}([0,M];\mathbf{H}^{-\delta})}(\Pi_{0,M}S(t)\mathscr{B}, \Pi_{0,M}\mathscr{K}_\varepsilon) \to 0, \quad t \to \infty,$$

where M *is an arbitrary positive number.*

3 Homogenization of Attractors of Reaction–Diffusion Equations in a Perforated Domain

In this section, we study the limit behavior of attractors $\mathfrak{A}_\varepsilon$ of the problem (2.1) as $\varepsilon \to 0+$ and the convergence of $\mathfrak{A}_\varepsilon$ to a trajectory attractor of the corresponding homogenized equation.

To determine a strange term (a potential of the limit equation), we consider the problem

$$\begin{aligned} &-\Delta_y v = 0, \quad y \in \mathbb{R}^n \setminus G_0, \\ &\frac{\partial v}{\partial \nu_y} + B(x,y)v = \overline{B}(x,y), \quad y \in \partial G_0, \\ &v \to 0, \quad |y| \to \infty, \end{aligned}$$

where $B(x,y)$ and $\overline{B}(x,y)$ are introduced above. In this problem, the variable x plays the role of the slow parameter. We define the limit potential by

$$V^{kk}(x) = \int\limits_{\partial G_0} \frac{\partial}{\partial \nu_y} v^k(x,y)\, d\sigma_y, \quad k = 1, \ldots, N. \tag{3.1}$$

The homogenized (limit) problem has the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \lambda \Delta u - \overline{a}(x) f(u) - V(x) u + \overline{g}(x), \quad x \in \Omega, \\
&u = 0, \quad x \in \partial\Omega, \\
&u = U(x), \quad t = 0,
\end{aligned}
\tag{3.2}
$$

where $V(x)$ is a diagonal matrix with entries $V^{kk}(x)$, $k = 1, \dots, N$.

The following assertion is similar to Lemma 2.1.

Lemma 3.1. *Let* $u(x,t) \in \mathbf{L}_2^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_{\mathbf{p}}^{\mathrm{loc}}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p}})$ *be a weak solution to the problem* (3.2). *Then the following assertions hold*:

(i) $u \in \mathbf{C}(\mathbb{R}_+; \mathbf{H})$,

(ii) $\|u(\cdot,t)\|^2$ *is absolutely continuous on* $\mathbb{R}_+$ *and*

$$
\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 + \int\limits_\Omega \lambda \nabla u(x,t) \cdot \nabla u(x,t) dx + \int\limits_\Omega \overline{a}(x) f(u(x,t)) \cdot u(x,t) dx
$$

$$
+ \int\limits_\Omega V(x) u(x,t) \cdot u(x,t) dx = \int\limits_\Omega \overline{g}(x) \cdot u(x,t) dx.
$$

The problem (3.2) has a trajectory attractor $\overline{\mathfrak{A}}$ in the trajectory space $\overline{\mathscr{K}}^+$ corresponding to the problem (3.2); moreover, $\overline{\mathfrak{A}} = \Pi_+ \overline{\mathscr{K}}$, where $\overline{\mathscr{K}}$ is the kernel of the problem (3.2) in $\mathscr{F}^b$.

We formulate the main homogenization theorem for the reaction–diffusion system.

Theorem 3.1. *The following limit relation holds in the topological space* Θ_+^{loc} :

$$
\mathfrak{A}_\varepsilon \to \overline{\mathfrak{A}}, \quad \varepsilon \to 0+ . \tag{3.3}
$$

Furthermore,

$$
\mathscr{K}_\varepsilon \to \overline{\mathscr{K}}, \quad \varepsilon \to 0+ \qquad \textit{in } \Theta^{\mathrm{loc}}. \tag{3.4}
$$

Remark 3.1. The functions $\mathfrak{A}_\varepsilon$ and $\mathscr{K}_\varepsilon$ are defined in the perforated domain Ω_ε. However, all these functions can be extended inside the holes in such a way that the norms of the extended functions in the spaces $\mathbf{H}$, $\mathbf{V}$, and $\mathbf{L}_p$ (without perforation) coincide with the corresponding norms in the perforated spaces $\mathbf{H}_\varepsilon, \mathbf{V}_\varepsilon$, and $\mathbf{L}_{p,\varepsilon}$. Therefore, in Theorem 3.1, all the distances are measured in the spaces without perforation with the extension inside the holes taken into account.

Proof of Theorem 3.1. It is clear that (3.4) implies (3.3). Therefore, it suffices to prove (3.4), i.e., to show that for any neighborhood $\mathscr{O}(\overline{\mathscr{K}})$ in Θ^{loc} there is $\varepsilon_1 = \varepsilon_1(\mathscr{O}) > 0$ such that

$$
\mathscr{K}_\varepsilon \subset \mathscr{O}(\overline{\mathscr{K}}) \quad \forall\, \varepsilon < \varepsilon_1. \tag{3.5}
$$

We assume that (3.5) fails. Then there exists a neighborhood $\mathscr{O}'(\overline{\mathscr{K}})$ in Θ^{loc}, a sequence $\varepsilon_k \to 0+$ as $k \to \infty$, and a sequence $u_{\varepsilon_k}(\cdot) = u_{\varepsilon_k}(s) \in \mathscr{K}_{\varepsilon_k}$ such that

$$
u_{\varepsilon_k} \notin \mathscr{O}'(\overline{\mathscr{K}}) \quad \forall\, k \in \mathbb{N}. \tag{3.6}
$$

The functions $u_{\varepsilon_k}(s)$, $s \in \mathbb{R}$, satisfy the relations

$$\begin{aligned} &\frac{\partial u_{\varepsilon_k}}{\partial t} = \lambda \Delta u_{\varepsilon_k} - a_{\varepsilon_k}(x) f(u_{\varepsilon_k}) + g_{\varepsilon_k}(x), \quad x \in \Omega_{\varepsilon_k}, \\ &\frac{\partial u_{\varepsilon_k}}{\partial \nu} + \varepsilon_k^{n/(2-n)} B^j_{\varepsilon_k}(x) u_{\varepsilon_k} = 0, \quad x \in \partial G^j_{\varepsilon_k}, \quad j \in \Upsilon_{\varepsilon_k}, \\ &u_{\varepsilon_k} = 0, \quad x \in \partial\Omega \end{aligned} \tag{3.7}$$

on the whole time-axis, $t \in \mathbb{R}$.

To derive an ε-uniform estimate, we use the following lemmas (cf. [12, Chapter III, Section 5] and [13] respectively).

Lemma 3.2. *Let*

$$W(f, g) = \int\limits_{\Omega_\varepsilon} \nabla f \, \nabla g \, dx + \int\limits_{\Omega_\varepsilon} q f g \, dx + \int\limits_{\partial\Omega_\varepsilon} r f g \, ds \tag{3.8}$$

be a bilinear form on $\mathbf{V}_\varepsilon$, and let $q(x) \geqslant 0$ and $r(x) \geqslant 0$ ($q \not\equiv 0$ or $r \not\equiv 0$). Then the bilinear form $W(f, g)$ determines the inner product on $\mathbf{V}_\varepsilon$ which is equivalent to the inner product

$$(f, g)_{\mathbf{H}^1(\Omega_\varepsilon)} = \int\limits_{\Omega_\varepsilon} (\nabla f \, \nabla g + f g) \, dx.$$

Lemma 3.3. *The coercitivity of the problem* (3.2) *implies the coercitivity of the problem* (2.1)*.*

Then, applying the integral identity (2.13) and using Lemmas 2.1, 3.2, 3.3, we obtain the required estimate. More exactly, the sequence $\{u_{\varepsilon_k}(s)\}$ is bounded in $\mathscr{F}^b$, i.e.,

$$\begin{aligned} \|u_{\varepsilon_k}\|_{\mathscr{F}^b} = \sup_{t \in \mathbb{R}} \|u_{\varepsilon_k}(t)\| + \sup_{t \in \mathbb{R}} \left(\int\limits_t^{t+1} \|u_{\varepsilon_k}(s)\|_1^2 ds \right)^{1/2} + \sup_{t \in \mathbb{R}} \|u_{\varepsilon_k}(s)\|_{\mathbf{L_p}(t,t+1;\mathbf{L_p})} \\ + \sup_{t \in \mathbb{R}} \left\| \frac{\partial u_{\varepsilon_k}}{\partial t}(s) \right\|_{\mathbf{L_q}(t,t+1;\mathbf{H^{-r}})} \leqslant C, \quad k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Consequently, there exists a subsequence $\{u_{\varepsilon'_k}(s)\} \subset \{u_{\varepsilon_k}(s)\}$ such that

$$u_{\varepsilon_k}(s) \to \overline{u}(s) \quad \text{in } \Theta^{\text{loc}}, \quad n \to \infty, \tag{3.10}$$

where $\overline{u}(s) \in \mathscr{F}^b$ satisfies (3.9) with the same constant C. From (3.9) we find

$$\begin{aligned} &u_{\varepsilon_k}(s) \rightharpoonup \overline{u}(s) \ \text{ weakly in } \mathbf{L}_2^{\text{loc}}(\mathbb{R}; \mathbf{V}_\varepsilon), \text{ weakly in } \mathbf{L}_\mathbf{p}^{\text{loc}}(\mathbb{R}; \mathbf{L}_{\mathbf{p},\varepsilon}), \ast\text{-weakly in } \mathbf{L}_\infty^{\text{loc}}(\mathbb{R}_+; \mathbf{H}_\varepsilon), \\ &\frac{\partial u_{\varepsilon_k}(s)}{\partial t} \rightharpoonup \frac{\partial \overline{u}(s)}{\partial t} \ \text{ weakly in } \mathbf{L}_{\mathbf{q},w}^{\text{loc}}(\mathbb{R}; \mathbf{H}_\varepsilon^{-\mathbf{r}}), \quad k \to \infty, \end{aligned}$$

as $n \to \infty$.

We assert that $\overline{u}(s) \in \overline{\mathscr{K}}$. As was already proved, $\|\overline{u}\|_{\mathscr{F}^b} \leqslant C$. It remains to verify that $\overline{u}(s)$ is a weak solution to the problem (3.2). Using (3.9) and (2.12), we find

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \lambda\Delta u_{\varepsilon_k} - g_{\varepsilon_k}(x) \longrightarrow \frac{\partial \overline{u}}{\partial t} - \lambda\Delta\overline{u} - \overline{g}(x), \quad k \to \infty \tag{3.11}$$

in the space $D'(\mathbb{R}; \mathbf{H}_\varepsilon^{-\mathbf{r}})$ since the derivative operator is continuous in the space of distributions.

We prove that

$$a\Big(x, \frac{x}{\varepsilon_n}\Big) f(u_{\varepsilon_n}) \rightharpoonup \overline{a}(x) f(\overline{u}), \quad n \to \infty, \tag{3.12}$$

weakly in $\mathbf{L}^{\rm loc}_{\mathbf{q},w}(\mathbb{R}; \mathbf{L}_\mathbf{q})$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_n}(s)\}$ is bounded in $\mathbf{L}_\mathbf{p}(-M, M; \mathbf{L}_{\mathbf{p},\varepsilon})$ (cf. (3.9)). By (2.5), the sequence $\{f(u_{\varepsilon_n}(s))\}$ is bounded in $\mathbf{L}_\mathbf{q}(-M, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. Since $\{u_{\varepsilon_n}(s)\}$ is bounded in $\mathbf{L}_2(-M, M; \mathbf{V}_\varepsilon)$ and $\{\partial_t u_{\varepsilon_n}(s)\}$ is bounded in $\mathbf{L}_\mathbf{q}(-M, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$, we can assume that $u_{\varepsilon_n}(s) \to \overline{u}(s)$ strongly in $\mathbf{L}_2(-M, M; \mathbf{L}_2) = \mathbf{L}_2(\Omega\times]-M, M[)$ as $n \to \infty$. Consequently, $u_{\varepsilon_n}(x, s) \to \overline{u}(x, s)$ for almost all $(x, s) \in \Omega\times]-M, M[$ as $n \to \infty$. Since $f(v)$ is continuous with respect to $v \in \mathbb{R}$, we conclude that

$$f(u_{\varepsilon_n}(x, s)) \to f(\overline{u}(x, s)) \quad \text{for almost all } (x, s) \in \Omega\times]-M, M[, \quad n \to \infty. \tag{3.13}$$

We have

$$a\Big(x, \frac{x}{\varepsilon_n}\Big) f(u_{\varepsilon_n}) - \overline{a}(x) f(\overline{u}) = a\Big(x, \frac{x}{\varepsilon_n}\Big)(f(u_{\varepsilon_n}) - f(\overline{u})) + \Big(a\Big(x, \frac{x}{\varepsilon_n}\Big) - \overline{a}(x)\Big) f(\overline{u}). \tag{3.14}$$

We show that both terms on the right-hand side of (3.14) weakly converge to zero in $\mathbf{L}_\mathbf{q}(-M, M; \mathbf{L}_\mathbf{q}) = \mathbf{L}_\mathbf{q}(\Omega\times]-M, M[)$ as $n \to \infty$. The sequence $a(x, x/\varepsilon_n)(f(u_{\varepsilon_n}) - f(\overline{u}))$ converges to zero for almost all $(x, s) \in \Omega\times]-M, M[$ (cf. (3.13)) as $n \to \infty$ and is bounded in $\mathbf{L}_\mathbf{q}(\Omega\times]-M, M[)$ (cf. (2.4)). Applying [14, Lemma 1.3], we conclude that

$$a\Big(x, \frac{x}{\varepsilon_n}\Big)(f(u_{\varepsilon_n}) - f(\overline{u})) \rightharpoonup 0, \quad n \to \infty,$$

weakly in $\mathbf{L}_\mathbf{q}(\Omega\times]-M, M[)$. The sequence $(a(x, x/\varepsilon_n) - \overline{a}(x)) f(\overline{u})$ also weakly converges to zero in $\mathbf{L}_\mathbf{q}(\Omega\times]-M, M[)$ as $n \to \infty$ since, by assumption, $a(x, x/\varepsilon_n) \rightharpoonup \overline{a}(x)$ $*$-weakly in $\mathbf{L}_{\infty,*w}(-M, M; \mathbf{L}_2)$ as $n \to \infty$ and $f(\overline{u}) \in \mathbf{L}_\mathbf{q}(\Omega\times]-M, M[)$. Hence (3.12) is proved.

Following [15, 16] (cf. also [17]), we can prove the following assertion.

Lemma 3.4. *For any $\varphi \in \mathbf{H}_\varepsilon$ and all t*

$$\Big|\varepsilon^{\frac{n}{n-2}} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} B_\varepsilon^j(x)\varphi \, ds - \int\limits_\Omega V(x)\overline{\varphi} \, dx\Big| \leqslant M\varepsilon \|\varphi\|_{\mathbf{H}_\varepsilon}, \tag{3.15}$$

and for any $\psi \in \mathscr{F}^b$ the following limit relation holds:

$$\varepsilon^{\frac{n}{n-2}} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} B_\varepsilon^j(x) u_\varepsilon \psi \, ds \longrightarrow \int\limits_\Omega V(x)\overline{u}\psi \, dx, \quad \varepsilon \to 0, \tag{3.16}$$

where $V(x)$ is defined by (3.1) *and the constant M is independent of ε.*

Proof. The inequality (3.15) is proved by using the same scheme as in [16, Lemma 2]. To prove (3.16), we substitute u_ε for a test function into (2.13) and obtain the uniform boundedness

$$\|\nabla u_\varepsilon\|_{\mathbf{H}_\varepsilon} \leqslant K, \quad \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} B_\varepsilon^j u_\varepsilon \psi \, dx \leqslant K\varepsilon^{-n/(2-n)},$$

where the constant K is independent of ε.

We consider the family of extension operators $P_\varepsilon : \mathbf{V}_\varepsilon \to \mathbf{V}$ such that $P_\varepsilon v = v$ almost everywhere in Ω_ε and

$$\|\nabla P_\varepsilon v\|_{\mathbf{H}} \leqslant \|\nabla v\|_{\mathbf{H}_\varepsilon} \quad \forall\, v \in \mathbf{V}_\varepsilon.$$

The construction of such operators is described in [18] in detail.

By Lemma 3.4, the sequence $\widetilde{u}_\varepsilon = P_\varepsilon u_\varepsilon$ is bounded in $\mathbf{V}$ and, consequently, weakly converges in $\mathbf{V}$. Therefore, there exists a function $u \in \mathbf{V}$ such that $\widetilde{u}_\varepsilon \rightharpoonup u$ in $\mathbf{V}$ as $\varepsilon \to 0$. In what follows, we write u_ε instead of $\widetilde{u}_\varepsilon$.

We set $T_r^j = \{x \in \mathbb{R}^n : |x - P_\varepsilon^j| \leqslant r\}$ and consider the auxiliary function v_ε^j solving the problem

$$\begin{aligned}
&\Delta v_\varepsilon^j = 0, \quad x \in T_{\frac{\varepsilon}{4}}^j \setminus \overline{G_\varepsilon^j},\\
&\frac{\partial v_\varepsilon^j}{\partial \nu} + \varepsilon^{n/(2-n)} B_\varepsilon^j(x) v_\varepsilon^j = \varepsilon^{n/(2-n)} \overline{B}_\varepsilon^j(x), \quad x \in \partial G_\varepsilon^j,\\
&v_\varepsilon^j = 0, \quad x \in \partial T_{\frac{\varepsilon}{4}}^j.
\end{aligned} \tag{3.17}$$

It is easy to show that

$$\varepsilon^{n/(2-n)} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} B_\varepsilon^j(x) u_\varepsilon \varphi ds = -\sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\frac{\varepsilon}{4}}^j} \frac{\partial v_\varepsilon^j}{\partial \nu} u_\varepsilon \varphi \, ds.$$

Thus, we have proved that

$$\mathscr{V}_\varepsilon(x) = \begin{cases} v_\varepsilon^j(x), & x \in T_{\frac{\varepsilon}{4}}^j \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon,\\ 0, & x \in \mathbb{R}^n \setminus \overline{T_{\frac{\varepsilon}{4}}^j}. \end{cases} \tag{3.18}$$

As proved in [15], $\|\mathscr{V}_\varepsilon\|^2_{\mathbf{V}_\varepsilon} \leqslant K\varepsilon^2$ and $\widetilde{\mathscr{V}_\varepsilon} \rightharpoonup 0$ weakly in $\mathbf{V}$ and $\widetilde{\mathscr{V}_\varepsilon} \to 0$ strongly in $\mathbf{H}$ as $\varepsilon \to 0$, where $\widetilde{\mathscr{V}_\varepsilon} = P_\varepsilon \mathscr{V}_\varepsilon$. Using [15, Lemmas 4.1 and 4.2], we find

$$\left| \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_{\frac{\varepsilon}{4}}^j} \frac{\partial v_\varepsilon^j}{\partial \nu} h_\varepsilon \, ds + \int_\Omega V(x) h \, dx \right| \to 0, \quad \varepsilon \to 0, \tag{3.19}$$

for $h_\varepsilon, h \in \mathbf{V}$ such that $h_\varepsilon \rightharpoonup h$ in $\mathbf{V}$. Finally, the convergence (3.16) is obtained from (3.19). $\square$

Using (3.11), (3.12), (3.16) and passing to the limit in the equation of the problem (3.7) in the space $D'(\mathbb{R}_+; \mathbf{H}^{-\mathbf{r}})$ as $k \to \infty$, we find that $\overline{u}(x, s)$ is a solution to the problem

$$\begin{aligned}
&\frac{\partial \overline{u}}{\partial t} = \lambda \Delta \overline{u} - \overline{a}(x) f(\overline{u}) - V(x)\,\overline{u} + \overline{g}(x), \quad x \in \Omega,\\
&\overline{u} = 0, \quad x \in \partial\Omega.
\end{aligned} \tag{3.20}$$

Consequently, $\overline{u} \in \overline{\mathscr{K}}$. As was already proved, $u_{\varepsilon_k}(s) \to \overline{u}(s)$ in $\Theta^{\rm loc}$ as $k \to \infty$. Since $u_{\varepsilon_k}(s) \notin \mathscr{O}'(\overline{\mathscr{K}})$, it follows that $\overline{u} \notin \mathscr{O}'(\overline{\mathscr{K}})$ and, consequently, $\overline{u} \notin \overline{\mathscr{K}}$. Thus, we arrive at a contradiction. □

Using the compact inclusions (2.17) and (2.18), we can improve the convergence (3.3).

Corollary 3.1. *For any* $0 < \delta \leqslant 1$ *and* $M > 0$

$$\operatorname{dist}_{\mathbf{L}_2([0,M];\,\mathbf{H}^{1-\delta})}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0, \quad \varepsilon \to 0+, \tag{3.21}$$

$$\operatorname{dist}_{\mathbf{C}([0,M];\,\mathbf{H}^{-\delta})}(\Pi_{0,M}\mathfrak{A}_\varepsilon, \Pi_{0,M}\overline{\mathfrak{A}}) \to 0, \quad \varepsilon \to 0+. \tag{3.22}$$

To prove (3.21) and (3.22), we repeat the proof of Theorem 3.1 with the topology in $\Theta^{\rm loc}$ replaced with that in $\mathbf{L}_2^{\rm loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta})$ or $\mathbf{C}^{\rm loc}(\mathbb{R}_+; \mathbf{H}^{-\delta})$.

Finally, we consider the reaction–diffusion systems for which the uniqueness theorem is true for the Cauchy problem. It suffices to assume that the nonlinear term $f(u)$ in (2.1) satisfies the condition

$$(f(v_1) - f(v_2), v_1 - v_2) \geqslant -C|v_1 - v_2|^2 \quad \forall\, v_1, v_2 \in \mathbb{R}^N \tag{3.23}$$

(cf. [11, 5]). As proved in [11], if (3.23) holds, then the problems (2.1) and (3.2) generate dynamical semigroups in $\mathbf{H}$ possessing global attractors $\mathscr{A}_\varepsilon$ and $\overline{\mathscr{A}}$ that are bounded in the space $\mathbf{V} = \mathbf{H}_0^1(\Omega)$. Moreover, $\mathscr{A}_\varepsilon = \{u(0) \mid u \in \mathfrak{A}_\varepsilon\}$, $\overline{\mathscr{A}} = \{u(0) \mid u \in \overline{\mathfrak{A}}\}$. In this case, (3.22) implies the following assertion.

Corollary 3.2. *Let the assumptions of Theorem* 3.1 *hold. Then* $\operatorname{dist}_{\mathbf{H}^{-\delta}}(\mathscr{A}_\varepsilon, \overline{\mathscr{A}}) \to 0$ *as* $\varepsilon \to 0+$.

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