The Inflation and Perturbation of Nonautonomous Difference Equations and Their Pullback Attractors [1]

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Abstract Nonautonomous difference equations are formulated as difference cocycles driven by an autonomous dynamical system. Pullback attractors are the appropriate generalization of autonomous attractors to cocycles and their existence follows when the difference cocycle has a pullback absorbing set. The effects of perturbing the driving autonomous system of a difference cocycle are considered here, in the weaker sense of perturbations of its input variable in the nonautonomous difference equation and in the stronger sense of the driving system being perturbed to a nearby driving system. The existence of a pullback attractor of the inflated cocycle dynamics and the shadowing of the driving system are important assumptions here.

Keywords Nonautonomous difference equation, Difference cocycle, Pullback attractor, Inflated attractors and dynamics, Driving system, Skew-product system, Total stability

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1 Introduction

In this article we discuss the perturbation and inflation of pullback attractors for nonautonomous difference equations which are formulated as a skew-product system, i.e., consisting of a cocycle mapping in the state space that is driven by an autonomous dynamical system [6]. The existence of a pullback attractor for a setvalued "inflation" of the cocycle dynamics implies the total stability of the pullback attractor of the original system, i.e. the persistence of the pullback attractor under perturbations of the cocycle mapping. However, if the driving system itself is perturbed to a new driving system, then an additional property such as the shadowing of the original driving system needs also to be assumed. These concepts are introduced and developed here.

For simplicity we shall restrict attention to the Euclidean state space $\mathbb{R}^d$, though our results can be extended to more general metric or Banach state spaces with appropriate modifications to assumptions. To describe the proximity and convergence of sets, we recall that the Hausdorff separation $H^*(A,B)$ of nonempty compact subsets A, B of $\mathbb{R}^d$ is defined as

$$H^*(A,B) := \max_{a\in A} \operatorname{dist}(a,B) = \max_{a\in A} \min_{b\in B} \|a-b\|$$

and that $H(A,B) = \max\{H^*(A,B), H^*(B,A)\}$ defines a metric, called the Hausdorff metric, on the space $\mathcal{H}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$. In addition, $B[A,\epsilon]$ denotes the closed neighbourhood of a compact set A with radius ϵ and $B(A,\epsilon)$ denotes the open neighbourhood of a compact set A with radius ϵ, i.e.,

$$B[A,\epsilon] = \{x \in \mathbb{R}^d : \operatorname{dist}(x,A) \le \epsilon\}, \qquad B(A,\epsilon) = \{x \in \mathbb{R}^d : \operatorname{dist}(x,A) < \epsilon\}.$$

2 The autonomous case

Successive iteration of an autonomous difference equation

$$x_{n+1} = f(x_n) \tag{1}$$

generates the forwards solution mapping $\phi : \mathbb{Z}^+ \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$x_n = \phi(n,x_0) = f^n(x_0) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(x_0),$$

which satisfies the initial condition $\phi(0,x_0) = x_0$ and the semigroup property

$$\begin{aligned} \phi(n,\phi(m,x_0)) &= f^n(\phi(m,x_0)) = \\ & f^n \circ f^m(x_0) = f^{n+m}(x_0) = \phi(n+m,x_0) \end{aligned} \tag{2}$$

for all $x_0 \in \mathbb{R}^d$, and integers $n, m \ge 0$. Property (2) says that the solution mapping ϕ forms a semigroup under composition; it is typically only a semigroup rather than a group since the mapping f need not be invertible. Assuming that the mapping f in the difference equation (1) is at least continuous, it follows that the mappings $\phi(n,\cdot)$ are continuous for every $n \in \mathbb{Z}^+$. The solution mapping ϕ then generates a discrete time autonomous semidynamical system on the state space $\mathbb{R}^d$ [1, 15].

A nonempty compact subset A of $\mathbb{R}^d$ is called invariant under ϕ, or ϕ–invariant, if $\phi(n,A) = A$ for all $n \in \mathbb{Z}^+$ or, equivalently, if $f(A) = A$. Simple examples are steady state solutions and periodic solutions; in the first case A consists of a single point, which must thus be a fixed point of the mapping f, and for a solution with period r it consists of a finite set of r distinct points $\{p_1, \ldots, p_r\}$ which are fixed point of the composite mapping f^r (but not for an f^j with j smaller than r). Invariant sets can also be much more complicated, for example fractal sets. Many are the ω–limit sets of some trajectory, i.e. defined by

$$\omega^+(x_0) = \left\{ y \in \mathbb{R}^d \;:\; \exists n_j \to \infty, \quad \phi(n_j, x_0) \to y \right\},$$

which is nonempty, compact and ϕ–invariant when the forwards trajectory $\{\phi(n,x_0);\ n \in \mathbb{Z}^+\}$ is bounded. The asymptotic behaviour of a autonomous semidynamical system is characterized by its ω–limit sets in general, and its attractors and their associated absorbing sets in particular. An attractor is a nonempty ϕ–invariant compact set A^* that attracts all trajectories starting in some neighbourhood $\mathcal{U}$ of A^*, that is with $\omega^+(x_0) \subset A^*$ for all $x_0 \in \mathcal{U}$ or, equivalently, with $\lim_{n\to\infty} \operatorname{dist}(\phi(n,x_0), A^*) = 0$ for all $x_0 \in \mathcal{U}$. A nonempty compact subset A^* of $\mathbb{R}^d$ is a global maximal attractor of the discrete time autonomous semidynamical system ϕ on $\mathbb{R}^d$ if it is ϕ–invariant and attracts bounded sets, i.e.

$$\lim_{n\to\infty} H^*\left(\phi\left(n,D\right), A^*\right) = 0 \quad \text{for any bounded subset } D \subset \mathbb{R}^d. \tag{3}$$

The existence and approximate location of such a maximal attractor follows from that of more easily found absorbing sets, which typically have a convenient simpler shape such as a ball or ellipsoid. A nonempty compact subset B of $\mathbb{R}^d$ is called an absorbing set of a discrete time autonomous semidynamical system ϕ on $\mathbb{R}^d$ if for every bounded subset D of $\mathbb{R}^d$ there exists a $N_D \in \mathbb{Z}^+$ such that $\phi(n,D) \subset B$ for all $n \geq N_D$ in $\mathbb{Z}^+$.

Theorem 2.1 *Suppose that a discrete time autonomous semidynamical system ϕ on $\mathbb{R}^d$ generated by a continuous mapping f has an absorbing set B. Then ϕ has a unique global maximal attractor $A^* \subset B$ given by*

$$A^* = \bigcap_{m\geq 0} \overline{\bigcup_{n\geq m} \phi(n,B)}. \tag{4}$$

A maximal attractor (but not always a local or point-attracting attractor, see [1]) is in fact uniformly Lyapunov asymptotically stable in that it is also

Lyapunov stable, i.e. for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\operatorname{dist}(\phi(n, x_0), A^*) < \epsilon \quad \text{for all} \quad n \in \mathbb{Z}^+ \quad \text{whenever} \quad \operatorname{dist}(x_0, A^*) < \delta,$$

as well as attracting bounded sets as in (3). Uniformly Lyapunov asymptotically stable sets can be characterized by a Lyapunov functions [1, 15], which can be used to establish the existence of an absorbing set and hence that of a nearby maximal attractor in a perturbed autonomous system, i.e., if the semidynamical system generated by the autonomous difference equation (1) has a global maximal attractor A, then for any continuous $g : \mathbb{R}^d \to \mathbb{R}^d$ with $\sup_{x\in\mathbb{R}^d} \|g(x)\| \leq 1$ and $\epsilon > 0$ small enough, the semidynamical system generated by the perturbed autonomous difference equation

$$x_{n+1} = f(x_n) + \epsilon\, g(x_n)$$

has a maximal autonomous attractor $A^{\epsilon,g}$ which converges upper semicontinuously to A in the sense that

$$H^*(A^{\epsilon,g}, A) \to 0 \quad \text{as } \epsilon \to 0+ .$$

This property is often known as <u>total stability</u>. Similarly, for sufficiently small $\epsilon > 0$, the <u>setvalued</u> semidynamical system Φ^ϵ generated by the “inflated” difference inclusion

$$x_{n+1} \in F^\epsilon(x_n),$$

where F^ϵ is defined by $F^\epsilon(x) := B[f(x), \epsilon] = f(x) + B[0, \epsilon]$, has a maximal attractor A^ϵ, which converges continuously to A (since it contains A as well as all above $A^{\epsilon,g}$), i.e., $H(A^\epsilon, A) \to 0$ as $\epsilon \to 0+$.

In fact, in the autonomous case under discussion, the uniform asymptotic stability of the maximal attractor A, its total stability, and the existence of an “inflated” attractor A^ϵ for sufficiently small $\epsilon > 0$ all imply each other.

3 Nonautonomous difference equations

Difference equations of the form

$$x_{n+1} = f_n(x_n), \tag{5}$$

in which the mappings on the right hand side are allowed to vary with the time instant n are called <u>nonautonomous</u> difference equations. Such nonautonomous difference equations arise quite naturally in many different ways. The mappings f_n in (5) may of course vary completely arbitrarily, but often there is some relationship between them or some regularity in the way in which they are chosen. For example, the mappings may all be the same as in the very special autonomous subcase or they may vary periodically within, or be chosen irregularly from, a finite family $\{g_1, \cdots, g_r\}$, in which case (5) can be rewritten as

$$x_{n+1} = g_{k_n}(x_n), \tag{6}$$

where the $k_n \in \{1, \cdots, r\}$ and $f_n = g_{k_n}$. More generally, a difference equation may involve a parameter $q \in Q$ which varies in time by choice or randomly, giving rise to the nonautonomous difference equation

$$x_{n+1} = f\left(x_n, q_n\right), \tag{7}$$

so $f_n(x) = f(x, q_n)$ here for the prescribed choice of $q_n \in Q$. Another example, the difference equation (5) may represent a variable time–step discretization method for an autonomous differential equation. See [6, 11, 12, 14] for more details and examples.

The nonautonomous difference equation (5) has the forwards solution mapping ϕ defined through iteration by

$$x_n = \phi(n, n_0; x_{n_0}) = f_{n-1} \circ \cdots \circ f_{n_0}(x_{n_0}) \qquad \text{for all} \quad n > n_0, \tag{8}$$

for the initial value $\phi(n_0, n_0; x_{n_0}) = x_{n_0}$ at time $n = n_0 \in \mathbb{Z}$.

As in the autonomous case, the long–term or asymptotic behaviour and related concepts such as asymptotic stability, limit sets and attractors are of major interest. However, the general nonautonomous case differs crucially from the autonomous case in that the starting time n_0 is just as important as the time that has elapsed since starting, i.e. $n - n_0$, and hence many of the concepts that have been developed and extensively investigated for autonomous dynamical systems in general and autonomous difference equations in particular are either too restrictive or no longer valid or meaningful. Moreover, the above formalism is often too general to allow useful assertions to be made about the dynamics of the nonautonomous system as it does not say anything explicitly about how the solution mapping changes in time. Such information can be incorporated through a driving system in the skew-product formalism of a nonautonomous dynamical system.

3.1 Skew-product formalism

Let (P, d_P) be a metric space, which we call the parameter set, and let $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ be a group of continuous mappings from P onto itself, i.e. with $\theta_0(p) = p$ and $\theta_n \circ \theta_m(p) = \theta_{n+m}(p)$ for all $p \in P$ and n, $m \in \mathbb{Z}$ (henceforth we write $\theta_n p$ instead of $\theta_n(p)$). Essentially, θ is a discrete time autonomous dynamical system on P that models the driving mechanism for the change in the mappings f_n on the right hand side of the nonautonomous difference equation (5), which we now write as

$$x_{n+1} = f\left(\theta_n p, x_n\right) \tag{9}$$

for $n \in \mathbb{Z}$, where $f : P \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous mapping. The corresponding solution mapping $\phi : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ is now defined by

$$\phi(0, p, x) := x, \qquad \phi(j, p, x) := f\left(\theta_{j-1}p, \cdot\right) \circ \cdots \circ f\left(p, x\right), \qquad j \in \mathbb{N},$$

for each $p \in P$ and $x \in \mathbb{R}^d$. The mapping ϕ satisfies the initial value property $\phi(0,p,x) := x$ and the cocycle property with respect to the driving system θ on P, i.e.

$$\phi(i+j,p,x) := \phi\left(i,\theta_j p,\phi\left(j,p,x\right)\right) \tag{10}$$

for all i, $j \in \mathbb{Z}^+$, $p \in P$ and $x \in \mathbb{R}^d$, and will be called a discrete time or difference cocycle with respect to the driving system θ on P. Note that each of the mappings $\phi(j,\cdot,\cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ here is continuous.

Remark 3.1 If ϕ be a difference cocycle on $\mathbb{R}^d$ with respect to a group $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ of mappings of metric space P into itself. Then the mapping $\Pi : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to P \times \mathbb{R}^d$ defined by

$$\Pi(j,p,x) := (\theta_j p, \phi(j,p,x))$$

for all $j \in \mathbb{Z}^+$, $(p,x) \in P \times \mathbb{R}^d$ forms an autonomous semidynamical system on the state space $P \times \mathbb{R}^d$, i.e. the set of mappings $\{\Pi(j,\cdot,\cdot)\}_{j\in\mathbb{Z}^+}$ of $P \times \mathbb{R}^d$ into itself is a semigroup, thus a discrete time autonomous semidynamical system, which is called a discrete time skew-product system [2, 15].

The above examples can be reformulated in the skew-product formalism with appropriate choices of parameter space P and θ. The nonautonomous difference equation (5) with continuous mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$ generates a difference cocycle ϕ over the parameter set $P = \mathbb{Z}$ with respect to the group of left shift mappings $\theta_j := \theta^j$ for $j \in \mathbb{Z}$, where $\theta n := n+1$ for $n \in \mathbb{Z}$. Here ϕ is defined by

$$\phi(0,n,x) := x \quad \text{and} \quad \phi(j,n,x) := f_{n+j-1} \circ \cdots \circ f_n(x), \qquad j \in \mathbb{N},$$

for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}^d$. The mappings $\phi(j,n,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ here are all continuous. (The autonomous case can be considered as a difference cocycle with respect to a singleton parameter set $P = \{p_0\}$ with θ consisting just of the identity mapping on P).

While $\mathbb{Z}$ appears to be the natural choice for the parameter set above, in the following example the use of sequence spaces is more advantageous as such spaces are often compact. As will be seen in Theorem 3.2, stronger assertions can then be made about the dynamical behaviour of the difference cocycle.

The nonautonomous difference equation (6) with continuous mappings $g_k : \mathbb{R}^d \to \mathbb{R}^d$ for $k \in \{1,\cdots,r\}$ generates a difference cocycle over the parameter set $P = \{1,\cdots,r\}^{\mathbb{Z}}$ of bi–infinite sequences $p = \{k_n, n \in \mathbb{Z}\}$ with $k_n \in \{1,\cdots,r\}$ with respect to the group of left shift operators $\theta_n := \theta^n$ for $n \in \mathbb{Z}$, where $\theta\{k_n, n \in \mathbb{Z}\} = \{k_{n+1}, n \in \mathbb{Z}\}$. The mapping ϕ defined by

$$\phi(0,p,x) := x \quad \text{and} \quad \phi(j,p,x) := g_{k_{j-1}} \circ \cdots \circ g_{k_0}(x), \qquad j \in \mathbb{N},$$

for all $x \in \mathbb{R}^d$, where $p = \{k_n, n \in \mathbb{Z}\}$, is a difference cocycle. Note that the parameter space $P = \{1, \cdots, r\}^{\mathbb{Z}}$ here is a compact metric space with the metric

$$d(p, p') = \sum_{n=-\infty}^{\infty} (r+1)^{-|n|} |k_n - k'_n| .$$

In addition, the mappings $\theta_n : P \to P$ and $\phi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ here are all continuous.

3.2 Pullback attractors for difference cocycles

The concept of an autonomous maximal attractor for the discrete time skew-product system is not always appropriate as the cocycle dynamics in the state space $\mathbb{R}^d$ are often of prime importance, with the driving system dynamics in the space P being of somewhat lesser direct interest. The concept of pullback attractor provides a useful analogue of an attractor for the nonautonomous cocycle dynamics. See [4, 6, 11, 12, 14].

A family $\widehat{A} = \{A_p : p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a pullback attractor of a difference cocycle ϕ on $\mathbb{R}^d$ if it is $\phi-$ invariant, i.e., $\phi(j, p, A_p) = A_{\theta_j p}$ for all $j \in \mathbb{Z}^+$, and pullback attracts bounded sets, i.e.

$$H^* \left(\phi(j, \theta_{-j} p, D), A_p\right) = 0 \qquad j \to \infty \tag{11}$$

for all $p \in P$ and all bounded subsets D of $\mathbb{R}^d$.

The pullback absorbing sets, in general, now depend on the parameter too. A family $\widehat{B} = \{B_p : p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called a pullback absorbing set family for a difference cocycle ϕ on $\mathbb{R}^d$ if for each $p \in P$ and every bounded subset D of $\mathbb{R}^d$ there exists an $N_{p,D} \in \mathbb{Z}^+$ such that $\phi(j, \theta_{-j} p, D) \subseteq B_p$ for all $j \geq N_{p,D}$ and $p \in P$. If $N_{p,D}$ is independent of p, then $\widehat{B}$ is said to be uniformly absorbing. A proof of the following theorem can be found in [11] (see also [12]).

Theorem 3.2 *Suppose that a difference cocycle ϕ with $\phi(j, p, \cdot) : \mathbb{R}^d \to \mathbb{R}^d$ continuous for each $j \in \mathbb{Z}^+$ and $p \in P$ has a pullback absorbing set family $\widehat{B}$ $= \{B_p : p \in P\}$. Then there exists a pullback attractor $\widehat{A} = \{A_p : p \in P\}$ with component sets determined uniquely by*

$$A_p = \bigcap_{n \geq 0} \overline{\bigcup_{j \geq n} \phi\left(j, \theta_{-j} p, B_{\theta_{-j} p)}\right)}. \tag{12}$$

If, in addition, P is a compact metric space, the θ_n are bijective and continuous, the mappings $\phi(j, \cdot, \cdot) : P \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous for all $j \in \mathbb{Z}^+$, and $\widehat{B}$ is uniformly absorbing, then

$$\lim_{n \to \infty} \sup_{p \in P} H^* \left(\phi(n, p, D), A(P)\right) = 0 \tag{13}$$

for any bounded subset D of $\mathbb{R}^d$, where $A(P) := \overline{\bigcup_{p \in P} A_p}$.

4 Inflated difference cocycles and pullback attractors

The "ϵ–inflation" of the of the nonautonomous difference equation (9) leads to the nonautonomous difference inclusion

$$x_{n+1} \in F^{\epsilon}(\theta_n p, x_n) \tag{14}$$

driven by the autonomous dynamical system $\theta = \{\theta_n\}_{n\in\mathbb{Z}}$ on P. The set

$$F^{\epsilon}(p,x) := B[f(p,x),\epsilon] = f(p,x) + B[0,\epsilon]$$

is compact and convex, and the setvalued mapping $(\epsilon,p,x) \mapsto F^{\epsilon}(p,x)$ is continuous in the variables (ϵ,p,x). The difference inclusion (14) thus generates a compact setvalued cocycle mapping $\Phi^{\epsilon}(n,p,x)$, which is continuous in the variables (ϵ,p,x). Φ^{ϵ} will be called ϵ–inflated difference cocycle of the single-valued cocycle ϕ of system (9). Note that $\phi(n,p,x) \in \Phi^{\epsilon}(n,p,x)$ for all n, p, x and $\epsilon > 0$.

Below we also consider the ϵ–internal inflation of a nonautonomous difference equation (9), which is a difference inclusion like (14) except now the set $F^{\epsilon}(p,x)$ is defined as

$$F^{\epsilon}(p,x) := f(B[0,\epsilon],x) \equiv \bigcup_{d_P(q,p)\le\epsilon} f(q,x).$$

The corresponding setvalued difference cocycle will be called the ϵ–internally inflated difference cocycle. See [8, 9, 14].

4.1 Inflated pullback attractors

Pullback attractors for a setvalued ϵ-inflated cocycle Φ^{ϵ} are defined analogously to the single-valued case. A family $\widehat{A}^{\epsilon} = \{A^{\epsilon}_p : p \in P\}$ of nonempty compact subsets of $\mathbb{R}^d$ is called pullback attractor of Φ^{ϵ}, or an ϵ–inflated pullback attractor of ϕ, if it is Φ^{ϵ}–invariant, i.e.,

$$\Phi^{\epsilon}(j,p,A^{\epsilon}_p) = A^{\epsilon}_{\theta_j p} \qquad \text{for all} \quad j \in \mathbb{Z}^+, p \in P,$$

and if it pullback attracts nonempty bounded subsets of $\mathbb{R}^d$, i.e.,

$$\lim_{j\to\infty} H^*\left(\Phi^{\epsilon}(j,\theta_{-j}p,D), A^{\epsilon}_p\right) = 0$$

for all $p \in P$ and nonempty bounded subsets D of $\mathbb{R}^d$.

The existence of an ϵ–inflated pullback attractor follows from that of a corresponding pullback absorbing family as in the single-valued case. However, it need not follow from the existence of a pullback attractor for the associated single-valued cocycle. Similarly, the pullback attractor (of the associated single-valued cocycle) is generally not totally stable, but the existence of an inflated pullback attractor for some $\epsilon > 0$ does imply totally stable.

The following theorem shows that if an ϵ_0–inflated pullback attractor exists for a particular value $\epsilon_0 > 0$, then an ϵ–inflated pullback attractor exists for all smaller values of ϵ, including $\epsilon = 0$, and are nested as ϵ decreases. The result also applies to internally inflated difference cocycles. The proof follows directly from definitions.

Theorem 4.1 *Suppose for some $\epsilon_0 > 0$ that the cocycle ϕ has an ϵ_0–inflated attractor $\widehat{A}^{\epsilon_0} = \{A_p^{\epsilon_0} : p \in P\}$. Then the cocycle ϕ has an ϵ–inflated attractor $\widehat{A}^{\epsilon} = \{A_p^{\epsilon} : p \in P\}$ for every $\epsilon \in [0, \epsilon_0]$ and these are related through*

$$A_p^{\epsilon} \subset A_p^{\epsilon'}, \qquad A_p^{\epsilon} = \bigcap_{\epsilon < \epsilon'} A_p^{\epsilon'}, \tag{15}$$

for any $0 \leq \epsilon < \epsilon' \leq \epsilon_0$ and each $p \in P$.

5 Perturbation of pullback attractors

There are various different ways in which a nonautonomous difference equation (9) can be perturbed. The most obvious way, given our remark on the importance of the cocycle dynamics, is by a direct perturbation to the mapping f in (9) resulting in a perturbed difference equation

$$x_{n+1} = f_{\epsilon}\left(\theta_n p, x_n\right), \tag{16}$$

where $f_{\epsilon} : P \times \mathbb{R}^d \to \mathbb{R}^d$ is a continuous mapping and the corresponding solution mapping $\phi_{\epsilon} : \mathbb{Z}^+ \times P \times \mathbb{R}^d \to \mathbb{R}^d$ is a difference cocycle with respect to the same driving system as in the unperturbed difference equation (9). Suppose that $f_{\epsilon}(p, x) \subset f(p, x) + B[0, \epsilon]$ for all $x \in \mathbb{R}^d$ and $p \in P$, i.e., if $\sup_{(p,x) \in P \times \mathbb{R}^d} \|f_{\epsilon}(x, p) - f(x, p)\| \leq \epsilon$, and suppose that the unperturbed difference equation (9) has an ϵ_0–inflated pullback attractor $\widehat{A}^{\epsilon_0} = \{A_p^{\epsilon_0} : p \in P\}$ for some $\epsilon_0 \geq \epsilon$. Hence (9) itself has a pullback attractor $\widehat{A} = \{A_p : p \in P\}$ with $A_p \subset A_p^{\epsilon_0}$ for each $p \in P$ and the perturbed difference equation (16) has a pullback attractor $\widehat{A}^{\epsilon,pert} = \{A_p^{\epsilon,pert} : p \in P\}$ with $A_p^{\epsilon,pert} \subset A_p^{\epsilon_0}$ for each $p \in P$. Moreover the upper semi continuous convergence of component subsets holds, i.e.

$$\lim_{\epsilon \to 0} H^*\left(A_p^{\epsilon,pert}, A_p\right) = 0$$

for each $p \in P$.

Another type of perturbation is through the driving system and thus indirectly on the driven cocycle dynamics: This might occur in the "weak" sense of perturbations to the p–variable in the f mapping, i.e. resulting in a perturbed difference equation

$$x_{n+1} = f\left(q_n, x_n\right)$$

with the same mapping f as before and an arbitrary sequence $\{q_n : n \in \mathbb{Z}\}$ with $d_P(q_n, \theta_n p) \leq \epsilon$ for all $n \in \mathbb{Z}$ with some sufficiently small ϵ. (Think of a round–off error or digitization error when inputing the driving system values $p_n = \theta_n p$ into the difference equation). This situation is covered in that just discussed involving the inflation of the mapping f in the original nonautonomous difference equation (9).

Alternatively, the driving system θ on P itself might be perturbed, resulting in a new driving system θ^ϵ on P, and hence the perturbed difference equation

$$x_{n+1} = f\left(\theta_n^\epsilon p, x_n\right) \tag{17}$$

with the same mapping f as before. This thus perturbs the effect of the driving system in a “strong” sense. It is reasonable to assume that $d_\infty\left(\theta^\epsilon, \theta\right) := \sup_{p\in P} d_P(\theta^\epsilon p, \theta p) \to 0$ as $\epsilon \to 0$ and hence, by continuity of composite functions, that $d_\infty(\theta_n^\epsilon, \theta_n) \to 0$ as $\epsilon \to 0$ for each $n \in \mathbb{Z}$. If this convergence were uniform in $n \in \mathbb{Z}$, then we would be in the previous situation with $q_n := \theta_n^\epsilon p$. But uniformity is too strong an assumption in most instances.

5.1 Perturbation of a shadowing driving system

An autonomous dynamical system θ on P is said to have the shadowing property if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that for any bi-infinite sequence $\{q_n, n \in \mathbb{Z}\}$ in P satisfying

$$d_P\left(q_{n+1}, \theta q_n\right) < \delta, \qquad \text{for almost all } n \in \mathbb{Z}, \tag{18}$$

there exists an exact solution $\{p_n, n \in \mathbb{Z}\}$ of θ, i.e. with $p_{n+1} = \theta p_n$ for all $n \in \mathbb{Z}$, such that

$$d_P\left(p_n, q_n\right) < \epsilon \qquad \text{for all} \quad n \in \mathbb{Z}. \tag{19}$$

holds.

In our context we consider the sequence $\{q_n, n \in \mathbb{Z}\}$ in P to be a solution of some perturbed driving system θ^ϵ satisfying $d_\infty\left(\theta^\epsilon, \theta\right) \to 0$ as $\epsilon \to 0$, i.e., with $q_{n+1} = \theta^\epsilon q_n$ for all $n \in \mathbb{Z}$.

Let ϕ^ϵ be the difference cocycle generated by (17) with the perturbed driving system θ^ϵ. We will show that ϕ^ϵ has a pullback attractor when the original difference cocycle has an inflated pullback attractor and a shadowing driven system. The following Lemma is needed in the proof.

Lemma 5.1 *Suppose that the driving system θ has the shadowing property and that the perturbed driving system θ^ϵ satisfies $d_\infty\left(\theta^\epsilon, \theta\right) < \delta$ for $\delta = \delta(\epsilon)$ as in the shadowing property. Then, for any nonempty compact subset D of $\mathbb{R}^d$,*

$$\phi^\epsilon(n, D, q) \subseteq \bigcup_{d_P(p,q)\leq\epsilon} \Phi^\epsilon(n, D, p), \qquad n \in \mathbb{Z}^+, \tag{20}$$

and

$$\phi^\epsilon(n, D, \theta^\epsilon_{-n} q) \subseteq \bigcup_{d_P(p,q)\le\epsilon} \Phi^\epsilon(n, D, \theta_{-n} p), \qquad n \in \mathbb{Z}^+, \tag{21}$$

where Φ^ϵ is the internally ϵ-inflated cocycle solution mapping of the original difference cocycle ϕ.

Proof. Fix an $\epsilon > 0$ and $q_0 \in P$. Let $x_n := \phi^\epsilon(n, x_0, q_0)$ denote the solution of the driven equation of

$$x_{n+1} = f(\theta^\epsilon_n q_0, x_n), \qquad x(0) = x_0,$$

with the perturbed driving system θ^ϵ. Then this solution satisfies the internally inflated difference inclusion

$$x_{n+1} \in F_\epsilon(\theta_n p_0, x_n), \qquad x(0) = x_0,$$

where p_0 corresponds to q_0 under the shadowing property since

$$d_P\left(q_{n+1}, \theta q_n\right) \le d_P\left(q_{n+1}, \theta^\epsilon q_n\right) + d_P\left(\theta^\epsilon q_n, \theta q_n\right) \le d_\infty\left(\theta^\epsilon, \theta\right) < \delta(\epsilon)$$

and thus $d_P\left(\theta_n p_0, \theta^\epsilon_n q_0\right) < \epsilon$ for all $n \in \mathbb{Z}$. It then follows from the definition of the solution mapping for the internally inflated system that

$$\phi^\epsilon(n, x_0, q_0) = x_n \in \Phi^\epsilon(n, x_0, p_0)$$

and the required forwards inclusion (20) is an immediate consequence of the fact that $d_P(p_0, q_0) < \epsilon$. The backwards inclusion (21) is proved analogously. $\square$

Remark 5.2 Although quite simple, Lemma 5.1 is important because it clearly shows the differing influences of the weak and strong forms of perturbations of the driving system on the behaviour of the system. The former manifests itself through the second ϵ in (20) and (21), i.e., in the Φ^ϵ term, and the latter through the first ϵ in (20) and (21), i.e., under the set union symbol. $\square$

Theorem 5.3 *Suppose that the original system 16, that the driven cocycle system ϕ possesses a uniform internally ϵ–inflated attractor $\widehat{A}^\epsilon := \{A^\epsilon_p : p \in P\}$ for each $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$, and that the driving system θ on P has the shadowing property. In addition, suppose that the perturbed driving system θ^ϵ satisfies $d_\infty(\theta^\epsilon, \theta) \le \delta$ for $\delta = \delta(\epsilon)$ of the shadowing property for some $\epsilon \in (0, \epsilon_0]$.*

Then the perturbed difference cocycle ϕ^ϵ has a pullback attractor $\widehat{A}^{\epsilon,pert} := \{A^{\epsilon,pert}_q : q \in P\}$ such that

$$A^{\epsilon,pert}_q \subseteq \bigcup \left\{A^\epsilon_p : d_P(p,q) \le \epsilon\right\}, \qquad q \in P. \tag{22}$$

Proof of Theorem 5.3. Let $\epsilon \in (0, \epsilon_0]$ correspond to the $\delta(\epsilon)$ of the shadowing property. Then by the uniformity assumption on the internally ϵ-inflated attractor $\widehat{A}^\epsilon$, for any $\sigma > 0$ and nonempty compact subset D of $\mathbb{R}^d$ there exists a $N = N(\epsilon, \sigma, D) \geq 0$ such that

$$\Phi^\epsilon(n, D, p) \subset B\left(A^\epsilon_{\theta_n p}, \sigma\right) \qquad \text{for all } \; n \geq N(\epsilon, \sigma, D), \; p \in P.$$

It follows immediately from this inclusion and from Lemma 5.1 that

$$\phi^\epsilon(n, D, q) \subseteq B\left(\bigcup \left\{A^\epsilon_{\theta_n p} : d_P(p, q) \leq \epsilon\right\}, \sigma\right) \tag{23}$$

and

$$\phi^\epsilon(n, D, \theta^\epsilon_{-n} q) \subseteq B\left(\bigcup \left\{A^\epsilon_p : d_P(p, q) \leq \epsilon\right\}, \sigma\right) \tag{24}$$

for all $n \geq N(\epsilon, \sigma, D)$ and $q \in P$. Now fix an arbitrary $\sigma > 0$ and define

$$B_\sigma := B\left[\bigcup \left\{A^{\epsilon_0}_p : p \in P\right\}, \sigma\right],$$

where $\widehat{A}^{\epsilon_0} := \{A^{\epsilon_0}_p : p \in P\}$ is the uniform internally ϵ_0-inflated attractor. This set B_σ is compact since the set P and the sets $A^{\epsilon_0}_p$ are compact and the mapping $p \mapsto A^{\epsilon_0}_p$ is upper semicontinuous. Moreover, by Theorem 4.1, B_σ contains any set A^ϵ_p with $\epsilon \in [0, \epsilon_0)$ and $p \in P$. Hence by (23) and (24), respectively,

$$\phi^\epsilon(n, D, q) \subseteq B\left(\bigcup \left\{A^\epsilon_{\theta_n p} : d_P(p, q) \leq \epsilon\right\}, \sigma\right) \subseteq B_\sigma$$

and

$$\phi^\epsilon(n, D, \theta^\epsilon_{-n} q) \subseteq B\left(\bigcup \left\{A^\epsilon_p : d_P(p, q) \leq \epsilon\right\}, \sigma\right) \subseteq B_\sigma \tag{25}$$

for all $n \geq N(\epsilon, \sigma, D)$ and $q \in P$.

The existence of an attractor (forwards and pullback) $\widehat{A}^{\epsilon,pert} := \{A^{\epsilon,pert}_p : p \in P\}$ of the perturbed difference cocycle ϕ^ϵ follows from the above inclusions by Theorem 3.2; see also Theorems 2.8 or 2.9 of [2]. In particular, for $(\phi^\epsilon, \theta^\epsilon)$ instead of (ϕ, θ) with the pullback absorbing system consisting of the same subset B_σ gives

$$A^{\epsilon,pert}_q = \bigcap_{\tau \geq 0} \overline{\bigcup_{n \geq \tau} \phi^\epsilon(n, B_\sigma, \theta^\epsilon_{-n} q)} \subseteq B\left(\bigcup \left\{A^\epsilon_p : d_P(p, q) \leq \epsilon\right\}, \sigma\right), \qquad q \in P,$$

where the set inclusion follows from (25) with $D = B_\sigma$. The desired inclusion (22) then follows since $\sigma > 0$ can be chosen arbitrarily small. □

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